Derivation of relation between the time derivatives

Here we derive the relation between the Lagrangian and Eulerian time derivatives. Consider the Lagrangian fluid particle that occupies the point O in some fixed Eulerian framework at time $t = 0$. If the vector velocity of the fluid at this point and time is denoted by $\underline{u}$ then a short time δt later the fluid particle will be displaced to the point O' where the vector OO' is given by $\underline{u}\delta t$. In a Cartesian framework with its origin at O, the coordinates of the point O' will be $(\delta x = u\delta t, \delta y = v\delta t, \delta z = w\delta t)$.

Now consider the Lagrangian and Eulerian time derivatives of some general transportable property in the fluid motion that we will denote by Q. By definition, the Eulerian time derivative of Q at O is simply $\partial Q/\partial t$. By contrast the Lagrangian time derivative, DQ/Dt will be given by

$$\frac{DQ}{Dt} = \left[\frac{\{Q\}_{O',t=\delta t} - \{Q\}_{O,t=0}}{\delta t} \right]_{\delta t \to 0} \tag{Bad1}$$

which, using the first two terms in a Taylor series expansion to write $\{Q\}_{O',t=\delta t}$ in terms of quantities evaluated at O at time $t = 0$, leads to

$$\frac{DQ}{Dt} = \left[\frac{Q + \frac{\partial Q}{\partial t}\delta t + \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta y + \frac{\partial Q}{\partial z}\delta z - Q}{\delta t} \right]_{\delta t \to 0} \tag{Bad2}$$

where all quantities are now evaluated at O and time $t = 0$; all second and higher order terms in the Taylor series expansion have been omitted since they disappear when $\delta t \to 0$.

Substituting for δx, δy, and δz using $\delta x = u\delta t$, $\delta y = v\delta t$, and $\delta z = w\delta t$ and then taking the limit as $\delta t \to 0$ leads to

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z} \tag{Bad3}$$

Consequently the fundamental relationship between the Lagrangian and Eulerian time derivatives is

$$\begin{aligned} \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \\ &\equiv \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} \\ &\equiv \frac{\partial}{\partial t} + (\underline{u} \, .\nabla) \end{aligned} \tag{Bad4}$$

where both the tensor and vector forms will be used in the material that follows.