EXAMEN ECUACIONES DIFERENCIALES

NOMBRE:____

1. Resuelva la ecuación diferencial ordinaria, de primer orden y su condición inicial, determine solución complementaria y particular (36pts)

$$\frac{dy}{dt} - \frac{2t}{2-t^2}y = e^t \qquad\qquad y(2) = 4$$

Ecuación Homogénea y Solución Complementaria

$$\frac{dy_c}{dt} - \frac{2t}{2-t^2}y_c = 0$$

$$\frac{dy_c}{dt} = \frac{2t}{2-t^2}y_c$$

$$\frac{dy_c}{y_c} = \frac{2t}{2-t^2}dt$$

$$\int \frac{dy_c}{y_c} = \int \frac{2t}{2-t^2}dt$$

$$ln|y_c| = -ln|2-t^2| + C$$

$$y_c(t) = \frac{C}{2-t^2}$$

Solución Particular

$$y_p(t) = u(t)y_1(t)$$

$$y_1(t) = \frac{1}{2-t^2}$$

$$u(t) = \int \frac{e^t}{\frac{1}{2-t^2}}\ dt$$

$$u(t) = \int e^t(2-t^2)\ dt$$

$$u(t) = e^t(2t - t^2)$$

$$y_p(t) = u(t)y_1(t)$$

$$y_p(t) = \frac{e^t(2t - t^2)}{2 - t^2}$$

$$y(t) = y_c(t) + y_p(t) = \frac{C}{2 - t^2} + \frac{e^t(2t - t^2)}{2 - t^2}$$

Condición Inicial

$$y(2) = \frac{C}{2 - 2^2} + \frac{e^2(2 \times 2 - 2^2)}{2 - 2^2} = 4$$

$$y(2) = \frac{C}{2 - 4} + \frac{e^2(4 - 4)}{2 - 4} = 4$$

$$-\frac{C}{2} = 4$$

$$C = -8$$

$$y(t) = y_p(t) + y_p(t) = -\frac{8}{2 - t^2} + \frac{e^t(2t - t^2)}{2 - t^2}$$

2. Resuelva la ecuación diferencial ordinaria de segundo orden y sus condiciones iniciales determine solución complementaria y particular (36pts)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = \sum_{n=-\infty}^{\infty}\left[\frac{sinh(\pi)}{\pi(1+n^2)}(-1)^n(1+jn)\right]e^{jnt} \qquad y(0) = -1 \qquad \frac{dy}{dt}(0) = 1$$

Ecuación Homogénea y Solución Complementaria

$$\frac{d^2y_c}{dt^2} + 2\frac{dy_c}{dt} + 10y_c = 0$$

Polinomio característico

$$s^2 + 2s + 10 = 0$$

Raíces

$$s_1 = \quad -1 + j3 \quad s_2 = -1 - j3$$

$$y_c(t) = e^{-t}[C_1 cos(3t) + C_2 sin(3t)]$$

Solución Particular

$$\frac{d^2 y_p}{dt^2} + 2\frac{dy_p}{dt} + 10y_p = \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)}{\pi(1+n^2)}(-1)^n(1+jn)\right] e^{jnt}$$

Asumimos

$$y_p(t) = \sum_{n=-\infty}^{\infty} y_{pn}(t)$$

Entonces

$$\frac{d^2 y_{pn}}{dt^2} + 2\frac{dy_{pn}}{dt} + 10y_{pn} = \left[\frac{sinh(\pi)}{\pi(1+n^2)}(-1)^n(1+jn)\right] e^{jnt}$$

Podemos asumir

$$y_{pn}(t) = A_n e^{jnt}$$

$$\frac{dy_{pn}}{dt}(t) = jnA_n e^{jnt}$$

$$\frac{d^2 y_{pn}}{dt^2}(t) = -n^2 A_n e^{jnt}$$

Reemplazamos

$$[-n^2 + 2jn + 10]A_n e^{jnt} = \left[\frac{sinh(\pi)}{\pi(1+n^2)}(-1)^n(1+jn)\right] e^{jnt}$$

$$A_n = \frac{sinh(\pi)(-1)^n(1+jn)}{-n^2 + 2jn + 10}$$

$$y_p(t) = \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right] e^{jnt}$$

$$y(t) = y_p(t) + y_p(t) = e^{-t}[C_1 cos(3t) + C_2 sin(3t)] + \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right] e^{jnt}$$

$$y(t) = e^{-t}[C_1 cos(3t) + C_2 sin(3t)] + \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right] e^{jnt}$$

$$\frac{dy}{dt}(t) = -e^{-t}[C_1 cos(3t) + C_2 sin(3t)] + 3e^{-t}[-C_1 sin(3t) + C_2 cos(3t)] + \sum_{n=-\infty}^{\infty} \left[jn\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right] e^{jnt}$$

$$y(0) = C_1 + C_2 = -1 - \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right]$$

$$\frac{dy}{dt}(0) = -C_1 + 3C_2 = 1 - \sum_{n=-\infty}^{\infty} \left[jn\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right]$$

Sistema de Ecuaciones

$$C_1 + C_2 = -1 - \mathcal{A}$$
$$-C_1 + 3C_2 = 1 - \mathcal{B}$$

$$C_1 = \frac{3}{4}\mathcal{A} - \frac{1}{4}\mathcal{B} - \frac{3}{4}$$

$$C_2 = \frac{1}{4}\mathcal{A} + \frac{1}{4}\mathcal{B} - \frac{1}{4}$$

$$y_c(t) + y_p(t) = e^{-t}\left[\left(\frac{3}{4}\mathcal{A} - \frac{1}{4}\mathcal{B} - \frac{3}{4}\right) cos(3t) + \left(\frac{1}{4}\mathcal{A} + \frac{1}{4}\mathcal{B} - \frac{1}{4}\right) sin(3t)\right] + \sum_{n=-\infty}^{\infty} \left[\frac{sinh(\pi)(-1)^n(1+jn)}{-n^2+2jn+10}\right] e^{jnt}$$

3. Resuelva la ecuación de onda con sus condiciones de contorno y sus condiciones iniciales

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + b\frac{\partial p}{\partial t}$$

$$p(0,t) = 0 \qquad\qquad \frac{\partial p}{\partial x}(L,t) = 0$$

$$p(x,0) = \begin{cases} 0 & 0 \le x < \frac{L}{4} - a \\ a^2 - x^2 & \frac{L}{4} - a \le x \le \frac{L}{4} + a \\ 0 & \frac{L}{4} + a < x \le L \end{cases} \qquad\qquad \frac{\partial p}{\partial x}(x,0) = 0$$

Usaremos el método de separación de variables, el cual asume que la solución es de la forma

$$p(x,t) = X(x)T(t)$$

Lo integrar a la ecuación de onda

$$\frac{\partial^2[X(x)T(t)]}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2[X(x)T(t)]}{\partial t^2} + b\frac{\partial[X(x)T(t)]}{\partial t}$$

$$T(t)\frac{d^2X(x)}{dx^2} = X(x)\frac{1}{c^2}\frac{d^2T(t)}{dt^2} + bX(x)\frac{dT(t)}{dt}$$

Se multiplica a ambos lados por el inverso de $X(x)T(t)$

$$T(t)\frac{d^2X(x)}{dx^2} = X(x)\frac{1}{c^2}\frac{d^2T(t)}{dt^2} + bX(x)\frac{dT(t)}{dt} \rightarrow \frac{1}{X(x)T(t)}$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{c^2}\frac{1}{T(t)}\left[\frac{d^2T(t)}{dt^2} + b\frac{dT(t)}{dt}\right]$$

La primera parte solamente depende de x y la segunda de t

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k^2$$

$$-k^2 = \frac{1}{c^2}\frac{1}{T(t)}\left[\frac{d^2T(t)}{dt^2} + b\frac{dT(t)}{dt}\right]$$

Entonces

$$\frac{1}{X(x)}\frac{\partial^2X(x)}{\partial x^2} = \frac{1}{c^2}\frac{1}{T(t)}\left[\frac{d^2T(t)}{dt^2} + b\frac{dT(t)}{dt}\right] = -k^2$$

La constante es la misma para ambos lados de la ecuación y se designa de manera conveniente $-k^2$. Como resultado podemos separar y desacoplar las ecuaciones y sus condiciones de contorno

$$\frac{1}{X(x)}\frac{dX(x)}{dx^2} = -k^2$$

$$X(0) = 0 \qquad \frac{dX}{dx}(L) = 0$$

Necesitamos arreglar la ecuación de forma más conveniente

$$\frac{dX(x)}{dx^2} = -k^2 X(x)$$

$$\frac{dX(x)}{dx^2} + k^2 X(x) = 0$$

Solución

$$X(x) = C_1 cos(kx) + C_2 sin(kx)$$

Primera condición de contorno

$$X(x) = C_1 cos(k \times 0) + C_2 sin(k \times 0) = C_1 = 0$$

Entonces tenemos

$$X(x) = C_2 sin(kx)$$

Derivamos

$$\frac{dX}{dx}(x) = kC_2 cos(kx)$$

Evaluamos

$$\frac{dX}{dx}(L) = kC_2 cos(kL) = 0$$

Se cumple para

$$k_n L = (2n-1)\frac{\pi}{2} \qquad n = 1,2,3,\dots$$

$$k_n = (2n-1)\frac{\pi}{2L} \qquad n = 1,2,3,\dots$$

Sin pérdida de generalidad $C_2 = 1$

$$X_n(x) = sin(k_n x)$$

$$X_n(x) = sin\left[(2n-1)\frac{\pi}{2L}x\right]$$

La ecuación asociada al tiempo

$$\frac{1}{c^2}\frac{1}{T(t)}\left[\frac{d^2T(t)}{dt^2} + b\frac{dT(t)}{dt}\right] = -k^2$$

La arreglamos considerando que tenemos infinitas soluciones para k_n

$$\frac{1}{c^2}\frac{1}{T_n(t)}\left[\frac{d^2T_n(t)}{dt^2} + b\frac{dT_n(t)}{dt}\right] = -k_n^2 \qquad n = 1,2,3, \ldots$$

$$\frac{d^2T_n(t)}{dt^2} + c^2 b\frac{dT_n(t)}{dt} = -c^2 k_n^2 T_n(t) \qquad n = 1,2,3, \ldots$$

Podemos arreglar

$$\frac{d^2T_n(t)}{dt^2} + c^2 b\frac{dT_n(t)}{dt} + c^2 k_n^2 T_n(t) = 0 \qquad n = 1,2,3, \ldots$$

$$\frac{d^2T_n(t)}{dt^2} + c^2 b\frac{dT_n(t)}{dt} + \omega_n^2 T_n(t) = 0 \qquad n = 1,2,3, \ldots$$

$$\omega_n = ck_n$$

Las soluciones son

$$T_n(t) = e^{-\delta t}[A_n cos(\omega_{nd}t) + B_n sin(\omega_{nd}t)] \qquad n = 1,2,3, \ldots$$

$$T_n(t) = e^{-\left(\frac{c^2 b}{2}\right)t}\left\{A_n cos\left[t\sqrt{\omega_n^2 - \delta^2}\right] + B_n sin\left[t\sqrt{\omega_n^2 - \delta^2}\right]\right\} \qquad n = 1,2,3, \ldots$$

$$\omega_{nd} = \sqrt{\omega_n^2 - \delta^2} = \sqrt{\left[(2n-1)\frac{\pi c}{2L}\right]^2 - \left[\frac{c^2 b}{2}\right]^2} \qquad n = 1,2,3, \ldots$$

$$\delta = \frac{c^2 b}{2}$$

Las múltiples soluciones para la presión sonora son

$$p_n(x,t) = X_n(x)T_n(t) \qquad n = 1,2,3,\ldots$$

$$p_n(x,t) = e^{-\delta t}[A_n cos(\omega_{nd} t) + B_n sin(\omega_{nd} t)] sin(k_n x) \qquad n = 1,2,3,\ldots$$

$$p_n(x,t) = e^{-\left(\frac{c^2 b}{2}\right)t} \left\{ A_n cos\left[t\sqrt{\omega_n^2 - \delta^2}\right] + B_n sin\left[t\sqrt{\omega_n^2 - \delta^2}\right] \right\} sin\left[(2n-1)\frac{\pi}{2L}x\right] \qquad n = 1,2,3,\ldots$$

La solución completa corresponde a la combinación de todas las soluciones

$$p(x,t) = \sum_{n=1}^{\infty} p_n(x,t)$$

$$p(x,t) = \sum_{n=1}^{\infty} e^{-\delta t}[A_n cos(\omega_{nd} t) + B_n sin(\omega_{nd} t)] sin(k_n x)$$

La derivada parcial con respecto al tiempo es

$$\frac{\partial p}{\partial t}(x,t) = \sum_{n=1}^{\infty} \{ -\delta e^{-\delta t}[A_n \cos(\omega_{nd} t) + B_n \sin(\omega_{nd} t)] sin(k_n x) + \omega_{nd} e^{-\delta t}[-A_n sin(\omega_{nd} t) + B_n cos(\omega_{nd} t)] sin(k_n x) \}$$

Evaluamos las condiciones iniciales

$$p(x,0) = \begin{cases} 0 & 0 \le x < \frac{L}{4} - a \\ a^2 - x^2 & \frac{L}{4} - a \le x \le \frac{L}{4} + a \\ 0 & \frac{L}{4} + a < x \le L \end{cases}$$

$$p(x,0) = \sum_{n=1}^{\infty} e^{-\delta \times 0}[A_n cos(\omega_{nd} \times 0) + B_n sin(\omega_{nd} \times 0)] sin(k_n x) = \begin{cases} 0 & 0 \le x < \frac{L}{4} - a \\ a^2 - x^2 & \frac{L}{4} - a \le x \le \frac{L}{4} + a \\ 0 & \frac{L}{4} + a < x \le L \end{cases}$$

$$p(x,0) = \sum_{n=1}^{\infty} A_n sin(k_n x) = \begin{cases} 0 & 0 \leq x < \frac{L}{4} - a \\ a^2 - x^2 & \frac{L}{4} - a \leq x \leq \frac{L}{4} + a \\ 0 & \frac{L}{4} + a < x \leq L \end{cases}$$

$$A_n = \frac{2}{L} \int_{\frac{L}{4}-a}^{\frac{L}{4}+a} (a^2 - x^2) sin\left[(2n-1)\frac{\pi}{2L}x\right] dx$$

$$A_n = \left[-\frac{a^2 cos\left[(2n-1)\frac{\pi}{2L}x\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} - \frac{2cos\left[(2n-1)\frac{\pi}{2L}x\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^3} - \frac{2x\, sin\left[(2n-1)\frac{\pi}{2L}x\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^2} + \frac{x^2 cos\left[(2n-1)\frac{\pi}{2L}x\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} \right]_{\frac{L}{4}-a}^{\frac{L}{4}+a}$$

$$A_n = \left[-\frac{a^2 cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}+a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} - \frac{2cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}+a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^3} - \frac{2\left(\frac{L}{4}+a\right)\, sin\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}+a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^2} + \frac{\left(\frac{L}{4}+a\right)^2 cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}+a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} \right] - \left[-\frac{a^2 cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}-a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} - \frac{2cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}-a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^3} - \frac{2\left(\frac{L}{4}-a\right)\, sin\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}-a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]^2} + \frac{\left(\frac{L}{4}-a\right)^2 cos\left[(2n-1)\frac{\pi}{2L}\left(\frac{L}{4}-a\right)\right]}{\left[(2n-1)\frac{\pi}{2L}\right]} \right]$$

Como la segunda condición inicial es nula entonces

$$A_n = \frac{\delta A_n}{\omega_{nd}}$$