Determine the radius of convergence of $f(z)$.

12. Let $\{a_n\}$ be the sequence of real numbers defined by the conditions:

$$a_0 = 1,\ a_1 = 2, \qquad \text{and} \qquad a_n = a_{n-1} + a_{n-2} \qquad \text{for} \quad n \geqq 2.$$

Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n.$$

[*Hint*: What is the general solution of a difference equation? Cf. Exercise 6 of §1.]

13. More generally, let u_1, u_2 be complex numbers such that the polynomial

$$P(T) = T^2 - u_1 T - u_2 = (T - \alpha_1)(T - \alpha_2)$$

has two distinct roots with $|\alpha_1| < |\alpha_2|$. Let a_0, a_1 be given, and let

$$a_n = u_1 a_{n-1} + u_2 a_{n-2} \qquad \text{for} \quad n \geqq 2.$$

What is the radius of convergence of the series $\sum a_n T^n$?

II, §3. RELATIONS BETWEEN FORMAL AND CONVERGENT SERIES

Sums and Products

Let $f = f(T)$ and $g = g(T)$ be formal power series. We may form their formal product and sum, $f + g$ and fg. If f converges absolutely for some complex number z, then we have the value $f(z)$, and similarly for $g(z)$.

Theorem 3.1. *If f, g are power series which converge absolutely on the disc $D(0, r)$, then $f + g$ and fg also converge absolutely on this disc. If α is a complex number, αf converges absolutely on this disc, and we have*

$$(f + g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z),$$
$$(\alpha f)(z) = \alpha \cdot f(z)$$

for all z in the disc.

Proof. We give the proof for the product, which is the hardest. Let

$$f = \sum a_n T^n \qquad \text{and} \qquad g = b_n T^n,$$