University of Leeds, School of Mathematics

MATH 2021 Complex Analysis: Problems 1.

Submit by Monday 14 February 2005

1. Find all the complex solutions to the following equations:

   (i) $e^z = 1 - i$; (ii) $\cos z = 5/4$; (iii) $\dfrac{i}{e^{3iz}} = 1 + i$.

2. Sketch the following subsets of $\mathbb{C}$ and say which of them is a domain:

   (i) $\{z \in \mathbb{C} : 1 < |z - 2| < 3, \quad \mathrm{Re}\,(z) \geq 0\}$; (ii) $\{z \in \mathbb{C} : |\exp(z)| < 1\}$.

3. Determine which of the following series converge:

   (i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{i}{n}\right)$;

   (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \left(\cos\left(\left(\frac{\pi}{e}\right)^n\right) + i \sin\left(\left(\frac{\pi}{e}\right)^n\right)\right)$.

4. Find the radius of convergence of the following power series:

   (i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n z^n$; (ii) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n^2} z^n$; (iii) $\displaystyle\sum_{n=0}^{\infty} z^{n^2}$.

5. Express $\cos^2(z)$ as a power series in z, giving all terms up to z^6.

6. Show that

   (i) $\sin z = \dfrac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ and $\cos z = \dfrac{1}{2}\left(e^{iz} + e^{-iz}\right)$;

   (ii) $\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$, for all $z, w \in \mathbb{C}$;

   (iii) $\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$, for all $z, w \in \mathbb{C}$.

   Hence, deduce that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

7. We define

   $$\sinh z = \frac{1}{2}(e^z - e^{-z}) \text{ and } \cosh z = \frac{1}{2}(e^z + e^{-z}).$$

   (i) Prove that $\sin iz = i \sinh z$ and $\cos iz = \cosh z$ for all $z \in \mathbb{C}$.

   (ii) Prove that $|\cos(x + iy)|^2 = \cos^2 x + \cosh^2 y - 1$ for all $x, y \in \mathbb{R}$. Hence, deduce that $|\cos(z)|$ is an unbounded function on $\mathbb{C}$.
   [Hint: For the first result, use 6(iii) and 7(i).]

Your answers should of course be written in sentences, notation should be defined, you should explain what you are doing, and symbols such as $=$ and $\implies$ should be used correctly.

Ex 1: Worked examples

1. Find the limit of $\dfrac{n}{n+i}$ as n tends to infinity.

   Solution: We have

$$\begin{aligned}\frac{n}{n+i} &= \frac{n/n}{n/n+i/n}\\ &= \frac{1}{1+i/n}\\ &\to 1,\ \text{as } n\to\infty.\end{aligned}$$

2. Does $\dfrac{1}{(1+i)^n}$ converge?

   Solution: We have

$$\frac{1}{(1+i)^n} = \left(\frac{1}{1+i}\right)^n = \left(\frac{1}{1+i}\frac{1-i}{1-i}\right)^n = \left(\frac{1-i}{2}\right)^n.$$

   Consider the modulus of $(1-i)/2$:

$$\left|\frac{1-i}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}.$$

   So we have

$$\left|\left(\frac{1-i}{2}\right)^n\right| = \left|\frac{1-i}{2}\right|^n = \left(\sqrt{\frac{1}{2}}\right)^n \to 0.$$

   As the modulus of the sequence tends to zero, the sequence must tend to zero.

3. Find the radius of convergence for $\displaystyle\sum_{n=1}^{\infty}\frac{(3z)^n}{n^2}$.

   Solution: We use the ratio test on the series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(3z)^n}{n^2}\right|$. Let $a_n = \left|\dfrac{(3z)^n}{n^2}\right|$. Then,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(3z)^{n+1}}{(n+1)^2}\right| \Big/ \left|\frac{(3z)^n}{n^2}\right|\\ &= \left|\frac{(3z)^{n+1}}{(3z)^n}\Big/\frac{n^2}{(n+1)^2}\right|\\ &= \left(\frac{n}{n+1}\right)^2|3z|\\ &= 3\left(\frac{n}{n+1}\right)^2|z|\\ &\to 3.1.|z|,\ \text{as } n\to\infty,\\ &= 3|z|.\end{aligned}$$

   The series converges absolutely, and hence converges, if $3|z|<1$, i.e. $|z|<1/3$.

   Therefore, the radius of convergence is $1/3$.

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MATH 2021 Complex Analysis: Problems 2.

Submit by Thursday 3 March 2005

1. Sketch the following contours:

   (i) $\gamma_1(t) = 2 + 2e^{2\pi it} \qquad (0 \le t \le 1)$;

   (ii) $\gamma_2(t) = \begin{cases} it, & (0 \le t \le 1), \\ t - 1 + i, & (1 \le t \le 2); \end{cases}$

   (iii) $\gamma_3(t) = i + e^{-it} \qquad (0 \le t \le \pi/2)$.

   (Check your answers carefully, because these contours will be used several times later.)

2. Using the contours above, evaluate the following contour integrals directly from the definition ($\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$):

   (i) $\displaystyle\int_{\gamma_1} \frac{dz}{z-2}$; (ii) $\displaystyle\int_{\gamma_2} \bar{z}\,dz$; (iii) $\displaystyle\int_{\gamma_3} \frac{dz}{(z-i)^2}$.

3. (Exercise 6.3 of notes) Show that the length of the contour given by traversing once round a circle, centred at z_0 and of radius r, is $2\pi r$.

4. Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a contour. Show that $L(\gamma) \ge |\gamma(b) - \gamma(a)|$. What does this mean geometrically?

5. Find an estimate for $\left|\frac{e^z}{z}\right|$, where $z = e^{i\theta}$, $\theta \in \mathbb{R}$. (I.e. Find an M such that $\left|\frac{e^z}{z}\right| \le M$, where $z = e^{i\theta}$, $\theta \in \mathbb{R}$.)

6. Set $\gamma(t) = 3e^{it} \quad (0 \le t \le \pi)$. Use the Estimation Lemma to show that
$$\left|\int_\gamma \frac{e^z}{z-1}\,dz\right| \le \frac{3}{2}\pi e^3.$$

7. Let $\gamma(t)$ describe the semi-circle Re^{it}, where $0 \le t \le \pi$, and $R > 3$. Show that
$$\left|\int_\gamma \frac{e^{3iz}}{(z^2+4)(z^2+9)}\,dz\right| \le \frac{\pi R}{(R^2-4)(R^2-9)}.$$

Ex 2: Worked examples

1. Find $\displaystyle\int_\gamma \frac{dz}{z}$ where γ describes the semi-circle from -1 to 1 in the upper half of the complex plane.

   Solution: We have $\gamma(t) = e^{-it}$, where $-\pi \le t \le 0$. Then,

$$\begin{aligned}\int_\gamma \frac{dz}{z} &= \int_{-\pi}^{0} \frac{1}{e^{-it}}\left(-ie^{-it}\right)\, dt\\ &= -i\int_{-\pi}^{0} dt\\ &= -\pi i.\end{aligned}$$

   [Note that other parametrisations of the path will give the same answer. Eg. Let $\gamma(t) = e^{-it+\pi i}$ for $0 \le t \le \pi$.]

2. Let $\gamma(t) = Re^{it}$, $0 \le t \le \pi$ describe a semi-circle, with $R > 1$.

   Show that
$$\left|\int_\gamma \frac{e^{2iz}}{(z^2+1)^2}\, dz\right| \le \frac{\pi R}{(R^2-1)^2}.$$

   Solution:

   We have $z^2 = z^2 + 1 - 1$. So

$$\begin{aligned}|z^2| &\le |z^2+1| + 1 \text{ by the triangle inequality,}\\ |z|^2 - 1 &\le |z^2+1|\\ \frac{1}{|z^2+1|} &\le \frac{1}{|z|^2-1}, \text{ for } |z| > 1,\\ \frac{1}{(|z^2+1|)^2} &\le \frac{1}{(|z|^2-1)^2}\\ \frac{1}{(|z^2+1|)^2} &\le \frac{1}{(R^2-1)^2}, \text{ since } |z| = R \text{ on the semi-circle.}\end{aligned}$$

   For the numerator of the integrand:

$$|e^{2iz}| = |e^{iz}|^2 = |e^{Re(iz)}|^2 = |e^{-Im(z)}|^2.$$

   But $Im(z) \ge 0$ because γ lies in the upper half-plane. Thus, $0 \le e^{-Im(z)} \le 1$ and so $|e^{2iz}| \le 1^2 = 1$.

   Hence,
$$\left|\frac{e^{2iz}}{(z^2+1)^2}\right| \le \frac{1}{(R^2-1)^2}.$$

   The length of γ is πR since it gives a semi-circle. Thus, the estimate follows from the Estimation Lemma.

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MATH 2021 Complex Analysis: Problems 3.

Submit by Friday 18 March 2003

1. Express each of the following functions of $z = x + iy$ in the form $u(x, y) + iv(x, y)$, and hence or otherwise determine the set of points on which it is differentiable:

   (i) $1/z \quad (z \in \mathbb{C}\backslash\{0\})$; (ii) $\bar{z} \quad (z \in \mathbb{C})$; (iii) $\arg(z) \quad (\mathrm{Re}\,(z) > 0)$.

2. Verify directly that $u(x, y) = x^3 - 3xy^2 + 1$ is a harmonic function. Find an analytic function $f(z)$ such that u is the real part of f.

3. Check that the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$u(x, y) = x^4 - 6x^2y^2 + y^4 - x^2 + y^2$$

   satisfies Laplace's equation at each point of $\mathbb{R}^2$.

   Find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that the function f given by $f(x+iy) = u(x, y) + iv(x, y)$ is complex differentiable at each point of $\mathbb{C}$.

4. Let f be differentiable on a domain D and suppose that $f(z)$ is always real. Use the Cauchy–Riemann equations to show that $f' = 0$ on D.

5. (Failure of the mean value theorem.) Give an example of a differentiable function $f : \mathbb{C} \to \mathbb{C}$ for which $f(0) = f(1)$, but such that $f'(z) = 0$ for *no* value of z.

6. Recall the following contours from Problems 2, which we use again below.

   $$\begin{aligned} \gamma_1(t) &= 2 + 2e^{2\pi it} \qquad (0 \le t \le 1), \\ \gamma_2(t) &= \begin{cases} it, & (0 \le t \le 1), \\ t - 1 + i, & (1 \le t \le 2), \end{cases} \\ \gamma_3(t) &= i + e^{-it} \qquad (0 \le t \le \pi/2). \end{aligned}$$

   Use the Fundamental Theorem of Calculus for contour integrals to evaluate the following:

   $$\int_{\gamma_1} z\,dz; \qquad \int_{\gamma_2} \cos(\pi z)\,dz; \qquad \int_{\gamma_3} \frac{dz}{(z-i)^2}.$$

7. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, and set $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$.

   (i) If $g(z) = f(z) + f(-z)$ find the value of $g^{(k)}(0)$ for each k (in terms of the a_n).

   (ii) Deduce that, if f is an odd function (i.e., $f(-z) = -f(z)$), then $a_n = 0$ for all even values of n.

Ex 3: Worked examples

1. Find a k such that $v(x, y) = y^3 - 4xy + kx^2 y$ could be the imaginary part of a complex differentiable function.

   Solution: For the function to be the imaginary part of a differentiable complex function it must satisfy Laplace's equation. We must have,

   $$0 = v_{xx} + v_{yy} = 2ky + 6y = y(k + 6),$$

   for all y, i.e $k = -3$.

   [Note that this does not imply v is the imaginary part of a differentiable function for $k = 3$. It is the only possibility that satisfies Laplace's equation. In fact, v is the imaginary part of $f(z) = -z^3 - 2z^2$.]

2. Let $f(z) = e^{\overline{z}}$, for all $z \in \mathbb{C}$. Determine the points where f is differentiable.

   Solution: We have,

   $$e^{\overline{z}} = e^{x-iy} = e^x(\cos y + i \sin y).$$

   So $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$. Thus,

   $$u_x = e^x \cos y \text{ and } v_y = -e^x \cos y$$

   The first CR equation, $u_x = v_y$, can therefore only hold if $\cos y = 0$, i.e. $y = \frac{1}{2}(2k + 1)\pi$ for some $k \in \mathbb{Z}$.

   Now,

   $$u_y = -e^x \sin y \text{ and } v_x = -e^x \sin y.$$

   The second CR equation, $u_y = -v_x$, can only hold if $\sin y = 0$, i.e. $y = 2k\pi$ for some $k \in \mathbb{Z}$.

   Thus, for the CR equations to hold we need $y = \frac{1}{2}(2k + 1)\pi$ for some $k \in \mathbb{Z}$, and $y = 2k\pi$ for some $k \in \mathbb{Z}$. No such y exist, and so the CR equations hold nowhere.

   Hence, f is differentiable nowhere.

3. Integrate $\displaystyle\int_\gamma \frac{dz}{(3z+2)^2}$, where $\gamma = \gamma_2$ from over the page.

   The function is the derivative of $F(z) = -\dfrac{1}{3(3z+2)}$ and so by the Fundamental Theorem, we have

   $$\begin{aligned}
   \int_\gamma \frac{dz}{(3z+2)^2} &= F(\gamma(2)) - F(\gamma(0)) = F(1+i) - F(0) \\
   &= -\frac{1}{3(3(1+i)+2)} - \left(-\frac{1}{3(3 \times 0 + 2)}\right) \\
   &= -\frac{1}{15+9i} + \frac{1}{6} \\
   &= \frac{2}{17} + \frac{1}{34} i.
   \end{aligned}$$

4. Integrate $\displaystyle\int_\gamma \sin z \, dz$, where $\gamma = \gamma_1$ from over the page.

   Solution: $\sin z$ is the derivative of $-\cos z$ so we can apply the Fundamental Theorem of Calculus. The contour is closed and hence the integral is zero.

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MATH 2021 Complex Analysis: Problems 4.

Submit by Friday 29 April 2005

1. Use Cauchy's integral formula to evaluate the following:

   (i) $\displaystyle\int_{\Gamma_1} \frac{z^4}{z-1}\,dz$, where $\Gamma_1(t) = 2e^{it} \qquad (0 \le t \le 2\pi)$;

   (ii) $\displaystyle\int_{\Gamma_2} \frac{z^4}{z-1}\,dz$, where $\Gamma_2(t) = 2e^{-it} \qquad (0 \le t \le 4\pi)$;

   (iii) $\displaystyle\int_{\Gamma_3} \frac{\cos(z)}{z^2-2z}\,dz$, where Γ_3 is the square with vertices at $\pm 1 \pm i$, traversed once anticlockwise.

2. Let $\gamma(t) = e^{it} \quad (0 \le t \le 2\pi)$. Evaluate the following integrals:

   (i) $\displaystyle\int_\gamma \frac{e^{\sin(z)}}{z(z-2)^2}\,dz$; (ii) $\displaystyle\int_\gamma \frac{e^{1/z}}{z^2}\,dz$; (iii) $\displaystyle\int_\gamma |z+1|^2\,dz$.

   [Hint: For part (iii) we have $z\overline{z} = 1$ for $z = e^{it}$.]

3. Suppose that f and g are differentiable functions on the domain D and $z_0 \in D$.

   (i) Suppose that $f(z)$ has a zero at z_0. Show that $f(z) = (z - z_0)h(z)$, where h is differentiable for $|z - z_0| < R$ for some $R > 0$. [Hint: Think Taylor!]

   (ii) Show that $h(z_0) = f'(z_0)$.

   (iii) Hence, deduce L'Hopital's rule for complex differentiable functions: Show that, if $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$, then

   $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

4. Let f be a function differentiable on all of $\mathbb{C}$ with Taylor expansion $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ about 0.

   (i) Let $g(z) = e^z f(z)$ for $z \in \mathbb{C}$. Show that the Taylor expansion of g is

   $$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{(n-k)!} \right) z^n.$$

   (ii) Show that, if $f(z) = \exp(\exp(z))$, then the sequence (a_n) satisfies the recurrence relation

   $$a_n = \frac{1}{n} \sum_{k=0}^{n-1} \frac{a_k}{(n-k-1)!} \qquad \text{for } n \ge 1.$$

   [Hint: Consider $f'(z)$.]

5. Let f be function complex differentiable on the whole of $\mathbb{C}$. Suppose that f satisfies

$$f(z+m+ni) = f(z) \qquad \text{for all } z \in \mathbb{C},\ m,n \in \mathbb{Z}. \qquad (*)$$

This question will demonstrate the surprising fact that f must be constant.

Let $\gamma : [a,b] \to \mathbb{C}$ be the contour which goes once anticlockwise round the square with corners $0, 1, 1+i, i$, and put $M = \sup\{|f(\gamma(t))| : t \in [a,b]\}$.

(i) Quote a theorem about real continuous functions implying that M is finite;

(ii) Apply a theorem from the course to show that $|f(z)| \le M$ for all z inside γ.

(iii) Deduce from (*) that $|f(z)| \le M$ for all $z \in \mathbb{C}$.

(iv) Prove that f is constant (applying another theorem from the course).

Ex 4: Worked examples

1. Evaluate the following integral using Cauchy's Integral Formula: $\displaystyle\int_\gamma \frac{e^z+z}{z-2}\,dz$ where γ is (i) a circle of radius 1 centred at the origin and (ii) a circle of radius 3 centred at the origin.

   Solution: For both cases Cauchy's Integral Formula gives, taking $f(z) = e^z + z$, which is differentiable on all of $\mathbb{C}$, (so $D = \mathbb{C}$),

$$\int_\gamma \frac{e^z+z}{z-2}\,dz = 2\pi i f(2) n(\gamma, 2) = 2\pi i(e^2+2)n(\gamma,2).$$

   For (i) we have $n(\gamma, 2) = 0$ as 2 lies outside γ, hence the integral is 0.

   For (ii) we have $n(\gamma, 2) = 1$, hence the integral is $2\pi i(e^2+2)$.

2. Evaluate the integral $\displaystyle\int_\gamma \frac{z^2}{z^2+1}\,dz$ where γ is the circle of radius 1 centred at i.

   Solution: The denominator factorises to give $\displaystyle\frac{z^2}{(z+i)(z-i)}$. Let $f(z) = \displaystyle\frac{z^2}{z+i}$. Then f is differentiable on $\mathbb{C}\backslash\{-i\}$, (i.e. $D = \mathbb{C}\backslash\{-i\}$) so f is differentiable on γ and all the interior points of γ. We have $n(\gamma, i) = 1$. Thus, by Cauchy's Integral Formula,

$$\int_\gamma \frac{z^2}{z^2+1}\,dz = 2\pi i f(i) n(\gamma, i) = 2\pi i \frac{i^2}{i+i} \times 1 = 2\pi i \frac{i}{2} = -\pi.$$

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MATH 2021 Complex Analysis: Problems 5.

Submit by Friday 13 May 2005

1. Use the methods given in the course to calculate the residues of the following functions at the points stated.

$$\text{(i)}\ \frac{e^{iz}}{z^2+4} \text{ at } 2i; \qquad \text{(ii)}\ \frac{e^{\sin z}}{z(z-2)^2} \text{ at } 0; \qquad \text{(iii)}\ \frac{\cos 2z}{z^3} \text{ at } 0;$$

$$\text{(iv)}\ \frac{1}{e^z-1} \text{ at } 0; \qquad \text{(v)}\ \frac{1+\cos z}{(z-\pi)^3} \text{ at } \pi; \qquad \text{(vi)}\ \frac{1}{(e^z-1)^2} \text{ at } 0.$$

2. Use Cauchy's residue theorem to calculate the following integrals (all contours traced anticlockwise):

   (i) $\displaystyle\int_{\gamma_1} \frac{\cos 2z}{z^3}\,dz$, where γ_1 is the square with vertices $\pm 1 \pm i$.

   (ii) $\displaystyle\int_{\gamma_2} \frac{e^{iz}}{z^2+4}\,dz$, where γ_2 is the boundary of the semicircle $\{|z| \leq 10, \mathrm{Im}(z) \geq 0\}$.

   (iii) $\displaystyle\int_{\gamma_3} \frac{e^{\sin z}}{z(z-2)^2}\,dz$, where γ_3 is the unit circle.

   (iv) $\displaystyle\int_{\gamma_4} \frac{dz}{e^z-1}$, where γ_4 is the diamond with vertices $1, 10i, -1, -10i$.

   (You may wish to use some of the results of an earlier exercise.)

3. Evaluate the following integrals.

$$\text{(i)}\ \int_{-\infty}^{\infty} \frac{dx}{x^4+1} \qquad \text{(ii)}\ \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx \qquad (0 < a < b)$$

4. (i) Calculate the residues of the function

$$f(z) = \frac{2}{9z^2+82z+9}$$

   at all its poles.

   (ii) Hence, using the calculus of residues, calculate

$$\int_0^{2\pi} \frac{1}{41+9\cos(\theta)}\,d\theta.$$

5. (i) Find the poles and their orders of the function

$$f(z) = \frac{e^{iz}}{z^4+18z^2+81}.$$

   (ii) Calculate the residues of the poles in the upper half-plane for this function and hence, using the calculus of residues, calculate

$$\int_{\infty}^{\infty} \frac{\cos x}{x^4+18x^2+81}\,dx.$$

Ex 5: Worked examples

1. Classify the poles of $f(z) = \dfrac{\sin(z+2i)}{z^3(z^2+4)}$.

   Solution: The denominator factorises to $z^3(z-2i)(z+2i)$ and hence has zeros at $z = 0$ (of multiplicity 3), $z = 2i$ and $z = -2i$. For the first two the numerator $\sin(z+2i)$ is non-zero. Hence, the pole at $z = 0$ has order 3, and the pole at $z = 2i$ has order 1, i.e. it is simple.

   However, $\sin(-2i+2i) = 0$, so we must analyse further for the pole at $z = -2i$. We have $\sin(z+2i) = (z+2i) - \dfrac{(z+2i)^3}{3!} + \dfrac{(z+2i)^5}{5!} - \dots$, and $z^3(z^2+4) = z^3(z-2i)(z+2i)$, so

$$\frac{\sin(z+2i)}{z^3(z^2+4)} = \frac{1}{z^3(z-2i)}\left(1 - \frac{(z+2i)^2}{3!} + \frac{(z+2i)^4}{5!} - \dots\right).$$

   Since at $z = -2i$, $z^3(z-2i) \neq 0$, by Corollary 15.7, there exists a Taylor series $g(z)$ for $z^3(z-2i)$ at $z = -2i$. So, the function f is the product of two power series at $z = -2i$, hence the pole is removable.

2. Classify the poles and calculate the corresponding residues of $\dfrac{e^{z\pi}}{z^2+1}$.

   Solution: The denominator factorises so we get $\dfrac{e^{z\pi}}{z^2+1} = \dfrac{e^{z\pi}}{(z+i)(z-i)}$. The numerator is not zero when the denominator is, (in fact it is never zero), and so $z = i$ and $z = -i$ are simple poles. By Method 2 we have

$$\operatorname{res}\left(\frac{e^{z\pi}}{z^2+1}, \pm i\right) = \left.\frac{e^{z\pi}}{2z}\right|_{z=\pm i} = \frac{e^{\pm i\pi}}{\pm 2i} = \frac{\mp i}{2}(-1) = \pm\frac{i}{2}.$$

3. Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{2+\cos t}$.

   Solution: Let $z = \gamma(t) = e^{it}$, $-\pi \leq t \leq \pi$, then $\cos t = \frac{1}{2}\left(z + \frac{1}{z}\right)$. So,

$$\frac{1}{2+\cos t} = \frac{dt}{2 + \frac{1}{2}\left(z+\frac{1}{z}\right)} = \frac{2z}{z^2+4z+1}, \text{ for } z = e^{it}.$$

   We have,

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{dt}{2+\cos t} &= -i\int_\gamma \frac{2z}{z^2+4z+1} z^{-1}\, dz \\
&= -2i\int_\gamma \frac{1}{z^2+4z+1}\, dz \\
&= -2i\int_\gamma \frac{1}{(z-a)(z-b)}\, dz
\end{aligned}$$

   where $a = -2+\sqrt{3}$ and $b = -2-\sqrt{3}$. We have simple poles at a and b. Only a lies within the unit circle and so by Cauchy's Residue Formula,

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{dt}{2+\cos t} &= -2i \times 2\pi i \lim_{z\to a}(z-a) \times \frac{1}{(z-a)(z-b)} \\
&= 4\pi \lim_{z\to a} \frac{1}{z-b} \\
&= 4\pi \frac{1}{a-b} \\
&= \frac{2\pi}{\sqrt{3}}.
\end{aligned}$$