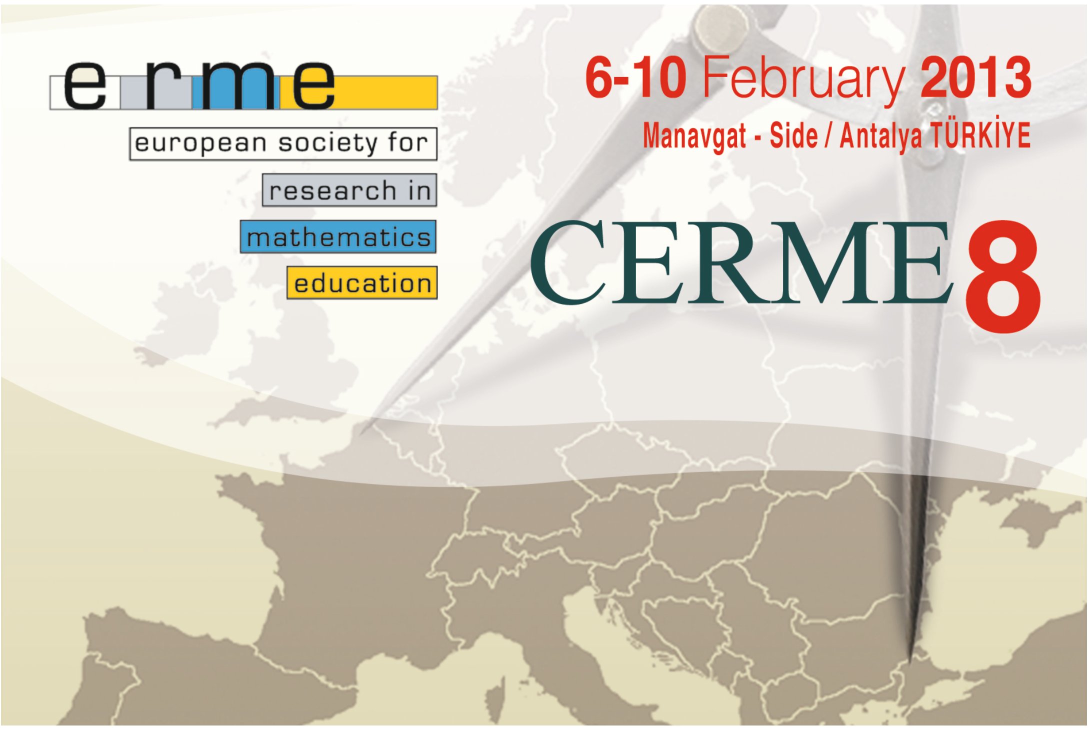

Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education

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PROCEEDINGS OF THE EIGHTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION: GENERAL INTRODUCTION

Maria Alessandra Mariotti Ferdinando Arzarello
University of Siena University of Turin

We are glad to present the Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education (ERME), which took place 6-10 February 2013, at Manavgat-Side/Antalya in Turkey.

The story of ERME begins at its first congress in Osnabrueck, Germany, in 1998 and develops all along the CERME congresses that have taken place every two years since CERME2 in 2001. The vision shared by the first group of founders was that of establishing a community to promote communication, cooperation and collaboration in mathematics education research in Europe. At the very beginning of the ERME story, considerable time was spent talking about the nature of our conferences. Especially we wondered how were we going to achieve the communicative, cooperative and collaborative spirit we envisaged. It was clear that the conference should offer more than just a platform for presenting and listening to papers, as many other conferences do. We wanted that CERME could allow groups of researchers in a particular scientific area really to *work* together on their area of research, with sufficient time to get to know each other, to share and discuss their research and to engage in deep scholarly debate. At the same time we wanted to support the scientific development of young researchers fostering their active participation to our research community.

Therefore it was decided that CERME should abandoned the common format of parallel research report presentations and adopt a new format based on Working Group activities where participation by all who attend the congress could be promoted. Such a format has been developed stating a clear policy for the organization and the management of thematic Working Groups. At CERME participants spent most of the time in discussion and debate within the thematic Working Groups (WGs), during 6 or 7 *working* sessions of 90-120. Each CERME participant selects the membership of just one such Group, on the base of her/his personal scientific interests. For each WG, a team of leaders is nominated by the Scientific Program Committee, the leaders have in charge the complex organization of the WGs, preparing and managing what will happen at the Conference. Though participation to the conference is completely free, prospective participants are encouraged to contribute with submitting a paper or a poster. The leaders' team organizes the peer review process among the member of the Group according significantly devolved and distributed responsibility in criticising but also supporting the elaboration of the single contributions. This process aims not only at rising the quality of the papers but also at developing a sense of belonging to a community, for all participants. At the end of this first phase all the accepted paper will be posted on the website of the conference and participants are expected to read all the papers related to their own WGs, before attending the conference. This corpus of papers will constitute the first working material for the WGs activities, and a great deal of time and intellectual efforts are spent by the leaders to outline the structure of the working sessions where the different contributions will be fully discussed and related to the other contributions.

At CERME 8, different ways of organizing the working group sessions were set up by the different leaders teams. The main objective was always that of fostering the discussion exploiting the richness of the contributions. In some cases the discussion was structured according to subthemes focussing on specific clusters that emerged from the variety of the papers. Other times the leaders proposed the participants specific questions that were sent in advance to the authors of the papers who were requested to focus a short contribution on this question. The variety in the organization structure witnesses of the complexity of the task that the leaders team are asked to face but also of their passion and commitment in accomplishing their work, for which the ERME community is highly grateful to them. In the introduction to the collection of papers of each WG, the reader will find a description of the different organizations that were adopted.

The particular format of the conference gives the participants the opportunity of getting fruitful feedbacks that can enlarge and enrich their own perspectives; thus, after the conference the authors have the possibility to further revise their papers, integrating significant elements emerged from their WG's discussion: this will be the latest form in which papers will pass the final review process and when accepted will appear in the proceedings. The double review process that is used at CERME congresses - papers are firstly accepted for discussion in the WGs and than their final version has to be accepted for being published in the proceedings - not only aims at raising the quality of the papers but also at assuring a fair balance between quality and inclusion, two goals that seem to pull in different directions, and may create tension, sometimes frustration. However, the attainment of a good balance between *quality* and *inclusion* constitutes the main challenge of our community according to our main objectie: to ensure the ERME spirit‘ of communication, cooperation and collaboration.

The number of WGs increased in the years and since CERME7 we have 17 WGs, and excepting of the WG 15 and WG 17, the number of participants in each is around 25-30 on average, including about 4 WG leaders.

The themes of the WGs are as follows:

WG1: Argumentation and proof

WG2: Arithmetic and number systems

WG3: Algebraic thinking

WG4: Geometrical thinking

WG5: Stochastic thinking

WG6: Applications and modelling

WG7: Mathematical potential, creativity and talent

WG8: Affect and mathematical thinking

WG9: Mathematics and language

WG10: Cultural diversity and Mathematics Education

WG11: Comparative studies in Mathematics Education

WG12: History in Mathematics Education

WG13: Early Years Mathematics

WG14: University mathematics education

WG15: Technologies and resources in mathematics education

WG16: Different theoretical perspectives / approaches in research in mathematics education

WG17: From a study of teaching practices to issues in teacher education

The success of the ERME Conferences is witnessed by the constant increasing number of participants and presentations. In Manavgat 520 participants attended the congress, from 45 countries within and beyond Europe.

In addition to the WG activities, the congress was enriched by in a number of plenary scientific activities, and a varied social and cultural program.

The opening session included a plenary address by Paolo Boero who proposed a deep reflection on how to deal, as researchers, with the unavoidable complexity of big problems concerning the teaching and learning of mathematics in our societies. On the base of a long personal elaboration, strictly and functionally interwoven with the evolution of the experimental activity in the school carried out with the Genoa research group since the seventies, Boero offered us some answers to those big questions emerging from complex phenomena, particularly those concerning societal needs and values and related educational choices.

As at previous CERME congresses, two other plenary talks were given by former WGs leaders, Alain Kuzniak and Candia Morgan.

Kuzniak presented a vivid account of what are today the core items and the contributions of researches in the didactics of geometry, and he did it in the light of the rich discussions which have been occurring in the CERME Working Group on geometry from its beginning in 1999. Candia Morgan delineated a superb survey of the complex field of study of language in mathematics education. As she said, she offered *her* map, her personal and critical account, on previous studies in this field, and especially a theoretical elaboration as it emerged from the active discussion taking place at the CERME Working Group on Language and Mathematics over the years.

Three papers corresponding to these three plenary addresses are included in these proceedings.

Though these proceeding do not contain any document related to it, let me mention another fundamental event that took place one day before the opening of the Congress: the YERME (Young European Researchers in Mathematics Education) day. This is now a constant appointment where young researchers – doctoral students or post-doctoral researchers - meet expert scholars in thematic discussion groups. This event, together with the YERME Summer School (YESS), is based on the volunteering of some members of the society. At CERME 8 the organization of the YERME day was coordinated by João Pedro da Ponte, Ferdinando Arzarello and Behiye Ubuz; the activities were led by professors Paolo Boero, Uffe Thomas Jankvist, Barbara Jaworski, Ester Levenson, Maria Alessandra Mariotti, João Pedro da Ponte, Susanne Prediger, Mario Sanchéz, Susanne Schnell, Behiye Ubuz. (http://cerme8.metu.edu.tr/yerme.html)

As said, our Conference has a very particular format, it promotes the active involvement of all the participants and its success highly depends on their contributions; however, success also depends on the commitment of those who made this involvement possible, to them we want to express our gratitude in behalf of the ERME community: to the members of the Scientific Program Committee, for the inspiration and support that they offered in the scientific planning of the conference, to the Leaders of the WGs, for the competence, the energy and the engagement that they invested in their responsibility, and last but not least to the President, Behiye Ubuz, and the members of the Local Organizing Committee, for the incredible work done in preparing and supervising the organization of the conference, they allowed all the participant enjoy the conference days of intensive intellectual work in a efficient, comfortable and delightful place. Their attentive support did not finished with the end of the conference but continued in the patient and competent work of editing these proceedings.

We are certain that the reader will appreciate the richness of the contributions collected in this text that we hope will offer the opportunity to share with us something of the exciting experience of our congress, and encourage interested researchers to meet us at the next CERMEs.

Maria Alessandra Mariotti (Chair of the program Committee)

Ferdinando Arzarello (ERME President)

Information on-line

The CERME website was at : http://cerme8.metu.edu.tr/

These proceedings can be accessed online from: http://www.mathematik.uni-dortmund.de/~erme/doc/cerme7/CERME7.pdf

EDITORIAL INTRODUCTION FOR THE EIGHT CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

http://www.cerme8.metu.edu.tr/

Behiye Ubuz and Çiğdem Haser
Middle East Technical University, Turkey

Maria Alessandra Mariotti
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The Eight Congress of the European Society for Research in Mathematics Education (CERME8) was held at the Starlight Convention Center, Thalasso & Spa Hotel in Manavgat-Side, Antalya, Turkey from 6th to 10th February, 2013, chaired by Prof. Dr. Behiye Ubuz (Local Organizer Chair) and Prof. Dr. Maria Alessandra Mariotti (International Program Committee Chair). It aimed to promote the development of mathematics education through intellectual communication and cooperation by attending thematic working groups, plenary talks, poster sessions, and so forth. At CERME8 there were 3 invited plenary talks given by Paolo Boero, Alain Kuzniak, and Candia Morgan together with 17 thematic working groups (WGs). The main work of the congress took place in these Thematic Working Groups, facilitated by some 73 Working Group leaders. The congress was preceded by a meeting of Young European Researchers in Mathematics Education (YERME) on 5th and 6th February 2013.

A total of 375 Research Papers and 90 poster proposal were submitted for the congress. Following peer review, 310 Research Papers and 57 poster proposals were accepted for publication in the proceeding.

A view from Starlight Convention Center, Thalasso & Spa Hotel

Around 520 participants attended the congress, from 45 countries within and beyond Europe. Many participants from European countries were there: UK (31), Portugal (29), Germany (98), Italy (22), Greece (10), Finland (6), Spain (30), Netherlands (8), Sweden (54), Cyprus (2), Denmark (16), Norway (24), Austria (1), Czech Republic (5), France (41), Ireland (5), Romania (1), Russia (4), Belgium (4), Iceland (4), Estonia (2), Latvia (1), Poland (2), and

Switzerland (4). Moreover, there were 34 researchers from Turkey, 12 from Israel, 20 from the US and 16 from Canada, and 1 from Australia, 1 from the Far East (Japan), 27 from Latin America (Brazil, Chile, Colombia, Mexico) and several others from non-affluent countries, such as Iran (2), South Africa (1), Saudi Arabia (3), Algeria (1), Kuwait (1), Tunisian (2), Lebanon (1), and Zaire (1).

In the first section of this proceeding, the plenary talks by Paolo Boero, Alain Kuzniak, and Candia Morgan are presented. They kindly accepted the invitation of the scientific program committee and provided a written account of the ideas they presented in their plenary talks. The second section documents the research papers and poster communications accepted for publication in the proceeding. The papers and the posters are presented under each WG following the introduction written by each WG's leaders. Introductions summarize the scope of the WGs' works and the value of the studies presented.

A view from Side-Antalya (Taken from http://www.resim11.com/Antalya.html)

CERME8 must surely be regarded as a great opportunity for teachers, mathematics educators, teacher educators, and policy makers around the world and in Turkey who are interested in mathematics education and its development. It also provided a valuable development opportunity for the young researchers through YERME.

The proceeding for CERME8 is produced electronically both in CD format and in the website http://www.mathematik.uni-dortmund.de/~erme/doc/cerme8/CERME8.pdf . You can access the individual research papers and poster contributions via the hyperlinks provided on the contents page. We hope that every participant enjoyed the conference and their stay in Turkey!

Behiye Ubuz
(Chair of Local Organising Committee)

Çiğdem Haser
(Congress Secretariat)

Maria Alessandra Mariotti
(Chair of the Program Committee)

METAPHORICAL RANDOM WALKS: A ROYAL ROAD TO STOCHASTIC THINKING?

Jorge Soto-Andrade[1]

CIAE and Department of Mathematics, Faculty of Science, University of Chile

We are interested in finding means to facilitating the access to stochastic thinking, especially for non-mathematically oriented learners. We argue that random walks are such means, since they allow a friendly visualisation of randomness, are amenable to experimental and enactive metaphorical study and provide "universal models" for a broad spectrum of stochastic problems. An explicit example of a didactical situation is given (Brownie's walk) that has been tested with students and in-service teachers of various backgrounds since 2007. The findings suggest that this sort of approach significantly facilitates the access of non-mathematically oriented learners to otherwise cryptic mathematical notions, allowing them to construct them while solving situated concrete problems.

INTRODUCTION

We are concerned about facilitating the access to stochastic thinking. We are especially interested in approaches helpful for "general" non-mathematically oriented students in school, college and university. For instance, students majoring in humanities and social sciences or prospective or in-service elementary school teachers.

We claim that random walks provide such an access, indeed a "royal road" so to say.

Why random walks? Because random walks cross boundaries, appearing in the "natural world" as well as in the "cultural world". They are a visual embodiment of randomness, that can be easily enacted and simulated, from primary school onwards. We can approach them in manifold ways: statistically, metaphorically, probabilistically… They provide "universal models" and metaphors for sundry phenomena. Indeed they facilitate the access of non-mathematically oriented learners to stochastic thinking, enabling them even to construct probabilistic notions while solving situated concrete problems. They show the way to a bottom-up approach to probability and statistics.

From the didactical viewpoint, it is crucial that the study of simple random walks may be undertaken "bare handed", with practically no previous statistic or probabilistic tools or concepts. Students may tackle the paradigmatic question: *Where is the walker, after a given number of steps?* equipped with sheer common sense. Most important, they may realize that there are *several levels of answers* to this sort

[1] Supported by PBCT- CONICYT, Proyecto CIE-05

of question, the 0^{th} level being: Nobody knows! For subsequent levels see the example of Brownie's walk below. Random walks also provide a propitious soil for the emergence of metaphors that make possible the construction of the concept of probability: for instance, the *probability* of finding the walker at given position at a given time appears as the *portion* of the walker found at that position and time.

Historically random walks appear rather late in science, as Brownian motion of pollen grains (Powles, 1978), although it might be argued that they were "anticipated" by Lucretius in 60 AD, who observed dust particles "skirmishing" under sunlight and thought this to be caused by "motions of matter latent and unseen at the bottom" (loc.cit). Contemporary examples include Brownian motion in metallic alloys (Preuss, 2002), foreseen by Einstein in 1905 (Einstein, 1956), besides fluctuations of stock markets! Interestingly, we find also instances of "cultural random walks", quite motivating for humanists. Recall for example Saint Francis of Assisi's friars walking along the road network of medieval Italy to preach the Gospel, trying to be just an instrument of God's will. So they wanted to avoid choosing rationally their direction at each crossroad. Seen from the outside, their goal was to random walk in the road network, choosing at random at each crossroad. Saint Francis devised a clever method to implement this random choice: He told a friar to spin and whirl nonstop, in spite of dizziness or nausea. As a result (friars not being whirling dervishes) the friar finally collapsed and fell to the ground. Then the whole company would choose the road closest to the direction shown by the friar's head (Anonymous, ca. 1600).

In this paper, we describe a specific example of the use of random walks to introduce the students to stochastic thinking, that may be regarded as a first avatar of a fundamental didactical situation (Brousseau, 1998).

This approach has been tested with students and teachers with various backgrounds (ranging from scientific and humanistic students to in-service elementary school teachers and juvenile offenders). An a priori and a posteriori analyses were carried out, in the sense of didactical engineering. Finally, results obtained are discussed and some conclusions are drawn.

THEORETICAL FRAMEWORK: METAPHORS AND DIDACTICAL SITUATIONS.

Nature and Role of Metaphors in Mathematics Education.

It has been progressively recognized during the last decade (Araya, 2000; Bills, 2003; Chiu, 2000, 2001; English, 1997; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Parzysz et al., 2007; Presmeg, 1997; Sfard, 1994, 1997, 2009, Soto-Andrade 2006, 2007, 2012, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by"

(Bills, 2003). See also Soto-Andrade (2012) for a recent survey. We make use of conceptual metaphors (Lakoff & Núñez, 2000), that appear as mappings from a "source domain" into a "target domain", carrying the inferential structure of the first domain into the one of the second, and enabling us to understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent. We notice than in the literature the terms representations, analogies or models are often loosely used as equivalent to metaphors.

Recall that Grundvorstellungen (better translated as "fundamental notions" than "basic ideas") for mathematical content, that are very often operationally equivalent to conceptual metaphors, have been developed for a couple of centuries in the German school of didactics of mathematics (vom Hofe, 1995, 1998). In the case of probability they are usually given in a more prescriptive way than we do here for metaphors (Malle & Malle, 2003).

Our approach regarding the role of metaphors in the teaching-learning of mathematics emphasizes their "poietic" role, that brings concepts into existence, described as "reification" by Sfard (2009).

Didactical Situations.

Instead of introducing first concept and tools that students will apply later to solving exercises and problems as in traditional teaching, in the spirit of Brousseau (1998), we aim at students constructing or discovering those tools and concepts, when trying to solve a problem or challenge, in the context of a didactical situation. The concepts or tools we intend to teach should emerge as "the" way to "save your life" in the given situation.

We describe the role of metaphors in didactical situations with the help of a "voltaic metaphor": Metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough "didactical tension" is built up in a didactical situation (Brousseau, 1998) for them. This requires setting up a suitable didactical situation and succeeding in having the students sustain and bear the necessary didactical tension. The notion of didactical tension deserves here further study.

OUR DIDACTICAL SITUATION: BROWNIE'S RANDOM WALK

Perhaps the most natural visual example of randomness is Brownian motion. So we begin by presenting Brownian motion to the students in an interactive form (using real time videos and also simulating applets). Then, the task of studying this phenomenon being complex, we try to settle for some "baby version" of it. Usually the students themselves have the idea of "simplifying" the phenomenon that we want to study, as much as possible, but without "throwing away the baby with the bath's water".

We consider here a specific "baby avatar" of Brownian motion: A 2D random walk, whose protagonist is a puppy, suitably called Brownie (as sometimes suggested by the students themselves). Also an even simpler version, a 1D random walk may appear, that we do not consider here. We discuss below, somewhat halfway between “bricolage” (Gravemeijr, 1998) and didactical design (Artigue, 2009), how this random walk has been tackled by the students with the help of suitable metaphors and without a previous knowledge of probability or statistics.

Where is Brownie?

Brownie is a little puppy that escapes randomly from home, when she smells the shampoo her master intends to give her. At each street corner, confused by the traffic's noise and smells, escaping barely from being overrun, she chooses equally likely any of the 4 cardinal direction and runs nonstop a whole block until the next corner. Exhausted, after 4 blocks, say, she lies at some corner. Her master would like to know where to look for Brownie and also to estimate how far she will end up from home…

Research background (the students)

a) primary school teachers enrolled in a 15 month professional development program, at the University of Chile, aiming at improving their mathematical competences, from 2007 to 2010 and in 2012.
b) 1st year University of Chile students majoring in social sciences and humanities, from 2007 to 2012 (1 semester mathematics course).
c) University of Chile undergraduates majoring in mathematics and physics (one semester probability and statistics course) in 2009-2011.
d) University of Chile prospective secondary school teachers (one semester probability and statistics course) in 2009 – 2011.
e) Secondary school teachers enrolled in various professional development programs, in Chile, in 2006-2008 and 2011-2012
f) elementary school students (6th to 8th grade), whose teachers were engaged in the professional development program described above (in progress).

Methodology

Learners were observed by the author and two assistants during work sessions (some of them videotaped), their written outputs were kept or scanned. They also answered questionnaires related to their use, appreciation and preferences regarding metaphors used in courses a) and b). Learners a) and e) did group work most of the time. Other learners participated in interactive lessons with a high degree of individual participation, horizontal interaction included.

We describe now the a priori and a posteriori analyses in the sense of didactical engineering or didactical design (Artigue, 2009) related to this experimentation.

A priori analysis.

The mathematical situation.

We have several approaches to the problem.

"Arid" theoretical approach: We calculate stepwise the probabilities of finding Brownie at the different crossings in the city map. We obtain in this way the probability distribution of the sequence of random variables X_n = "Brownie's position after n steps" and D_n = "distance of Brownie to the origin after n steps". To obtain a general expression for this distribution, a clever idea is to do harmonic analysis and synthesis of these processes with respect to the (non-commutative!) symmetry group of the random walk.

Statistical approach: To see what is going on, we can make a statistical simulation of the random walk, eventually using a worksheet.

Metaphorical approach: With the help of a "hydraulic metaphor" (Soto-Andrade, 2006, 2007), we replace the puppy's random walk on the city map by a fission or sharing process on a grid. There we may see, for instance, the grid as a system of ducts and imagine a litre of water at the origin that flows symmetrically and fairly to the 4 next neighbours, each time. We have then to calculate stepwise with a deterministic process, which is equivalent to the original stochastic process, but has the virtue to avoid probabilistic language.

The didactical situation.
Working in groups, the teachers or students should realize quickly that there are impossible corners (street crossings) for Brownie, even close to her home. There might be divergent opinions on whether returning home is possible after 1 or 3 blocks, for instance. Then, after having spotted the corners where it is possible to find Brownie (after a 4 block run), some (up to one half of the) students or teachers will believe that they are equally likely. Others would have the vague intuition that some corners are more likely because Brownie can get there by several different paths.

Quite late, some will have the idea of experimenting, by simulating a good number of puppies, to try to settle the question. The majority of them will now be convinced that some corners are more favorable to find Brownie; her home for instance. They will not have much trouble in setting up a corner *ranking.* But they will have some trouble in quantifying their feeling of bigger or smaller likelihood. How to assign "weights" to the different corners? Some will usually think of counting paths to quantify their vague feeling.

It is likely that the "Solomonic metaphor" (cut the puppy into four pieces) or the "hydraulic metaphor" (the puppy flows equitably to the four immediate neighbours) will not emerge spontaneously. But this could happen under some minimal prompting like: *What, more concretely, could you imagine instead of this (rather abstract) fair random choice between the 4 cardinal directions?* (using some discrete gesture language).

The students or teachers should not have much trouble quantifying the likelihood of presence of the puppy at the different corners, with the help of the hydraulic metaphor. They will realize quickly the conservation law of the puppy: putting together her pieces at each step you reconstruct the whole puppy. The idea might also emerge in the classroom to unleash a pack of puppies from home, that would spread out evenly, splitting into four equal groups at each corner (better begin with 16 puppies that will run 2 blocks each). We call this a "pedestrian metaphor".

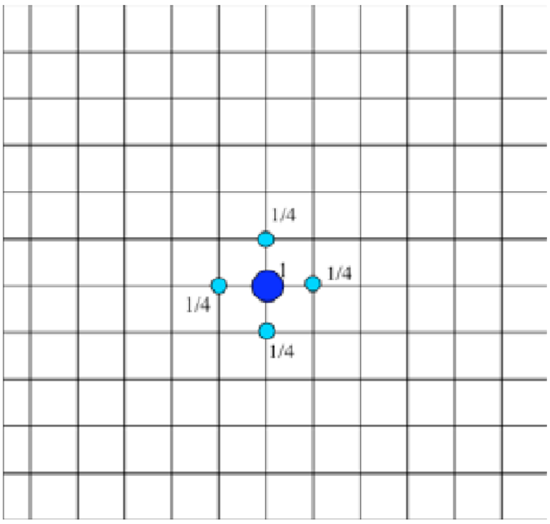
So we hypothesize that – eventually under some prompting – Solomonic, hydraulic and pedestrian metaphors may emerge, that will enable the students to solve the problem for a given (small) number of steps. Path counting will also emerge as a competing strategy to quantify likelihood, especially among students having being previously exposed to more mathematics.

A posteriori analysis.

Unexpectedly our primary school teachers had a strong tendency to see without hesitation the 9 possible corners for Brownie after a 2 block run as equally likely! Dissenting opinions emerged very slowly among them. On the contrary learners from groups b), c), d) quickly sensed that corners closer to her home were more likely.

We have witnessed the emergence in the classroom of metaphors like the Solomonic, hydraulic or pedestrian one (Soto-Andrade, 2006, 2007) right after the students or teachers were prompted to try to see the situation "otherwise", while working in groups, with discrete support from the facilitator. Typically in a class of 30, one or two immediately "see" the puppy split into four pieces. These metaphors emerged more easily from learners in groups a) and b). Initially however many teachers seemed to have feeling that they were violating some (unspoken) didactical contract when allowing themselves to metaphorize.

In cases where this sort of visualization seemed out of reach, the hydraulic metaphor enabled us to act out the situation: 13 in-service elementary school teachers distribute themselves suitably on the nodes of a virtual grid on the floor, one of them holding a container with 1 litre water that she shares equitably with her immediate 4 neighbours, and so on… Each one of them could estimate the amount of water that



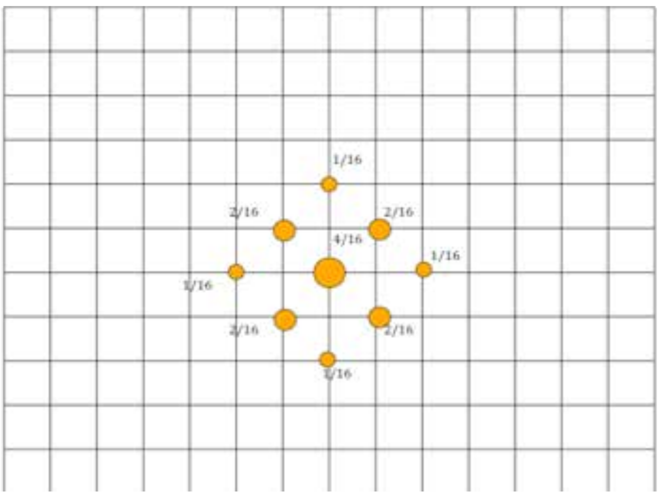


Fig. 1 Brownie's splitting (0 and 1 block). **Fig. 2. Brownie's splitting (2 blocks).**

she will have after a given number of steps. Eventually they simulate this (more easily although less dramatically) with a square of board that they partition into four pieces and so on. See figures 1 and 2 above.

Results

As hypothesized, Solomonic, hydraulic and pedestrian metaphors did indeed emerged among our students, eventually with the help of some prompting. Path counting appeared also, among more mathematically literate students, leading to an assignment of probabilistic weights based in Laplace rule. One obstacle to the emergence of metaphors was the current didactical contract saying that metaphorizing is not good mathematical manners. Enacting the walk with the help of a hydraulic metaphor was a big help for most learners, especially for elementary school teachers.

With this course of action, most students and teachers succeeded in constructing the concept of probability at the same time that they solved the problem in context, with the help of suitable conceptual metaphors, in the sense of Lakoff and Núñez (2000).

Discussion

Our voltaic metaphor seems suitable to describe what happened in the classroom, regarding the emergence of metaphors that enable the learners to solve the problem they are tackling. To build up the necessary didactical tension is here crucial but not easy, especially when working with big classes (more than 50 students).

On the other hand, we remarked that in our hydraulic acting out of the random walk, each participant does something very simple, but the outcome is the analogical solution of a non trivial problem. This could be seen as just one simple example of application of "swarm intelligence" to the didactics of mathematics.

Also the role of the teacher in this sort of "mise-en-scène" turns out to be quite delicate. Definitely better than the metaphor "a teacher is a technician" emerges here the metaphor "a teacher is a tightrope walker". In particular, experience shows that didactical micro-gestures of the teacher - tightrope walker may feed a butterfly effect in the classroom.

Notice that in our metaphoric approach the students are not given a "Grundvorstellung" (fundamental notion) for probability before they address the problem. On the contrary, they are prompted to tackle the problem "bare handed" first and eventually look for a friendly metaphor for the *concrete random walk* they want to study (e.g. "Brownie splits"). When trying to give pertinent answers to the questions asked, "poietic metaphors" may emerge that enable them to construct the abstract probability concept, like: "probabilities of finding Brownie at a given corner are pieces of Brownie". This fits the framework of Brousseau's didactical (better, adidactical) situations (Brousseau, 1998).

Brownie's walk seen from Google Earth (zooming out).

First, Brownie's walk can approximate efficiently Brownian motion: just make the grid denser and denser…If you zoom out, with the help of Google Earth, tracking Brownie with GPS, you will see the trajectories of pollen grains.

Second, the title of this section is also metaphoric: if we zoom out cognitively, we realize that Brownie's random walk is a paradigmatic example of a random walk (a metonymy, in fact) and that random walks play very often the role of universal models in probabilistic as well as statistical problems. For example:

1. The famous Italian unfinished tournament problem: Two players of equal strength compete in a tournament that consists of a series of games. The winner, who will get the one million euros prize, is the one who completes 10 wins first. Now, the tournament must be interrupted by "force majeure" when one player had won 8 times and the other 7 times. How should the prize be divided fairly between the two players?
This problem can solved very easily by modelling or metaphorizing it by a random walk (a 2D one!) and solving the random walk metaphorically. This is friendlier than Pascal classical solution (apparently he was not very fond of random walks). If we recall that "mathematics is the art of seeing the invisible" (Soto-Andrade, 2008) we would agree that students are doing real mathematics when they see the evolution of the tournament as a random walk on a 2D grid with absorbing barriers.

2. Evaluating screening tests (e.g. false positive problems) is a tough task for experienced physicians (Zhu & Gigerenzer, 2006; Gigerenzer, 2011). The typical question that physicians answer wrongly is: "I have got a positive HIV test, how likely is that I am really a carrier?"

Natural frequencies have been suggested (loc. cit.) as a means to get a correct answer without much toil, in a way a 10 year old could do. But if you look at this problem as a question about a (2 step) random walk, that you can solve with the help of a pedestrian metaphor, your recover exactly natural frequencies.

It is interesting to compare the relative popularity, among students in various levels and countries, of the hydraulic and pedestrian metaphors. According to Gigerenzer (loc. cit.), pedestrian metaphors (i.e. natural frequencies) should be much more popular, because you just manipulate whole integers and you compute a fraction only at the very end, or even you get your result in the form "*m* out of *n*", as in pre-fraction days, that is all the same efficient for practical purposes. In the hydraulic metaphor however you need to manipulate fractions all the way. We have found experimentally however, consistently year after year, that 1st year university students majoring in Social Sciences and Humanities (group a) above) tend to prefer the hydraulic metaphor (8 out of 10 approx.), even if computing with fractions is not all that smooth for them. Apparently one reason for that is the *conceptual* impact of seeing one litre of grapefruit juice (or probabilistic fluid, if you like) draining downwards through a graph of ducts, something that helps them tackle Zeno's paradox, for instance.

3. The classical fundamental didactical situation of Brousseau (1998) for statistics where 7th graders try to find out the composition of a bottle containing 5 marbles of 2 different colours by looking inside through a tiny opening (that lets them see just one marble at a time), can be modelled by a random walk in the plane that "ends up" engaging into *one* region out of four possible wedge-like regions (corresponding to the 1-4, 2-3, 3-2 and 4-1 compositions)

CONCLUSIONS.

We claimed that random walks are a means to facilitate access to stochastic thinking, especially for non-mathematically oriented learners. As grist to the mill of this claim, we described here an explicit didactical engineering for a concrete example of a random walk (Brownie's walk) whose bare handed study facilitates the construction of the concept of probability, thanks to the likely emergence of various helpful (mainly enactive) metaphors.

We have tested this didactical engineering with learners of several backgrounds, students as well as in service teachers across Chile. According to their performance (assessed by tests and group work) and answers to questionnaires and interviews, this approach (random walks metaphorically tackled) did help them to understand and even construct otherwise cryptic mathematical notions and to solve problems involving randomness in a friendly and efficient way (e.g. false positive problems). We noticed some initial resistance to metaphorizing, mainly among university students majoring in mathematics and secondary school teachers. University students majoring in humanities and social sciences and primary school teachers were more prone to metaphorizing, especially after being granted permission to do so ("you are not supposed to metaphorize in mathematics" seems to be the implicit current didactical contract). Among university students remarkable metaphorical capacities were detected in students coming from alternative schools, like Montessori or Waldorf. Otherwise it seems clear that traditional teaching in Chile, especially from grade 7th onwards tends to thwart metaphorizing. This suggests promoting this approach already at the beginning of elementary school and exploring related approaches emphasizing embodiment and enaction. Further research on this didactical phenomenon, quantitative as well as qualitative, seems commendable.

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