4 Sequences. Convergence

Now that we have defined the notion of distance between points, we are ready to discuss the notion of convergence. This notion is familiar from analysis (see MAB141), where is has been studied for the simplest example of a metric: the distance $\rho_{\mathbb{R}}$ between real numbers.

4.1 Sequences and subsequences

Let A be any set. Suppose that for any natural number n we have specified an element x_n in A. Then we say that a *sequence* is given. In other words, a sequence is merely an infinite list $x_1, x_2, \ldots$ of elements of A. We denote a sequence by (x_n) or simply by writing down its first several terms: $x_1, x_2, \ldots$. One should distinguish between a sequence and a set of all its elements, since some elements in a sequence may be listed repeatedly. For example, $0, 1, 0, 1, 0, \ldots$ is a sequence, whereas the set of all its elements is merely $\{0, 1\}$.

Let $k_1, k_2, \ldots$ be a strictly increasing sequence of natural numbers: $1 \leq k_1 < k_2 < k_2, \ldots$. Then, for any sequence (x_n), we can consider a *subsequence* (x_{k_n}), which consists of the terms $x_{k_1}, x_{k_2}, \ldots$.

For example, let $x_n = 1/n$ and $k_n = n^2$; then the subsequence (x_{k_n}) of a sequence (x_n) is $1, 1/2^2, 1/3^2, 1/4^2, \ldots$.

4.2 Convergence

Definition 4.1 (!!). Let M be a metric space and (x_n) be a sequence of elements of M. The sequence (x_n) is said to *converge* to a point $x \in M$ if

$$\lim_{n\to\infty} \rho(x_n, x) = 0. \tag{4.1}$$

The point x is called a *limit* of the sequence (x_n) and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If we want to emphasize that the convergence is considered with respect to the metric ρ of the space M, we shall write M-$\lim_{n\to\infty} x_n$ or ρ-$\lim_{n\to\infty} x_n$.

One can reformulate the definition of convergence in the following way. The sequence (x_n) converges to x if and only if for any neighbourhood $B(x;\varepsilon)$ of x, there exists a natural number $N = N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ one has $x_n \in B(x;\varepsilon)$. Indeed, by the definition of the limit, (4.1) means that for any $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ one has $\rho(x_n, x) < \varepsilon$. But the last inequality means exactly that x_n belongs to the ball $B(x;\varepsilon)$.

Yet another formulation of the definition of convergence is the following one. The sequence (x_n) converges to x if and only if every neighbourhood $B(x;\varepsilon)$ contains all but finitely many of the terms of (x_n). Indeed, the expression 'all but finitely many' means precisely 'all starting from some $N = N(\varepsilon)$'.

The limit of a sequence is unique. Indeed, if a sequence (x_n) converged to a and to b, then we should have

$$\rho(a, b) \leq \rho(a, x_n) + \rho(x_n, b) \to 0 \text{ as } n \to \infty.$$

Therefore, $\rho(a, b) = 0$ which, by the axiom (i) of the metric, implies that $a = b$.

Proposition 4.2 (!). *If a sequence (x_n) converges to x, than any of its subsequences also converges to x.*

Proof.

$$\begin{aligned} \underline{\text{We know:}} \quad & \forall \varepsilon > 0 \quad \exists N(\varepsilon) \quad \forall n \geq N(\varepsilon): \quad x_n \in B(x; \varepsilon); \quad (4.2) \\ \underline{\text{we need to prove:}} \quad & \forall \varepsilon > 0 \quad \exists K(\varepsilon) \quad \forall n \geq K(\varepsilon): \quad x_{k_n} \in B(x; \varepsilon). \quad (4.3) \end{aligned}$$

But since $k_n \geq n$, by (4.2) we see that for any $n \geq N(\varepsilon)$ one has $x_{k_n} \in B(x; \varepsilon)$. Therefore (4.3) holds with $K(\varepsilon) = N(\varepsilon)$. ∎

Proposition 4.3. *If $\lim_{n\to\infty} x_n = x$, then for any $y \in M$ one has*

$$\lim_{n\to\infty} \rho(x_n, y) = \rho(x, y). \tag{4.4}$$

Proof. 1. First let us prove the following useful inequality:

$$|\rho(a, c) - \rho(b, c)| \leq \rho(a, b). \tag{4.5}$$

Indeed, by the triangle inequality,

$$\begin{aligned} \rho(a, c) \leq \rho(a, b) + \rho(b, c) \quad &\Rightarrow \quad \rho(a, c) - \rho(b, c) \leq \rho(a, b); \\ \rho(b, c) \leq \rho(b, a) + \rho(a, c) \quad &\Rightarrow \quad \rho(b, c) - \rho(a, c) \leq \rho(a, b), \end{aligned}$$

which gives (4.5).

2. Taking $a = x_n$, $b = x$, $c = y$ in (4.5), we get

$$|\rho(x_n, y) - \rho(x, y)| \leq \rho(x_n, x) \to 0 \text{ as } n \to \infty,$$

which implies (4.4). ∎

In a similar way, one can prove that if $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then one has $\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x, y)$.

Similarly to subsets, we call a sequence (x_n) *bounded* if there exists $R > 0$ such that for any $m, n \in \mathbb{N}$ one has $\rho(x_n, x_m) \leq R$. Equivalently, a sequence (x_n) is bounded if and only if there exists a ball B such that $x_n \in B$ for all $n \in \mathbb{N}$.

Theorem 4.4 (!). *Every convergent sequence is bounded.*

Proof. Let $x_n \to x$ in the metric space M. By the definition of convergence, there exists such $N \in \mathbb{N}$ that for all $n \geq N$ one has $x_n \in B(x; 1)$. Let us take

$$R_0 = \max_{n=1,\ldots,N-1} \rho(x, x_n), \quad R = \max\{1, R_0\}.$$

We see that $x_n \in B(x; R)$ for any $n \in \mathbb{N}$. Thus, (x_n) is bounded. ∎

The concept of convergence looks simpler in normed linear spaces.

Theorem 4.5. *Let L be a normed linear space. A sequence (x_n) converges to x if and only if the sequence $(x_n - x)$ converges to 0.*

Proof. By the definition,

$$x = \lim_{n\to\infty} x_n \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho(x, x_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \|x - x_n\| = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho(x - x_n, 0) = 0$$
$$\Leftrightarrow \quad \lim_{n\to\infty} (x_n - x) = 0.$$

∎

4.3 Examples

1. Let us discuss convergence in the spaces $\mathbb{R}^n_1$, $\mathbb{R}^n_2$, $\mathbb{R}^n_\infty$.

Theorem 4.6 (!*). *Let $(x^{(k)})$ be a sequence of points in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then the following conditions are equivalent:*
(i) $\left\|x^{(k)} - x\right\|_{\mathbb{R}^n_2} \to 0$ *as* $k \to \infty$;
(ii) $\left\|x^{(k)} - x\right\|_{\mathbb{R}^n_1} \to 0$ *as* $k \to \infty$;
(iii) $\left\|x^{(k)} - x\right\|_{\mathbb{R}^n_\infty} \to 0$ *as* $k \to \infty$;
(iv) $x^{(k)}_j \to x_j$ *as* $k \to \infty$ *for all coordinates* $j = 1, \dots, n$.

Proof. First note that (iii) and (iv) are equivalent by the definition of the norm in $\mathbb{R}^n_\infty$. Next, let us prove that for any $y \in \mathbb{R}^n$, one has

$$\|y\|_{\mathbb{R}^n_\infty} \le \|y\|_{\mathbb{R}^n_2} \le \|y\|_{\mathbb{R}^n_1} \le n \|y\|_{\mathbb{R}^n_\infty}. \tag{4.6}$$

The first two inequalities in (4.6) have been already proven above (see (3.8)). Let us prove the last one. We have:

$$\|y\|_{\mathbb{R}^n_1} = \sum_{k=1}^n |y_k| \le \sum_{k=1}^n \max_{j=1,\dots,n} |y_j| = \|y\|_{\mathbb{R}^n_\infty} \sum_{k=1}^n 1 = n \|y\|_{\mathbb{R}^n_\infty}.$$

It remains to note that by (4.6), (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) ∎

By the above theorem, a sequence $(x^{(k)})$ converges or not converges to x in $\mathbb{R}^n_1$, $\mathbb{R}^n_2$ or $\mathbb{R}^n_\infty$ simultaneously. The convergence means that all the n coordinates of $x^{(k)}$ converge to the corresponding coordinates of x as $k \to \infty$.

2. Let us discuss convergence in the sequence spaces l_p, l_∞. Consider the following sequence:

$$x^{(n)} = \frac{1}{n}(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots).$$

Then $\|x^{(n)}\|_{l_\infty} = 1/n$, $\|x^{(n)}\|_{l_1} = 1$. Thus, $x^{(n)}$ converges to zero in l_∞, but does not converge to zero in l_1.

What about convergence in l_p for $p \neq 1$? One has $\|x^{(n)}\|_{l_p} = n^{\frac{1}{p}-1}$, so $\|x^{(n)}\|_{l_p} \to 0$ for $p > 1$.

3. Let us discuss convergence in $C(a, b)$. Convergence of functions in $C(a, b)$ is called *uniform convergence*. By definition, the sequence of functions f_n converges uniformly to the function f on a (finite, infinite or semi-infinite) interval Δ, if

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \text{ such that } \forall n \geq N(\varepsilon), \text{ one has } \sup_{x \in \Delta} |f(x) - f_n(x)| \leq \varepsilon.$$

This is equivalent to:

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \text{ such that } \forall n \geq N(\varepsilon), \text{ and } \forall x \in \Delta, \text{ one has } |f(x) - f_n(x)| \leq \varepsilon.$$

Note that in the last line, N does not depend on x, i.e. is chosen *uniformly* for all x. Hence the term uniform convergence.

Example 4.7. The sequence $f_n(x) = \frac{1}{n^2+x^2}$ converges to the zero function $f(x) = 0$ in $C(\mathbb{R})$ (i.e., converges uniformly on $\mathbb{R}$). Indeed,

$$\sup_{x \leq 0} |f_n(x)| = f_n(0) = n^{-2} \to 0 \text{ as } n \to \infty.$$

One should distinguish between uniform convergence and *pointwise convergence*. A sequence of functions f_n is said to converge *pointwise* to a function $f(x)$ on a (finite, infinite or semi-infinite) interval Δ, if for any $x \in \Delta$ one has $f_n(x) \to f(x)$ as $n \to \infty$. Note that pointwise convergence means precisely the following:

$$\forall x \in [a, b] \text{ and } \forall \varepsilon > 0 \quad \exists N = N(\varepsilon, x) \text{ such that } \forall n \geq N(\varepsilon, x), \text{ one has } |f(x) - f_n(x)| \leq \varepsilon,$$

which differs from the definition of the uniform convergence 'only' by the order of statements and by the fact that now N depends on x.

If a sequence of functions f_n converges uniformly to f, then it converges to f pointwise on the same interval (this follows directly from the definitions above: one simply has to take $N(\varepsilon, x)$ to be equal to $N(\varepsilon)$). However, the converse is not true, as the following example shows.

Example 4.8. Consider the sequence of functions $f_n(x) = \frac{1}{1+(x-n)^2}$ in $C(\mathbb{R})$ (draw the graph of f_0, f_1, f_2). On the one hand, for any fixed $x \in \mathbb{R}$, $f_n(x) \to 0$. Thus $f_n(x)$ *pointwise converges* to the zero function $f(x) = 0$. However, $\|f_n\|_{C(\mathbb{R})} = 1$ and thus f_n does *not* converge to zero *uniformly* on $\mathbb{R}$.

These considerations depend on the interval of the real line that we are discussing. Indeed, the above sequence converges uniformly to the zero function oin the interval $(-\infty, 0]$:

$$\sup_{x \leq 0} |f_n(x)| = f_n(0) = (1 + n^2)^{-1} \to 0 \text{ as } n \to \infty.$$

Example 4.9. The sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges to zero in $C(-1,1)$. Indeed,

$$\|f_n\|_{C(-1,1)} = \sup_{x\in[-1,1]} |f_n(x)| \le \frac{1}{n} \to 0, \quad \text{as } n \to \infty.$$

Note that this sequence does not converge to zero in $C^1(-1,1)$:

$$\|f_n\|_{C^1} = \sup_{x\in[-1,1]} (|f_n(x)| + |f_n'(x)|) \ge (|f_n(0)| + |f_n'(0)|) = 1.$$

4. Consider the space with a discrete metric ρ_d. Here $x_n \to x$ if and only if $x_n = x$ for all large enough n.