7.9 Corollary
If f is analytic in a region symmetric with respect to the real axis and if f is real for real z, then

$$f(z) = \overline{f(\bar{z})}.$$

Exercises

1. Show that if f is analytic and non-constant on a compact domain, $\operatorname{Re} f$ and $\operatorname{Im} f$ assume their maxima and minima and on the boundary.

2. Prove that the image of a region under a non-constant analytic function is also a region.

3. a. Suppose f is nonconstant and analytic on S and $f(S) = T$. Show that if $f(z)$ is a boundary point of T, z is a boundary point of S.
   b. Let $f(z) = z^2$ on the set S which is the union of the semi-discs $S_1 = \{z : |z| \le 2; \operatorname{Re} z \le 0\}$ and $S_2 = \{z : |z| \le 1; \operatorname{Re} z \ge 0\}$. Show that there are points z on the boundary of S for which $f(z)$ is an interior point of $f(S)$.

*4. Suppose f is C-analytic in $D(0; 1)$ and maps the unit circle into itself. Show then that f maps the entire disc onto itself. [*Hint*: Use the Maximum-Modulus Theorem to show that f maps $D(0; 1)$ into itself. Then apply the previous exercise to conclude that the mapping is onto.]

5. Suppose f is *entire* and $|f| = 1$ on $|z| = 1$. Prove $f(z) = Cz^n$. [*Hint*: First use the maximum and minimum modulus theorem to show

$$f(z) = C\prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}.]$$

6. Suppose that f is analytic in the annulus: $1 \le |z| \le 2$, that $|f| \le 1$ for $|z| = 1$ and that $|f| \le 4$ for $|z| = 2$. Prove $|f(z)| \le |z|^2$ throughout the annulus.

7. Given f analytic in $|z| < 2$, bounded there by 10, and such that $f(1) = 0$. Find the best possible upper bound for $|f(\frac{1}{2})|$.

8. Suppose that f is analytic and bounded by 1 in the unit disc with $f(\alpha) \ne 0$ for some $\alpha \ll 1$. Show that there exists a function g, analytic and bounded by 1 in the unit disc, with $|g'(\alpha)| > |f'(\alpha)|$.

9. Find $\max_f |f'(\alpha)|$ where f ranges over the class of analytic functions bounded by 1 in the unit disc, and α is a fixed point of $|z| < 1$. [*Hint*: By the previous exercise, you may assume $f(\alpha) = 0$.]
   Show that

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z)}{z - \alpha} \ll \lim_{z\to\alpha} \frac{B_\alpha(z)}{z - \alpha} = B'_\alpha(\alpha).$$

10. Suppose f is entire and $|f(z)| \le 1/|\operatorname{Re} z|^2$ for all z. Show that $f \equiv 0$.