Lagrangian and Eulerian Time Derivatives

The first tool that we will need for this conversion process is a relation between the time derivatives within each of these descriptions of motion. The Lagrangian time derivative (often called the material time derivative) is denoted by the operator D/Dt and, as its name implies, is defined as the rate of change with time of some property of the fluid (denoted here by Q which could be the velocity, density, pressure, etc.) within some particular fluid element; that is to say as we move along with the fluid. On the other hand the Eulerian time derivative, denoted here by $\partial/\partial t$ is the rate of change with time of some fluid property at a fixed point within the coordinate frame of reference that we have chosen for that Eulerian view.

By way of an illustrative example, think of the fluid flow as a highway filled with rush-hour traffic and consider Q in this example to be the velocity of the vehicles. Then the Lagrangian acceleration, DQ/Dt, is the acceleration of an individual vehicle as it speeds up or slows down during its journey. On the other hand, the Eulerian acceleration, $\partial/\partial t$, would be the rate of increase or decrease in velocity of the vehicles at one particular location on the highway. Clearly these two accelerations are not necessarily the same. Though they must be related in some way, that relationship is not immediately obvious.

Therefore one of the first tasks we face in developing the basic equations of fluid motion is to find the fundamental relationship between the Lagrangian and Eulerian time derivatives. It can be shown that

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} \equiv \frac{\partial}{\partial t} + (\underline{u}\ .\nabla) \tag{Bab1}$$

In cylindrical coordinates, r, θ, z, with velocities u_r, u_θ, u_z the Lagrangian derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \tag{Bab2}$$

In spherical coordinates, r, θ, ϕ with velocities u_r, u_θ, u_ϕ the Lagrangian derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{Bab3}$$

To illustrate the above result and its consequences let us consider, as an example, the accelerations that can occur in a fluid flow. For this purpose, we examine the time derivatives of the velocity, u_i, namely the Lagrangian accelerations, Du/Dt, and the Eulerian acceleration, $\partial u_i/\partial t$. The former is the acceleration of a particle fluid element within a flow, the rate of increase of the velocity as we travel with the fluid element. On the other hand, the latter is the rate of increase of the velocity at a particular, fixed point in the flow. From the above relation:

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} \tag{Bab4}$$

or in terms of its Cartesian components:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{Bab5}$$

Note, most obviously that the Lagrangian and Eulerian accelerations are, in general, different and the differences are related to the spatial velocity gradients in the flow.

At this point, it is appropriate to define what we mean by the phrase "steady flow". A steady flow in a particular coordinate system is one in which the velocities at every point are not changing with time so that

$$\frac{\partial u_i}{\partial t} = 0 \quad ; \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 \tag{Bab6}$$

In other words the Eulerian time derivatives of the velocity are zero. Note from the above that this does not mean that there is no acceleration in the flow since the Lagrangian accelerations are not necessarily zero. Indeed in a steady flow it follows that

$$\begin{aligned} \frac{Du_i}{Dt} &= u_j\frac{\partial u_i}{\partial x_j} \\ \frac{Du}{Dt} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \frac{Dv}{Dt} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \frac{Dw}{Dt} &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned} \tag{Bab7}$$

A very simple example is to consider the steady flow through a duct as shown in figure 1. By definition, since the flow is steady the velocity at some fixed point in the nozzle such as A is unchanging with time. But since the velocity is decreasing from left to right, the Lagrangian acceleration, the acceleration of a fluid element passing through the point A is not zero. Thus we demonstrate that Newton's law applied to that fluid element must utilize the Lagrangian rather than the Eulerian acceleration.

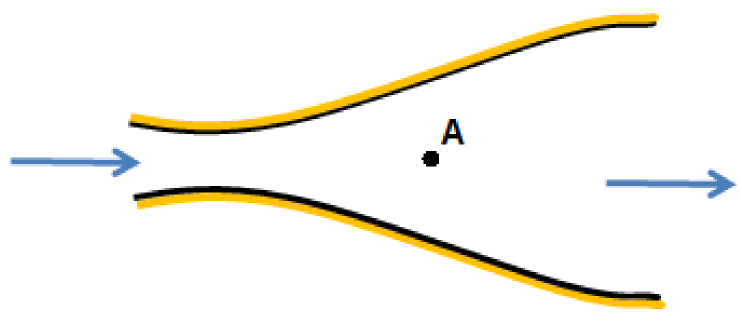

Figure 1: Duct Flow