5

Analytic Trigonometry

5.1 Using Fundamental Identities

5.2 Verifying Trigonometric Identities

5.3 Solving Trigonometric Equations

5.4 Sum and Difference Formulas

5.5 Multiple-Angle and Product-to-Sum Formulas

In Mathematics

Analytic trigonometry is used to simplify trigonometric expressions and solve trigonometric equations.

In Real Life

Analytic trigonometry is used to model real-life phenomena. For instance, when an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane. Concepts of trigonometry can be used to describe the apex angle of the cone. (See Exercise 137, page 415.)

Christopher Pasatier/Reuters/Landov

IN CAREERS

There are many careers that use analytic trigonometry. Several are listed below.

- Mechanical Engineer
  Exercise 89, page 396
- Physicist
  Exercise 90, page 403
- Athletic Trainer
  Exercise 135, page 415
- Physical Therapist
  Exercise 8, page 425

5.1 Using Fundamental Identities

What you should learn

- Recognize and write the fundamental trigonometric identities.
- Use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions.

Why you should learn it

Fundamental trigonometric identities can be used to simplify trigonometric expressions. For instance, in Exercise 123 on page 379, you can use trigonometric identities to simplify an expression for the coefficient of friction.

Introduction

In Chapter 4, you studied the basic definitions, properties, graphs, and applications of the individual trigonometric functions. In this chapter, you will learn how to use the fundamental identities to do the following.

1. Evaluate trigonometric functions.
2. Simplify trigonometric expressions.
3. Develop additional trigonometric identities.
4. Solve trigonometric equations.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \cos(-u) = \cos u \qquad \tan(-u) = -\tan u$$

$$\csc(-u) = -\csc u \qquad \sec(-u) = \sec u \qquad \cot(-u) = -\cot u$$

Study Tip

You should learn the fundamental trigonometric identities well, because they are used frequently in trigonometry and they will also appear later in calculus. Note that u can be an angle, a real number, or a variable.

Pythagorean identities are sometimes used in radical form such as

$$\sin u = \pm\sqrt{1 - \cos^2 u}$$

or

$$\tan u = \pm\sqrt{\sec^2 u - 1}$$

where the sign depends on the choice of u.

Using the Fundamental Identities

One common application of trigonometric identities is to use given values of trigonometric functions to evaluate other trigonometric functions.

Example 1 Using Identities to Evaluate a Function

Use the values $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

Solution

Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\sin^2 u = 1 - \cos^2 u \qquad \text{Pythagorean identity}$$

$$= 1 - \left(-\frac{2}{3}\right)^2 \qquad \text{Substitute } -\tfrac{2}{3} \text{ for } \cos u.$$

$$= 1 - \frac{4}{9} = \frac{5}{9}. \qquad \text{Simplify.}$$

Because $\sec u < 0$ and $\tan u > 0$, it follows that u lies in Quadrant III. Moreover, because $\sin u$ is negative when u is in Quadrant III, you can choose the negative root and obtain $\sin u = -\sqrt{5}/3$. Now, knowing the values of the sine and cosine, you can find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3} \qquad \csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\cos u = -\frac{2}{3} \qquad \sec u = \frac{1}{\cos u} = -\frac{3}{2}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2} \qquad \cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

CHECK*Point* Now try Exercise 21.

Example 2 Simplifying a Trigonometric Expression

Simplify $\sin x \cos^2 x - \sin x$.

Solution

First factor out a common monomial factor and then use a fundamental identity.

$$\sin x \cos^2 x - \sin x = \sin x(\cos^2 x - 1) \qquad \text{Factor out common monomial factor.}$$

$$= -\sin x(1 - \cos^2 x) \qquad \text{Factor out } -1.$$

$$= -\sin x(\sin^2 x) \qquad \text{Pythagorean identity}$$

$$= -\sin^3 x \qquad \text{Multiply.}$$

CHECK*Point* Now try Exercise 59.

TECHNOLOGY

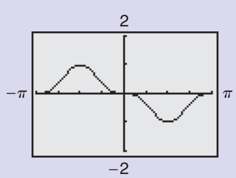
You can use a graphing utility to check the result of Example 2. To do this, graph

$$y_1 = \sin x \cos^2 x - \sin x$$

and

$$y_2 = -\sin^3 x$$

in the same viewing window, as shown below. Because Example 2 shows the equivalence algebraically and the two graphs appear to coincide, you can conclude that the expressions are equivalent.



When factoring trigonometric expressions, it is helpful to find a special polynomial factoring form that fits the expression, as shown in Example 3.

Example 3 Factoring Trigonometric Expressions

Factor each expression.

a. $\sec^2\theta - 1$ **b.** $4\tan^2\theta + \tan\theta - 3$

Solution

a. This expression has the form $u^2 - v^2$, which is the difference of two squares. It factors as

$$\sec^2\theta - 1 = (\sec\theta - 1)(\sec\theta + 1).$$

b. This expression has the polynomial form $ax^2 + bx + c$, and it factors as

$$4\tan^2\theta + \tan\theta - 3 = (4\tan\theta - 3)(\tan\theta + 1).$$

CHECKPoint Now try Exercise 61.

Algebra Help

In Example 3, you need to be able to factor the difference of two squares and factor a trinomial. You can review the techniques for factoring in Appendix A.3.

On occasion, factoring or simplifying can best be done by first rewriting the expression in terms of just *one* trigonometric function or in terms of *sine and cosine only*. These strategies are shown in Examples 4 and 5, respectively.

Example 4 Factoring a Trigonometric Expression

Factor $\csc^2 x - \cot x - 3$.

Solution

Use the identity $\csc^2 x = 1 + \cot^2 x$ to rewrite the expression in terms of the cotangent.

$$\csc^2 x - \cot x - 3 = (1 + \cot^2 x) - \cot x - 3 \quad \text{Pythagorean identity}$$
$$= \cot^2 x - \cot x - 2 \quad \text{Combine like terms.}$$
$$= (\cot x - 2)(\cot x + 1) \quad \text{Factor.}$$

CHECKPoint Now try Exercise 65.

Example 5 Simplifying a Trigonometric Expression

Simplify $\sin t + \cot t \cos t$.

Solution

Begin by rewriting $\cot t$ in terms of sine and cosine.

$$\sin t + \cot t\cos t = \sin t + \left(\frac{\cos t}{\sin t}\right)\cos t \quad \text{Quotient identity}$$
$$= \frac{\sin^2 t + \cos^2 t}{\sin t} \quad \text{Add fractions.}$$
$$= \frac{1}{\sin t} \quad \text{Pythagorean identity}$$
$$= \csc t \quad \text{Reciprocal identity}$$

CHECKPoint Now try Exercise 71.

Study Tip

Remember that when adding rational expressions, you must first find the least common denominator (LCD). In Example 5, the LCD is $\sin t$.

Example 6 Adding Trigonometric Expressions

Perform the addition and simplify.

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}$$

Solution

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{(\sin\theta)(\sin\theta)+(\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)}$$

$$=\frac{\sin^2\theta+\cos^2\theta+\cos\theta}{(1+\cos\theta)(\sin\theta)}$$ Multiply.

$$=\frac{1+\cos\theta}{(1+\cos\theta)(\sin\theta)}$$ Pythagorean identity: $\sin^2\theta+\cos^2\theta=1$

$$=\frac{1}{\sin\theta}$$ Divide out common factor.

$$=\csc\theta$$ Reciprocal identity

CHECK*Point* Now try Exercise 75.

The next two examples involve techniques for rewriting expressions in forms that are used in calculus.

Example 7 Rewriting a Trigonometric Expression

Rewrite $\frac{1}{1+\sin x}$ so that it is *not* in fractional form.

Solution

From the Pythagorean identity $\cos^2 x=1-\sin^2 x=(1-\sin x)(1+\sin x)$, you can see that multiplying both the numerator and the denominator by $(1-\sin x)$ will produce a monomial denominator.

$$\frac{1}{1+\sin x}=\frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}$$ Multiply numerator and denominator by $(1-\sin x)$.

$$=\frac{1-\sin x}{1-\sin^2 x}$$ Multiply.

$$=\frac{1-\sin x}{\cos^2 x}$$ Pythagorean identity

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}$$ Write as separate fractions.

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$ Product of fractions

$$=\sec^2 x-\tan x\sec x$$ Reciprocal and quotient identities

CHECK*Point* Now try Exercise 81.

Example 8 Trigonometric Substitution

Use the substitution $x = 2\tan\theta$, $0 < \theta < \pi/2$, to write

$$\sqrt{4 + x^2}$$

as a trigonometric function of θ.

Solution

Begin by letting $x = 2\tan\theta$. Then, you can obtain

$$\sqrt{4 + x^2} = \sqrt{4 + (2\tan\theta)^2} \quad \text{Substitute } 2\tan\theta \text{ for } x.$$

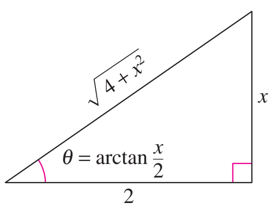
$$= \sqrt{4 + 4\tan^2\theta} \quad \text{Rule of exponents}$$

$$= \sqrt{4(1 + \tan^2\theta)} \quad \text{Factor.}$$

$$= \sqrt{4\sec^2\theta} \quad \text{Pythagorean identity}$$

$$= 2\sec\theta. \quad \sec\theta > 0 \text{ for } 0 < \theta < \pi/2$$

CHECK*Point* Now try Exercise 93.



Angle whose tangent is $\pi/2$.
FIGURE 5.1

Figure 5.1 shows the right triangle illustration of the trigonometric substitution $x = 2\tan\theta$ in Example 8. You can use this triangle to check the solution of Example 8. For $0 < \theta < \pi/2$, you have

$$\text{opp} = x, \quad \text{adj} = 2, \quad \text{and} \quad \text{hyp} = \sqrt{4 + x^2}.$$

With these expressions, you can write the following.

$$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\sec\theta = \frac{\sqrt{4 + x^2}}{2}$$

$$2\sec\theta = \sqrt{4 + x^2}$$

So, the solution checks.

Example 9 Rewriting a Logarithmic Expression

Rewrite $\ln|\csc\theta| + \ln|\tan\theta|$ as a single logarithm and simplify the result.

Solution

$$\ln|\csc\theta| + \ln|\tan\theta| = \ln|\csc\theta\tan\theta| \quad \text{Product Property of Logarithms}$$

$$= \ln\left|\frac{1}{\sin\theta} \cdot \frac{\sin\theta}{\cos\theta}\right| \quad \text{Reciprocal and quotient identities}$$

$$= \ln\left|\frac{1}{\cos\theta}\right| \quad \text{Simplify.}$$

$$= \ln|\sec\theta| \quad \text{Reciprocal identity}$$

CHECK*Point* Now try Exercise 113.

Algebra Help

Recall that for positive real numbers u and v,

$$\ln u + \ln v = \ln(uv).$$

You can review the properties of logarithms in Section 3.3.

5.1 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blank to complete the trigonometric identity.

1. $\frac{\sin u}{\cos u} = $ ________
2. $\frac{1}{\csc u} = $ ________
3. $\frac{1}{\tan u} = $ ________
4. $\frac{1}{\cos u} = $ ________
5. $1 + $ ________ $= \csc^2 u$
6. $1 + \tan^2 u = $ ________
7. $\sin\left(\frac{\pi}{2} - u\right) = $ ________
8. $\sec\left(\frac{\pi}{2} - u\right) = $ ________
9. $\cos(-u) = $ ________
10. $\tan(-u) = $ ________

SKILLS AND APPLICATIONS

In Exercises 11–24, use the given values to evaluate (if possible) all six trigonometric functions.

11. $\sin x = \frac{1}{2}, \quad \cos x = \frac{\sqrt{3}}{2}$
12. $\tan x = \frac{\sqrt{3}}{3}, \quad \cos x = -\frac{\sqrt{3}}{2}$
13. $\sec \theta = \sqrt{2}, \quad \sin \theta = -\frac{\sqrt{2}}{2}$
14. $\csc \theta = \frac{25}{7}, \quad \tan \theta = \frac{7}{24}$
15. $\tan x = \frac{8}{15}, \quad \sec x = -\frac{17}{15}$
16. $\cot \phi = -3, \quad \sin \phi = \frac{\sqrt{10}}{10}$
17. $\sec \phi = \frac{3}{2}, \quad \csc \phi = -\frac{3\sqrt{5}}{5}$
18. $\cos\left(\frac{\pi}{2} - x\right) = \frac{3}{5}, \quad \cos x = \frac{4}{5}$
19. $\sin(-x) = -\frac{1}{3}, \quad \tan x = -\frac{\sqrt{2}}{4}$
20. $\sec x = 4, \quad \sin x > 0$
21. $\tan \theta = 2, \quad \sin \theta < 0$
22. $\csc \theta = -5, \quad \cos \theta < 0$
23. $\sin \theta = -1, \quad \cot \theta = 0$
24. $\tan \theta$ is undefined, $\quad \sin \theta > 0$

In Exercises 25–30, match the trigonometric expression with one of the following.

(a) $\sec x$ (b) -1 (c) $\cot x$
(d) 1 (e) $-\tan x$ (f) $\sin x$

25. $\sec x \cos x$
26. $\tan x \csc x$
27. $\cot^2 x - \csc^2 x$
28. $(1 - \cos^2 x)(\csc x)$
29. $\frac{\sin(-x)}{\cos(-x)}$
30. $\frac{\sin[(\pi/2) - x]}{\cos[(\pi/2) - x]}$

In Exercises 31–36, match the trigonometric expression with one of the following.

(a) $\csc x$ (b) $\tan x$ (c) $\sin^2 x$
(d) $\sin x \tan x$ (e) $\sec^2 x$ (f) $\sec^2 x + \tan^2 x$

31. $\sin x \sec x$
32. $\cos^2 x(\sec^2 x - 1)$
33. $\sec^4 x - \tan^4 x$
34. $\cot x \sec x$
35. $\frac{\sec^2 x - 1}{\sin^2 x}$
36. $\frac{\cos^2[(\pi/2) - x]}{\cos x}$

In Exercises 37–58, use the fundamental identities to simplify the expression. There is more than one correct form of each answer.

37. $\cot \theta \sec \theta$
38. $\cos \beta \tan \beta$
39. $\tan(-x) \cos x$
40. $\sin x \cot(-x)$
41. $\sin \phi(\csc \phi - \sin \phi)$
42. $\sec^2 x(1 - \sin^2 x)$
43. $\frac{\cot x}{\csc x}$
44. $\frac{\csc \theta}{\sec \theta}$
45. $\frac{1 - \sin^2 x}{\csc^2 x - 1}$
46. $\frac{1}{\tan^2 x + 1}$
47. $\frac{\tan \theta \cot \theta}{\sec \theta}$
48. $\frac{\sin \theta \csc \theta}{\tan \theta}$
49. $\sec \alpha \cdot \frac{\sin \alpha}{\tan \alpha}$
50. $\frac{\tan^2 \theta}{\sec^2 \theta}$
51. $\cos\left(\frac{\pi}{2} - x\right) \sec x$
52. $\cot\left(\frac{\pi}{2} - x\right) \cos x$
53. $\frac{\cos^2 y}{1 - \sin y}$
54. $\cos t(1 + \tan^2 t)$
55. $\sin \beta \tan \beta + \cos \beta$
56. $\csc \phi \tan \phi + \sec \phi$
57. $\cot u \sin u + \tan u \cos u$
58. $\sin \theta \sec \theta + \cos \theta \csc \theta$

In Exercises 59–70, factor the expression and use the fundamental identities to simplify. There is more than one correct form of each answer.

59. $\tan^2 x - \tan^2 x \sin^2 x$

60. $\sin^2 x \csc^2 x - \sin^2 x$

61. $\sin^2 x \sec^2 x - \sin^2 x$

62. $\cos^2 x + \cos^2 x \tan^2 x$

63. $\dfrac{\sec^2 x - 1}{\sec x - 1}$

64. $\dfrac{\cos^2 x - 4}{\cos x - 2}$

65. $\tan^4 x + 2\tan^2 x + 1$

66. $1 - 2\cos^2 x + \cos^4 x$

67. $\sin^4 x - \cos^4 x$

68. $\sec^4 x - \tan^4 x$

69. $\csc^3 x - \csc^2 x - \csc x + 1$

70. $\sec^3 x - \sec^2 x - \sec x + 1$

In Exercises 71–74, perform the multiplication and use the fundamental identities to simplify. There is more than one correct form of each answer.

71. $(\sin x + \cos x)^2$

72. $(\cot x + \csc x)(\cot x - \csc x)$

73. $(2\csc x + 2)(2\csc x - 2)$

74. $(3 - 3\sin x)(3 + 3\sin x)$

In Exercises 75–80, perform the addition or subtraction and use the fundamental identities to simplify. There is more than one correct form of each answer.

75. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x}$

76. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$

77. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x}$

78. $\dfrac{\tan x}{1 + \sec x} + \dfrac{1 + \sec x}{\tan x}$

79. $\tan x + \dfrac{\cos x}{1 + \sin x}$

80. $\tan x - \dfrac{\sec^2 x}{\tan x}$

In Exercises 81–84, rewrite the expression so that it is not in fractional form. There is more than one correct form of each answer.

81. $\dfrac{\sin^2 y}{1 - \cos y}$

82. $\dfrac{5}{\tan x + \sec x}$

83. $\dfrac{3}{\sec x - \tan x}$

84. $\dfrac{\tan^2 x}{\csc x + 1}$

NUMERICAL AND GRAPHICAL ANALYSIS In Exercises 85–88, use a graphing utility to complete the table and graph the functions. Make a conjecture about y_1 and y_2.

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1							
y_2							

85. $y_1 = \cos\left(\dfrac{\pi}{2} - x\right), \quad y_2 = \sin x$

86. $y_1 = \sec x - \cos x, \quad y_2 = \sin x \tan x$

87. $y_1 = \dfrac{\cos x}{1 - \sin x}, \quad y_2 = \dfrac{1 + \sin x}{\cos x}$

88. $y_1 = \sec^4 x - \sec^2 x, \quad y_2 = \tan^2 x + \tan^4 x$

In Exercises 89–92, use a graphing utility to determine which of the six trigonometric functions is equal to the expression. Verify your answer algebraically.

89. $\cos x \cot x + \sin x$

90. $\sec x \csc x - \tan x$

91. $\dfrac{1}{\sin x}\left(\dfrac{1}{\cos x} - \cos x\right)$

92. $\dfrac{1}{2}\left(\dfrac{1 + \sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta}\right)$

In Exercises 93–104, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

93. $\sqrt{9 - x^2}, \quad x = 3\cos\theta$

94. $\sqrt{64 - 16x^2}, \quad x = 2\cos\theta$

95. $\sqrt{16 - x^2}, \quad x = 4\sin\theta$

96. $\sqrt{49 - x^2}, \quad x = 7\sin\theta$

97. $\sqrt{x^2 - 9}, \quad x = 3\sec\theta$

98. $\sqrt{x^2 - 4}, \quad x = 2\sec\theta$

99. $\sqrt{x^2 + 25}, \quad x = 5\tan\theta$

100. $\sqrt{x^2 + 100}, \quad x = 10\tan\theta$

101. $\sqrt{4x^2 + 9}, \quad 2x = 3\tan\theta$

102. $\sqrt{9x^2 + 25}, \quad 3x = 5\tan\theta$

103. $\sqrt{2 - x^2}, \quad x = \sqrt{2}\sin\theta$

104. $\sqrt{10 - x^2}, \quad x = \sqrt{10}\sin\theta$

In Exercises 105–108, use the trigonometric substitution to write the algebraic equation as a trigonometric equation of θ, where $-\pi/2 < \theta < \pi/2$. Then find $\sin\theta$ and $\cos\theta$.

105. $3 = \sqrt{9 - x^2}, \quad x = 3\sin\theta$

106. $3 = \sqrt{36 - x^2}, \quad x = 6\sin\theta$

107. $2\sqrt{2} = \sqrt{16 - 4x^2}, \quad x = 2\cos\theta$

108. $-5\sqrt{3} = \sqrt{100 - x^2}, \quad x = 10\cos\theta$

In Exercises 109–112, use a graphing utility to solve the equation for θ, where $0 \le \theta < 2\pi$.

109. $\sin\theta = \sqrt{1 - \cos^2\theta}$

110. $\cos\theta = -\sqrt{1 - \sin^2\theta}$

111. $\sec\theta = \sqrt{1 + \tan^2\theta}$

112. $\csc\theta = \sqrt{1 + \cot^2\theta}$

In Exercises 113–118, rewrite the expression as a single logarithm and simplify the result.

113. $\ln|\cos x| - \ln|\sin x|$ **114.** $\ln|\sec x| + \ln|\sin x|$

115. $\ln|\sin x| + \ln|\cot x|$ **116.** $\ln|\tan x| + \ln|\csc x|$

117. $\ln|\cot t| + \ln(1 + \tan^2 t)$

118. $\ln(\cos^2 t) + \ln(1 + \tan^2 t)$

In Exercises 119–122, use a calculator to demonstrate the identity for each value of *θ*.

119. $\csc^2 \theta - \cot^2 \theta = 1$

(a) $\theta = 132°$ (b) $\theta = \dfrac{2\pi}{7}$

120. $\tan^2 \theta + 1 = \sec^2 \theta$

(a) $\theta = 346°$ (b) $\theta = 3.1$

121. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin \theta$

(a) $\theta = 80°$ (b) $\theta = 0.8$

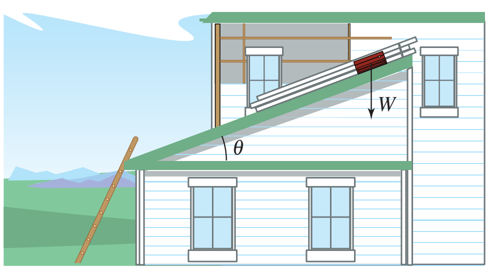
122. $\sin(-\theta) = -\sin \theta$

(a) $\theta = 250°$ (b) $\theta = \frac{1}{2}$

123. FRICTION The forces acting on an object weighing W units on an inclined plane positioned at an angle of θ with the horizontal (see figure) are modeled by

$$\mu W \cos \theta = W \sin \theta$$

where μ is the coefficient of friction. Solve the equation for μ and simplify the result.



124. RATE OF CHANGE The rate of change of the function $f(x) = -x + \tan x$ is given by the expression $-1 + \sec^2 x$. Show that this expression can also be written as $\tan^2 x$.

125. RATE OF CHANGE The rate of change of the function $f(x) = \sec x + \cos x$ is given by the expression $\sec x \tan x - \sin x$. Show that this expression can also be written as $\sin x \tan^2 x$.

126. RATE OF CHANGE The rate of change of the function $f(x) = -\csc x - \sin x$ is given by the expression $\csc x \cot x - \cos x$. Show that this expression can also be written as $\cos x \cot^2 x$.

EXPLORATION

TRUE OR FALSE? **In Exercises 127 and 128, determine whether the statement is true or false. Justify your answer.**

127. The even and odd trigonometric identities are helpful for determining whether the value of a trigonometric function is positive or negative.

128. A cofunction identity can be used to transform a tangent function so that it can be represented by a cosecant function.

In Exercises 129–132, fill in the blanks. (*Note:* The notation $x \to c^+$ indicates that x approaches c from the right and $x \to c^-$ indicates that x approaches c from the left.)

129. As $x \to \dfrac{\pi}{2}^-$, $\sin x \to$ ______ and $\csc x \to$ ______.

130. As $x \to 0^+$, $\cos x \to$ ______ and $\sec x \to$ ______.

131. As $x \to \dfrac{\pi}{2}^-$, $\tan x \to$ ______ and $\cot x \to$ ______.

132. As $x \to \pi^+$, $\sin x \to$ ______ and $\csc x \to$ ______.

In Exercises 133–138, determine whether or not the equation is an identity, and give a reason for your answer.

133. $\cos \theta = \sqrt{1 - \sin^2 \theta}$ **134.** $\cot \theta = \sqrt{\csc^2 \theta + 1}$

135. $\dfrac{(\sin k\theta)}{(\cos k\theta)} = \tan \theta$, k is a constant.

136. $\dfrac{1}{(5 \cos \theta)} = 5 \sec \theta$

137. $\sin \theta \csc \theta = 1$ **138.** $\csc^2 \theta = 1$

139. Use the trigonometric substitution $u = a \sin \theta$, where $-\pi/2 < \theta < \pi/2$ and $a > 0$, to simplify the expression $\sqrt{a^2 - u^2}$.

140. Use the trigonometric substitution $u = a \tan \theta$, where $-\pi/2 < \theta < \pi/2$ and $a > 0$, to simplify the expression $\sqrt{a^2 + u^2}$.

141. Use the trigonometric substitution $u = a \sec \theta$, where $0 < \theta < \pi/2$ and $a > 0$, to simplify the expression $\sqrt{u^2 - a^2}$.

142. CAPSTONE

(a) Use the definitions of sine and cosine to derive the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.

(b) Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to derive the other Pythagorean identities, $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$. Discuss how to remember these identities and other fundamental identities.

5.2 Verifying Trigonometric Identities

What you should learn

- Verify trigonometric identities.

Why you should learn it

You can use trigonometric identities to rewrite trigonometric equations that model real-life situations. For instance, in Exercise 70 on page 386, you can use trigonometric identities to simplify the equation that models the length of a shadow cast by a gnomon (a device used to tell time).

Robert W. Ginn/PhotoEdit

Introduction

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to verifying identities *and* solving equations is the ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a *conditional equation* is an equation that is true for only some of the values in its domain. For example, the conditional equation

$$\sin x = 0 \qquad \text{Conditional equation}$$

is true only for $x = n\pi$, where n is an integer. When you find these values, you are *solving* the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$$\sin^2 x = 1 - \cos^2 x \qquad \text{Identity}$$

is true for all real numbers x. So, it is an identity.

Verifying Trigonometric Identities

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, and the process is best learned by practice.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. If the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Always try *something*. Even paths that lead to dead ends provide insights.

Verifying trigonometric identities is a useful process if you need to convert a trigonometric expression into a form that is more useful algebraically. When you verify an identity, you cannot *assume* that the two sides of the equation are equal because you are trying to verify that they *are* equal. As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.

Example 1 Verifying a Trigonometric Identity

Verify the identity $(\sec^2 \theta - 1)/\sec^2 \theta = \sin^2 \theta$.

Solution

The left side is more complicated, so start with it.

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{(\tan^2 \theta + 1) - 1}{\sec^2 \theta} \qquad \text{Pythagorean identity}$$

$$= \frac{\tan^2 \theta}{\sec^2 \theta} \qquad \text{Simplify.}$$

$$= \tan^2 \theta(\cos^2 \theta) \qquad \text{Reciprocal identity}$$

$$= \frac{\sin^2 \theta}{(\cos^2 \theta)}(\cos^2 \theta) \qquad \text{Quotient identity}$$

$$= \sin^2 \theta \qquad \text{Simplify.}$$

Notice how the identity is verified. You start with the left side of the equation (the more complicated side) and use the fundamental trigonometric identities to simplify it until you obtain the right side.

CHECKPoint Now try Exercise 15.

WARNING / CAUTION

Remember that an identity is only true for all real values in the domain of the variable. For instance, in Example 1 the identity is not true when $\theta = \pi/2$ because $\sec^2 \theta$ is not defined when $\theta = \pi/2$.

There can be more than one way to verify an identity. Here is another way to verify the identity in Example 1.

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{\sec^2 \theta}{\sec^2 \theta} - \frac{1}{\sec^2 \theta} \qquad \text{Rewrite as the difference of fractions.}$$

$$= 1 - \cos^2 \theta \qquad \text{Reciprocal identity}$$

$$= \sin^2 \theta \qquad \text{Pythagorean identity}$$

Example 2 Verifying a Trigonometric Identity

Verify the identity $2 \sec^2 \alpha = \dfrac{1}{1 - \sin \alpha} + \dfrac{1}{1 + \sin \alpha}$.

Algebraic Solution

The right side is more complicated, so start with it.

$$\frac{1}{1 - \sin \alpha} + \frac{1}{1 + \sin \alpha} = \frac{1 + \sin \alpha + 1 - \sin \alpha}{(1 - \sin \alpha)(1 + \sin \alpha)} \qquad \text{Add fractions.}$$

$$= \frac{2}{1 - \sin^2 \alpha} \qquad \text{Simplify.}$$

$$= \frac{2}{\cos^2 \alpha} \qquad \text{Pythagorean identity}$$

$$= 2 \sec^2 \alpha \qquad \text{Reciprocal identity}$$

CHECKPoint Now try Exercise 31.

Numerical Solution

Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = 2/\cos^2 x$ and $y_2 = 1/(1 - \sin x) + 1/(1 + \sin x)$ for different values of x, as shown in Figure 5.2. From the table, you can see that the values appear to be identical, so $2 \sec^2 x = 1/(1 - \sin x) + 1/(1 + \sin x)$ appears to be an identity.

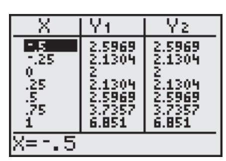


FIGURE 5.2

Example 3 Verifying a Trigonometric Identity

Verify the identity $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$.

Algebraic Solution

By applying identities before multiplying, you obtain the following.

$$(\tan^2 x + 1)(\cos^2 x - 1) = (\sec^2 x)(-\sin^2 x) \quad \text{Pythagorean identities}$$

$$= -\frac{\sin^2 x}{\cos^2 x} \quad \text{Reciprocal identity}$$

$$= -\left(\frac{\sin x}{\cos x}\right)^2 \quad \text{Rule of exponents}$$

$$= -\tan^2 x \quad \text{Quotient identity}$$

Graphical Solution

Use a graphing utility set in *radian* mode to graph the left side of the identity $y_1 = (\tan^2 x + 1)(\cos^2 x - 1)$ and the right side of the identity $y_2 = -\tan^2 x$ in the same viewing window, as shown in Figure 5.3. (Select the *line* style for y_1 and the *path* style for y_2.) Because the graphs appear to coincide, $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$ appears to be an identity.

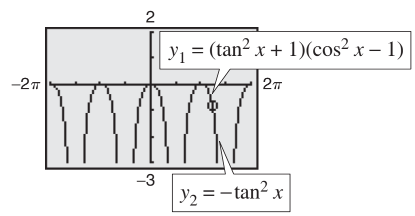


FIGURE 5.3

CHECKPoint Now try Exercise 53.

Example 4 Converting to Sines and Cosines

Verify the identity $\tan x + \cot x = \sec x \csc x$.

Solution

Try converting the left side into sines and cosines.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \quad \text{Quotient identities}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \quad \text{Add fractions.}$$

$$= \frac{1}{\cos x \sin x} \quad \text{Pythagorean identity}$$

$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x} \quad \text{Product of fractions.}$$

$$= \sec x \csc x \quad \text{Reciprocal identities}$$

CHECKPoint Now try Exercise 25.

WARNING / CAUTION

Although a graphing utility can be useful in helping to verify an identity, you must use algebraic techniques to produce a *valid* proof.

Recall from algebra that *rationalizing the denominator* using conjugates is, on occasion, a powerful simplification technique. A related form of this technique, shown below, works for simplifying trigonometric expressions as well.

$$\frac{1}{1 - \cos x} = \frac{1}{1 - \cos x}\left(\frac{1 + \cos x}{1 + \cos x}\right) = \frac{1 + \cos x}{1 - \cos^2 x} = \frac{1 + \cos x}{\sin^2 x}$$

$$= \csc^2 x(1 + \cos x)$$

This technique is demonstrated in the next example.

Study Tip

As shown at the right, $\csc^2 x(1 + \cos x)$ is considered a simplified form of $1/(1 - \cos x)$ because the expression does not contain any fractions.

Example 5 Verifying a Trigonometric Identity

Verify the identity $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

Algebraic Solution

Begin with the *right* side because you can create a monomial denominator by multiplying the numerator and denominator by $1 + \sin x$.

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x}{1 - \sin x}\left(\frac{1 + \sin x}{1 + \sin x}\right) \qquad \text{Multiply numerator and denominator by } 1 + \sin x.$$

$$= \frac{\cos x + \cos x \sin x}{1 - \sin^2 x} \qquad \text{Multiply.}$$

$$= \frac{\cos x + \cos x \sin x}{\cos^2 x} \qquad \text{Pythagorean identity}$$

$$= \frac{\cos x}{\cos^2 x} + \frac{\cos x \sin x}{\cos^2 x} \qquad \text{Write as separate fractions.}$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \qquad \text{Simplify.}$$

$$= \sec x + \tan x \qquad \text{Identities}$$

CHECK*Point* Now try Exercise 59.

Graphical Solution

Use a graphing utility set in the *radian* and *dot* modes to graph $y_1 = \sec x + \tan x$ and $y_2 = \cos x/(1 - \sin x)$ in the same viewing window, as shown in Figure 5.4. Because the graphs appear to coincide, $\sec x + \tan x = \cos x/(1 - \sin x)$ appears to be an identity.

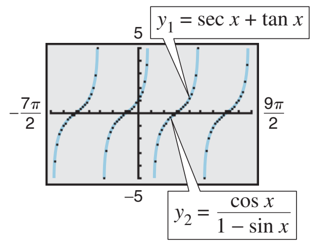


FIGURE 5.4

In Examples 1 through 5, you have been verifying trigonometric identities by working with one side of the equation and converting to the form given on the other side. On occasion, it is practical to work with each side *separately*, to obtain one common form equivalent to both sides. This is illustrated in Example 6.

Example 6 Working with Each Side Separately

Verify the identity $\dfrac{\cot^2 \theta}{1 + \csc \theta} = \dfrac{1 - \sin \theta}{\sin \theta}$.

Algebraic Solution

Working with the left side, you have

$$\frac{\cot^2 \theta}{1 + \csc \theta} = \frac{\csc^2 \theta - 1}{1 + \csc \theta} \qquad \text{Pythagorean identity}$$

$$= \frac{(\csc \theta - 1)(\cancel{\csc \theta + 1})}{\cancel{1 + \csc \theta}} \qquad \text{Factor.}$$

$$= \csc \theta - 1. \qquad \text{Simplify.}$$

Now, simplifying the right side, you have

$$\frac{1 - \sin \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} \qquad \text{Write as separate fractions.}$$

$$= \csc \theta - 1. \qquad \text{Reciprocal identity}$$

The identity is verified because both sides are equal to $\csc \theta - 1$.

CHECK*Point* Now try Exercise 19.

Numerical Solution

Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = \cot^2 x/(1 + \csc x)$ and $y_2 = (1 - \sin x)/\sin x$ for different values of x, as shown in Figure 5.5. From the table you can see that the values appear to be identical, so $\cot^2 x/(1 + \csc x) = (1 - \sin x)/\sin x$ appears to be an identity.

X	Y1	Y2
-.5	-3.086	-3.086
-.25	-5.042	-5.042
0	ERROR	ERROR
.25	3.042	3.042
.5	1.0858	1.0858
.75	.46705	.46705
	.1884	.1884

X=1

FIGURE 5.5

In Example 7, powers of trigonometric functions are rewritten as more complicated sums of products of trigonometric functions. This is a common procedure used in calculus.

Example 7 Three Examples from Calculus

Verify each identity.

a. $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$

b. $\sin^3 x \cos^4 x = (\cos^4 x - \cos^6 x) \sin x$

c. $\csc^4 x \cot x = \csc^2 x(\cot x + \cot^3 x)$

Solution

a. $\tan^4 x = (\tan^2 x)(\tan^2 x)$ — Write as separate factors.

$= \tan^2 x(\sec^2 x - 1)$ — Pythagorean identity

$= \tan^2 x \sec^2 x - \tan^2 x$ — Multiply.

b. $\sin^3 x \cos^4 x = \sin^2 x \cos^4 x \sin x$ — Write as separate factors.

$= (1 - \cos^2 x) \cos^4 x \sin x$ — Pythagorean identity

$= (\cos^4 x - \cos^6 x) \sin x$ — Multiply.

c. $\csc^4 x \cot x = \csc^2 x \csc^2 x \cot x$ — Write as separate factors.

$= \csc^2 x(1 + \cot^2 x) \cot x$ — Pythagorean identity

$= \csc^2 x(\cot x + \cot^3 x)$ — Multiply.

CHECK*Point* Now try Exercise 63.

Classroom Discussion

Error Analysis You are tutoring a student in trigonometry. One of the homework problems your student encounters asks whether the following statement is an identity.

$$\tan^2 x \sin^2 x \stackrel{?}{=} \frac{5}{6} \tan^2 x$$

Your student does not attempt to verify the equivalence algebraically, but mistakenly uses only a graphical approach. Using range settings of

Xmin $= -3\pi$	Ymin $= -20$
Xmax $= 3\pi$	Ymax $= 20$
Xscl $= \pi/2$	Yscl $= 1$

your student graphs both sides of the expression on a graphing utility and concludes that the statement is an identity.

What is wrong with your student's reasoning? Explain. Discuss the limitations of verifying identities graphically.

5.2 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY

In Exercises 1 and 2, fill in the blanks.

1. An equation that is true for all real values in its domain is called an ________.
2. An equation that is true for only some values in its domain is called a ________ ________.

In Exercises 3–8, fill in the blank to complete the trigonometric identity.

3. $\dfrac{1}{\cot u} =$ ________

4. $\dfrac{\cos u}{\sin u} =$ ________

5. $\sin^2 u +$ ________ $= 1$

6. $\cos\left(\dfrac{\pi}{2} - u\right) =$ ________

7. $\csc(-u) =$ ________

8. $\sec(-u) =$ ________

SKILLS AND APPLICATIONS

In Exercises 9–50, verify the identity.

9. $\tan t \cot t = 1$
10. $\sec y \cos y = 1$
11. $\cot^2 y(\sec^2 y - 1) = 1$
12. $\cos x + \sin x \tan x = \sec x$
13. $(1 + \sin \alpha)(1 - \sin \alpha) = \cos^2 \alpha$
14. $\cos^2 \beta - \sin^2 \beta = 2\cos^2 \beta - 1$
15. $\cos^2 \beta - \sin^2 \beta = 1 - 2\sin^2 \beta$
16. $\sin^2 \alpha - \sin^4 \alpha = \cos^2 \alpha - \cos^4 \alpha$
17. $\dfrac{\tan^2 \theta}{\sec \theta} = \sin \theta \tan \theta$
18. $\dfrac{\cot^3 t}{\csc t} = \cos t(\csc^2 t - 1)$
19. $\dfrac{\cot^2 t}{\csc t} = \dfrac{1 - \sin^2 t}{\sin t}$
20. $\dfrac{1}{\tan \beta} + \tan \beta = \dfrac{\sec^2 \beta}{\tan \beta}$
21. $\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \cos^3 x\sqrt{\sin x}$
22. $\sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) = \sec^5 x \tan^3 x$
23. $\dfrac{\cot x}{\sec x} = \csc x - \sin x$
24. $\dfrac{\sec \theta - 1}{1 - \cos \theta} = \sec \theta$
25. $\csc x - \sin x = \cos x \cot x$
26. $\sec x - \cos x = \sin x \tan x$
27. $\dfrac{1}{\tan x} + \dfrac{1}{\cot x} = \tan x + \cot x$
28. $\dfrac{1}{\sin x} - \dfrac{1}{\csc x} = \csc x - \sin x$
29. $\dfrac{1 + \sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta} = 2 \sec \theta$
30. $\dfrac{\cos \theta \cot \theta}{1 - \sin \theta} - 1 = \csc \theta$
31. $\dfrac{1}{\cos x + 1} + \dfrac{1}{\cos x - 1} = -2 \csc x \cot x$
32. $\cos x - \dfrac{\cos x}{1 - \tan x} = \dfrac{\sin x \cos x}{\sin x - \cos x}$
33. $\tan\left(\dfrac{\pi}{2} - \theta\right) \tan \theta = 1$
34. $\dfrac{\cos[(\pi/2) - x]}{\sin[(\pi/2) - x]} = \tan x$
35. $\dfrac{\tan x \cot x}{\cos x} = \sec x$
36. $\dfrac{\csc(-x)}{\sec(-x)} = -\cot x$
37. $(1 + \sin y)[1 + \sin(-y)] = \cos^2 y$
38. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$
39. $\dfrac{\tan x + \cot y}{\tan x \cot y} = \tan y + \cot x$
40. $\dfrac{\cos x - \cos y}{\sin x + \sin y} + \dfrac{\sin x - \sin y}{\cos x + \cos y} = 0$
41. $\sqrt{\dfrac{1 + \sin \theta}{1 - \sin \theta}} = \dfrac{1 + \sin \theta}{|\cos \theta|}$
42. $\sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}} = \dfrac{1 - \cos \theta}{|\sin \theta|}$
43. $\cos^2 \beta + \cos^2\left(\dfrac{\pi}{2} - \beta\right) = 1$
44. $\sec^2 y - \cot^2\left(\dfrac{\pi}{2} - y\right) = 1$
45. $\sin t \csc\left(\dfrac{\pi}{2} - t\right) = \tan t$
46. $\sec^2\left(\dfrac{\pi}{2} - x\right) - 1 = \cot^2 x$
47. $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$
48. $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$
49. $\tan\left(\sin^{-1} \dfrac{x - 1}{4}\right) = \dfrac{x - 1}{\sqrt{16 - (x - 1)^2}}$
50. $\tan\left(\cos^{-1} \dfrac{x + 1}{2}\right) = \dfrac{\sqrt{4 - (x + 1)^2}}{x + 1}$

ERROR ANALYSIS In Exercises 51 and 52, describe the error(s).

51. $(1 + \tan x)[1 + \cot(-x)]$
$= (1 + \tan x)(1 + \cot x)$
$= 1 + \cot x + \tan x + \tan x \cot x$
$= 1 + \cot x + \tan x + 1$
$= 2 + \cot x + \tan x$

52. $\dfrac{1 + \sec(-\theta)}{\sin(-\theta) + \tan(-\theta)} = \dfrac{1 - \sec \theta}{\sin \theta - \tan \theta}$
$= \dfrac{1 - \sec \theta}{(\sin \theta)[1 - (1/\cos \theta)]}$
$= \dfrac{1 - \sec \theta}{\sin \theta(1 - \sec \theta)}$
$= \dfrac{1}{\sin \theta} = \csc \theta$

In Exercises 53–60, (a) use a graphing utility to graph each side of the equation to determine whether the equation is an identity, (b) use the *table* feature of a graphing utility to determine whether the equation is an identity, and (c) confirm the results of parts (a) and (b) algebraically.

53. $(1 + \cot^2 x)(\cos^2 x) = \cot^2 x$

54. $\csc x(\csc x - \sin x) + \dfrac{\sin x - \cos x}{\sin x} + \cot x = \csc^2 x$

55. $2 + \cos^2 x - 3\cos^4 x = \sin^2 x(3 + 2\cos^2 x)$

56. $\tan^4 x + \tan^2 x - 3 = \sec^2 x(4\tan^2 x - 3)$

57. $\csc^4 x - 2\csc^2 x + 1 = \cot^4 x$

58. $(\sin^4 \beta - 2\sin^2 \beta + 1)\cos \beta = \cos^5 \beta$

59. $\dfrac{1 + \cos x}{\sin x} = \dfrac{\sin x}{1 - \cos x}$ **60.** $\dfrac{\cot \alpha}{\csc \alpha + 1} = \dfrac{\csc \alpha + 1}{\cot \alpha}$

In Exercises 61–64, verify the identity.

61. $\tan^5 x = \tan^3 x \sec^2 x - \tan^3 x$

62. $\sec^4 x \tan^2 x = (\tan^2 x + \tan^4 x)\sec^2 x$

63. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x)\cos x$

64. $\sin^4 x + \cos^4 x = 1 - 2\cos^2 x + 2\cos^4 x$

In Exercises 65–68, use the cofunction identities to evaluate the expression without using a calculator.

65. $\sin^2 25° + \sin^2 65°$ **66.** $\cos^2 55° + \cos^2 35°$

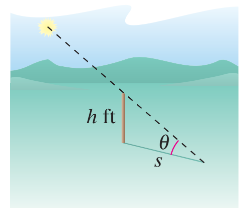
67. $\cos^2 20° + \cos^2 52° + \cos^2 38° + \cos^2 70°$

68. $\tan^2 63° + \cot^2 16° - \sec^2 74° - \csc^2 27°$

69. RATE OF CHANGE The rate of change of the function $f(x) = \sin x + \csc x$ with respect to change in the variable x is given by the expression $\cos x - \csc x \cot x$. Show that the expression for the rate of change can also be $-\cos x \cot^2 x$.

70. SHADOW LENGTH The length s of a shadow cast by a vertical gnomon (a device used to tell time) of height h when the angle of the sun above the horizon is θ (see figure) can be modeled by the equation

$$s = \frac{h \sin(90° - \theta)}{\sin \theta}.$$



(a) Verify that the equation for s is equal to $h \cot \theta$.

(b) Use a graphing utility to complete the table. Let $h = 5$ feet.

θ	15°	30°	45°	60°	75°	90°
s						

(c) Use your table from part (b) to determine the angles of the sun that result in the maximum and minimum lengths of the shadow.

(d) Based on your results from part (c), what time of day do you think it is when the angle of the sun above the horizon is 90°?

EXPLORATION

TRUE OR FALSE? In Exercises 71 and 72, determine whether the statement is true or false. Justify your answer.

71. There can be more than one way to verify a trigonometric identity.

72. The equation $\sin^2 \theta + \cos^2 \theta = 1 + \tan^2 \theta$ is an identity because $\sin^2(0) + \cos^2(0) = 1$ and $1 + \tan^2(0) = 1$.

THINK ABOUT IT In Exercises 73–77, explain why the equation is not an identity and find one value of the variable for which the equation is not true.

73. $\sin \theta = \sqrt{1 - \cos^2 \theta}$ **74.** $\tan \theta = \sqrt{\sec^2 \theta - 1}$

75. $1 - \cos \theta = \sin \theta$ **76.** $\csc \theta - 1 = \cot \theta$

77. $1 + \tan \theta = \sec \theta$

78. CAPSTONE Write a short paper in your own words explaining to a classmate the difference between a trigonometric identity and a conditional equation. Include suggestions on how to verify a trigonometric identity.

5.3 SOLVING TRIGONOMETRIC EQUATIONS

What you should learn

- Use standard algebraic techniques to solve trigonometric equations.
- Solve trigonometric equations of quadratic type.
- Solve trigonometric equations involving multiple angles.
- Use inverse trigonometric functions to solve trigonometric equations.

Why you should learn it

You can use trigonometric equations to solve a variety of real-life problems. For instance, in Exercise 92 on page 396, you can solve a trigonometric equation to help answer questions about monthly sales of skiing equipment.

Tom Stillo/Index Stock Imagery/Photo Library

Introduction

To solve a trigonometric equation, use standard algebraic techniques such as collecting like terms and factoring. Your preliminary goal in solving a trigonometric equation is to *isolate* the trigonometric function in the equation. For example, to solve the equation $2 \sin x = 1$, divide each side by 2 to obtain

$$\sin x = \frac{1}{2}.$$

To solve for x, note in Figure 5.6 that the equation $\sin x = \frac{1}{2}$ has solutions $x = \pi/6$ and $x = 5\pi/6$ in the interval $[0, 2\pi)$. Moreover, because $\sin x$ has a period of 2π, there are infinitely many other solutions, which can be written as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi \qquad \text{General solution}$$

where n is an integer, as shown in Figure 5.6.

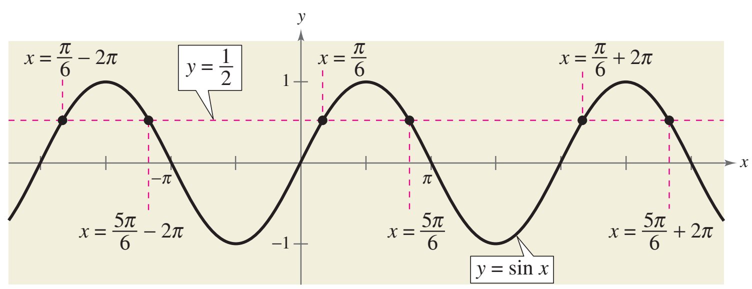


FIGURE 5.6

Another way to show that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions is indicated in Figure 5.7. Any angles that are coterminal with $\pi/6$ or $5\pi/6$ will also be solutions of the equation.

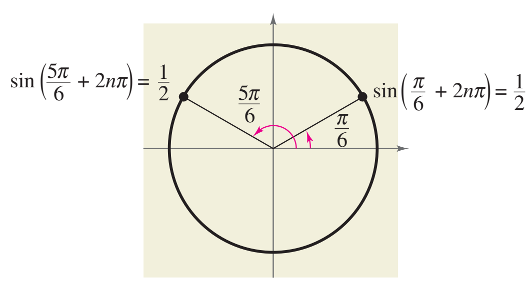


FIGURE 5.7

When solving trigonometric equations, you should write your answer(s) using exact values rather than decimal approximations.

Example 1 Collecting Like Terms

Solve $\sin x + \sqrt{2} = -\sin x$.

Solution

Begin by rewriting the equation so that $\sin x$ is isolated on one side of the equation.

$$\sin x + \sqrt{2} = -\sin x \quad \text{Write original equation.}$$

$$\sin x + \sin x + \sqrt{2} = 0 \quad \text{Add } \sin x \text{ to each side.}$$

$$\sin x + \sin x = -\sqrt{2} \quad \text{Subtract } \sqrt{2} \text{ from each side.}$$

$$2 \sin x = -\sqrt{2} \quad \text{Combine like terms.}$$

$$\sin x = -\frac{\sqrt{2}}{2} \quad \text{Divide each side by 2.}$$

Because $\sin x$ has a period of 2π, first find all solutions in the interval $[0, 2\pi)$. These solutions are $x = 5\pi/4$ and $x = 7\pi/4$. Finally, add multiples of 2π to each of these solutions to get the general form

$$x = \frac{5\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{7\pi}{4} + 2n\pi \quad \text{General solution}$$

where n is an integer.

CHECKPoint Now try Exercise 11.

Example 2 Extracting Square Roots

Solve $3 \tan^2 x - 1 = 0$.

Solution

Begin by rewriting the equation so that $\tan x$ is isolated on one side of the equation.

$$3 \tan^2 x - 1 = 0 \quad \text{Write original equation.}$$

$$3 \tan^2 x = 1 \quad \text{Add 1 to each side.}$$

$$\tan^2 x = \frac{1}{3} \quad \text{Divide each side by 3.}$$

$$\tan x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3} \quad \text{Extract square roots.}$$

Because $\tan x$ has a period of π, first find all solutions in the interval $[0, \pi)$. These solutions are $x = \pi/6$ and $x = 5\pi/6$. Finally, add multiples of π to each of these solutions to get the general form

$$x = \frac{\pi}{6} + n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi \quad \text{General solution}$$

where n is an integer.

CHECKPoint Now try Exercise 15.

WARNING / CAUTION

When you extract square roots, make sure you account for both the positive and negative solutions.

The equations in Examples 1 and 2 involved only one trigonometric function. When two or more functions occur in the same equation, collect all terms on one side and try to separate the functions by factoring or by using appropriate identities. This may produce factors that yield no solutions, as illustrated in Example 3.

Example 3 Factoring

Solve $\cot x \cos^2 x = 2 \cot x$.

Solution

Begin by rewriting the equation so that all terms are collected on one side of the equation.

$$\cot x \cos^2 x = 2 \cot x \qquad \text{Write original equation.}$$

$$\cot x \cos^2 x - 2 \cot x = 0 \qquad \text{Subtract } 2 \cot x \text{ from each side.}$$

$$\cot x(\cos^2 x - 2) = 0 \qquad \text{Factor.}$$

By setting each of these factors equal to zero, you obtain

$$\cot x = 0 \quad \text{and} \quad \cos^2 x - 2 = 0$$

$$x = \frac{\pi}{2} \qquad \cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}.$$

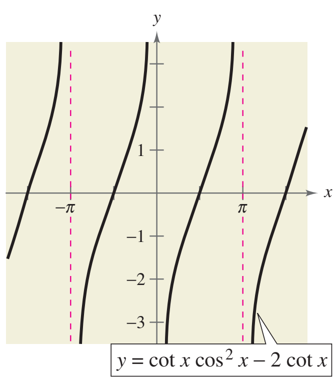


FIGURE 5.8

The equation $\cot x = 0$ has the solution $x = \pi/2$ [in the interval $(0, \pi)$]. No solution is obtained for $\cos x = \pm\sqrt{2}$ because $\pm\sqrt{2}$ are outside the range of the cosine function. Because $\cot x$ has a period of π, the general form of the solution is obtained by adding multiples of π to $x = \pi/2$, to get

$$x = \frac{\pi}{2} + n\pi \qquad \text{General solution}$$

where n is an integer. You can confirm this graphically by sketching the graph of $y = \cot x \cos^2 x - 2 \cot x$, as shown in Figure 5.8. From the graph you can see that the x-intercepts occur at $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$, and so on. These x-intercepts correspond to the solutions of $\cot x \cos^2 x - 2 \cot x = 0$.

CHECK*Point* Now try Exercise 19.

Equations of Quadratic Type

Many trigonometric equations are of quadratic type $ax^2 + bx + c = 0$. Here are a couple of examples.

Quadratic in sin x	*Quadratic in sec x*
$2 \sin^2 x - \sin x - 1 = 0$	$\sec^2 x - 3 \sec x - 2 = 0$
$2(\sin x)^2 - \sin x - 1 = 0$	$(\sec x)^2 - 3(\sec x) - 2 = 0$

To solve equations of this type, factor the quadratic or, if this is not possible, use the Quadratic Formula.

Algebra Help

You can review the techniques for solving quadratic equations in Appendix A.5.

Example 4 Factoring an Equation of Quadratic Type

Find all solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

Algebraic Solution

Begin by treating the equation as a quadratic in $\sin x$ and factoring.

$$2\sin^2 x - \sin x - 1 = 0 \qquad \text{Write original equation.}$$

$$(2\sin x + 1)(\sin x - 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi)$.

$$2\sin x + 1 = 0 \qquad \text{and} \qquad \sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \qquad \sin x = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad x = \frac{\pi}{2}$$

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = 2\sin^2 x - \sin x - 1$ for $0 \le x < 2\pi$, as shown in Figure 5.9. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts to be

$$x \approx 1.571 \approx \frac{\pi}{2}, \quad x \approx 3.665 \approx \frac{7\pi}{6}, \quad \text{and} \quad x \approx 5.760 \approx \frac{11\pi}{6}.$$

These values are the approximate solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

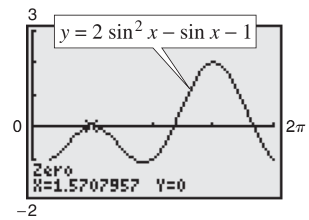


FIGURE 5.9

CHECKPoint Now try Exercise 33.

Example 5 Rewriting with a Single Trigonometric Function

Solve $2\sin^2 x + 3\cos x - 3 = 0$.

Solution

This equation contains both sine and cosine functions. You can rewrite the equation so that it has only cosine functions by using the identity $\sin^2 x = 1 - \cos^2 x$.

$$2\sin^2 x + 3\cos x - 3 = 0 \qquad \text{Write original equation.}$$

$$2(1 - \cos^2 x) + 3\cos x - 3 = 0 \qquad \text{Pythagorean identity}$$

$$2\cos^2 x - 3\cos x + 1 = 0 \qquad \text{Multiply each side by } -1.$$

$$(2\cos x - 1)(\cos x - 1) = 0 \qquad \text{Factor.}$$

Set each factor equal to zero to find the solutions in the interval $[0, 2\pi)$.

$$2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\cos x - 1 = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0$$

Because $\cos x$ has a period of 2π, the general form of the solution is obtained by adding multiples of 2π to get

$$x = 2n\pi, \quad x = \frac{\pi}{3} + 2n\pi, \quad x = \frac{5\pi}{3} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

CHECKPoint Now try Exercise 35.

Sometimes you must square each side of an equation to obtain a quadratic, as demonstrated in the next example. Because this procedure can introduce extraneous solutions, you should check any solutions in the original equation to see whether they are valid or extraneous.

Example 6 Squaring and Converting to Quadratic Type

Find all solutions of $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution

It is not clear how to rewrite this equation in terms of a single trigonometric function. Notice what happens when you square each side of the equation.

$$\cos x + 1 = \sin x \qquad \text{Write original equation.}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x \qquad \text{Square each side.}$$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x \qquad \text{Pythagorean identity}$$

$$\cos^2 x + \cos^2 x + 2\cos x + 1 - 1 = 0 \qquad \text{Rewrite equation.}$$

$$2\cos^2 x + 2\cos x = 0 \qquad \text{Combine like terms.}$$

$$2\cos x(\cos x + 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero produces

$$2\cos x = 0 \quad \text{and} \quad \cos x + 1 = 0$$

$$\cos x = 0 \qquad \cos x = -1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \pi.$$

Because you squared the original equation, check for extraneous solutions.

Check $x = \pi/2$

$$\cos\frac{\pi}{2} + 1 \stackrel{?}{=} \sin\frac{\pi}{2} \qquad \text{Substitute } \pi/2 \text{ for } x.$$

$$0 + 1 = 1 \qquad \text{Solution checks. } \checkmark$$

Check $x = 3\pi/2$

$$\cos\frac{3\pi}{2} + 1 \stackrel{?}{=} \sin\frac{3\pi}{2} \qquad \text{Substitute } 3\pi/2 \text{ for } x.$$

$$0 + 1 \neq -1 \qquad \text{Solution does not check.}$$

Check $x = \pi$

$$\cos\pi + 1 \stackrel{?}{=} \sin\pi \qquad \text{Substitute } \pi \text{ for } x.$$

$$-1 + 1 = 0 \qquad \text{Solution checks. } \checkmark$$

Of the three possible solutions, $x = 3\pi/2$ is extraneous. So, in the interval $[0, 2\pi)$, the only two solutions are $x = \pi/2$ and $x = \pi$.

CHECKPoint Now try Exercise 37.

Study Tip

You square each side of the equation in Example 6 because the squares of the sine and cosine functions are related by a Pythagorean identity. The same is true for the squares of the secant and tangent functions and for the squares of the cosecant and cotangent functions.

Functions Involving Multiple Angles

The next two examples involve trigonometric functions of multiple angles of the forms $\sin ku$ and $\cos ku$. To solve equations of these forms, first solve the equation for ku, then divide your result by k.

Example 7 Functions of Multiple Angles

Solve $2 \cos 3t - 1 = 0$.

Solution

$$2 \cos 3t - 1 = 0 \qquad \text{Write original equation.}$$

$$2 \cos 3t = 1 \qquad \text{Add 1 to each side.}$$

$$\cos 3t = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3} \qquad \text{General solution}$$

where n is an integer.

CHECKPoint Now try Exercise 39.

Example 8 Functions of Multiple Angles

Solve $3 \tan \frac{x}{2} + 3 = 0$.

Solution

$$3 \tan \frac{x}{2} + 3 = 0 \qquad \text{Write original equation.}$$

$$3 \tan \frac{x}{2} = -3 \qquad \text{Subtract 3 from each side.}$$

$$\tan \frac{x}{2} = -1 \qquad \text{Divide each side by 3.}$$

In the interval $[0, \pi)$, you know that $x/2 = 3\pi/4$ is the only solution, so, in general, you have

$$\frac{x}{2} = \frac{3\pi}{4} + n\pi.$$

Multiplying this result by 2, you obtain the general solution

$$x = \frac{3\pi}{2} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

CHECKPoint Now try Exercise 43.

Using Inverse Functions

In the next example, you will see how inverse trigonometric functions can be used to solve an equation.

Example 9 Using Inverse Functions

Solve $\sec^2 x - 2\tan x = 4$.

Solution

$$\sec^2 x - 2\tan x = 4 \quad \text{Write original equation.}$$

$$1 + \tan^2 x - 2\tan x - 4 = 0 \quad \text{Pythagorean identity}$$

$$\tan^2 x - 2\tan x - 3 = 0 \quad \text{Combine like terms.}$$

$$(\tan x - 3)(\tan x + 1) = 0 \quad \text{Factor.}$$

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [Recall that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$$\tan x - 3 = 0 \quad \text{and} \quad \tan x + 1 = 0$$

$$\tan x = 3 \qquad \tan x = -1$$

$$x = \arctan 3 \qquad x = -\frac{\pi}{4}$$

Finally, because $\tan x$ has a period of π, you obtain the general solution by adding multiples of π

$$x = \arctan 3 + n\pi \quad \text{and} \quad x = -\frac{\pi}{4} + n\pi \quad \text{General solution}$$

where n is an integer. You can use a calculator to approximate the value of arctan 3.

CHECK*Point* Now try Exercise 63.

Classroom Discussion

Equations with No Solutions One of the following equations has solutions and the other two do not. Which two equations do not have solutions?

a. $\sin^2 x - 5\sin x + 6 = 0$

b. $\sin^2 x - 4\sin x + 6 = 0$

c. $\sin^2 x - 5\sin x - 6 = 0$

Find conditions involving the constants b and c that will guarantee that the equation

$$\sin^2 x + b\sin x + c = 0$$

has at least one solution on some interval of length 2π.

5.3 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. When solving a trigonometric equation, the preliminary goal is to ________ the trigonometric function involved in the equation.

2. The equation $2 \sin \theta + 1 = 0$ has the solutions $\theta = \frac{7\pi}{6} + 2n\pi$ and $\theta = \frac{11\pi}{6} + 2n\pi$, which are called ________ solutions.

3. The equation $2 \tan^2 x - 3 \tan x + 1 = 0$ is a trigonometric equation that is of ________ type.

4. A solution of an equation that does not satisfy the original equation is called an ________ solution.

SKILLS AND APPLICATIONS

In Exercises 5–10, verify that the x-values are solutions of the equation.

5. $2 \cos x - 1 = 0$

(a) $x = \frac{\pi}{3}$ (b) $x = \frac{5\pi}{3}$

6. $\sec x - 2 = 0$

(a) $x = \frac{\pi}{3}$ (b) $x = \frac{5\pi}{3}$

7. $3 \tan^2 2x - 1 = 0$

(a) $x = \frac{\pi}{12}$ (b) $x = \frac{5\pi}{12}$

8. $2 \cos^2 4x - 1 = 0$

(a) $x = \frac{\pi}{16}$ (b) $x = \frac{3\pi}{16}$

9. $2 \sin^2 x - \sin x - 1 = 0$

(a) $x = \frac{\pi}{2}$ (b) $x = \frac{7\pi}{6}$

10. $\csc^4 x - 4 \csc^2 x = 0$

(a) $x = \frac{\pi}{6}$ (b) $x = \frac{5\pi}{6}$

In Exercises 11–24, solve the equation.

11. $2 \cos x + 1 = 0$ **12.** $2 \sin x + 1 = 0$

13. $\sqrt{3} \csc x - 2 = 0$ **14.** $\tan x + \sqrt{3} = 0$

15. $3 \sec^2 x - 4 = 0$ **16.** $3 \cot^2 x - 1 = 0$

17. $\sin x(\sin x + 1) = 0$

18. $(3 \tan^2 x - 1)(\tan^2 x - 3) = 0$

19. $4 \cos^2 x - 1 = 0$ **20.** $\sin^2 x = 3 \cos^2 x$

21. $2 \sin^2 2x = 1$ **22.** $\tan^2 3x = 3$

23. $\tan 3x(\tan x - 1) = 0$ **24.** $\cos 2x(2 \cos x + 1) = 0$

In Exercises 25–38, find all solutions of the equation in the interval $[0, 2\pi)$.

25. $\cos^3 x = \cos x$ **26.** $\sec^2 x - 1 = 0$

27. $3 \tan^3 x = \tan x$ **28.** $2 \sin^2 x = 2 + \cos x$

29. $\sec^2 x - \sec x = 2$ **30.** $\sec x \csc x = 2 \csc x$

31. $2 \sin x + \csc x = 0$ **32.** $\sec x + \tan x = 1$

33. $2 \cos^2 x + \cos x - 1 = 0$

34. $2 \sin^2 x + 3 \sin x + 1 = 0$

35. $2 \sec^2 x + \tan^2 x - 3 = 0$

36. $\cos x + \sin x \tan x = 2$

37. $\csc x + \cot x = 1$ **38.** $\sin x - 2 = \cos x - 2$

In Exercises 39–44, solve the multiple-angle equation.

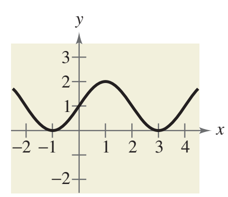
39. $\cos 2x = \frac{1}{2}$ **40.** $\sin 2x = -\frac{\sqrt{3}}{2}$

41. $\tan 3x = 1$ **42.** $\sec 4x = 2$

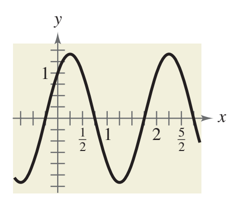
43. $\cos \frac{x}{2} = \frac{\sqrt{2}}{2}$ **44.** $\sin \frac{x}{2} = -\frac{\sqrt{3}}{2}$

In Exercises 45–48, find the x-intercepts of the graph.

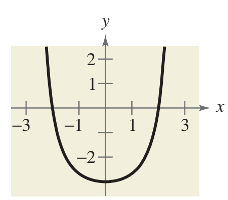
45. $y = \sin \frac{\pi x}{2} + 1$



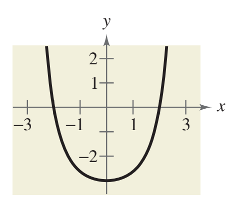
46. $y = \sin \pi x + \cos \pi x$



47. $y = \tan^2\left(\frac{\pi x}{6}\right) - 3$



48. $y = \sec^4\left(\frac{\pi x}{8}\right) - 4$



In Exercises 49–58, use a graphing utility to approximate the solutions (to three decimal places) of the equation in the interval $[0, 2\pi)$.

49. $2\sin x + \cos x = 0$

50. $4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$

51. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 4$

52. $\dfrac{\cos x \cot x}{1 - \sin x} = 3$

53. $x\tan x - 1 = 0$

54. $x\cos x - 1 = 0$

55. $\sec^2 x + 0.5\tan x - 1 = 0$

56. $\csc^2 x + 0.5\cot x - 5 = 0$

57. $2\tan^2 x + 7\tan x - 15 = 0$

58. $6\sin^2 x - 7\sin x + 2 = 0$

In Exercises 59–62, use the Quadratic Formula to solve the equation in the interval $[0, 2\pi)$. Then use a graphing utility to approximate the angle x.

59. $12\sin^2 x - 13\sin x + 3 = 0$

60. $3\tan^2 x + 4\tan x - 4 = 0$

61. $\tan^2 x + 3\tan x + 1 = 0$

62. $4\cos^2 x - 4\cos x - 1 = 0$

In Exercises 63–74, use inverse functions where needed to find all solutions of the equation in the interval $[0, 2\pi)$.

63. $\tan^2 x + \tan x - 12 = 0$

64. $\tan^2 x - \tan x - 2 = 0$

65. $\tan^2 x - 6\tan x + 5 = 0$

66. $\sec^2 x + \tan x - 3 = 0$

67. $2\cos^2 x - 5\cos x + 2 = 0$

68. $2\sin^2 x - 7\sin x + 3 = 0$

69. $\cot^2 x - 9 = 0$

70. $\cot^2 x - 6\cot x + 5 = 0$

71. $\sec^2 x - 4\sec x = 0$

72. $\sec^2 x + 2\sec x - 8 = 0$

73. $\csc^2 x + 3\csc x - 4 = 0$

74. $\csc^2 x - 5\csc x = 0$

In Exercises 75–78, use a graphing utility to approximate the solutions (to three decimal places) of the equation in the given interval.

75. $3\tan^2 x + 5\tan x - 4 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

76. $\cos^2 x - 2\cos x - 1 = 0, \quad [0, \pi]$

77. $4\cos^2 x - 2\sin x + 1 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

78. $2\sec^2 x + \tan x - 6 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

In Exercises 79–84, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval $[0, 2\pi)$, and (b) solve the trigonometric equation and demonstrate that its solutions are the x-coordinates of the maximum and minimum points of f. (Calculus is required to find the trigonometric equation.)

	Function	*Trigonometric Equation*
79.	$f(x) = \sin^2 x + \cos x$	$2\sin x\cos x - \sin x = 0$
80.	$f(x) = \cos^2 x - \sin x$	$-2\sin x\cos x - \cos x = 0$
81.	$f(x) = \sin x + \cos x$	$\cos x - \sin x = 0$
82.	$f(x) = 2\sin x + \cos 2x$	$2\cos x - 4\sin x\cos x = 0$
83.	$f(x) = \sin x\cos x$	$-\sin^2 x + \cos^2 x = 0$
84.	$f(x) = \sec x + \tan x - x$	$\sec x\tan x + \sec^2 x - 1 = 0$

FIXED POINT In Exercises 85 and 86, find the smallest positive fixed point of the function f. [A *fixed point* of a function f is a real number c such that $f(c) = c$.]

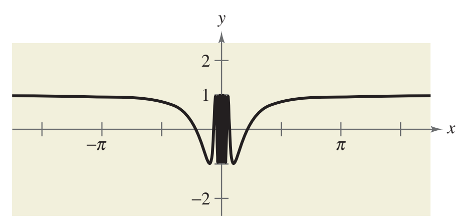
85. $f(x) = \tan\dfrac{\pi x}{4}$

86. $f(x) = \cos x$

87. GRAPHICAL REASONING Consider the function given by

$$f(x) = \cos\frac{1}{x}$$

and its graph shown in the figure.



(a) What is the domain of the function?

(b) Identify any symmetry and any asymptotes of the graph.

(c) Describe the behavior of the function as $x \to 0$.

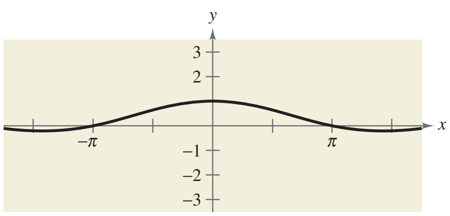
(d) How many solutions does the equation

$$\cos\frac{1}{x} = 0$$

have in the interval $[-1, 1]$? Find the solutions.

(e) Does the equation $\cos(1/x) = 0$ have a greatest solution? If so, approximate the solution. If not, explain why.

88. GRAPHICAL REASONING Consider the function given by $f(x) = (\sin x)/x$ and its graph shown in the figure.



(a) What is the domain of the function?

(b) Identify any symmetry and any asymptotes of the graph.

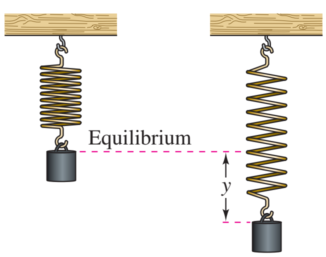
(c) Describe the behavior of the function as $x \to 0$.

(d) How many solutions does the equation

$$\frac{\sin x}{x} = 0$$

have in the interval $[-8, 8]$? Find the solutions.

89. HARMONIC MOTION A weight is oscillating on the end of a spring (see figure). The position of the weight relative to the point of equilibrium is given by $y = \frac{1}{12}(\cos 8t - 3 \sin 8t)$, where y is the displacement (in meters) and t is the time (in seconds). Find the times when the weight is at the point of equilibrium $(y = 0)$ for $0 \le t \le 1$.



90. DAMPED HARMONIC MOTION The displacement from equilibrium of a weight oscillating on the end of a spring is given by $y = 1.56e^{-0.22t} \cos 4.9t$, where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function for $0 \le t \le 10$. Find the time beyond which the displacement does not exceed 1 foot from equilibrium.

91. SALES The monthly sales S (in thousands of units) of a seasonal product are approximated by

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

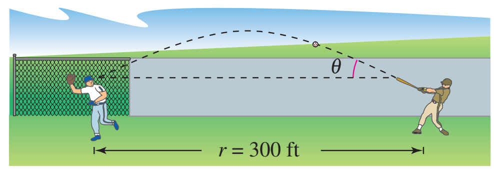
where t is the time (in months), with $t = 1$ corresponding to January. Determine the months in which sales exceed 100,000 units.

92. SALES The monthly sales S (in hundreds of units) of skiing equipment at a sports store are approximated by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. Determine the months in which sales exceed 7500 units.

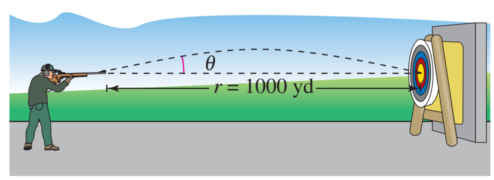
93. PROJECTILE MOTION A batted baseball leaves the bat at an angle of θ with the horizontal and an initial velocity of $v_0 = 100$ feet per second. The ball is caught by an outfielder 300 feet from home plate (see figure). Find θ if the range r of a projectile is given by $r = \frac{1}{32} v_0^2 \sin 2\theta$.



Not drawn to scale

94. PROJECTILE MOTION A sharpshooter intends to hit a target at a distance of 1000 yards with a gun that has a muzzle velocity of 1200 feet per second (see figure). Neglecting air resistance, determine the gun's minimum angle of elevation θ if the range r is given by

$$r = \frac{1}{32} v_0^2 \sin 2\theta.$$



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95. FERRIS WHEEL A Ferris wheel is built such that the height h (in feet) above ground of a seat on the wheel at time t (in minutes) can be modeled by

$$h(t) = 53 + 50 \sin\left(\frac{\pi}{16}t - \frac{\pi}{2}\right).$$

The wheel makes one revolution every 32 seconds. The ride begins when $t = 0$.

(a) During the first 32 seconds of the ride, when will a person on the Ferris wheel be 53 feet above ground?

(b) When will a person be at the top of the Ferris wheel for the first time during the ride? If the ride lasts 160 seconds, how many times will a person be at the top of the ride, and at what times?

96. DATA ANALYSIS: METEOROLOGY The table shows the average daily high temperatures in Houston H (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. (Source: National Climatic Data Center)

Month, t	Houston, H
1	62.3
2	66.5
3	73.3
4	79.1
5	85.5
6	90.7
7	93.6
8	93.5
9	89.3
10	82.0
11	72.0
12	64.6

(a) Create a scatter plot of the data.

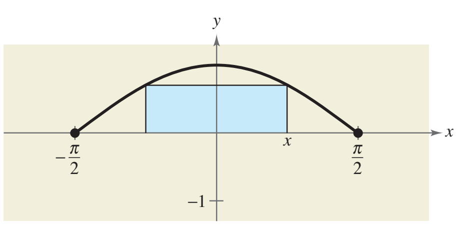
(b) Find a cosine model for the temperatures in Houston.

(c) Use a graphing utility to graph the data points and the model for the temperatures in Houston. How well does the model fit the data?

(d) What is the overall average daily high temperature in Houston?

(e) Use a graphing utility to describe the months during which the average daily high temperature is above 86°F and below 86°F.

97. GEOMETRY The area of a rectangle (see figure) inscribed in one arc of the graph of $y = \cos x$ is given by $A = 2x \cos x, 0 < x < \pi/2$.



(a) Use a graphing utility to graph the area function, and approximate the area of the largest inscribed rectangle.

(b) Determine the values of x for which $A \geq 1$.

98. QUADRATIC APPROXIMATION Consider the function given by $f(x) = 3 \sin(0.6x - 2)$.

(a) Approximate the zero of the function in the interval $[0, 6]$.

(b) A quadratic approximation agreeing with f at $x = 5$ is $g(x) = -0.45x^2 + 5.52x - 13.70$. Use a graphing utility to graph f and g in the same viewing window. Describe the result.

(c) Use the Quadratic Formula to find the zeros of g. Compare the zero in the interval $[0, 6]$ with the result of part (a).

EXPLORATION

TRUE OR FALSE? In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.

99. The equation $2 \sin 4t - 1 = 0$ has four times the number of solutions in the interval $[0, 2\pi)$ as the equation $2 \sin t - 1 = 0$.

100. If you correctly solve a trigonometric equation to the statement $\sin x = 3.4$, then you can finish solving the equation by using an inverse function.

101. THINK ABOUT IT Explain what would happen if you divided each side of the equation $\cot x \cos^2 x = 2 \cot x$ by $\cot x$. Is this a correct method to use when solving equations?

102. GRAPHICAL REASONING Use a graphing utility to confirm the solutions found in Example 6 in two different ways.

(a) Graph both sides of the equation and find the x-coordinates of the points at which the graphs intersect.

Left side: $y = \cos x + 1$

Right side: $y = \sin x$

(b) Graph the equation $y = \cos x + 1 - \sin x$ and find the x-intercepts of the graph. Do both methods produce the same x-values? Which method do you prefer? Explain.

103. Explain in your own words how knowledge of algebra is important when solving trigonometric equations.

104. CAPSTONE Consider the equation $2 \sin x - 1 = 0$. Explain the similarities and differences between finding all solutions in the interval $\left[0, \frac{\pi}{2}\right)$, finding all solutions in the interval $[0, 2\pi)$, and finding the general solution.

PROJECT: METEOROLOGY To work an extended application analyzing the normal daily high temperatures in Phoenix and in Seattle, visit this text's website at *academic.cengage.com*. (Data Source: NOAA)

5.4 Sum and Difference Formulas

What you should learn

- Use sum and difference formulas to evaluate trigonometric functions, verify identities, and solve trigonometric equations.

Why you should learn it

You can use identities to rewrite trigonometric expressions. For instance, in Exercise 89 on page 403, you can use an identity to rewrite a trigonometric expression in a form that helps you analyze a harmonic motion equation.

Richard Megna/Fundamental Photographs

Using Sum and Difference Formulas

In this and the following section, you will study the uses of several trigonometric identities and formulas.

Sum and Difference Formulas

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad \tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

For a proof of the sum and difference formulas, see Proofs in Mathematics on page 422.

Examples 1 and 2 show how **sum and difference formulas** can be used to find exact values of trigonometric functions involving sums or differences of special angles.

Example 1 Evaluating a Trigonometric Function

Find the exact value of $\sin \frac{\pi}{12}$.

Solution

To find the *exact* value of $\sin \frac{\pi}{12}$, use the fact that

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}.$$

Consequently, the formula for $\sin(u - v)$ yields

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

Try checking this result on your calculator. You will find that $\sin \frac{\pi}{12} \approx 0.259$.

CHECKPoint Now try Exercise 7.

Study Tip

Another way to solve Example 2 is to use the fact that $75° = 120° - 45°$ together with the formula for $\cos(u - v)$.

Example 2 Evaluating a Trigonometric Function

Find the exact value of $\cos 75°$.

Solution

Using the fact that $75° = 30° + 45°$, together with the formula for $\cos(u + v)$, you obtain

$$\begin{aligned}\cos 75° &= \cos(30° + 45°)\\ &= \cos 30° \cos 45° - \sin 30° \sin 45°\\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

CHECKPoint Now try Exercise 11.

Example 3 Evaluating a Trigonometric Expression

Find the exact value of $\sin(u + v)$ given

$$\sin u = \frac{4}{5}, \text{ where } 0 < u < \frac{\pi}{2}, \quad \text{and} \quad \cos v = -\frac{12}{13}, \text{ where } \frac{\pi}{2} < v < \pi.$$

Solution

Because $\sin u = 4/5$ and u is in Quadrant I, $\cos u = 3/5$, as shown in Figure 5.10. Because $\cos v = -12/13$ and v is in Quadrant II, $\sin v = 5/13$, as shown in Figure 5.11. You can find $\sin(u + v)$ as follows.

$$\begin{aligned}\sin(u + v) &= \sin u \cos v + \cos u \sin v\\ &= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)\\ &= -\frac{48}{65} + \frac{15}{65}\\ &= -\frac{33}{65}\end{aligned}$$

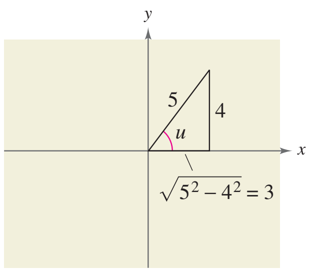


FIGURE 5.10

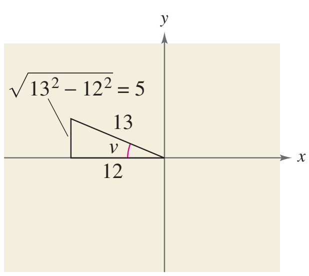


FIGURE 5.11

CHECKPoint Now try Exercise 43.

Example 4 An Application of a Sum Formula

Write $\cos(\arctan 1 + \arccos x)$ as an algebraic expression.

Solution

This expression fits the formula for $\cos(u + v)$. Angles $u = \arctan 1$ and $v = \arccos x$ are shown in Figure 5.12. So

$$\begin{aligned}\cos(u + v) &= \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x)\\ &= \frac{1}{\sqrt{2}} \cdot x - \frac{1}{\sqrt{2}} \cdot \sqrt{1 - x^2}\\ &= \frac{x - \sqrt{1 - x^2}}{\sqrt{2}}.\end{aligned}$$

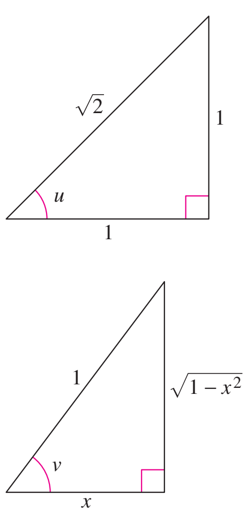


FIGURE 5.12

CHECKPoint Now try Exercise 57.

HISTORICAL NOTE

Hipparchus, considered the most eminent of Greek astronomers, was born about 190 B.C. in Nicaea. He was credited with the invention of trigonometry. He also derived the sum and difference formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$.

Example 5 shows how to use a difference formula to prove the cofunction identity

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

Example 5 Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$.

Solution

Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x \\ &= (0)(\cos x) + (1)(\sin x) \\ &= \sin x.\end{aligned}$$

CHECKPoint Now try Exercise 61.

Sum and difference formulas can be used to rewrite expressions such as

$$\sin\left(\theta + \frac{n\pi}{2}\right) \quad \text{and} \quad \cos\left(\theta + \frac{n\pi}{2}\right), \quad \text{where } n \text{ is an integer}$$

as expressions involving only $\sin\theta$ or $\cos\theta$. The resulting formulas are called **reduction formulas.**

Example 6 Deriving Reduction Formulas

Simplify each expression.

a. $\cos\left(\theta - \dfrac{3\pi}{2}\right)$ **b.** $\tan(\theta + 3\pi)$

Solution

a. Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\theta - \frac{3\pi}{2}\right) &= \cos\theta\cos\frac{3\pi}{2} + \sin\theta\sin\frac{3\pi}{2} \\ &= (\cos\theta)(0) + (\sin\theta)(-1) \\ &= -\sin\theta.\end{aligned}$$

b. Using the formula for $\tan(u + v)$, you have

$$\begin{aligned}\tan(\theta + 3\pi) &= \frac{\tan\theta + \tan 3\pi}{1 - \tan\theta\tan 3\pi} \\ &= \frac{\tan\theta + 0}{1 - (\tan\theta)(0)} \\ &= \tan\theta.\end{aligned}$$

CHECKPoint Now try Exercise 73.

Example 7 Solving a Trigonometric Equation

Find all solutions of $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = -1$ in the interval $[0, 2\pi)$.

Algebraic Solution

Using sum and difference formulas, rewrite the equation as

$$\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} + \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} = -1$$

$$2 \sin x \cos \frac{\pi}{4} = -1$$

$$2(\sin x)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$\sin x = -\frac{\sqrt{2}}{2}.$$

So, the only solutions in the interval $[0, 2\pi)$ are

$$x = \frac{5\pi}{4} \quad \text{and} \quad x = \frac{7\pi}{4}.$$

Graphical Solution

Sketch the graph of

$$y = \sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) + 1 \text{ for } 0 \le x < 2\pi.$$

as shown in Figure 5.13. From the graph you can see that the x-intercepts are $5\pi/4$ and $7\pi/4$. So, the solutions in the interval $[0, 2\pi)$ are

$$x = \frac{5\pi}{4} \quad \text{and} \quad x = \frac{7\pi}{4}.$$

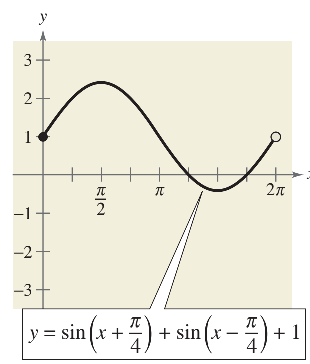


FIGURE 5.13

CHECKPoint Now try Exercise 79.

The next example was taken from calculus. It is used to derive the derivative of the sine function.

Example 8 An Application from Calculus

Verify that $\frac{\sin(x + h) - \sin x}{h} = (\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1 - \cos h}{h}\right)$ where $h \neq 0$.

Solution

Using the formula for $\sin(u + v)$, you have

$$\frac{\sin(x + h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \frac{\cos x \sin h - \sin x(1 - \cos h)}{h}$$

$$= (\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1 - \cos h}{h}\right).$$

CHECKPoint Now try Exercise 105.

5.4 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blank.

1. $\sin(u - v) =$ ________
2. $\cos(u + v) =$ ________
3. $\tan(u + v) =$ ________
4. $\sin(u + v) =$ ________
5. $\cos(u - v) =$ ________
6. $\tan(u - v) =$ ________

SKILLS AND APPLICATIONS

In Exercises 7–12, find the exact value of each expression.

7. (a) $\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$ (b) $\cos\frac{\pi}{4} + \cos\frac{\pi}{3}$
8. (a) $\sin\left(\frac{3\pi}{4} + \frac{5\pi}{6}\right)$ (b) $\sin\frac{3\pi}{4} + \sin\frac{5\pi}{6}$
9. (a) $\sin\left(\frac{7\pi}{6} - \frac{\pi}{3}\right)$ (b) $\sin\frac{7\pi}{6} - \sin\frac{\pi}{3}$
10. (a) $\cos(120° + 45°)$ (b) $\cos 120° + \cos 45°$
11. (a) $\sin(135° - 30°)$ (b) $\sin 135° - \cos 30°$
12. (a) $\sin(315° - 60°)$ (b) $\sin 315° - \sin 60°$

In Exercises 13–28, find the exact values of the sine, cosine, and tangent of the angle.

13. $\frac{11\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{6}$
14. $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$
15. $\frac{17\pi}{12} = \frac{9\pi}{4} - \frac{5\pi}{6}$
16. $-\frac{\pi}{12} = \frac{\pi}{6} - \frac{\pi}{4}$
17. $105° = 60° + 45°$
18. $165° = 135° + 30°$
19. $195° = 225° - 30°$
20. $255° = 300° - 45°$
21. $\frac{13\pi}{12}$
22. $-\frac{7\pi}{12}$
23. $-\frac{13\pi}{12}$
24. $\frac{5\pi}{12}$
25. $285°$
26. $-105°$
27. $-165°$
28. $15°$

In Exercises 29–36, write the expression as the sine, cosine, or tangent of an angle.

29. $\sin 3 \cos 1.2 - \cos 3 \sin 1.2$
30. $\cos\frac{\pi}{7}\cos\frac{\pi}{5} - \sin\frac{\pi}{7}\sin\frac{\pi}{5}$
31. $\sin 60° \cos 15° + \cos 60° \sin 15°$
32. $\cos 130° \cos 40° - \sin 130° \sin 40°$
33. $\dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$
34. $\dfrac{\tan 140° - \tan 60°}{1 + \tan 140° \tan 60°}$
35. $\dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$
36. $\cos 3x \cos 2y + \sin 3x \sin 2y$

In Exercises 37–42, find the exact value of the expression.

37. $\sin\frac{\pi}{12}\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\sin\frac{\pi}{4}$
38. $\cos\frac{\pi}{16}\cos\frac{3\pi}{16} - \sin\frac{\pi}{16}\sin\frac{3\pi}{16}$
39. $\sin 120° \cos 60° - \cos 120° \sin 60°$
40. $\cos 120° \cos 30° + \sin 120° \sin 30°$
41. $\dfrac{\tan(5\pi/6) - \tan(\pi/6)}{1 + \tan(5\pi/6)\tan(\pi/6)}$
42. $\dfrac{\tan 25° + \tan 110°}{1 - \tan 25° \tan 110°}$

In Exercises 43–50, find the exact value of the trigonometric function given that $\sin u = \frac{5}{13}$ and $\cos v = -\frac{3}{5}$. (Both u and v are in Quadrant II.)

43. $\sin(u + v)$
44. $\cos(u - v)$
45. $\cos(u + v)$
46. $\sin(v - u)$
47. $\tan(u + v)$
48. $\csc(u - v)$
49. $\sec(v - u)$
50. $\cot(u + v)$

In Exercises 51–56, find the exact value of the trigonometric function given that $\sin u = -\frac{7}{25}$ and $\cos v = -\frac{4}{5}$. (Both u and v are in Quadrant III.)

51. $\cos(u + v)$
52. $\sin(u + v)$
53. $\tan(u - v)$
54. $\cot(v - u)$
55. $\csc(u - v)$
56. $\sec(v - u)$

In Exercises 57–60, write the trigonometric expression as an algebraic expression.

57. $\sin(\arcsin x + \arccos x)$
58. $\sin(\arctan 2x - \arccos x)$
59. $\cos(\arccos x + \arcsin x)$
60. $\cos(\arccos x - \arctan x)$

In Exercises 61–70, prove the identity.

61. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ **62.** $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

63. $\sin\left(\frac{\pi}{6} + x\right) = \frac{1}{2}(\cos x + \sqrt{3}\sin x)$

64. $\cos\left(\frac{5\pi}{4} - x\right) = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

65. $\cos(\pi - \theta) + \sin\left(\frac{\pi}{2} + \theta\right) = 0$

66. $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

67. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

68. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

69. $\sin(x + y) + \sin(x - y) = 2\sin x\cos y$

70. $\cos(x + y) + \cos(x - y) = 2\cos x\cos y$

In Exercises 71–74, simplify the expression algebraically and use a graphing utility to confirm your answer graphically.

71. $\cos\left(\frac{3\pi}{2} - x\right)$ **72.** $\cos(\pi + x)$

73. $\sin\left(\frac{3\pi}{2} + \theta\right)$ **74.** $\tan(\pi + \theta)$

In Exercises 75–84, find all solutions of the equation in the interval $[0, 2\pi)$.

75. $\sin(x + \pi) - \sin x + 1 = 0$

76. $\sin(x + \pi) - \sin x - 1 = 0$

77. $\cos(x + \pi) - \cos x - 1 = 0$

78. $\cos(x + \pi) - \cos x + 1 = 0$

79. $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

80. $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$

81. $\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) = 1$

82. $\tan(x + \pi) + 2\sin(x + \pi) = 0$

83. $\sin\left(x + \frac{\pi}{2}\right) - \cos^2 x = 0$

84. $\cos\left(x - \frac{\pi}{2}\right) + \sin^2 x = 0$

In Exercises 85–88, use a graphing utility to approximate the solutions in the interval $[0, 2\pi)$.

85. $\cos\left(x + \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right) = 1$

86. $\tan(x + \pi) - \cos\left(x + \frac{\pi}{2}\right) = 0$

87. $\sin\left(x + \frac{\pi}{2}\right) + \cos^2 x = 0$

88. $\cos\left(x - \frac{\pi}{2}\right) - \sin^2 x = 0$

89. HARMONIC MOTION A weight is attached to a spring suspended vertically from a ceiling. When a driving force is applied to the system, the weight moves vertically from its equilibrium position, and this motion is modeled by

$$y = \frac{1}{3}\sin 2t + \frac{1}{4}\cos 2t$$

where y is the distance from equilibrium (in feet) and t is the time (in seconds).

(a) Use the identity

$$a\sin B\theta + b\cos B\theta = \sqrt{a^2 + b^2}\sin(B\theta + C)$$

where $C = \arctan(b/a)$, $a > 0$, to write the model in the form $y = \sqrt{a^2 + b^2}\sin(Bt + C)$.

(b) Find the amplitude of the oscillations of the weight.

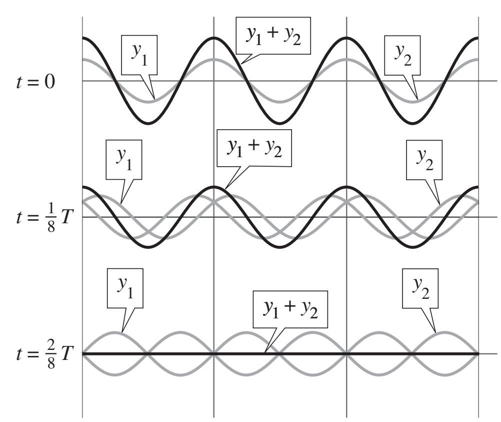
(c) Find the frequency of the oscillations of the weight.

90. STANDING WAVES The equation of a standing wave is obtained by adding the displacements of two waves traveling in opposite directions (see figure). Assume that each of the waves has amplitude A, period T, and wavelength λ. If the models for these waves are

$$y_1 = A\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad \text{and} \quad y_2 = A\cos 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

show that

$$y_1 + y_2 = 2A\cos\frac{2\pi t}{T}\cos\frac{2\pi x}{\lambda}.$$



EXPLORATION

TRUE OR FALSE? In Exercises 91–94, determine whether the statement is true or false. Justify your answer.

91. $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

92. $\cos(u \pm v) = \cos u \cos v \pm \sin u \sin v$

93. $\tan\left(x - \dfrac{\pi}{4}\right) = \dfrac{\tan x + 1}{1 - \tan x}$

94. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

In Exercises 95–98, verify the identity.

95. $\cos(n\pi + \theta) = (-1)^n \cos \theta,\quad n$ is an integer

96. $\sin(n\pi + \theta) = (-1)^n \sin \theta,\quad n$ is an integer

97. $a \sin B\theta + b \cos B\theta = \sqrt{a^2 + b^2} \sin(B\theta + C)$, where $C = \arctan(b/a)$ and $a > 0$

98. $a \sin B\theta + b \cos B\theta = \sqrt{a^2 + b^2} \cos(B\theta - C)$, where $C = \arctan(a/b)$ and $b > 0$

In Exercises 99–102, use the formulas given in Exercises 97 and 98 to write the trigonometric expression in the following forms.

(a) $\sqrt{a^2 + b^2} \sin(B\theta + C)$ (b) $\sqrt{a^2 + b^2} \cos(B\theta - C)$

99. $\sin \theta + \cos \theta$

100. $3 \sin 2\theta + 4 \cos 2\theta$

101. $12 \sin 3\theta + 5 \cos 3\theta$

102. $\sin 2\theta + \cos 2\theta$

In Exercises 103 and 104, use the formulas given in Exercises 97 and 98 to write the trigonometric expression in the form $a \sin B\theta + b \cos B\theta$.

103. $2 \sin\left(\theta + \dfrac{\pi}{4}\right)$

104. $5 \cos\left(\theta - \dfrac{\pi}{4}\right)$

105. Verify the following identity used in calculus.

$$\frac{\cos(x + h) - \cos x}{h} = \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

106. Let $x = \pi/6$ in the identity in Exercise 105 and define the functions f and g as follows.

$$f(h) = \frac{\cos[(\pi/6) + h] - \cos(\pi/6)}{h}$$

$$g(h) = \cos\frac{\pi}{6}\left(\frac{\cos h - 1}{h}\right) - \sin\frac{\pi}{6}\left(\frac{\sin h}{h}\right)$$

(a) What are the domains of the functions f and g?

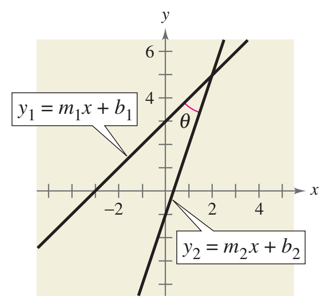
(b) Use a graphing utility to complete the table.

h	0.5	0.2	0.1	0.05	0.02	0.01
$f(h)$						
$g(h)$						

(c) Use a graphing utility to graph the functions f and g.

(d) Use the table and the graphs to make a conjecture about the values of the functions f and g as $h \to 0$.

In Exercises 107 and 108, use the figure, which shows two lines whose equations are $y_1 = m_1 x + b_1$ and $y_2 = m_2 x + b_2$. Assume that both lines have positive slopes. Derive a formula for the angle between the two lines. Then use your formula to find the angle between the given pair of lines.



107. $y = x$ and $y = \sqrt{3}x$

108. $y = x$ and $y = \dfrac{1}{\sqrt{3}}x$

In Exercises 109 and 110, use a graphing utility to graph y_1 and y_2 in the same viewing window. Use the graphs to determine whether $y_1 = y_2$. Explain your reasoning.

109. $y_1 = \cos(x + 2),\quad y_2 = \cos x + \cos 2$

110. $y_1 = \sin(x + 4),\quad y_2 = \sin x + \sin 4$

111. PROOF

(a) Write a proof of the formula for $\sin(u + v)$.

(b) Write a proof of the formula for $\sin(u - v)$.

112. CAPSTONE Give an example to justify each statement.

(a) $\sin(u + v) \neq \sin u + \sin v$

(b) $\sin(u - v) \neq \sin u - \sin v$

(c) $\cos(u + v) \neq \cos u + \cos v$

(d) $\cos(u - v) \neq \cos u - \cos v$

(e) $\tan(u + v) \neq \tan u + \tan v$

(f) $\tan(u - v) \neq \tan u - \tan v$

5.5 MULTIPLE-ANGLE AND PRODUCT-TO-SUM FORMULAS

What you should learn

- Use multiple-angle formulas to rewrite and evaluate trigonometric functions.
- Use power-reducing formulas to rewrite and evaluate trigonometric functions.
- Use half-angle formulas to rewrite and evaluate trigonometric functions.
- Use product-to-sum and sum-to-product formulas to rewrite and evaluate trigonometric functions.
- Use trigonometric formulas to rewrite real-life models.

Why you should learn it

You can use a variety of trigonometric formulas to rewrite trigonometric functions in more convenient forms. For instance, in Exercise 135 on page 415, you can use a double-angle formula to determine at what angle an athlete must throw a javelin.

Mark Dadswell/Getty Images

Multiple-Angle Formulas

In this section, you will study four other categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as $\sin ku$ and $\cos ku$.
2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$.
3. The third category involves *functions of half-angles* such as $\sin(u/2)$.
4. The fourth category involves *products of trigonometric functions* such as $\sin u \cos v$.

You should learn the **double-angle formulas** because they are used often in trigonometry and calculus. For proofs of these formulas, see Proofs in Mathematics on page 423.

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\cos 2u = \cos^2 u - \sin^2 u$$
$$= 2\cos^2 u - 1$$
$$= 1 - 2\sin^2 u$$

Example 1 Solving a Multiple-Angle Equation

Solve $2 \cos x + \sin 2x = 0$.

Solution

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve.

$$2\cos x + \sin 2x = 0 \quad \text{Write original equation.}$$

$$2\cos x + 2\sin x \cos x = 0 \quad \text{Double-angle formula}$$

$$2\cos x(1 + \sin x) = 0 \quad \text{Factor.}$$

$$2\cos x = 0 \quad \text{and} \quad 1 + \sin x = 0 \quad \text{Set factors equal to zero.}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2} \quad \text{Solutions in } [0, 2\pi)$$

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$

where n is an integer. Try verifying these solutions graphically.

CHECKPoint Now try Exercise 19.

Example 2 Using Double-Angle Formulas to Analyze Graphs

Use a double-angle formula to rewrite the equation

$$y = 4\cos^2 x - 2.$$

Then sketch the graph of the equation over the interval $[0, 2\pi]$.

Solution

Using the double-angle formula for $\cos 2u$, you can rewrite the original equation as

$$y = 4\cos^2 x - 2 \quad \text{Write original equation.}$$

$$= 2(2\cos^2 x - 1) \quad \text{Factor.}$$

$$= 2\cos 2x. \quad \text{Use double-angle formula.}$$

Using the techniques discussed in Section 4.5, you can recognize that the graph of this function has an amplitude of 2 and a period of π. The key points in the interval $[0, \pi]$ are as follows.

Maximum	Intercept	Minimum	Intercept	Maximum
$(0, 2)$	$\left(\frac{\pi}{4}, 0\right)$	$\left(\frac{\pi}{2}, -2\right)$	$\left(\frac{3\pi}{4}, 0\right)$	$(\pi, 2)$

Two cycles of the graph are shown in Figure 5.14.

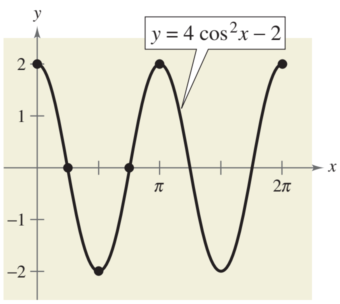


FIGURE 5.14

CHECK*Point* Now try Exercise 33.

Example 3 Evaluating Functions Involving Double Angles

Use the following to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\cos\theta = \frac{5}{13}, \qquad \frac{3\pi}{2} < \theta < 2\pi$$

Solution

From Figure 5.15, you can see that $\sin\theta = y/r = -12/13$. Consequently, using each of the double-angle formulas, you can write

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{25}{169}\right) - 1 = -\frac{119}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119}.$$

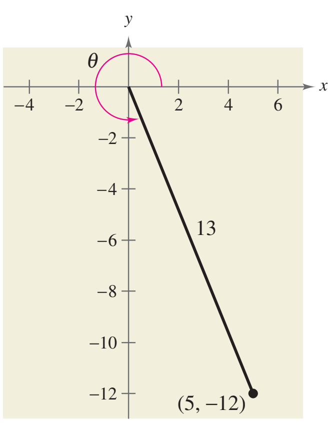


FIGURE 5.15

CHECK*Point* Now try Exercise 37.

The double-angle formulas are not restricted to angles 2θ and θ. Other *double* combinations, such as 4θ and 2θ or 6θ and 3θ, are also valid. Here are two examples.

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta \qquad \text{and} \qquad \cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$$

By using double-angle formulas together with the sum formulas given in the preceding section, you can form other multiple-angle formulas.

Example 4 Deriving a Triple-Angle Formula

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2 \sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x\\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x\\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

CHECKPoint Now try Exercise 117.

Power-Reducing Formulas

The double-angle formulas can be used to obtain the following **power-reducing formulas.** Example 5 shows a typical power reduction that is used in calculus.

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

For a proof of the power-reducing formulas, see Proofs in Mathematics on page 423.

Example 5 Reducing a Power

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution

Note the repeated use of power-reducing formulas.

$$\begin{aligned}\sin^4 x &= (\sin^2 x)^2 && \text{Property of exponents}\\ &= \left(\frac{1 - \cos 2x}{2}\right)^2 && \text{Power-reducing formula}\\ &= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) && \text{Expand.}\\ &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) && \text{Power-reducing formula}\\ &= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x && \text{Distributive Property}\\ &= \frac{1}{8}(3 - 4\cos 2x + \cos 4x) && \text{Factor out common factor.}\end{aligned}$$

CHECKPoint Now try Exercise 43.

Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas.**

Half-Angle Formulas

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos \frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The signs of $\sin \frac{u}{2}$ and $\cos \frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example 6 Using a Half-Angle Formula

Find the exact value of $\sin 105°$.

Solution

Begin by noting that $105°$ is half of $210°$. Then, using the half-angle formula for $\sin(u/2)$ and the fact that $105°$ lies in Quadrant II, you have

$$\begin{aligned}\sin 105° &= \sqrt{\frac{1 - \cos 210°}{2}}\\ &= \sqrt{\frac{1 - (-\cos 30°)}{2}}\\ &= \sqrt{\frac{1 + \left(\sqrt{3}/2\right)}{2}}\\ &= \frac{\sqrt{2 + \sqrt{3}}}{2}.\end{aligned}$$

The positive square root is chosen because $\sin \theta$ is positive in Quadrant II.

CHECK*Point* Now try Exercise 59.

Use your calculator to verify the result obtained in Example 6. That is, evaluate $\sin 105°$ and $\left(\sqrt{2 + \sqrt{3}}\right)/2$.

$$\sin 105° \approx 0.9659258$$

$$\frac{\sqrt{2 + \sqrt{3}}}{2} \approx 0.9659258$$

You can see that both values are approximately 0.9659258.

Study Tip

To find the exact value of a trigonometric function with an angle measure in D°M′S″ form using a half-angle formula, first convert the angle measure to decimal degree form. Then multiply the resulting angle measure by 2.

Example 7 Solving a Trigonometric Equation

Find all solutions of $2 - \sin^2 x = 2\cos^2 \frac{x}{2}$ in the interval $[0, 2\pi)$.

Algebraic Solution

$$2 - \sin^2 x = 2\cos^2 \frac{x}{2}$$ Write original equation.

$$2 - \sin^2 x = 2\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2$$ Half-angle formula

$$2 - \sin^2 x = 2\left(\frac{1 + \cos x}{2}\right)$$ Simplify.

$$2 - \sin^2 x = 1 + \cos x$$ Simplify.

$$2 - (1 - \cos^2 x) = 1 + \cos x$$ Pythagorean identity

$$\cos^2 x - \cos x = 0$$ Simplify.

$$\cos x(\cos x - 1) = 0$$ Factor.

By setting the factors $\cos x$ and $\cos x - 1$ equal to zero, you find that the solutions in the interval $[0, 2\pi)$ are

$$x = \frac{\pi}{2}, \quad x = \frac{3\pi}{2}, \quad \text{and} \quad x = 0.$$

CHECKPoint Now try Exercise 77.

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = 2 - \sin^2 x - 2\cos^2(x/2)$, as shown in Figure 5.16. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts in the interval $[0, 2\pi)$ to be

$$x = 0, x \approx 1.571 \approx \frac{\pi}{2}, \text{ and } x \approx 4.712 \approx \frac{3\pi}{2}.$$

These values are the approximate solutions of $2 - \sin^2 x - 2\cos^2(x/2) = 0$ in the interval $[0, 2\pi)$.

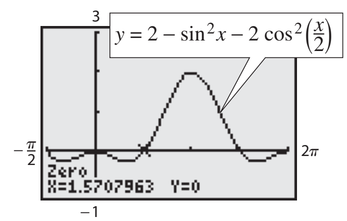


FIGURE 5.16

Product-to-Sum Formulas

Each of the following **product-to-sum formulas** can be verified using the sum and difference formulas discussed in the preceding section.

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Product-to-sum formulas are used in calculus to evaluate integrals involving the products of sines and cosines of two different angles.

Example 8 Writing Products as Sums

Rewrite the product $\cos 5x \sin 4x$ as a sum or difference.

Solution

Using the appropriate product-to-sum formula, you obtain

$$\cos 5x \sin 4x = \tfrac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]$$
$$= \tfrac{1}{2}\sin 9x - \tfrac{1}{2}\sin x.$$

CHECK*Point* Now try Exercise 85.

Occasionally, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. This can be accomplished with the following **sum-to-product formulas.**

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

For a proof of the sum-to-product formulas, see Proofs in Mathematics on page 424.

Example 9 Using a Sum-to-Product Formula

Find the exact value of $\cos 195° + \cos 105°$.

Solution

Using the appropriate sum-to-product formula, you obtain

$$\cos 195° + \cos 105° = 2\cos\left(\frac{195° + 105°}{2}\right)\cos\left(\frac{195° - 105°}{2}\right)$$
$$= 2\cos 150° \cos 45°$$
$$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{6}}{2}.$$

CHECK*Point* Now try Exercise 99.

Example 10 Solving a Trigonometric Equation

Solve $\sin 5x + \sin 3x = 0$.

Algebraic Solution

$$\sin 5x + \sin 3x = 0 \qquad \text{Write original equation.}$$

$$2 \sin\left(\frac{5x + 3x}{2}\right) \cos\left(\frac{5x - 3x}{2}\right) = 0 \qquad \text{Sum-to-product formula}$$

$$2 \sin 4x \cos x = 0 \qquad \text{Simplify.}$$

By setting the factor $2 \sin 4x$ equal to zero, you can find that the solutions in the interval $[0, 2\pi)$ are

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

The equation $\cos x = 0$ yields no additional solutions, so you can conclude that the solutions are of the form

$$x = \frac{n\pi}{4}$$

where n is an integer.

Graphical Solution

Sketch the graph of

$$y = \sin 5x + \sin 3x,$$

as shown in Figure 5.17. From the graph you can see that the x-intercepts occur at multiples of $\pi/4$. So, you can conclude that the solutions are of the form

$$x = \frac{n\pi}{4}$$

where n is an integer.

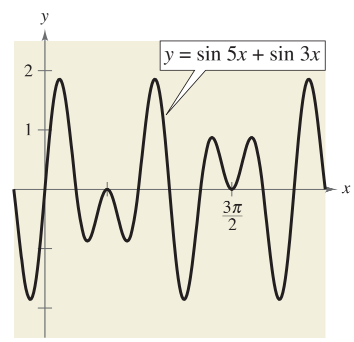


FIGURE 5.17

CHECKPoint Now try Exercise 103.

Example 11 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos x + \cos 3x} = \tan x$.

Solution

Using appropriate sum-to-product formulas, you have

$$\frac{\sin 3x - \sin x}{\cos x + \cos 3x} = \frac{2 \cos\left(\frac{3x + x}{2}\right) \sin\left(\frac{3x - x}{2}\right)}{2 \cos\left(\frac{x + 3x}{2}\right) \cos\left(\frac{x - 3x}{2}\right)}$$

$$= \frac{2 \cos(2x) \sin x}{2 \cos(2x) \cos(-x)}$$

$$= \frac{\sin x}{\cos(-x)}$$

$$= \frac{\sin x}{\cos x} = \tan x.$$

CHECKPoint Now try Exercise 121.

Application

Example 12 Projectile Motion

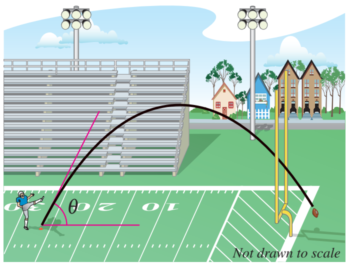


FIGURE 5.18

Ignoring air resistance, the range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is given by

$$r = \frac{1}{16}v_0^2 \sin \theta \cos \theta$$

where r is the horizontal distance (in feet) that the projectile will travel. A place kicker for a football team can kick a football from ground level with an initial velocity of 80 feet per second (see Figure 5.18).

a. Write the projectile motion model in a simpler form.

b. At what angle must the player kick the football so that the football travels 200 feet?

c. For what angle is the horizontal distance the football travels a maximum?

Solution

a. You can use a double-angle formula to rewrite the projectile motion model as

$$r = \frac{1}{32}v_0^2(2 \sin \theta \cos \theta) \qquad \text{Rewrite original projectile motion model.}$$

$$= \frac{1}{32}v_0^2 \sin 2\theta. \qquad \text{Rewrite model using a double-angle formula.}$$

b.

$$r = \frac{1}{32}v_0^2 \sin 2\theta \qquad \text{Write projectile motion model.}$$

$$200 = \frac{1}{32}(80)^2 \sin 2\theta \qquad \text{Substitute 200 for } r \text{ and 80 for } v_0.$$

$$200 = 200 \sin 2\theta \qquad \text{Simplify.}$$

$$1 = \sin 2\theta \qquad \text{Divide each side by 200.}$$

You know that $2\theta = \pi/2$, so dividing this result by 2 produces $\theta = \pi/4$. Because $\pi/4 = 45°$, you can conclude that the player must kick the football at an angle of 45° so that the football will travel 200 feet.

c. From the model $r = 200 \sin 2\theta$ you can see that the amplitude is 200. So the maximum range is $r = 200$ feet. From part (b), you know that this corresponds to an angle of 45°. Therefore, kicking the football at an angle of 45° will produce a maximum horizontal distance of 200 feet.

CHECK*Point* Now try Exercise 135.

Classroom Discussion

Deriving an Area Formula Describe how you can use a double-angle formula or a half-angle formula to derive a formula for the area of an isosceles triangle. Use a labeled sketch to illustrate your derivation. Then write two examples that show how your formula can be used.

5.5 EXERCISES

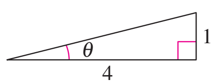
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blank to complete the trigonometric formula.

1. $\sin 2u =$ ________
2. $\dfrac{1 + \cos 2u}{2} =$ ________
3. $\cos 2u =$ ________
4. $\dfrac{1 - \cos 2u}{1 + \cos 2u} =$ ________
5. $\sin \dfrac{u}{2} =$ ________
6. $\tan \dfrac{u}{2} =$ ________
7. $\cos u \cos v =$ ________
8. $\sin u \cos v =$ ________
9. $\sin u + \sin v =$ ________
10. $\cos u - \cos v =$ ________

SKILLS AND APPLICATIONS

In Exercises 11–18, use the figure to find the exact value of the trigonometric function.



11. $\cos 2\theta$
12. $\sin 2\theta$
13. $\tan 2\theta$
14. $\sec 2\theta$
15. $\csc 2\theta$
16. $\cot 2\theta$
17. $\sin 4\theta$
18. $\tan 4\theta$

In Exercises 19–28, find the exact solutions of the equation in the interval $[0, 2\pi)$.

19. $\sin 2x - \sin x = 0$
20. $\sin 2x + \cos x = 0$
21. $4 \sin x \cos x = 1$
22. $\sin 2x \sin x = \cos x$
23. $\cos 2x - \cos x = 0$
24. $\cos 2x + \sin x = 0$
25. $\sin 4x = -2 \sin 2x$
26. $(\sin 2x + \cos 2x)^2 = 1$
27. $\tan 2x - \cot x = 0$
28. $\tan 2x - 2 \cos x = 0$

In Exercises 29–36, use a double-angle formula to rewrite the expression.

29. $6 \sin x \cos x$
30. $\sin x \cos x$
31. $6 \cos^2 x - 3$
32. $\cos^2 x - \frac{1}{2}$
33. $4 - 8 \sin^2 x$
34. $10 \sin^2 x - 5$
35. $(\cos x + \sin x)(\cos x - \sin x)$
36. $(\sin x - \cos x)(\sin x + \cos x)$

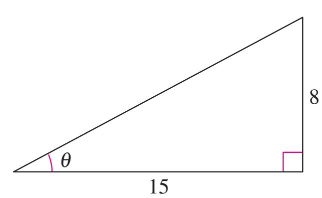
In Exercises 37–42, find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$ using the double-angle formulas.

37. $\sin u = -\dfrac{3}{5}, \quad \dfrac{3\pi}{2} < u < 2\pi$
38. $\cos u = -\dfrac{4}{5}, \quad \dfrac{\pi}{2} < u < \pi$
39. $\tan u = \dfrac{3}{5}, \quad 0 < u < \dfrac{\pi}{2}$
40. $\cot u = \sqrt{2}, \quad \pi < u < \dfrac{3\pi}{2}$
41. $\sec u = -2, \quad \dfrac{\pi}{2} < u < \pi$
42. $\csc u = 3, \quad \dfrac{\pi}{2} < u < \pi$

In Exercises 43–52, use the power-reducing formulas to rewrite the expression in terms of the first power of the cosine.

43. $\cos^4 x$
44. $\sin^4 2x$
45. $\cos^4 2x$
46. $\sin^8 x$
47. $\tan^4 2x$
48. $\sin^2 x \cos^4 x$
49. $\sin^2 2x \cos^2 2x$
50. $\tan^2 2x \cos^4 2x$
51. $\sin^4 x \cos^2 x$
52. $\sin^4 x \cos^4 x$

In Exercises 53–58, use the figure to find the exact value of the trigonometric function.



53. $\cos \dfrac{\theta}{2}$
54. $\sin \dfrac{\theta}{2}$
55. $\tan \dfrac{\theta}{2}$
56. $\sec \dfrac{\theta}{2}$
57. $\csc \dfrac{\theta}{2}$
58. $\cot \dfrac{\theta}{2}$

In Exercises 59–66, use the half-angle formulas to determine the exact values of the sine, cosine, and tangent of the angle.

59. $75°$ **60.** $165°$

61. $112° 30'$ **62.** $67° 30'$

63. $\pi/8$ **64.** $\pi/12$

65. $3\pi/8$ **66.** $7\pi/12$

In Exercises 67–72, (a) determine the quadrant in which $u/2$ lies, and (b) find the exact values of $\sin(u/2)$, $\cos(u/2)$, and $\tan(u/2)$ using the half-angle formulas.

67. $\cos u = \frac{7}{25}, \quad 0 < u < \frac{\pi}{2}$

68. $\sin u = \frac{5}{13}, \quad \frac{\pi}{2} < u < \pi$

69. $\tan u = -\frac{5}{12}, \quad \frac{3\pi}{2} < u < 2\pi$

70. $\cot u = 3, \quad \pi < u < \frac{3\pi}{2}$

71. $\csc u = -\frac{5}{3}, \quad \pi < u < \frac{3\pi}{2}$

72. $\sec u = \frac{7}{2}, \quad \frac{3\pi}{2} < u < 2\pi$

In Exercises 73–76, use the half-angle formulas to simplify the expression.

73. $\sqrt{\frac{1 - \cos 6x}{2}}$ **74.** $\sqrt{\frac{1 + \cos 4x}{2}}$

75. $-\sqrt{\frac{1 - \cos 8x}{1 + \cos 8x}}$ **76.** $-\sqrt{\frac{1 - \cos(x - 1)}{2}}$

In Exercises 77–80, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to graph the equation and verify the solutions.

77. $\sin \frac{x}{2} + \cos x = 0$ **78.** $\sin \frac{x}{2} + \cos x - 1 = 0$

79. $\cos \frac{x}{2} - \sin x = 0$ **80.** $\tan \frac{x}{2} - \sin x = 0$

In Exercises 81–90, use the product-to-sum formulas to write the product as a sum or difference.

81. $\sin \frac{\pi}{3} \cos \frac{\pi}{6}$ **82.** $4 \cos \frac{\pi}{3} \sin \frac{5\pi}{6}$

83. $10 \cos 75° \cos 15°$ **84.** $6 \sin 45° \cos 15°$

85. $\sin 5\theta \sin 3\theta$ **86.** $3 \sin(-4\alpha) \sin 6\alpha$

87. $7 \cos(-5\beta) \sin 3\beta$ **88.** $\cos 2\theta \cos 4\theta$

89. $\sin(x + y) \sin(x - y)$ **90.** $\sin(x + y) \cos(x - y)$

In Exercises 91–98, use the sum-to-product formulas to write the sum or difference as a product.

91. $\sin 3\theta + \sin \theta$ **92.** $\sin 5\theta - \sin 3\theta$

93. $\cos 6x + \cos 2x$ **94.** $\cos x + \cos 4x$

95. $\sin(\alpha + \beta) - \sin(\alpha - \beta)$ **96.** $\cos(\phi + 2\pi) + \cos \phi$

97. $\cos\left(\theta + \frac{\pi}{2}\right) - \cos\left(\theta - \frac{\pi}{2}\right)$

98. $\sin\left(x + \frac{\pi}{2}\right) + \sin\left(x - \frac{\pi}{2}\right)$

In Exercises 99–102, use the sum-to-product formulas to find the exact value of the expression.

99. $\sin 75° + \sin 15°$ **100.** $\cos 120° + \cos 60°$

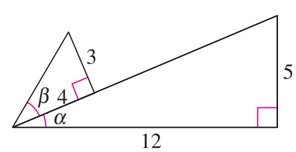
101. $\cos \frac{3\pi}{4} - \cos \frac{\pi}{4}$ **102.** $\sin \frac{5\pi}{4} - \sin \frac{3\pi}{4}$

In Exercises 103–106, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to graph the equation and verify the solutions.

103. $\sin 6x + \sin 2x = 0$ **104.** $\cos 2x - \cos 6x = 0$

105. $\frac{\cos 2x}{\sin 3x - \sin x} - 1 = 0$ **106.** $\sin^2 3x - \sin^2 x = 0$

In Exercises 107–110, use the figure to find the exact value of the trigonometric function.



107. $\sin 2\alpha$ **108.** $\cos 2\beta$

109. $\cos(\beta/2)$ **110.** $\sin(\alpha + \beta)$

In Exercises 111–124, verify the identity.

111. $\csc 2\theta = \frac{\csc \theta}{2 \cos \theta}$ **112.** $\sec 2\theta = \frac{\sec^2 \theta}{2 - \sec^2 \theta}$

113. $\sin \frac{\alpha}{3} \cos \frac{\alpha}{3} = \frac{1}{2} \sin \frac{2\alpha}{3}$ **114.** $\frac{\cos 3\beta}{\cos \beta} = 1 - 4 \sin^2 \beta$

115. $1 + \cos 10y = 2 \cos^2 5y$

116. $\cos^4 x - \sin^4 x = \cos 2x$

117. $\cos 4\alpha = \cos^2 2\alpha - \sin^2 2\alpha$

118. $(\sin x + \cos x)^2 = 1 + \sin 2x$

119. $\tan \frac{u}{2} = \csc u - \cot u$

120. $\sec \frac{u}{2} = \pm\sqrt{\frac{2 \tan u}{\tan u + \sin u}}$

121. $\dfrac{\cos 4x + \cos 2x}{\sin 4x + \sin 2x} = \cot 3x$

122. $\dfrac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \dfrac{x \pm y}{2}$

123. $\sin\left(\dfrac{\pi}{6} + x\right) + \sin\left(\dfrac{\pi}{6} - x\right) = \cos x$

124. $\cos\left(\dfrac{\pi}{3} + x\right) + \cos\left(\dfrac{\pi}{3} - x\right) = \cos x$

In Exercises 125–128, use a graphing utility to verify the identity. Confirm that it is an identity algebraically.

125. $\cos 3\beta = \cos^3 \beta - 3 \sin^2 \beta \cos \beta$

126. $\sin 4\beta = 4 \sin \beta \cos \beta(1 - 2 \sin^2 \beta)$

127. $(\cos 4x - \cos 2x)/(2 \sin 3x) = -\sin x$

128. $(\cos 3x - \cos x)/(\sin 3x - \sin x) = -\tan 2x$

In Exercises 129 and 130, graph the function by hand in the interval $[0, 2\pi]$ by using the power-reducing formulas.

129. $f(x) = \sin^2 x$ **130.** $f(x) = \cos^2 x$

In Exercises 131–134, write the trigonometric expression as an algebraic expression.

131. $\sin(2 \arcsin x)$ **132.** $\cos(2 \arccos x)$

133. $\cos(2 \arcsin x)$ **134.** $\sin(2 \arccos x)$

135. PROJECTILE MOTION The range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is

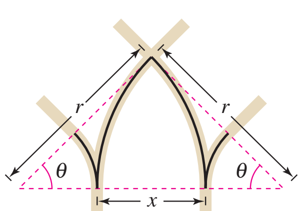
$$r = \frac{1}{32} v_0^2 \sin 2\theta$$

where r is measured in feet. An athlete throws a javelin at 75 feet per second. At what angle must the athlete throw the javelin so that the javelin travels 130 feet?

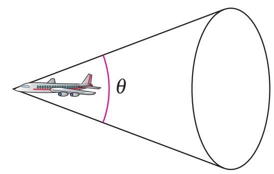
136. RAILROAD TRACK When two railroad tracks merge, the overlapping portions of the tracks are in the shapes of circular arcs (see figure). The radius of each arc r (in feet) and the angle θ are related by

$$\frac{x}{2} = 2r \sin^2 \frac{\theta}{2}.$$

Write a formula for x in terms of $\cos \theta$.



137. MACH NUMBER The mach number M of an airplane is the ratio of its speed to the speed of sound. When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane (see figure). The mach number is related to the apex angle θ of the cone by $\sin(\theta/2) = 1/M$.



(a) Find the angle θ that corresponds to a mach number of 1.

(b) Find the angle θ that corresponds to a mach number of 4.5.

(c) The speed of sound is about 760 miles per hour. Determine the speed of an object with the mach numbers from parts (a) and (b).

(d) Rewrite the equation in terms of θ.

EXPLORATION

138. CAPSTONE Consider the function given by $f(x) = \sin^4 x + \cos^4 x$.

(a) Use the power-reducing formulas to write the function in terms of cosine to the first power.

(b) Determine another way of rewriting the function. Use a graphing utility to rule out incorrectly rewritten functions.

(c) Add a trigonometric term to the function so that it becomes a perfect square trinomial. Rewrite the function as a perfect square trinomial minus the term that you added. Use a graphing utility to rule out incorrectly rewritten functions.

(d) Rewrite the result of part (c) in terms of the sine of a double angle. Use a graphing utility to rule out incorrectly rewritten functions.

(e) When you rewrite a trigonometric expression, the result may not be the same as a friend's. Does this mean that one of you is wrong? Explain.

TRUE OR FALSE? **In Exercises 139 and 140, determine whether the statement is true or false. Justify your answer.**

139. Because the sine function is an odd function, for a negative number u, $\sin 2u = -2 \sin u \cos u$.

140. $\sin \dfrac{u}{2} = -\sqrt{\dfrac{1 - \cos u}{2}}$ when u is in the second quadrant.

5 Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 5.1	Recognize and write the fundamental trigonometric identities *(p. 372)*.	**Reciprocal Identities** $\sin u = 1/\csc u$ $\cos u = 1/\sec u$ $\tan u = 1/\cot u$ $\csc u = 1/\sin u$ $\sec u = 1/\cos u$ $\cot u = 1/\tan u$ **Quotient Identities:** $\tan u = \dfrac{\sin u}{\cos u}$, $\cot u = \dfrac{\cos u}{\sin u}$ **Pythagorean Identities:** $\sin^2 u + \cos^2 u = 1$, $1 + \tan^2 u = \sec^2 u$, $1 + \cot^2 u = \csc^2 u$ **Cofunction Identities** $\sin[(\pi/2) - u] = \cos u$ $\cos[(\pi/2) - u] = \sin u$ $\tan[(\pi/2) - u] = \cot u$ $\cot[(\pi/2) - u] = \tan u$ $\sec[(\pi/2) - u] = \csc u$ $\csc[(\pi/2) - u] = \sec u$ **Even/Odd Identities** $\sin(-u) = -\sin u$ $\cos(-u) = \cos u$ $\tan(-u) = -\tan u$ $\csc(-u) = -\csc u$ $\sec(-u) = \sec u$ $\cot(-u) = -\cot u$	1–6
	Use the fundamental trigonometric identities to evaluate trigonometric functions, and simplify and rewrite trigonometric expressions *(p. 373)*.	In some cases, when factoring or simplifying trigonometric expressions, it is helpful to rewrite the expression in terms of just *one* trigonometric function or in terms of *sine and cosine only.*	7–28
Section 5.2	Verify trigonometric identities *(p. 380)*.	**Guidelines for Verifying Trigonometric Identities** **1.** Work with one side of the equation at a time. **2.** Look to factor an expression, add fractions, square a binomial, or create a monomial denominator. **3.** Look to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents. **4.** If the preceding guidelines do not help, try converting all terms to sines and cosines. **5.** Always try *something*.	29–36
Section 5.3	Use standard algebraic techniques to solve trigonometric equations *(p. 387)*.	Use standard algebraic techniques such as collecting like terms, extracting square roots, and factoring to solve trigonometric equations.	37–42
	Solve trigonometric equations of quadratic type *(p. 389)*.	To solve trigonometric equations of quadratic type $ax^2 + bx + c = 0$, factor the quadratic or, if this is not possible, use the Quadratic Formula.	43–46
	Solve trigonometric equations involving multiple angles *(p. 392)*.	To solve equations that contain forms such as $\sin ku$ or $\cos ku$, first solve the equation for ku, then divide your result by k.	47–52
	Use inverse trigonometric functions to solve trigonometric equations *(p. 393)*.	After factoring an equation and setting the factors equal to 0, you may get an equation such as $\tan x - 3 = 0$. In this case, use inverse trigonometric functions to solve. (See Example 9.)	53–56

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 5.4	Use sum and difference formulas to evaluate trigonometric functions, verify identities, and solve trigonometric equations *(p. 398).*	**Sum and Difference Formulas** $\sin(u + v) = \sin u \cos v + \cos u \sin v$ $\sin(u - v) = \sin u \cos v - \cos u \sin v$ $\cos(u + v) = \cos u \cos v - \sin u \sin v$ $\cos(u - v) = \cos u \cos v + \sin u \sin v$ $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$ $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$	57–80
Section 5.5	Use multiple-angle formulas to rewrite and evaluate trigonometric functions *(p. 405).*	**Double-Angle Formulas** $\sin 2u = 2 \sin u \cos u$ $\quad \cos 2u = \cos^2 u - \sin^2 u$ $\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u}$ $\quad = 2\cos^2 u - 1$ $= 1 - 2\sin^2 u$	81–86
	Use power-reducing formulas to rewrite and evaluate trigonometric functions *(p. 407).*	**Power-Reducing Formulas** $\sin^2 u = \dfrac{1 - \cos 2u}{2}, \quad \cos^2 u = \dfrac{1 + \cos 2u}{2}$ $\tan^2 u = \dfrac{1 - \cos 2u}{1 + \cos 2u}$	87–90
	Use half-angle formulas to rewrite and evaluate trigonometric functions *(p. 408).*	**Half-Angle Formulas** $\sin \dfrac{u}{2} = \pm\sqrt{\dfrac{1 - \cos u}{2}}, \quad \cos \dfrac{u}{2} = \pm\sqrt{\dfrac{1 + \cos u}{2}}$ $\tan \dfrac{u}{2} = \dfrac{1 - \cos u}{\sin u} = \dfrac{\sin u}{1 + \cos u}$ The signs of $\sin(u/2)$ and $\cos(u/2)$ depend on the quadrant in which $u/2$ lies.	91–100
	Use product-to-sum formulas *(p. 409)* and sum-to-product formulas *(p. 410)* to rewrite and evaluate trigonometric functions.	**Product-to-Sum Formulas** $\sin u \sin v = (1/2)[\cos(u - v) - \cos(u + v)]$ $\cos u \cos v = (1/2)[\cos(u - v) + \cos(u + v)]$ $\sin u \cos v = (1/2)[\sin(u + v) + \sin(u - v)]$ $\cos u \sin v = (1/2)[\sin(u + v) - \sin(u - v)]$ **Sum-to-Product Formulas** $\sin u + \sin v = 2 \sin\left(\dfrac{u + v}{2}\right)\cos\left(\dfrac{u - v}{2}\right)$ $\sin u - \sin v = 2 \cos\left(\dfrac{u + v}{2}\right)\sin\left(\dfrac{u - v}{2}\right)$ $\cos u + \cos v = 2 \cos\left(\dfrac{u + v}{2}\right)\cos\left(\dfrac{u - v}{2}\right)$ $\cos u - \cos v = -2 \sin\left(\dfrac{u + v}{2}\right)\sin\left(\dfrac{u - v}{2}\right)$	101–108
	Use trigonometric formulas to rewrite real-life models *(p. 412).*	A trigonometric formula can be used to rewrite the projectile motion model $r = (1/16)v_0^2 \sin \theta \cos \theta$. (See Example 12.)	109–114

5 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

5.1 In Exercises 1–6, name the trigonometric function that is equivalent to the expression.

1. $\dfrac{\sin x}{\cos x}$ **2.** $\dfrac{1}{\sin x}$

3. $\dfrac{1}{\sec x}$ **4.** $\dfrac{1}{\tan x}$

5. $\sqrt{\cot^2 x + 1}$ **6.** $\sqrt{1 + \tan^2 x}$

In Exercises 7–10, use the given values and trigonometric identities to evaluate (if possible) all six trigonometric functions.

7. $\sin x = \frac{5}{13}, \quad \cos x = \frac{12}{13}$

8. $\tan \theta = \dfrac{2}{3}, \quad \sec \theta = \dfrac{\sqrt{13}}{3}$

9. $\sin\left(\dfrac{\pi}{2} - x\right) = \dfrac{\sqrt{2}}{2}, \quad \sin x = -\dfrac{\sqrt{2}}{2}$

10. $\csc\left(\dfrac{\pi}{2} - \theta\right) = 9, \quad \sin \theta = \dfrac{4\sqrt{5}}{9}$

In Exercises 11–24, use the fundamental trigonometric identities to simplify the expression.

11. $\dfrac{1}{\cot^2 x + 1}$ **12.** $\dfrac{\tan \theta}{1 - \cos^2 \theta}$

13. $\tan^2 x(\csc^2 x - 1)$ **14.** $\cot^2 x(\sin^2 x)$

15. $\dfrac{\sin\left(\dfrac{\pi}{2} - \theta\right)}{\sin \theta}$ **16.** $\dfrac{\cot\left(\dfrac{\pi}{2} - u\right)}{\cos u}$

17. $\dfrac{\sin^2 \theta + \cos^2 \theta}{\sin \theta}$ **18.** $\dfrac{\sec^2(-\theta)}{\csc^2 \theta}$

19. $\cos^2 x + \cos^2 x \cot^2 x$ **20.** $\tan^2 \theta \csc^2 \theta - \tan^2 \theta$

21. $(\tan x + 1)^2 \cos x$ **22.** $(\sec x - \tan x)^2$

23. $\dfrac{1}{\csc \theta + 1} - \dfrac{1}{\csc \theta - 1}$ **24.** $\dfrac{\tan^2 x}{1 + \sec x}$

In Exercises 25 and 26, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

25. $\sqrt{25 - x^2}, x = 5 \sin \theta$ **26.** $\sqrt{x^2 - 16}, x = 4 \sec \theta$

27. RATE OF CHANGE The rate of change of the function $f(x) = \csc x - \cot x$ is given by the expression $\csc^2 x - \csc x \cot x$. Show that this expression can also be written as

$$\frac{1 - \cos x}{\sin^2 x}.$$

28. RATE OF CHANGE The rate of change of the function $f(x) = 2\sqrt{\sin x}$ is given by the expression $\sin^{-1/2} x \cos x$. Show that this expression can also be written as $\cot x\sqrt{\sin x}$.

5.2 In Exercises 29–36, verify the identity.

29. $\cos x(\tan^2 x + 1) = \sec x$

30. $\sec^2 x \cot x - \cot x = \tan x$

31. $\sec\left(\dfrac{\pi}{2} - \theta\right) = \csc \theta$ **32.** $\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

33. $\dfrac{1}{\tan \theta \csc \theta} = \cos \theta$ **34.** $\dfrac{1}{\tan x \csc x \sin x} = \cot x$

35. $\sin^5 x \cos^2 x = (\cos^2 x - 2\cos^4 x + \cos^6 x) \sin x$

36. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x) \cos x$

5.3 In Exercises 37–42, solve the equation.

37. $\sin x = \sqrt{3} - \sin x$ **38.** $4 \cos \theta = 1 + 2 \cos \theta$

39. $3\sqrt{3} \tan u = 3$ **40.** $\frac{1}{2} \sec x - 1 = 0$

41. $3 \csc^2 x = 4$ **42.** $4 \tan^2 u - 1 = \tan^2 u$

In Exercises 43–52, find all solutions of the equation in the interval $[0, 2\pi)$.

43. $2\cos^2 x - \cos x = 1$ **44.** $2\sin^2 x - 3 \sin x = -1$

45. $\cos^2 x + \sin x = 1$ **46.** $\sin^2 x + 2 \cos x = 2$

47. $2 \sin 2x - \sqrt{2} = 0$ **48.** $2 \cos \dfrac{x}{2} + 1 = 0$

49. $3 \tan^2\left(\dfrac{x}{3}\right) - 1 = 0$ **50.** $\sqrt{3} \tan 3x = 0$

51. $\cos 4x(\cos x - 1) = 0$ **52.** $3 \csc^2 5x = -4$

In Exercises 53–56, use inverse functions where needed to find all solutions of the equation in the interval $[0, 2\pi)$.

53. $\sin^2 x - 2 \sin x = 0$

54. $2\cos^2 x + 3 \cos x = 0$

55. $\tan^2 \theta + \tan \theta - 6 = 0$

56. $\sec^2 x + 6 \tan x + 4 = 0$

5.4 In Exercises 57–60, find the exact values of the sine, cosine, and tangent of the angle.

57. $285° = 315° - 30°$ **58.** $345° = 300° + 45°$

59. $\dfrac{25\pi}{12} = \dfrac{11\pi}{6} + \dfrac{\pi}{4}$ **60.** $\dfrac{19\pi}{12} = \dfrac{11\pi}{6} - \dfrac{\pi}{4}$

In Exercises 61–64, write the expression as the sine, cosine, or tangent of an angle.

61. $\sin 60° \cos 45° - \cos 60° \sin 45°$

62. $\cos 45° \cos 120° - \sin 45° \sin 120°$

63. $\dfrac{\tan 25° + \tan 10°}{1 - \tan 25° \tan 10°}$

64. $\dfrac{\tan 68° - \tan 115°}{1 + \tan 68° \tan 115°}$

In Exercises 65–70, find the exact value of the trigonometric function given that $\tan u = \frac{3}{4}$ and $\cos v = -\frac{4}{5}$. (u is in Quadrant I and v is in Quadrant III.)

65. $\sin(u + v)$ **66.** $\tan(u + v)$

67. $\cos(u - v)$ **68.** $\sin(u - v)$

69. $\cos(u + v)$ **70.** $\tan(u - v)$

In Exercises 71–76, verify the identity.

71. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$ **72.** $\sin\left(x - \frac{3\pi}{2}\right) = \cos x$

73. $\tan\left(x - \frac{\pi}{2}\right) = -\cot x$ **74.** $\tan(\pi - x) = -\tan x$

75. $\cos 3x = 4\cos^3 x - 3\cos x$

76. $\dfrac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = \dfrac{\tan \alpha - \tan \beta}{\tan \alpha + \tan \beta}$

In Exercises 77–80, find all solutions of the equation in the interval $[0, 2\pi)$.

77. $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right) = 1$

78. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = 1$

79. $\sin\left(x + \frac{\pi}{2}\right) - \sin\left(x - \frac{\pi}{2}\right) = \sqrt{3}$

80. $\cos\left(x + \frac{3\pi}{4}\right) - \cos\left(x - \frac{3\pi}{4}\right) = 0$

5.5 In Exercises 81–84, find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$ using the double-angle formulas.

81. $\sin u = -\frac{4}{5}, \quad \pi < u < \frac{3\pi}{2}$

82. $\cos u = -\frac{2}{\sqrt{5}}, \quad \frac{\pi}{2} < u < \pi$

83. $\sec u = -3, \quad \frac{\pi}{2} < u < \pi$

84. $\cot u = 2, \quad \pi < u < \frac{3\pi}{2}$

In Exercises 85 and 86, use double-angle formulas to verify the identity algebraically and use a graphing utility to confirm your result graphically.

85. $\sin 4x = 8\cos^3 x \sin x - 4\cos x \sin x$

86. $\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

In Exercises 87–90, use the power-reducing formulas to rewrite the expression in terms of the first power of the cosine.

87. $\tan^2 2x$ **88.** $\cos^2 3x$

89. $\sin^2 x \tan^2 x$ **90.** $\cos^2 x \tan^2 x$

In Exercises 91–94, use the half-angle formulas to determine the exact values of the sine, cosine, and tangent of the angle.

91. $-75°$ **92.** $15°$

93. $\frac{19\pi}{12}$ **94.** $-\frac{17\pi}{12}$

In Exercises 95–98, (a) determine the quadrant in which $u/2$ lies, and (b) find the exact values of $\sin(u/2)$, $\cos(u/2)$, and $\tan(u/2)$ using the half-angle formulas.

95. $\sin u = \frac{7}{25}, \quad 0 < u < \pi/2$

96. $\tan u = \frac{4}{3}, \quad \pi < u < 3\pi/2$

97. $\cos u = -\frac{2}{7}, \quad \pi/2 < u < \pi$

98. $\sec u = -6, \quad \pi/2 < u < \pi$

In Exercises 99 and 100, use the half-angle formulas to simplify the expression.

99. $-\sqrt{\dfrac{1 + \cos 10x}{2}}$ **100.** $\dfrac{\sin 6x}{1 + \cos 6x}$

In Exercises 101–104, use the product-to-sum formulas to write the product as a sum or difference.

101. $\cos \frac{\pi}{6} \sin \frac{\pi}{6}$ **102.** $6 \sin 15° \sin 45°$

103. $\cos 4\theta \sin 6\theta$ **104.** $2 \sin 7\theta \cos 3\theta$

In Exercises 105–108, use the sum-to-product formulas to write the sum or difference as a product.

105. $\sin 4\theta - \sin 8\theta$

106. $\cos 6\theta + \cos 5\theta$

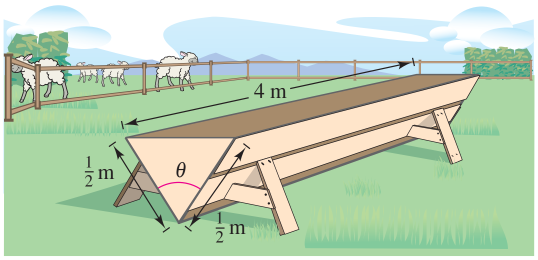
107. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)$

108. $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right)$

109. PROJECTILE MOTION A baseball leaves the hand of the player at first base at an angle of θ with the horizontal and at an initial velocity of $v_0 = 80$ feet per second. The ball is caught by the player at second base 100 feet away. Find θ if the range r of a projectile is

$$r = \frac{1}{32} v_0^2 \sin 2\theta.$$

110. GEOMETRY A trough for feeding cattle is 4 meters long and its cross sections are isosceles triangles with the two equal sides being $\frac{1}{2}$ meter (see figure). The angle between the two sides is θ.



(a) Write the trough's volume as a function of $\theta/2$.

(b) Write the volume of the trough as a function of θ and determine the value of θ such that the volume is maximum.

HARMONIC MOTION In Exercises 111–114, use the following information. A weight is attached to a spring suspended vertically from a ceiling. When a driving force is applied to the system, the weight moves vertically from its equilibrium position, and this motion is described by the model $y = 1.5 \sin 8t - 0.5 \cos 8t$, where y is the distance from equilibrium (in feet) and t is the time (in seconds).

111. Use a graphing utility to graph the model.

112. Write the model in the form

$$y = \sqrt{a^2 + b^2} \sin(Bt + C).$$

113. Find the amplitude of the oscillations of the weight.

114. Find the frequency of the oscillations of the weight.

EXPLORATION

TRUE OR FALSE? In Exercises 115–118, determine whether the statement is true or false. Justify your answer.

115. If $\frac{\pi}{2} < \theta < \pi$, then $\cos \frac{\theta}{2} < 0$.

116. $\sin(x + y) = \sin x + \sin y$

117. $4 \sin(-x) \cos(-x) = -2 \sin 2x$

118. $4 \sin 45° \cos 15° = 1 + \sqrt{3}$

119. List the reciprocal identities, quotient identities, and Pythagorean identities from memory.

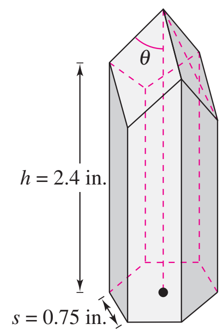
120. THINK ABOUT IT If a trigonometric equation has an infinite number of solutions, is it true that the equation is an identity? Explain.

121. THINK ABOUT IT Explain why you know from observation that the equation $a \sin x - b = 0$ has no solution if $|a| < |b|$.

122. SURFACE AREA The surface area of a honeycomb is given by the equation

$$S = 6hs + \frac{3}{2}s^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \quad 0 < \theta \le 90°$$

where $h = 2.4$ inches, $s = 0.75$ inch, and θ is the angle shown in the figure.



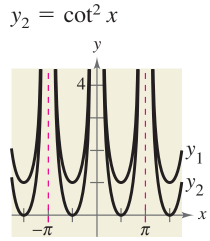
(a) For what value(s) of θ is the surface area 12 square inches?

(b) What value of θ gives the minimum surface area?

In Exercises 123 and 124, use the graphs of y_1 and y_2 to determine how to change one function to form the identity $y_1 = y_2$.

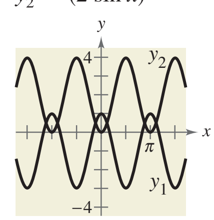
123. $y_1 = \sec^2\left(\frac{\pi}{2} - x\right)$

$y_2 = \cot^2 x$



124. $y_1 = \frac{\cos 3x}{\cos x}$

$y_2 = (2 \sin x)^2$



In Exercises 125 and 126, use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function.

125. $y = \sqrt{x + 3} + 4 \cos x$

126. $y = 2 - \frac{1}{2}x^2 + 3 \sin \frac{\pi x}{2}$

5 CHAPTER TEST

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. If $\tan\theta = \frac{6}{5}$ and $\cos\theta < 0$, use the fundamental identities to evaluate all six trigonometric functions of θ.
2. Use the fundamental identities to simplify $\csc^2\beta(1 - \cos^2\beta)$.
3. Factor and simplify $\dfrac{\sec^4 x - \tan^4 x}{\sec^2 x + \tan^2 x}$.
4. Add and simplify $\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$.
5. Determine the values of θ, $0 \le \theta < 2\pi$, for which $\tan\theta = -\sqrt{\sec^2\theta - 1}$ is true.
6. Use a graphing utility to graph the functions $y_1 = \cos x + \sin x \tan x$ and $y_2 = \sec x$. Make a conjecture about y_1 and y_2. Verify the result algebraically.

In Exercises 7–12, verify the identity.

7. $\sin\theta \sec\theta = \tan\theta$
8. $\sec^2 x \tan^2 x + \sec^2 x = \sec^4 x$
9. $\dfrac{\csc\alpha + \sec\alpha}{\sin\alpha + \cos\alpha} = \cot\alpha + \tan\alpha$
10. $\tan\left(x + \dfrac{\pi}{2}\right) = -\cot x$
11. $\sin(n\pi + \theta) = (-1)^n \sin\theta$, n is an integer.
12. $(\sin x + \cos x)^2 = 1 + \sin 2x$

13. Rewrite $\sin^4 \dfrac{x}{2}$ in terms of the first power of the cosine.
14. Use a half-angle formula to simplify the expression $\sin 4\theta/(1 + \cos 4\theta)$.
15. Write $4 \sin 3\theta \cos 2\theta$ as a sum or difference.
16. Write $\cos 3\theta - \cos\theta$ as a product.

In Exercises 17–20, find all solutions of the equation in the interval $[0, 2\pi)$.

17. $\tan^2 x + \tan x = 0$
18. $\sin 2\alpha - \cos\alpha = 0$
19. $4\cos^2 x - 3 = 0$
20. $\csc^2 x - \csc x - 2 = 0$

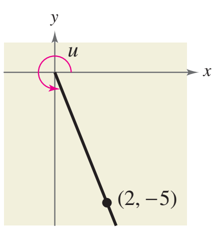


FIGURE FOR 23

21. Use a graphing utility to approximate the solutions of the equation $5 \sin x - x = 0$ accurate to three decimal places.
22. Find the exact value of $\cos 105°$ using the fact that $105° = 135° - 30°$.
23. Use the figure to find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$.
24. Cheyenne, Wyoming has a latitude of 41°N. At this latitude, the position of the sun at sunrise can be modeled by

$$D = 31 \sin\left(\frac{2\pi}{365}t - 1.4\right)$$

where t is the time (in days) and $t = 1$ represents January 1. In this model, D represents the number of degrees north or south of due east that the sun rises. Use a graphing utility to determine the days on which the sun is more than 20° north of due east at sunrise.

25. The heights h (in feet) of two people in different seats on a Ferris wheel can be modeled by

$$h_1 = 28 \cos 10t + 38 \quad \text{and} \quad h_2 = 28 \cos\left[10\left(t - \frac{\pi}{6}\right)\right] + 38, \ 0 \le t \le 2$$

where t is the time (in minutes). When are the two people at the same height?

PROOFS IN MATHEMATICS

Sum and Difference Formulas (p. 398)

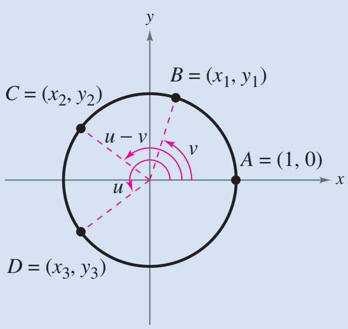
$$\sin(u+v) = \sin u \cos v + \cos u \sin v \qquad \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

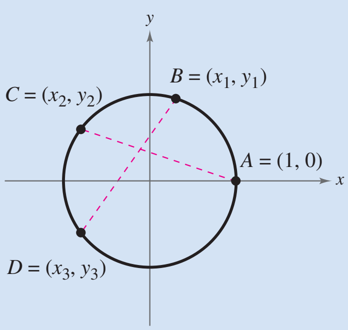
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v \qquad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

Proof





You can use the figures at the left for the proofs of the formulas for $\cos(u \pm v)$. In the top figure, let A be the point $(1, 0)$ and then use u and v to locate the points $B = (x_1, y_1)$, $C = (x_2, y_2)$, and $D = (x_3, y_3)$ on the unit circle. So, $x_i^2 + y_i^2 = 1$ for $i = 1, 2$, and 3. For convenience, assume that $0 < v < u < 2\pi$. In the bottom figure, note that arcs AC and BD have the same length. So, line segments AC and BD are also equal in length, which implies that

$$\sqrt{(x_2 - 1)^2 + (y_2 - 0)^2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$x_2^2 - 2x_2 + 1 + y_2^2 = x_3^2 - 2x_1x_3 + x_1^2 + y_3^2 - 2y_1y_3 + y_1^2$$

$$(x_2^2 + y_2^2) + 1 - 2x_2 = (x_3^2 + y_3^2) + (x_1^2 + y_1^2) - 2x_1x_3 - 2y_1y_3$$

$$1 + 1 - 2x_2 = 1 + 1 - 2x_1x_3 - 2y_1y_3$$

$$x_2 = x_3x_1 + y_3y_1.$$

Finally, by substituting the values $x_2 = \cos(u - v)$, $x_3 = \cos u$, $x_1 = \cos v$, $y_3 = \sin u$, and $y_1 = \sin v$, you obtain $\cos(u - v) = \cos u \cos v + \sin u \sin v$. The formula for $\cos(u + v)$ can be established by considering $u + v = u - (-v)$ and using the formula just derived to obtain

$$\cos(u+v) = \cos[u - (-v)] = \cos u \cos(-v) + \sin u \sin(-v)$$

$$= \cos u \cos v - \sin u \sin v.$$

You can use the sum and difference formulas for sine and cosine to prove the formulas for $\tan(u \pm v)$.

$$\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)}$$ Quotient identity

$$= \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}$$ Sum and difference formulas

$$= \frac{\dfrac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v}}{\dfrac{\cos u \cos v \mp \sin u \sin v}{\cos u \cos v}}$$ Divide numerator and denominator by $\cos u \cos v$.

$$= \frac{\dfrac{\sin u \cos v}{\cos u \cos v} \pm \dfrac{\cos u \sin v}{\cos u \cos v}}{\dfrac{\cos u \cos v}{\cos u \cos v} \mp \dfrac{\sin u \sin v}{\cos u \cos v}}$$ Write as separate fractions.

$$= \frac{\dfrac{\sin u}{\cos u} \pm \dfrac{\sin v}{\cos v}}{1 \mp \dfrac{\sin u}{\cos u} \cdot \dfrac{\sin v}{\cos v}}$$ Product of fractions

$$= \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$ Quotient identity

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Trigonometry and Astronomy

Trigonometry was used by early astronomers to calculate measurements in the universe. Trigonometry was used to calculate the circumference of Earth and the distance from Earth to the moon. Another major accomplishment in astronomy using trigonometry was computing distances to stars.

Double-Angle Formulas *(p. 405)*

$$\sin 2u = 2 \sin u \cos u \qquad \cos 2u = \cos^2 u - \sin^2 u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u} \qquad = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

Proof

To prove all three formulas, let $v = u$ in the corresponding sum formulas.

$$\sin 2u = \sin(u + u) = \sin u \cos u + \cos u \sin u = 2 \sin u \cos u$$

$$\cos 2u = \cos(u + u) = \cos u \cos u - \sin u \sin u = \cos^2 u - \sin^2 u$$

$$\tan 2u = \tan(u + u) = \frac{\tan u + \tan u}{1 - \tan u \tan u} = \frac{2 \tan u}{1 - \tan^2 u}$$

Power-Reducing Formulas *(p. 407)*

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Proof

To prove the first formula, solve for $\sin^2 u$ in the double-angle formula $\cos 2u = 1 - 2\sin^2 u$, as follows.

$$\cos 2u = 1 - 2\sin^2 u$$ Write double-angle formula.

$$2\sin^2 u = 1 - \cos 2u$$ Subtract $\cos 2u$ from and add $2 \sin^2 u$ to each side.

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$ Divide each side by 2.

In a similar way you can prove the second formula, by solving for $\cos^2 u$ in the double-angle formula

$$\cos 2u = 2\cos^2 u - 1.$$

To prove the third formula, use a quotient identity, as follows.

$$\begin{aligned}\tan^2 u &= \frac{\sin^2 u}{\cos^2 u}\\ &= \frac{\frac{1-\cos 2u}{2}}{\frac{1+\cos 2u}{2}}\\ &= \frac{1-\cos 2u}{1+\cos 2u}\end{aligned}$$

Sum-to-Product Formulas *(p. 410)*

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Proof

To prove the first formula, let $x = u + v$ and $y = u - v$. Then substitute $u = (x + y)/2$ and $v = (x - y)/2$ in the product-to-sum formula.

$$\begin{aligned}\sin u \cos v &= \frac{1}{2}[\sin(u+v) + \sin(u-v)]\\ \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \frac{1}{2}(\sin x + \sin y)\\ 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \sin x + \sin y\end{aligned}$$

The other sum-to-product formulas can be proved in a similar manner.

PROBLEM SOLVING

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. (a) Write each of the other trigonometric functions of θ in terms of $\sin \theta$.

(b) Write each of the other trigonometric functions of θ in terms of $\cos \theta$.

2. Verify that for all integers n,

$$\cos\left[\frac{(2n+1)\pi}{2}\right] = 0.$$

3. Verify that for all integers n,

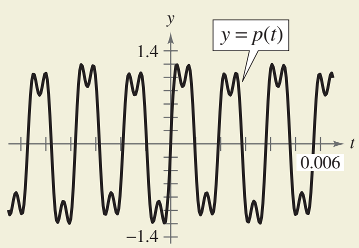
$$\sin\left[\frac{(12n+1)\pi}{6}\right] = \frac{1}{2}.$$

4. A particular sound wave is modeled by

$$p(t) = \frac{1}{4\pi}(p_1(t) + 30p_2(t) + p_3(t) + p_5(t) + 30p_6(t))$$

where $p_n(t) = \frac{1}{n}\sin(524n\pi t)$, and t is the time (in seconds).

(a) Find the sine components $p_n(t)$ and use a graphing utility to graph each component. Then verify the graph of p that is shown.



(b) Find the period of each sine component of p. Is p periodic? If so, what is its period?

(c) Use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to find the t-intercepts of the graph of p over one cycle.

(d) Use the *maximum* and *minimum* features of a graphing utility to approximate the absolute maximum and absolute minimum values of p over one cycle.

5. Three squares of side s are placed side by side (see figure). Make a conjecture about the relationship between the sum $u + v$ and w. Prove your conjecture by using the identity for the tangent of the sum of two angles.

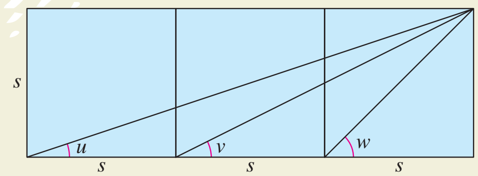


FIGURE FOR 5

6. The path traveled by an object (neglecting air resistance) that is projected at an initial height of h_0 feet, an initial velocity of v_0 feet per second, and an initial angle θ is given by

$$y = -\frac{16}{v_0^2 \cos^2 \theta}x^2 + (\tan \theta)x + h_0$$

where x and y are measured in feet. Find a formula for the maximum height of an object projected from ground level at velocity v_0 and angle θ. To do this, find half of the horizontal distance

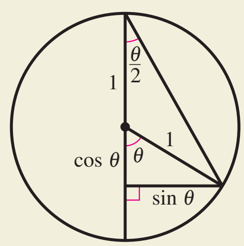
$$\frac{1}{32}v_0^2 \sin 2\theta$$

and then substitute it for x in the general model for the path of a projectile (where $h_0 = 0$).

7. Use the figure to derive the formulas for

$$\sin\frac{\theta}{2},\ \cos\frac{\theta}{2},\ \text{and } \tan\frac{\theta}{2}$$

where θ is an acute angle.



8. The force F (in pounds) on a person's back when he or she bends over at an angle θ is modeled by

$$F = \frac{0.6W\sin(\theta + 90°)}{\sin 12°}$$

where W is the person's weight (in pounds).

(a) Simplify the model.

(b) Use a graphing utility to graph the model, where $W = 185$ and $0° < \theta < 90°$.

(c) At what angle is the force a maximum? At what angle is the force a minimum?

9. The number of hours of daylight that occur at any location on Earth depends on the time of year and the latitude of the location. The following equations model the numbers of hours of daylight in Seward, Alaska (60° latitude) and New Orleans, Louisiana (30° latitude).

$$D = 12.2 - 6.4 \cos\left[\frac{\pi(t + 0.2)}{182.6}\right] \quad \text{Seward}$$

$$D = 12.2 - 1.9 \cos\left[\frac{\pi(t + 0.2)}{182.6}\right] \quad \text{New Orleans}$$

In these models, D represents the number of hours of daylight and t represents the day, with $t = 0$ corresponding to January 1.

(a) Use a graphing utility to graph both models in the same viewing window. Use a viewing window of $0 \le t \le 365$.

(b) Find the days of the year on which both cities receive the same amount of daylight.

(c) Which city has the greater variation in the number of daylight hours? Which constant in each model would you use to determine the difference between the greatest and least numbers of hours of daylight?

(d) Determine the period of each model.

10. The tide, or depth of the ocean near the shore, changes throughout the day. The water depth d (in feet) of a bay can be modeled by

$$d = 35 - 28 \cos \frac{\pi}{6.2} t$$

where t is the time in hours, with $t = 0$ corresponding to 12:00 A.M.

(a) Algebraically find the times at which the high and low tides occur.

(b) Algebraically find the time(s) at which the water depth is 3.5 feet.

(c) Use a graphing utility to verify your results from parts (a) and (b).

11. Find the solution of each inequality in the interval $[0, 2\pi]$.

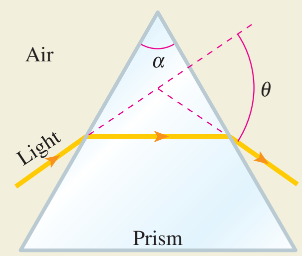
(a) $\sin x \ge 0.5$ (b) $\cos x \le -0.5$

(c) $\tan x < \sin x$ (d) $\cos x \ge \sin x$

12. The index of refraction n of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. Some common materials and their indices are air (1.00), water (1.33), and glass (1.50). Triangular prisms are often used to measure the index of refraction based on the formula

$$n = \frac{\sin\left(\dfrac{\theta}{2} + \dfrac{\alpha}{2}\right)}{\sin \dfrac{\theta}{2}}.$$

For the prism shown in the figure, $\alpha = 60°$.



(a) Write the index of refraction as a function of $\cot(\theta/2)$.

(b) Find θ for a prism made of glass.

13. (a) Write a sum formula for $\sin(u + v + w)$.

(b) Write a sum formula for $\tan(u + v + w)$.

14. (a) Derive a formula for $\cos 3\theta$.

(b) Derive a formula for $\cos 4\theta$.

15. The heights h (in inches) of pistons 1 and 2 in an automobile engine can be modeled by

$$h_1 = 3.75 \sin 733t + 7.5$$

and

$$h_2 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$$

where t is measured in seconds.

(a) Use a graphing utility to graph the heights of these two pistons in the same viewing window for $0 \le t \le 1$.

(b) How often are the pistons at the same height?