Ayudantía 10

10/10/2019

P1] Calcule las siguientes primitivas.

a) $\int \tan(x)\sec^3(x)\,dx$.

Sol: Lo hacemos de 2 formas:

- Forma 1: Sabemos que $\sec(x)$ y $\tan(x)$ $\underbrace{\text{se relacionan al derivar}}_{\text{esto nos da la idea de un posible cambio de variable}}$:

$$(\sec(x))' = \sec(x)\tan(x)$$
$$(\tan(x))' = \sec^2(x)$$

luego, si hacemos un cambio de variable:

→ (idealmente, el cambio de variable escogido debe ser de tal forma que dentro de la integral aparezca su derivada)

$u = \sec^3(x)$

$du = 3\sec^2(x) \cdot (\sec(x))' = 3\sec^2(x)\sec(x)\tan(x)\,dx$
$= 3\sec^3(x)\tan(x)\,dx$

$\Rightarrow \quad \tan(x)\,dx = \dfrac{du}{3\sec^3(x)} = \dfrac{du}{3u}$

Reemplazando en la integral:

$\int \tan(x)\sec^3(x)\,dx = \int \sec^3(x)\tan(x)\,dx$ / cambiamos el orden para que se vea el cambio de var

$= \int \cancel{u} \cdot \dfrac{du}{3\cancel{u}}$

$= \int \dfrac{du}{3}$

$= \dfrac{1}{3}\int du$

$= \dfrac{u}{3} + C$

$= \dfrac{\sec^3(x)}{3} + C$ {OJO} No olvidar volver a la variable original cuando son integrales indefinidas

· Forma 2: Note que $\tan(x) = \dfrac{\operatorname{sen}(x)}{\cos(x)}$ y $\sec^3(x) = \dfrac{1}{\cos^3(x)}$

$\Rightarrow \int \tan(x)\sec^3(x)\,dx = \int \dfrac{\operatorname{sen}(x)}{\cos(x)} \cdot \dfrac{1}{\cos^3(x)}\,dx$

$= \int \dfrac{\operatorname{sen}(x)}{\cos^4(x)}\,dx$

Como sen(x) y cos(x) se relacionan al derivar, pensamos en un posible cambio de variable.

Considere: $u = \cos(x)$

$du = -\operatorname{sen}(x)\,dx \Rightarrow \operatorname{sen}(x)\,dx = -du$

Reemplazando en la integral:

$$\int \frac{\operatorname{sen}(x)\,dx}{\cos^4(x)} = \int \frac{-du}{u^4}$$

$$= -\int u^{-4}\,du$$

$$= -\left(\frac{u^{-3}}{-3} + C_1\right)$$

$$= \frac{u^{-3}}{3} - C_1$$

$$= \frac{u^{-3}}{3} + C \quad \text{con } C = -C_1$$

$$= \frac{1}{3u^3} + C$$

volvemos a la variable original

$$= \frac{1}{3\cos^3(x)} + C$$

$$= \frac{\sec^3(x)}{3} + C \quad //$$

b) $\int \frac{1}{x}\ln(x)\,dx$

Sol: Note que $(\ln(x))' = \frac{1}{x}$, luego, el cambio de variable evidente en este caso es:

$u = \ln(x)$

$du = \frac{dx}{x}$

Así, $\int \frac{1}{x}\ln(x)\,dx = \int u\,du$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln(x))^2}{2} + C$$

c) $\int \arctan\left(\frac{2}{t}\right) dt$

Sol: Nos molesta el $2/t$ dentro del arctan( ), luego hagamos un cambio de variable:

$u = \frac{2}{t}$

$du = -\frac{2}{t^2}dt \Rightarrow dt = -\frac{t^2}{2}du$, pero $u = \frac{2}{t} \Leftrightarrow t = \frac{2}{u}$

$\Leftrightarrow t^2 = \frac{4}{u^2}$

luego, $dt = -\frac{\frac{4}{u^2}}{2}du = -\frac{2}{u^2}du$

Con esto, podemos reemplazar TODO lo que estaba en función de t por la variable u:

$$\int \arctan\left(\frac{2}{t}\right)dt = \int \arctan(u)\cdot -\frac{2}{u^2}du$$

$$= 2\int \arctan(u)\cdot -\frac{1}{u^2}\,du$$

Ahora, para resolver esta integral un cambio de variable no es buena idea, pues no podemos relacionar $\arctan(u)$ con $-\frac{1}{u^2}$ derivando, así, pensamos en una posible integración por partes, puesto que conocemos la derivada de $\arctan(u)$ y la primitiva de $-1/u^2$

Así, considere $f(u) = \arctan(u) \longrightarrow f'(u) = \frac{1}{1+u^2}\,du$

$g'(u) = -\frac{1}{u^2} \longrightarrow g(u) = \frac{1}{u}$

Integrando por partes:

$$\int \arctan(u) \cdot -\frac{1}{u^2}\,du = f(u) \cdot g(u) - \int f'(u)\,g(u)$$

$$= \frac{\arctan(u)}{u} - \underbrace{\int \frac{1}{u(1+u^2)}\,du}_{1} \quad (*)$$

1: $\int \frac{du}{u(1+u^2)}$ (1)

Usaremos fracciones parciales para separar $\frac{1}{u(1+u^2)}$

polinomios de un grado menos que el denominador

$$\frac{1}{u(1+u^2)} = \frac{A}{u} + \frac{Bu+C}{1+u^2}$$

$$= \frac{A(1+u^2) + u(Bu+C)}{u(1+u^2)}$$

Luego $A(1+u^2) + u(Bu+C) = 1$, buscamos A, B, C

Si $u=0$: $A(1+0^2) + 0 = 1 \Rightarrow \boxed{A=1}$

Si $u=1$: $1(1+1^2) + 1(B\cdot 1 + C) = 1$

$\Rightarrow 2 + B + C = 1$

$\Rightarrow \boxed{B = -C-1}$

luego $A(1+u^2)+u(Bu+C) = 1(1+u^2)+u((-C-1)u+C)$
$= 1+u^2+u(-Cu-u+C)$

Si $u=-1$: $1+(-1)^2+(-1)(-C(-1)-(-1)+C)=1$
$\Rightarrow 2-1(C+1+C)=1$
$\Rightarrow 2-1(2C+1)=1$
$\Rightarrow 2-2C-1=1$
$\Rightarrow \cancel{1}-2C=\cancel{1}$
$\Rightarrow 0=2C$
$\Rightarrow \boxed{C=0}$

y así $B=-0-1=-1$

$$\therefore \frac{1}{u(1+u^2)} = \frac{1}{u} - \frac{u}{1+u^2}$$

Reemplazando en (1):

$$\int \frac{du}{u(1+u^2)} = \underbrace{\int \frac{1}{u}\,du}_{(2)} - \underbrace{\int \frac{u\,du}{1+u^2}}_{3}$$

(2): $\int \frac{du}{u} = \ln(u) + C_1$

(3): $\int \frac{u\,du}{1+u^2}$, si hacemos un cambio de variable: $v=1+u^2$
$dv = 2u\,du \Rightarrow \frac{dv}{2}=u\,du$

aparece en la integral

$$\Rightarrow \int \frac{u\,du}{1+u^2} = \int \frac{dv}{2v} = \frac{1}{2}\left(\int \frac{dv}{v}\right)$$
$$= \frac{1}{2}(\ln(v)+C_2) = \frac{1}{2}\ln(1+u^2)+C_3$$

donde $C_3 = \frac{1}{2}C_2$

$$\Rightarrow \textcircled{1} = \ln(u) + C_1 - \left(\frac{1}{2}\ln(1+u^2) + C_3\right)$$

$$= \ln(u) - \frac{1}{2}\ln(1+u^2) + C_1 - C_3$$

$$= \ln(u) - \frac{1}{2}\ln(1+u^2) + C_4 \quad \text{con} \quad C_4 = C_1 - C_3$$

Así, reemplazando en $(*)$:

$$\int \arctan(u)\,\frac{-1}{u^2}\,du = \frac{\arctan(u)}{u} - \left(\ln(u) - \frac{1}{2}\ln(1+u^2) + C_4\right)$$

$$\Rightarrow 2\int \arctan(u)\cdot\frac{-1}{u^2}\,du = 2\,\frac{\arctan(u)}{u} - 2\ln(u) + \ln(1+u^2) - 2C_4$$

$$= 2\,\frac{\arctan(2/t)}{2/t} - 2\ln\left(\frac{2}{t}\right) + \ln\left(1+\frac{4}{t^2}\right) - 2C_4$$

$$= t\arctan(2/t) - 2\ln\left(\frac{2}{t}\right) + \ln\left(1+\frac{4}{t^2}\right) - 2C_4$$

$$= t\arctan(2/t) - \ln\left(\frac{4}{t^2}\right) + \ln\left(1+\frac{4}{t^2}\right) - 2C_4$$

$$\left\{\begin{aligned} &\ln\left(1+\frac{4}{t^2}\right) - \ln\left(\frac{4}{t^2}\right) \\ &= \ln\left(\frac{t^2+4}{t^2}\right) - \ln\left(\frac{4}{t^2}\right) \\ &= \ln\left(\frac{t^2+4}{4}\right) = \ln(t^2+4) - \ln(4) \end{aligned}\right.$$

$$= t\arctan\left(\frac{2}{t}\right) + \ln(t^2+4) + C \quad \text{con} \quad C = -2C_4 - \ln(4)$$

d) $\displaystyle\int \cos(\sqrt{s})\,ds$

Sol: Nuevamente, nos molesta $\sqrt{s}$, así:

C.V: $u = \sqrt{s}$

$$du = \frac{1}{2\sqrt{s}}\,ds \Rightarrow ds = 2\sqrt{s}\,du \quad \text{pero } u = \sqrt{s}$$

$$\Rightarrow ds = 2u\,du$$

Reemplazando:

$$\int \cos(\sqrt{s})\, ds = \int \cos(u)\, 2u\, du$$

No tenemos como relacionar $\cos(u)$ con $2u$ derivando, pero si usamos integración por partes:

$$f(u) = 2u \longrightarrow f'(u) = 2\,du$$
$$g'(u) = \cos(u) \longrightarrow g(u) = \operatorname{sen}(u)$$

[Tomé $f(u) = 2u$ para bajarle el grado a u al derivarlo)

Así: $\displaystyle\int \cos(u) 2u\, du = 2u\operatorname{sen}(u) - \int 2\operatorname{sen}(u)\, du$

$$= 2u\operatorname{sen}(u) + 2\int -\operatorname{sen}(u)\, du$$

$$= 2u\operatorname{sen}(u) + 2\left(\cos(u) + C_1\right)$$

$$= 2u\operatorname{sen}(u) + 2\cos(u) + 2C_1$$
$$= 2u\operatorname{sen}(u) + 2\cos(u) + C \quad \text{con } C = 2C_1$$
$$= 2\sqrt{s}\operatorname{sen}(\sqrt{s}) + 2\cos(\sqrt{s}) + C \,//$$

e) $\displaystyle\int \frac{e^y}{e^{2y}-4}\, dy$

Sol: Considere el c.v: $u = e^y$

$du = \underbrace{e^y\, dy}_{\text{aparece en la integral}}$

$$\Rightarrow \int \frac{e^y}{e^{2y}-4}\, dy = \int \frac{du}{u^2-4} \quad \text{y} \quad u^2-4 = (u+2)(u-2)$$

$$\Rightarrow \int \frac{du}{u^2-4} = \underbrace{\int \frac{du}{(u+2)(u-2)}}_{\text{nos gustaría separarla en 2 fracciones}} \quad (*)$$

• Fracciones parciales: $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$

$$= \frac{A(u-2) + B(u+2)}{(u+2)(u-2)}$$

$$\Rightarrow A(u-2) + B(u+2) = 1$$

Si $u=2$: $A(2-2) + B(2+2) = 1$

$$\Rightarrow 4B = 1$$

$$\Rightarrow \boxed{B = 1/4}$$

Si $u=-2$: $A(-2-2) + \frac{1}{4}\overset{0}{\cancel{(-2+2)}} = 1$

$$\Rightarrow -4A = 1$$

$$\Rightarrow \boxed{A = -1/4}$$

Así, $\frac{1}{(u+2)(u-2)} = \frac{-1/4}{(u+2)} + \frac{1/4}{(u-2)}$

$$\Rightarrow (*) = \int \left(\frac{-1/4}{(u+2)} + \frac{1/4}{(u-2)} \right) du$$

$$= \frac{1}{4}\left(\int \frac{-1}{u+2} du + \int \frac{1\, du}{u-2} \right)$$

$$= \frac{1}{4}\left(-\int \frac{du}{u+2} + \int \frac{du}{u-2} \right)$$

$$= \frac{1}{4}\left(-\left(\ln(u+2)+C_1\right)+\ln(u-2)+C_2\right)$$

$$= \frac{1}{4}\left(\ln(u-2)+C_2-\ln(u+2)-C_1\right)$$

$$= \frac{1}{4}\left(\ln(u-2)-\ln(u+2)+C_3\right) \quad \text{con } C_3 = C_2 - C_1$$

$$= \frac{1}{4}\left(\ln\left(\frac{u-2}{u+2}\right)+C_3\right) \quad / \ \ln(a)-\ln(b) = \ln\left(\frac{a}{b}\right)$$

$$= \frac{1}{4}\ln\left(\frac{u-2}{u+2}\right)+\frac{C_3}{4}$$

$$= \frac{1}{4}\ln\left(\frac{e^y-2}{e^y+2}\right)+C \ /\!/ \quad \text{con } C = \frac{C_3}{4}$$

f) $\displaystyle\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx$

Sol: Note que $x^3-3x^2+3x-1 = (x-1)^3$

$$\Rightarrow \int \frac{2x+1}{x^3-3x^2+3x-1}\,dx = \int \frac{2x+1}{(x-1)^3}\,dx$$

$$= \underbrace{\int \frac{2x\,dx}{(x-1)^3}}_{1} + \underbrace{\int \frac{dx}{(x-1)^3}}_{2}$$

Veamos cada una:

2 : $\displaystyle\int \frac{dx}{(x-1)^3}$, podemos integrarla directamente o hacer cambio de variable

C.V.: $u = x - 1$
$du = dx$

$$\Rightarrow \int \frac{dx}{(x-1)^3} = \int \frac{du}{u^3}$$
$$= \int u^{-3}\, du$$
$$= \frac{u^{-2}}{-2} + C_1$$
$$= \frac{-1}{2u^2} + C_1$$
$$= \frac{-1}{2(x-1)^2} + C_1$$

①: $\int \frac{2x\,dx}{(x-1)^3}$, considere nuevamente $u = x - 1 \Rightarrow x = u + 1$
$du = dx$

Así, $\int \frac{2x\,dx}{(x-1)^3} = 2 \int \frac{x\,dx}{(x-1)^3}$
$$= 2 \int \frac{(u+1)\,du}{u^3}$$
$$= 2\left(\int \frac{u\,du}{u^3} + \int \frac{du}{u^3} \right)$$
$$= 2\left(\int \frac{du}{u^2} + \int \frac{du}{u^3} \right)$$
$$= 2\left(\int u^{-2}\,du + \int u^{-3}\,du \right)$$
$$= 2\left(\frac{u^{-1}}{-1} + C_2 + \frac{u^{-2}}{-2} + C_3 \right)$$
$$= 2\left(\frac{-1}{u} + C_2 - \frac{1}{2u^2} + C_3 \right)$$

$$= \frac{-2}{u} + 2C_2 - \frac{1}{u^2} + 2C_3$$

$$= \frac{-2}{u} - \frac{1}{u^2} + 2C_2 + 2C_3$$

$$= \frac{-2u-1}{u^2} + 2C_2 + 2C_3$$

$$= \frac{-(2(x-1)+1)}{(x-1)^2} + 2C_2 + 2C_3$$

$\Rightarrow$

$$\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx = \int \frac{2x\,dx}{(x-1)^3} + \int \frac{dx}{(x-1)^3}$$

$$= \frac{-(2(x-1)+1)}{(x-1)^2} - \frac{1}{2(x-1)^2} + 2C_2 + 2C_3 + C_1$$

$$= \frac{-(2x-2+1)}{(x-1)^2} - \frac{1}{2(x-1)^2} + C \quad \text{con } C = 2C_2 + 2C_3 + C_1$$

$$= \frac{-(2x-1)}{(x-1)^2} - \frac{1}{2(x-1)^2} + C$$

$$= \frac{-2(2x-1)-1}{2(x-1)^2} + C$$

$$= \frac{-4x+2-1}{2(x-1)^2} + C$$

$$= \frac{-4x+1}{2(x-1)^2} + C$$

P21 Calcule las siguientes integrales definidas.

a) $\int_0^3 x^3\sqrt{1+x^2}\,dx$.

Sol: Nos molesta $\sqrt{1+x^2}$, si hacemos un cambio de variable:

$u = 1+x^2$ y si $x=0 \to u = 1+0^2 = 1$

$du = 2x\,dx$ $\quad x=3 \to u = 1+3^2 = 10$

así,

$$\int_0^3 x^3\sqrt{1+x^2}\,dx = \int_0^3 \sqrt{1+x^2}\,x^2\cdot x\,dx$$

$$= \frac{1}{2}\int_0^3 \sqrt{1+x^2}\cdot x^2\,\underbrace{2x\,dx}_{du} \quad y \quad x^2 = u-1$$

$$= \frac{1}{2}\int_1^{10} \sqrt{u}\cdot(u-1)\,du$$

$$= \frac{1}{2}\left(\int_1^{10} (\sqrt{u}\cdot u - \sqrt{u})\,du\right)$$

$$= \frac{1}{2}\left(\int_1^{10} u^{3/2}\,du - \int_1^{10} u^{1/2}\,du\right)$$

$$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2}\Big|_1^{10} - \frac{u^{3/2}}{3/2}\Big|_1^{10}\right)$$

$$= \cancel{\frac{1}{2}}\left(\frac{\cancel{2}}{5}u^{5/2}\Big|_1^{10} - \frac{\cancel{2}}{3}u^{3/2}\Big|_1^{10}\right)$$

$$= \frac{1}{5}\left(u^{5/2}\Big|_1^{10}\right) - \frac{1}{3}\left(u^{3/2}\Big|_1^{10}\right)$$

$$= \frac{1}{5}\left(10^{5/2} - 1^{5/2}\right) - \frac{1}{3}\left(10^{3/2} - 1^{3/2}\right)$$

$$= \frac{1}{5}\left((10^4\cdot 10)^{1/2} - 1\right) - \frac{1}{3}\left((10^2\cdot 10)^{1/2} - 1\right)$$

$$= \frac{1}{5}\left(10^2\cdot\sqrt{10} - 1\right) - \frac{1}{3}\left(10\sqrt{10} - 1\right)$$

$$= \frac{100\sqrt{10}}{5} - \frac{1}{5} - \frac{10\sqrt{10}}{3} + \frac{1}{3}$$

$$= \frac{100\sqrt{10}}{5} - \frac{10\sqrt{10}}{3} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{300\sqrt{10} - 50\sqrt{10}}{15} + \frac{2}{15}$$

$$= \frac{250\sqrt{10}}{15} + \frac{2}{15}$$

$$= \frac{250\sqrt{10}+2}{15}$$

$$= \frac{2(125\sqrt{10}+1)}{15} \,//$$

b) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{4-x^4}}$

Sol: $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^4}}\,dx = \int_0^1 \frac{x\,dx}{\sqrt{4\left(1-\frac{x^4}{4}\right)}}$

$$= \int_0^1 \frac{x\,dx}{2\sqrt{1-\left(\frac{x^2}{2}\right)^2}}$$

$$= \frac{1}{2}\int_0^1 \frac{x\,dx}{\sqrt{1-\left(\frac{x^2}{2}\right)^2}}$$

Luego, hacemos el siguiente cambio de variable:

$u = \frac{x^2}{2}$ y si $x = 0 \rightarrow u = 0$

$x = 1 \rightarrow u = 1/2$

$du = \frac{2x}{2}\,dx = x\,dx$

Así:

$$\frac{1}{2}\int_0^1 \frac{x\,dx}{\sqrt{1-\left(\frac{x^2}{2}\right)^2}} = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\left(\operatorname{arcsen}(u)\Big|_0^{1/2} \right)$$

$$= \frac{1}{2}\left(\operatorname{arcsen}\left(\frac{1}{2}\right) - \cancel{\operatorname{arcsen}(0)}^{0} \right)$$

$$= \frac{1}{2}\left(\operatorname{arcsen}\left(\frac{1}{2}\right) \right)$$

y $\operatorname{arcsen}\left(\frac{1}{2}\right) = y \iff \frac{1}{2} = \operatorname{sen}(y)$

$\iff y = \pi/6$.

$$\Rightarrow \frac{1}{2}\left(\operatorname{arcsen}\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

$$\therefore \int_0^1 \frac{x\,dx}{\sqrt{4-x^4}} = \frac{\pi}{12}$$

c) $\displaystyle\int_0^{\ln(2)} \frac{1-e^u}{1+e^u}\,du$

Sol: $\displaystyle\int_0^{\ln(2)} \frac{1-e^u}{1+e^u}\,du = \underbrace{\int_0^{\ln(2)} \frac{du}{1+e^u}}_{1} - \underbrace{\int_0^{\ln(2)} \frac{e^u\,du}{1+e^u}}_{2}$

1 : $\displaystyle\int_0^{\ln(2)} \frac{du}{1+e^u}$

Considere el c.v: $v = 1+e^u$ si $u=0$: $v=2$, si $u=\ln(2) \Rightarrow v=3$

$dv = e^u du$ $\Rightarrow dv = (v-1)du$

$\Rightarrow \frac{dv}{v-1} = du$

Así,

$$\int_0^{\ln(2)} \frac{du}{1+e^u} = \int_2^3 \frac{dv}{v(v-1)}$$

y $$\frac{1}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$$
$$= \frac{A(v-1)+B(v)}{v(v-1)}$$

$\Leftrightarrow A(v-1)+B(v)=1$

Si $v=1$: $\boxed{B=1}$

Si $v=0$: $A(0-1)+1\cdot 0 = 1$

$\Rightarrow -A = 1$

$\Rightarrow A = -1$

Así, $$\frac{1}{v(v-1)} = \frac{-1}{v} + \frac{1}{v-1}$$

Luego: $$\int_2^3 \frac{dv}{v(v-1)} = \int_2^3 \frac{-1}{v}dv + \int_2^3 \frac{dv}{v-1}$$
$$= -\int_2^3 \frac{dv}{v} + \ln(v-1)\Big|_2^3$$
$$= -\left(\ln(v)\Big|_2^3\right) + \ln(3-1) + \ln(2-1)$$

$$= -(\ln(3) - \ln(2)) + \ln(2) + \cancel{\ln(1)}^{\,0}$$
$$= \ln(2) - \ln(3) + \ln(2)$$
$$= 2\ln(2) - \ln(3)$$

(2): $\displaystyle\int_0^{\ln(2)} \frac{e^u\,du}{1+e^u}$

C.V: $v = 1+e^u$ si $u=0 \to v = 2$

$dv = e^u\,du$ $\quad u=\ln(2) \to v = 1+e^{\ln(2)} = 1+2=3$

$$= \int_2^3 \frac{dv}{v}$$

$$= \ln(v)\Big|_2^3$$

$$= \ln(3) - \ln(2)$$

Finalmente:

$$\int_0^{\ln(2)} \frac{1-e^u}{1+e^u}\,du = 2\ln(2) - \ln(3) - (\ln(3) - \ln(2))$$
$$= 3\ln(2) - 2\ln(3)$$ //

d) $\displaystyle\int_0^{0.5} \frac{x^2}{\sqrt{1-x^2}}\,dx$

Sol Considere el c.v: $x = \operatorname{sen}(u)$

$dx = \cos(u)\,du$

y si $x=0 \Rightarrow \operatorname{sen}(u)=0 \Rightarrow u=0$

$x=1/2 \Rightarrow \operatorname{sen}(u)=1/2 \Rightarrow u=\pi/6$

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Así:

$$\int_0^{0.5} \frac{x^2 dx}{\sqrt{1-x^2}} = \int_0^{\pi/6} \frac{\operatorname{sen}^2(u)\cdot\cos(u)\,du}{\sqrt{1-\operatorname{sen}^2(u)}}$$

y $1-\operatorname{sen}^2(u) = \cos^2(u)$ pues $\left(\operatorname{sen}^2(u)+\cos^2(u) = 1\right)$

$$\Rightarrow \int_0^{\pi/6} \frac{\operatorname{sen}^2(u)\,\cos(u)\,du}{\sqrt{\cos^2(u)}}$$

$$= \int_0^{\pi/6} \frac{\operatorname{sen}^2(u)\,\cancel{\cos(u)}\,du}{\cancel{\cos(u)}}$$

$$= \int_0^{\pi/6} \operatorname{sen}^2(u)\,du \qquad (*)$$

Ahora, sabemos que: $\operatorname{sen}^2(u) = 1-\cos^2(u)$

y $\cos^2(u) - \operatorname{sen}^2(u) = \cos(2u) \Rightarrow \cos^2(u) = \operatorname{sen}^2(u) + \cos(2u)$

Así, $\operatorname{sen}^2(u) = 1-\cos^2(u)$

$$= 1 - (\operatorname{sen}^2(u) + \cos(2u))$$

$$= 1 - \operatorname{sen}^2(u) - \cos(2u)$$

$$\Rightarrow 2\operatorname{sen}^2(u) = 1-\cos(2u)$$

$$\Rightarrow \operatorname{sen}^2(u) = \frac{1-\cos(2u)}{2}$$

Reemplazando en $(*)$:

$$\int_0^{\pi/6} sen^2(u)\,du = \int_0^{\pi/6} \frac{1-\cos(2u)}{2}\,du$$

$$= \int_0^{\pi/6} \frac{du}{2} - \int_0^{\pi/6} \frac{\cos(2u)\,du}{2}$$

$$= \frac{1}{2}\left(\int_0^{\pi/6} du\right) - \frac{1}{2}\left(\int_0^{\pi/6} \cos(2u)\,du\right)$$

$$= \frac{1}{2}\left(u\Big|_0^{\pi/6}\right) - \frac{1}{2}\left(\frac{sen(2u)}{2}\Big|_0^{\pi/6}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}-0\right) - \frac{1}{4}\left(sen\left(2\cdot\frac{\pi}{6}\right) - \cancel{sen(0)}^{\,0}\right)$$

$$= \frac{\pi}{12} - \frac{1}{4}\,sen\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{12} - \frac{1}{4}\,\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

$$\therefore \int_0^{0.5} \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

Bonus: Haremos el P1|c) y el P2|b) de otra forma:

P1 c) $\int \arctan\left(\frac{2}{t}\right)dt = \int 1\cdot\arctan\left(\frac{2}{t}\right)dt$

Sol: Integramos por partes: $f(t) = \arctan\left(\frac{2}{t}\right)$, $g'(t) = 1\,dt$

$f'(t) = \dfrac{(2/t)'}{1+\left(\frac{2}{t}\right)^2}$, $g(t) = t$

Recuerde que $\left(\arctan(h(x))\right)' = \dfrac{h'(x)}{1+h^2(x)}$

Así: $$\int \arctan\left(\frac{2}{t}\right) dt = t\arctan\left(\frac{2}{t}\right) - \underbrace{\int \frac{t\,(2/t)'}{1+\left(\frac{2}{t}\right)^2} dt}_{A}$$

y $$\frac{t\,(2/t)'}{1+\left(\frac{2}{t}\right)^2} = \frac{t\cdot\left(-\frac{2}{t^2}\right)}{1+\frac{4}{t^2}} = \frac{\frac{-2t}{\cancel{t^2}}}{\frac{t^2+4}{\cancel{t^2}}} = \frac{-2t}{t^2+4}$$

$$\Rightarrow A = \int \frac{-2t}{4+t^2} dt = -\int \frac{2t\,dt}{4+t^2} = -\left(\ln(t^2+4) + C\right)$$

Así, $$\int \arctan\left(\frac{2}{t}\right) dt = t\arctan\left(\frac{2}{t}\right) + \ln(t^2+4) + C_1 \;//$$

P2 b)| $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{4-x^4}}$

Sol: $$\int_0^1 \frac{x\,dx}{\sqrt{4-x^4}} = \int_0^1 \frac{x\,dx}{\sqrt{4\left(1-\frac{x^4}{4}\right)}}$$
$$= \int_0^1 \frac{x\,dx}{2\sqrt{1-\left(\frac{x^2}{2}\right)^2}}$$
$$= \frac{1}{2}\int_0^1 \frac{x\,dx}{\sqrt{1-\left(\frac{x^2}{2}\right)^2}}$$

CV: $\frac{x^2}{2} = \operatorname{sen}(u)$ límites de integración: si $x=0 \to \operatorname{sen}(u)=0 \to u=0$; si $x=1 \to \operatorname{sen}(u)=\frac{1}{2} \to u=\frac{\pi}{6}$

$$\frac{\cancel{2}x\,dx}{\cancel{2}} = \cos(u)\,du \iff x\,dx = \cos(u)\,du$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{\cos(u)\,du}{\sqrt{1-\operatorname{sen}^2(u)}}$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{\cancel{\cos(u)}\,du}{\cancel{\cos(u)}}$$

$$= \frac{1}{2}\int_0^{\pi/6} du$$

$$= \frac{1}{2}\left(u \Big|_0^{\pi/6} \right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - 0 \right)$$

$$= \frac{\pi}{12} //$$