Universidad de Chile
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MA22A: Cálculo en Varias Variables
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Clase Auxiliar 5

Problema 1
Sea $f : I\!R^2 \mapsto I\!R$. Se define $g : I\!R^+ \times [0, 2\pi) \mapsto I\!R$ por $g(r, \phi) = f(r \cos\phi, r \,\mathrm{sen}\,\phi)$.
Demuestre que:

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \phi^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Sol:
Calculamos la derivada con respecto a r de g:

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\cos\phi + \frac{\partial f}{\partial y}\sin\phi$$

calculamos ahora, la segunda derivada de g con respecto a r:

$$\frac{\partial^2 g}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\phi + \frac{\partial^2 f}{\partial y \partial x}\cos\phi\sin\phi + \frac{\partial^2 f}{\partial x \partial y}\cos\phi\sin\phi + \frac{\partial^2 f}{\partial y^2}\sin^2\phi$$

(pues recordemos que las derivadas estan igualmente evaluadas en $(r\cos\phi, r\sin\phi)$.
Ahora, calculamos:

$$\begin{array}{rcl} \frac{\partial g}{\partial \phi} & = & -\frac{\partial f}{\partial x} r\sin\phi + \frac{\partial f}{\partial y} r\sin\phi \\ \frac{\partial^2 g}{\partial \phi^2} & = & \frac{\partial^2 f}{\partial x^2} r^2\sin^2\phi - \frac{\partial^2 f}{\partial y\partial x} r^2 \sin\phi\cos\phi - \frac{\partial f}{\partial x} r\cos\phi \\ & & -\frac{\partial^2 f}{\partial x\partial y} r^2\sin\phi\cos\phi + \frac{\partial^2 f}{\partial y^2} r^2\cos^2\phi - \frac{\partial f}{\partial y} r\sin\phi \end{array}$$

esto último pues debemos usar la regla del producto para derivar (primero derivamos las derivadas que estan evaluadas en $(r\cos\phi, r\sin\phi)$, y luego los $\sin\phi$ y $\cos\phi$ que le acompañan).
Sumando todo, y ponderando por $\frac{1}{r}$ y $\frac{1}{r^2}$ donde corresponda, tenemos:

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \phi^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Problema 2
Sea $g : I\!R \mapsto I\!R$ una función de clase $\mathcal{C}^2$ y sea $f : I\!R^n \mapsto I\!R$, con $n \geq 3$, la función $f(x) := g(||x||)$.

- Probar que:

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x_i}^2} = \frac{n-1}{r} g^{'}(r) + g^{''}(r)$$

con $r = ||x||$, $x \neq 0$.
Sol:
Calculamos la derivada de f con respecto a x_i:

$$\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \frac{\partial g}{\partial r}(r) \cdot \frac{1}{2} \frac{2x_i}{||x||} = \frac{\partial g}{\partial r}(r) \cdot \frac{x_i}{||x||}$$

Ahora, derivamos nuevamente:

$$\begin{aligned} \frac{\partial^2 f}{\partial {x_i}^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial g}{\partial r}(r) \cdot \frac{x_i}{||x||}\right) &= \frac{\partial^2 g}{\partial r^2}(r) \cdot \frac{x_i}{||x||} \frac{x_i}{||x||} + \frac{\partial g}{\partial r}(r)\left[\frac{||x|| - \frac{{x_i}^2}{||x||}}{||x||^2}\right] \\ &= \frac{\partial^2 g}{\partial r^2}(r) \cdot \frac{{x_i}^2}{||x||^2} + \frac{\partial g}{\partial r}(r)\left[\frac{||x||^2 - {x_i}^2}{||x||^3}\right] \end{aligned}$$

Con esto, tenemos que:

$$\begin{aligned} \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x_i}^2} &= \sum_{i=1}^{n} \frac{\partial^2 g}{\partial r^2}(r) \cdot \frac{{x_i}^2}{||x||^2} + \frac{\partial g}{\partial r}(r)\left[\frac{||x||^2 - {x_i}^2}{||x||^3}\right] \\ &= \frac{\partial^2 g}{\partial r^2}(r) \cdot \frac{||x||^2}{||x||^2} + \frac{\partial g}{\partial r}(r) \frac{n||x||^2 - ||x||^2}{||x||^3} \\ &= g''(r) + \frac{n-1}{r} g'(r) \end{aligned}$$

- b) Probar que si $\Delta f = 0$, entonces:

  $$f(x) = \frac{a}{||x||^{n-2}} + b, \quad x \neq 0$$

  Sol:
  Ejercicio Propuesto , pues es mas bien de MA26A que de MA22A :), (nada más que integrar...).

Problema 3
Suponiendo que f es de clase $\mathcal{C}^2$, transformar la expresión:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x \partial y}$$

mediante el cambio de variable $u = x + y$, $u = x - y$.
Sol:

$$u = x + y \quad v = x - y \quad \Longrightarrow \frac{u+v}{2} = x \quad \frac{u-v}{2} = y$$

Así:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2} \tag{1}$$

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{1}{4} + 2\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{1}{4} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{1}{4} \tag{2}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} - \frac{\partial f}{\partial y} \cdot \frac{1}{2} \tag{3}$$

$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{1}{4} - 2\frac{\partial^2 f}{\partial y \partial x} \cdot \frac{1}{4} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{1}{4} \tag{4}$$

$$\frac{\partial^2 f}{\partial u \partial v} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{1}{4} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{1}{4} - \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{1}{4} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{1}{4} \tag{5}$$

de este modo, tenemos:

$$2(2) + 2(4) \Longrightarrow 2(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

y como f es $\mathcal{C}^2$:

$$2(2) - 2(4) \Longrightarrow \frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 f}{\partial x \partial y}$$

juntando estas últimas dos ecuaciones, se tiene:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x \partial y} = 3\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$$