absolutely for $\operatorname{Re} z \geqq c$, and therefore defines a holomorphic function for $\operatorname{Re} z > c$. This is true for every $c > 1$, and hence f is holomorphic for $\operatorname{Re} z > 1$.

In the same example, we have

$$f_n'(z) = \frac{-\log n}{n^z}.$$

By Theorem 1.2, it follows that

$$f'(z) = \sum_{n=1}^{\infty} \frac{-\log n}{n^z}$$

in this same region.

V, §1. EXERCISES

1. Let f be analytic on an open set U, let $z_0 \in U$ and $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)}\, dz,$$

where C is a small circle centered at z_0.

2. Weierstrass' theorem for a real interval $[a, b]$ states that a continuous function can be uniformly approximated by polynomials. Is this conclusion still true for the closed unit disc, i.e. can every continuous function on the disc be uniformly approximated by polynomials?

3. Let $a > 0$. Show that each of the following series represents a holomorphic function:

(a) $\displaystyle\sum_{n=1}^{\infty} e^{-an^2 z}$ for $\operatorname{Re} z > 0$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-anz}}{(a+n)^2}$ for $\operatorname{Re} z > 0$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(a+n)^z}$ for $\operatorname{Re} z > 1$.

4. Show that each of the two series converges uniformly on each closed disc $|z| \leqq c$ with $0 < c < 1$:

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{z^n}{(1 - z^n)^2}.$$

5. Prove that the two series in Exercise 4 are actually equal. [*Hint*: Write each one in a double series and reverse the order of summation.]