z_1, z_2, z_3, z_4. It will be easy if you do this separately for translations, inversions, and multiplications.

(b) Prove that the four numbers lie on the same straight line or on the same circle if and only if their cross ratio is a real number.

(c) Let z_1, z_2, z_3, z_4 be distinct complex numbers. Assume that they lie on the same circle, in that order. Prove that

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$

Fixed Points and Linear Algebra

13. Find the fixed points of the following functions:

(a) $f(z) = \dfrac{z-3}{z+1}$ (b) $f(z) = \dfrac{z-4}{z+2}$

(c) $f(z) = \dfrac{z-i}{z+1}$ (d) $f(z) = \dfrac{2z-3}{z+1}$

For the next two exercises, we assume that you know the terminology of eigenvalues from an elementary course in linear algebra.

14. Let M be a 2×2 complex matrix with non-zero determinant,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad ad - bc \neq 0.$$

Define $M(z) = (az + b)/(cz + d)$ as in the text for $z \neq -d/c$ $(c \neq 0)$. If $z = -d/c$ $(c \neq 0)$ we put $M(z) = \infty$. We define $M(\infty) = a/c$ if $c \neq 0$, and ∞ if $c = 0$.

(a) If L, M are two complex matrices as above, show directly that

$$L(M(z)) = (LM)(z)$$

for $z \in \mathbf{C}$ or $z = \infty$. Here LM is the product of matrices from linear algebra.

(b) Let λ, λ' be the eigenvalues of M viewed as a linear map on $\mathbf{C}^2$. Let

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{and} \quad W' = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}.$$

be the corresponding eigenvectors, so

$$MW = \lambda W \quad \text{and} \quad MW' = \lambda' W'$$

By a **fixed point** of M on $\mathbf{C}$ we mean a complex number z such that $M(z) = z$. Assume that M has two distinct fixed points in $\mathbf{C}$. Show that these fixed points are $w = w_1/w_2$ and $w' = w_1'/w_2'$.