Ecuaciones en derivadas parciales

P1 Determinar la serie de fourier de $f(x)=x^2$ en $[-\pi,\pi]$

Sol)
$$f(x)=x^2=\frac{a_0}{2}+\sum_{n\in\mathbb{N}\setminus\{0\}}\left(a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\operatorname{sen}\left(\frac{n\pi x}{l}\right)\right)$$

donde $l=\pi$, pues es $\forall x\in[-l,l]$

Usando que
- $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx \overset{par}{=} \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi}=\frac{2\pi^2}{3}$$
- $$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos(nx)dx = \frac{1}{\pi}\left[\frac{x^2\operatorname{sen}nx}{n}\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{2x}{n}\operatorname{sen}nx\,dx\right]$$

$dv=\cos nx$ $u=x^2$
$v=\frac{\operatorname{sen}nx}{n}$ $du=2x\,dx$

$$=-\frac{2}{n\pi}\int_{-\pi}^{\pi}x\operatorname{sen}nx\,dx \underset{par}{=} \frac{-4}{n\pi}\int_0^{\pi}x\operatorname{sen}x\,dx$$ → $u=x$ $dv=\operatorname{sen}nx$, $du=dx$ $v=-\frac{\cos nx}{n}$

$$=\frac{-4}{n\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi}+\int_0^{\pi}\frac{\cos nx}{n}dx\right]$$
$$=\frac{-4}{n\pi}\left[\frac{-\pi(-1)^n}{n}+\frac{\operatorname{sen}nx}{n^2}\Big|_0^{\pi}\right]$$
$$\Rightarrow a_n=\frac{4}{n^2}(-1)^n$$

- $$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\operatorname{sen}(nx)dx \overset{impar}{=} \emptyset$$

$$\Rightarrow x^2=\frac{1}{3}\pi^2+\sum_{n\geq 1}\frac{4(-1)^n}{n^2}\cos(nx)$$ //

De aquí podemos deducir que

$$\pi^2=\frac{1}{3}\pi^2+4\left(\frac{-1}{1}\cos(\pi)+\frac{1}{4}\cos(2\pi)-\frac{1}{9}\cos(3\pi)+\dots\right)$$
$$\Rightarrow \pi^2-\frac{1}{3}\pi^2=\frac{2}{3}\pi^2=4\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right)$$
$$\Rightarrow \sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Propuesto: $3x^4$, el hecho en clases.. (se procede de manera similar)

P2) Resolver la siguiente EDP con el método de variables separables

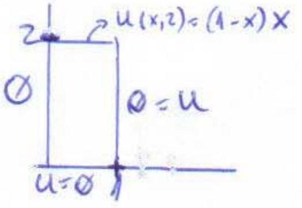
$$\Delta u(x,y) + u(x,y) = 0 \quad \text{en} \quad (1,0)\times(0,2) = \Omega$$

$$u(0,y) = u(x,0) = u(1,y) = 0$$

$$u(x,2) = x(1-x)$$

donde $\Delta u : \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Sol)



Planteamos la solución

$u(x,y) = X(x)\,Y(y)$ con el met de separación de variables

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = X''\cdot Y \quad y \quad \frac{\partial^2 u}{\partial y^2} = X\cdot Y'' \quad \text{en} \quad \Delta u + u = 0$$

$$\Rightarrow X''\cdot Y + X Y'' + X\cdot Y = 0 \qquad \Big/ \frac{1}{X\cdot Y}$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + 1 = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} + 1 = \frac{-Y''(y)}{Y(y)} \qquad \forall x\in(0,1) \quad \forall y\in(0,2)$$

$\Rightarrow \exists \lambda = cte \in \mathbb{C}$ tal que

$$\underset{(1)}{\frac{X''}{X} + 1 = -\lambda} \qquad y \qquad \underset{(2)}{\frac{-Y''}{Y} = \lambda}$$

(1) $X'' + (1+\lambda)X = 0$

$$\Rightarrow r^2 + (1+\lambda) = 0 \Rightarrow r = \pm\sqrt{-(1+\lambda)} = \pm i\sqrt{1+\lambda}$$

$$\Rightarrow \left[X(x) = A e^{i\sqrt{1+\lambda}\,x} + B e^{-i\sqrt{1+\lambda}\,x}\right]$$

(2) $Y'' - \lambda Y = 0$

$$\Rightarrow r^2 = \lambda \Rightarrow r = \pm\sqrt{\lambda}$$

$$\Rightarrow \left[Y(y) = C e^{\sqrt{\lambda}\,y} + D e^{-\sqrt{\lambda}\,y}\right]$$

- Vemos inmediato que si $\lambda = -1 \Rightarrow X(x) = A + B$ no cumple $u(x,2) = x(1-x)$
y si $\lambda = 0$, $Y(y) = C + D$,

$u(0,y) = (A+B)\cdot(C+D) = 0$ ssi $C+D = 0$ pero
$u(1,y) = (Ae^{i} + Be^{-i})(C+D) = 0$ $\quad u(x,2) = X(x)\cdot(C+D) \neq 0$.
$\Rightarrow$ no cumple cond. de borde

- $u(0,y) = 0$
$u(0,y) = (A+B)\,Y(y) \quad \forall y$, pero $Y(y) \neq 0$ (de la función cero)

$\Rightarrow A = -B$

- $u(1,y) = 0$
$u(1,y) = (Ae^{i\sqrt{1+\lambda}} + Be^{-i\sqrt{1+\lambda}})\cdot Y(y) \quad \forall y$

$$\Rightarrow Ae^{i\sqrt{1+\lambda}} + Be^{-i\sqrt{1+\lambda}} = 0 \overset{A=-B}{\Rightarrow} e^{i\sqrt{1+\lambda}} - e^{-i\sqrt{1+\lambda}} = 0$$

$$\Leftrightarrow \underbrace{e^{2i\sqrt{1+\lambda}}}_{\cos(2\sqrt{1+\lambda}) + i\,\mathrm{sen}(2\sqrt{1+\lambda})} = 1 \Rightarrow \left.\begin{array}{l} 2\sqrt{1+\lambda} = 2k_1\pi \\ 2\sqrt{1+\lambda} = k_2\pi \\ k_1, k_2 \in \mathbb{Z} \end{array}\right\} \Rightarrow \sqrt{1+\lambda} = k\pi,\ k \in \mathbb{Z}$$

$$\cos(2\sqrt{1+\lambda}) = 1, \quad \mathrm{sen}(2\sqrt{1+\lambda}) = 0$$

$$\Rightarrow \lambda = k^2\pi^2 - 1, \quad k \in \mathbb{Z}$$

Con esto obtenemos una sucesión $\{\lambda_k\}_{k\in\mathbb{Z}}$ lo que genera una familia de soluciones $u_k = X_k(x)\,Y_k(x)$
y ocupando el criterio de superposición de soluciones

$$u(x,y) = \sum_{k\in\mathbb{Z}} C_k u_k(x,y)$$ es solución de $\Delta u + u = 0$

Las otras condiciones de borde...

- $u(x,0) = 0$
$u(x,0) = X(x)\cdot(C+D) \quad \forall x$, pero $X(x) \neq 0$

$\Rightarrow C = -D$

- $u(x,2) = x(1-x)$

$$u(x,2) = \sum_{k\in\mathbb{Z}} C_k \overbrace{(e^{ik\pi x} - e^{-ik\pi x})}^{X(x)} \overbrace{(e^{2\sqrt{k^2\pi^2-1}} - e^{-2\sqrt{k^2\pi^2-1}})}^{Y(2)}$$

$$\Rightarrow u(x,2) = \sum_{k\in\mathbb{Z}} C_k\,\mathrm{sen}(k\pi x)\,\mathrm{senh}(2\sqrt{k^2\pi^2-1}) = x(1-x) \quad \forall x \in [0,1]$$

Aplicando $\int_{-1}^{1} * \cdot \operatorname{sen}(n\pi x)dx$ (Podemos hacerlo entre $(-1,1)$ pues para algún n

C_k corresponde al coeficiente de la extensión impar $\bar{F}:[-1,1]\to\mathbb{R}$ de F, es decir $(f = x(1-x))$

$$\bar{F}(x) = \begin{cases} x(1-x) & x\in[0,1] \\ x(x-1) & x\in[-1,0] \end{cases}$$

$$\Rightarrow C_n \cdot \operatorname{senh}(2\sqrt{n^2\pi^2-1}) \int_{-1}^{1} \operatorname{sen}^2(n\pi x)dx = \int_{-1}^{1} x(1-x)\operatorname{sen}(n\pi x)dx$$

$$\Rightarrow C_n \cdot \operatorname{senh}(2\sqrt{n^2\pi^2-1}) \int_{-1}^{1} \frac{1-\cos(2\pi n x)}{2}dx = \int_{-1}^{1} x\operatorname{sen}(n\pi x) - \overbrace{\int_{-1}^{1} x^2 \operatorname{sen}(n\pi x)dx}^{\emptyset \text{ (impar)}}$$

$$\Rightarrow C_n \operatorname{senh}(2\sqrt{n^2\pi^2-1}) \left[\frac{x}{2} - \cancel{\frac{\operatorname{sen}(2\pi n x)}{4\pi n}}^{\emptyset}\right]_{-1}^{1} = 2\int_{0}^{1} x\operatorname{sen}(n\pi x)$$

$u = x \quad w = \operatorname{sen} n\pi x$
$du = dx \quad v = -\frac{\cos n\pi x}{n\pi}$

$$\Rightarrow C_n \operatorname{senh}(2\sqrt{n^2\pi^2-1}) = 2\left[-x\frac{\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}dx\right]$$

$$\Rightarrow C_n \operatorname{senh}(2\sqrt{n^2\pi^2-1}) = \frac{2}{n\pi}\left[-1\cdot\underbrace{\cos(n\pi)}_{(-1)^n} + \cancel{\frac{\operatorname{sen} n\pi x}{n^2\pi^2}}^{\emptyset}\Big|_0^1\right]$$

$$\Rightarrow C_n = \frac{2(-1)^{n+1}}{n\pi \operatorname{senh}(2\sqrt{n^2\pi^2-1})}$$

$$\Rightarrow u(x,y) = \sum_{k\in\mathbb{Z}} \frac{2(-1)^{k+1}}{k\pi\operatorname{senh}(2\sqrt{k^2\pi^2-1})} \operatorname{senh}(y\sqrt{n^2\pi^2-1})\cdot\operatorname{sen}(n\pi x)$$

//

P3) Hallar la función armónica $u(x,t) \to u(x,t)$ en la banda infinita $0<x<\pi$, $t>0$, tq

$$u(0,t) = u(\pi,t) = 0 \quad (t>0)$$
$$u(x,0) = 1 \quad (0<x<\pi)$$
$$|u(x,y)| < M = cte.$$

Sol) Función armónica

$$\Rightarrow \Delta u = 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2}$$

Planteamos $u(x,t) = X(x)\cdot T(t)$

$$\Rightarrow X''\cdot T + X\cdot T'' = 0 \quad \Big/ \frac{1}{X\cdot T}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} \quad \forall x \in (0,\pi],\ \forall t>0$$
$\Rightarrow$ son constantes

$\Rightarrow$ 1) $\frac{X''}{X} = \lambda$

$$\Rightarrow X'' - \lambda X = 0 \qquad \Rightarrow X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$
$$\Rightarrow r^2 - \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}$$

2) $\frac{T''}{T} = -\lambda$

$$\Rightarrow T'' + \lambda T = 0 \qquad \Rightarrow T(t) = Ce^{i\sqrt{\lambda}t} + De^{-i\sqrt{\lambda}t}$$
$$\Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}$$

Si $\lambda > 0 \Rightarrow X(x)$ es exponencial e $T(t)$ es periódica
$\lambda < 0 \rightarrow X(x)$ es periódica e $T(t)$ es exponencial

Condiciones de borde:

- $u(0,t) = (A+B)\cdot T(t) = 0 \overset{\forall t}{\Rightarrow} A+B = 0 \Rightarrow A = -B$
- $u(\pi,t) = A\left(e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}\right)\cdot T(t) = 0 \overset{\forall t}{\Rightarrow} e^{2\sqrt{\lambda}\pi} = 1$

Como $\lambda \neq 0 \Rightarrow \lambda < 0$ con $\lambda = -\omega^2$

$$\Rightarrow e^{2iw\pi} = 1 = \underset{1}{\underset{\parallel}{\cos 2w\pi}} + i\,\underset{\emptyset}{\underset{\parallel}{\operatorname{sen} 2w\pi}} \Rightarrow \left.\begin{array}{l} 2w\pi = 2k_1\pi \\ 2w\pi = k_2\pi \\ k_1, k_2 \in \mathbb{Z} \end{array}\right\} \Rightarrow \underline{w = k \in \mathbb{Z}}$$

$\Rightarrow$ familia de soluciones

$$u(x,y) = \sum_{k\in\mathbb{Z}} C_k\, u_k(x,y) \quad \text{donde } u_k(x,y) = X_k(x)\, T_k(t)$$

Tenemos entonces

$$X(x) = A\left(e^{ikx} - e^{-ikx}\right) = 2iA \operatorname{sen} kx$$

$$T(t) = C e^{-kt} + D e^{kt}$$

- $|u(x,y)| < M \Rightarrow D = \emptyset$ (de lo contrario $\lim_{t\to\infty} T(t) = +\infty$)

- $u(x,0) = \sum_{k\in\mathbb{Z}} C_k \operatorname{sen} kx\, e^{-kt}\Big|_{t=0} = \sum_{k\in\mathbb{Z}} C_k \operatorname{sen} kx = 1$

  C_k contiene a $A\cdot C\cdot 2i$

Aplicando $\int_0^{\pi} * \operatorname{sen}\left(\frac{n\pi x}{\pi}\right) dx$ (l = largo del intervalo), para calcular C_n

$$\Rightarrow C_n \int_0^{\pi} \operatorname{sen}^2 nx\, dx = \int_0^{\pi} 1\cdot \operatorname{sen} nx\, dx = -\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{1-(-1)^n}{n}$$

$$\Rightarrow C_n \int_0^{\pi} \frac{1-\cos 2nx}{2} dx = \frac{C_n}{2}\left[x - \frac{\operatorname{sen} 2nx}{2n}\right]_0^{\pi} = \frac{\pi}{2} C_n = \frac{1-(-1)^n}{n}$$

Si n es par $\Rightarrow C_n = 0$
Si n es impar $\Rightarrow C_n = \dfrac{4}{\pi n}$

Luego

$$u(x,y) = \sum_{k\in\mathbb{Z}} C_{2k} \operatorname{sen} 2kx \cdot e^{-2kt} = \sum_{k\in\mathbb{Z}} \frac{2}{\pi k} \operatorname{sen} 2kx \cdot e^{-2kt}$$

P4) Se define la transformada de Fourier como ($\hat{f}$):

Sea $f:\mathbb{R}\to\mathbb{C}$ integrable ($\int|f|<+\infty$),

$$\hat{f}:\mathbb{R}\to\mathbb{C}$$
$$s\to\hat{f}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)e^{-iys}dy = Tf(s)$$

y la antitransformada de Fourier como: ($\check{g}$):

Sea $g:\mathbb{R}\to\mathbb{C}$ integrable,

$$\check{g}:\mathbb{R}\to\mathbb{C}$$
$$x\to\check{g}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(s)\,e^{ixs}ds = T^{-1}g(x)$$

$$\Rightarrow f(x) = T^{-1}(Tf)(x) = \check{\hat{f}}(x)$$

Calcule entonces la TF de $f(x)=e^{-px^2}$ tq $p>0$.

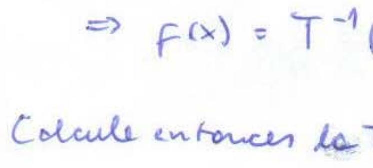
Sol)

$$\hat{F}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixs}f(x)\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixs}e^{-px^2}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-p\left(x^2+\frac{ixs}{p}\right)}dx$$

pero $x^2+\frac{ixs}{p}=\left(x+\frac{is}{2p}\right)^2+\frac{s^2}{4p^2}$

$$\Rightarrow\hat{F}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-p\left[\left(x+\frac{is}{2p}\right)^2+\frac{s^2}{4p^2}\right]}=\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4p}}\underbrace{\int_{-\infty}^{\infty}e^{-p\left(x+\frac{is}{2p}\right)^2}dx}_{I}$$

I lo calculamos en el plano complejo usando teo. Cauchy Goursat



$$\Rightarrow\int_{\Gamma}f(z)dz=0\ ,\quad \text{con } f(z)=e^{-pz^2}$$
con $\Gamma=\bigcup_{i}^{4}\Gamma_i$

① $\int_{\Gamma_1}f(z)dz=\int_{-R}^{R}e^{-px^2}\Rightarrow\lim_{R\to\infty}\int_{-R}^{R}e^{-px^2}=\sqrt{\frac{\pi}{p}}$

$z=x\in[-R,R]$
$dz=dx$

② $\int_{\Gamma_2}f(z)dz=i\int_{0}^{s/2p}e^{-p(R+iy)^2}dy=i\,e^{-pR^2}\int_{0}^{s/2p}e^{-p(2iRy-y^2)}dy\xrightarrow[R\to\infty]{}0$

$z=R+iy$
$dz=i\,dy$
$y\in[0,s/2p]$

③ $\int_{\Gamma_3} f(z)\,dz = \int_{R}^{-R} e^{-p\left(x+\frac{is}{2p}\right)^2} dx \xrightarrow{R\to\infty} -\int_{-\infty}^{\infty} e^{-p\left(x+\frac{is}{2p}\right)^2} dx$

$z = x + \frac{is}{2p}$
$dz = dx$
$x \in [R, -R]$

④ $\int_{\Gamma_4} f(z)\,dz = i\int_{s/2p}^{0} e^{-p(-R+iy)^2} dy = i\,e^{-pR^2}\int_{s/2p}^{0} e^{p(2Riy+y^2)} dy \xrightarrow{R\to\infty} 0$

$z = -R + iy$
$dz = i\,dy$
$y \in [\frac{s}{2p}, 0]$ y $\lim_{R\to\infty}$ ① + ②$^{\to 0}$ + ③ + ④$^{\to 0}$ = 0

$\Rightarrow \hat{F}(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4p}} \cdot \sqrt{\frac{\pi}{p}} \Rightarrow \boxed{\hat{F}(s) = \frac{1}{\sqrt{2p}} e^{-\frac{s^2}{4p}}}$

P5) Calcule la TF de $f(x) = \begin{cases} V_0 & |x| < x_0 \\ 0 & \sim \end{cases}$ (barrera potencial)

Sol)

$\hat{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{-x_0} f(x)\, e^{isx} dx}_{\equiv 0 \text{ pues } f(x)=0} + \int_{-x_0}^{x_0} f(x)\, e^{isx} dx + \underbrace{\int_{x_0}^{\infty} f(x)\, e^{isx} dx}_{\equiv 0 \text{ pues } f(x)=0} \right]$

$\Rightarrow \hat{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-x_0}^{x_0} V_0\, e^{isx} dx = \frac{V_0}{\sqrt{2\pi}} \int_{-x_0}^{x_0} \frac{1}{is} \frac{d}{dx}\left(e^{isx}\right) dx = \frac{V_0}{\sqrt{2\pi}\, is} \left(e^{isx_0} - e^{-isx_0}\right)$

pero $\operatorname{sen}(sx_0) = \frac{e^{isx_0} - e^{-isx_0}}{2i}$

$\Rightarrow \hat{F}(s) = \frac{2V_0}{\sqrt{2\pi}\, s} \operatorname{sen}(sx_0)$

$\Rightarrow \boxed{\hat{F}(s) = \sqrt{\frac{2}{\pi}} \frac{V_0}{s} \operatorname{sen}(sx_0)}$

Resúmen Anexo:

I Separación de Variables

I.1) Se plantea como solución de la ecn, $u(x,t) = X(x)\cdot T(t)$

I.2) Se reemplaza en la EDP, y se obtienen ecuaciones desacopladas para X y T

I.3) Se aplican condiciones de borde, y se estudia la constante del problema (de la separación de ecuaciones)

I.4) Se ocupa el ppio de superposición, es decir,

$$u(x,t) = \sum_{k=-N}^{N} A_k \phi_k(x,t) \quad , \ \phi_k(x,t) = X_k(x)\cdot T_k(t)$$

I.5) Se aplican el resto de las condiciones, y se busca el valor de la constante A_k

II. Series de Fourier

El desarrollo en SF de $f:[-\ell,\ell] \to \mathbb{R}$ es

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

cuya forma compleja sería

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{\ell}} \qquad \text{con } c_k = \begin{cases} a_0/2 & k = 0 \\ \frac{1}{2}(a_k - ib_k) & k \geq 1 \\ \frac{1}{2}(a_k + ib_k) & k \leq -1 \end{cases}$$

y los coeficientes

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\cos\left(\frac{n\pi\xi}{\ell}\right)d\xi$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(\eta)\,\text{sen}\left(\frac{n\pi\eta}{\ell}\right)d\eta$$