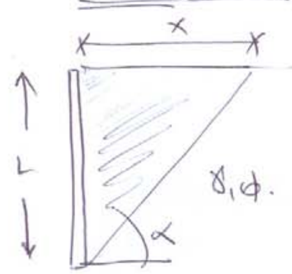
AUXILIAR # 7 GEOMECÁNICA

Abelardo Julio A.

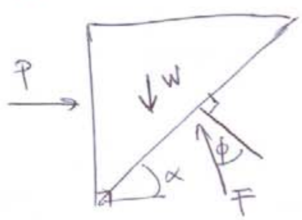
P1| Demostrar utilizando el método de la cuña que el ángulo que se forma en el suelo al desarrollarse empuje activo o pasivo corresponde a $45+\frac{\Phi}{2}$ y $45-\frac{\Phi}{2}$ respectivamente. Considerar el caso que $\delta=0°$ y $c'=0$. Además, considerar terraza Horizontal.

(i) Caso activo.

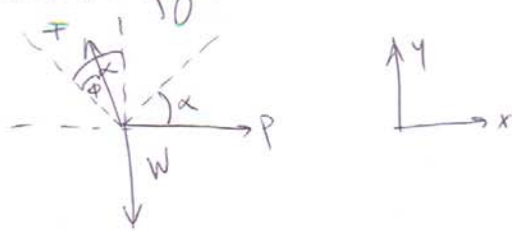


Se considera una cuña arbitraria de suelo limitada por el muro. La fza P existente entre el muro y la cuña se obtiene por equilibrio de Fzas (P es Horizontal ya que $\delta=0°$)

Luego:



Sistema de fzas:



a) $\Sigma F_x=0$: $P = F \operatorname{sen}(\alpha-\Phi)$

b) $\Sigma F_y=0$: $W = F\cos(\alpha-\Phi)$

$$\boxed{P = W\cdot tg(\alpha-\Phi)} \quad (*)$$

Pero $W = \gamma\cdot \text{Area}$, $\text{Area} = L\cdot\frac{x}{2} = \frac{L}{2}\cdot L\, tg(\frac{\pi}{2}-\alpha) = \frac{L^2}{2}\cdot ctg(\alpha)$

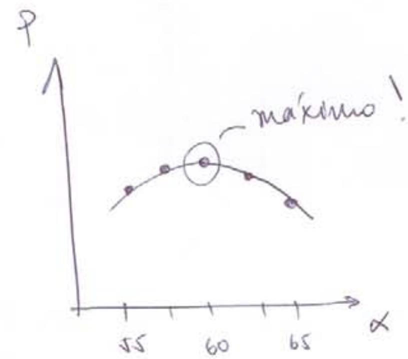
$$\therefore \boxed{P = \frac{1}{2}\gamma\cdot L^2\cdot ctg(\alpha)\, tg(\alpha-\Phi).}$$

El empuje activo se desarrollará para el valor de α que haga P máximo. Esto lo vemos en un ejemplo:

Supongamos $\phi = 30^\circ$, $L = 6$ m, $\gamma = 2\ t/m^3$.

$$\Rightarrow P = \frac{1}{2} \cdot 2 \cdot 6^2 \cdot \operatorname{ctg}(\alpha)\operatorname{tg}(\alpha - 30) = \boxed{36 \cdot \operatorname{ctg}(\alpha)\operatorname{tg}(\alpha-30)}$$

α	① $\operatorname{ctg}\alpha$	② $\operatorname{tg}(\alpha-30)$	①·②	P
55°	0,700	0,466	0,328	10,62
57,5°	0,637	0,520	0,331	10,72
60°	0,577	0,577	0,333	10,78
62,5°	0,521	0,637	0,331	10,72
65°	0,467	0,700	0,328	10,62



Analíticamente encontraremos el valor de α que hace máxima la fn $P = P(\alpha)$:

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma \cdot L^2 \cdot \left[-\operatorname{cosec}^2(\alpha) \cdot \operatorname{tg}(\alpha-\phi) + \operatorname{ctg}(\alpha) \sec^2(\alpha-\phi) \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\cos^2(\alpha-\phi) \cdot \operatorname{tg}(\alpha-\phi) + \operatorname{sen}^2(\alpha) \cdot \operatorname{ctg}(\alpha)}{\operatorname{sen}^2(\alpha) \cdot \cos^2(\alpha-\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\cos(\alpha-\phi) \cdot \operatorname{sen}(\alpha-\phi) + \operatorname{sen}(\alpha)\cos(\alpha)}{\operatorname{sen}^2(\alpha) \cdot \cos^2(\alpha-\phi)} \right]$$

$$= \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\left(\operatorname{sen}(\alpha)\cos(\phi)\left(\cos(\alpha)\cos(\phi) + \operatorname{sen}(\alpha)\operatorname{sen}(\phi)\right) - \operatorname{sen}(\phi)\cos(\alpha)\left(\cos(\alpha)\cos(\phi) + \ldots \ldots + \operatorname{sen}(\alpha)\operatorname{sen}(\phi)\right)\right) + \operatorname{sen}(\alpha) \cdot \cos(\alpha)}{\operatorname{sen}^2(\alpha) \cdot \cos^2(\alpha-\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\operatorname{sen}(\alpha) \cdot \cos(\alpha) \cdot \left(\cos^2(\phi) - \operatorname{sen}^2(\phi) - 1\right) - \operatorname{sen}(\phi)\cos(\phi) \cdot \left(\operatorname{sen}^2(\alpha) - \cos^2(\alpha)\right)}{\operatorname{sen}^2(\alpha)\cos^2(\alpha-\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\cancel{\tfrac{1}{2}}\, sen(2\alpha) \cdot (-) \cancel{2}\, sen^2(\phi) + sen(\phi)\cos(\phi)\cdot \cos(2\alpha)}{sen^2(\alpha)\cdot \cos^2(\alpha-\phi)} \right]$$

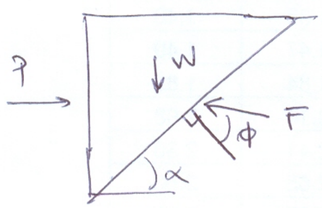
$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{sen(\phi)\left(sen(2\alpha)\cdot sen(\phi) + \cos(\phi)\cdot \cos(2\alpha)\right)}{sen^2(\alpha)\cdot \cos^2(\alpha-\phi)} \right]$$

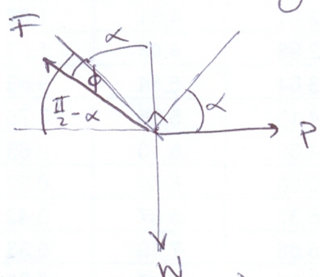
$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{sen(\phi)\cdot \cos(2\alpha-\phi)}{sen^2(\alpha)\cdot \cos^2(\alpha-\phi)} \right].$$

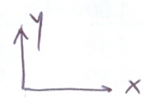
Se anula cuando $\cos(2\alpha-\phi) = 0 \iff 2\alpha - \phi = \frac{\pi}{2} \implies \boxed{\alpha = \frac{\pi}{4} + \frac{\phi}{2}}$

(ii) <u>Caso Pasivo</u>:

Sistema de fzas:







$$\Sigma F_x = 0: \quad P = \cos\left(\frac{\pi}{2} - \alpha - \phi\right)\cdot F$$

$$\Sigma F_y = 0: \quad W = F \cdot sen\left(\frac{\pi}{2} - \alpha - \phi\right)$$

$$P = W \cdot ctg\left(\frac{\pi}{2} - (\alpha+\phi)\right)$$

$$\boxed{P = W \cdot tg(\alpha+\phi)}.$$

Como $W = \gamma \cdot Area = \frac{\gamma \cdot L^2}{2}\cdot ctg(\alpha)$.

$$\implies \boxed{P = \frac{1}{2}\gamma L^2 \cdot ctg(\alpha)\cdot tg(\alpha+\phi)}$$

El empuje pasivo se desarrollará para el valor de α que haga P mínimo.

$$\implies \frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[-cosec^2(\alpha)\cdot tg(\alpha+\phi) + ctg(\alpha)\, sec^2(\alpha+\phi) \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\cos^2(\alpha+\phi)\cdot tg(\alpha+\phi) + ctg(\alpha)\cdot sen^2(\alpha)}{sen^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right].$$

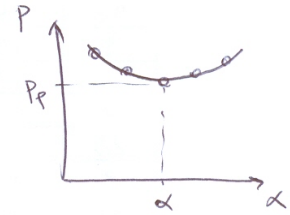
$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\cos(\alpha+\phi)\cdot \operatorname{sen}(\alpha+\phi) + \cos(\alpha)\cdot \operatorname{sen}(\alpha)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\left(\operatorname{sen}(\alpha)\cos(\phi)\left(\cos(\alpha)\cos(\phi) - \operatorname{sen}(\alpha)\cdot\operatorname{sen}(\phi)\right) + \operatorname{sen}(\phi)\cos(\alpha)\left(\cos(\alpha)\cos(\phi) - \operatorname{sen}(\alpha)\cdot\operatorname{sen}(\phi)\right)\right) + \operatorname{sen}(\alpha)\cos(\alpha)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{-\operatorname{sen}(\alpha)\cdot\cos(\alpha)\cdot\left(\cos^2(\phi) - \operatorname{sen}^2(\phi) - 1\right) - \operatorname{sen}(\phi)\cos(\phi)\left(\cos^2(\alpha) - \operatorname{sen}^2(\alpha)\right)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{\operatorname{sen}(2\alpha)\cdot\operatorname{sen}^2(\phi) - \operatorname{sen}(\phi)\cdot\cos(\phi)\cdot\cos(2\alpha)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$

$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[\frac{\operatorname{sen}(\phi)\cdot\left(\operatorname{sen}(2\alpha)\cdot\operatorname{sen}(\phi) - \cos(\phi)\cdot\cos(2\alpha)\right)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$



$$\frac{\partial P}{\partial \alpha} = \frac{1}{2}\gamma L^2 \cdot \left[-\frac{\operatorname{sen}(\phi)\cdot\cos(2\alpha+\phi)}{\operatorname{sen}^2(\alpha)\cdot \cos^2(\alpha+\phi)} \right]$$

se anula cuando $\cos(2\alpha+\phi) = 0 \iff 2\alpha+\phi = \frac{\pi}{2} \implies \boxed{\alpha = \frac{\pi}{4} - \frac{\phi}{2}}$

si se toman los valores de los ángulos encontrados y los reemplazo en las ecuaciones de empuje, obtenemos lo sgte:

i) ACTIVO:

$$P_a = \frac{1}{2}\gamma L^2 \cdot \operatorname{ctg}\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\cdot \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

$$P_a = \frac{1}{2}\gamma L^2 \cdot \underbrace{\operatorname{tg}^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}_{K_a} \longrightarrow \boxed{P_a = \frac{1}{2}\gamma L^2 \cdot K_a}$$

ii) PASIVO:

$$P_p = \frac{1}{2}\gamma L^2 \cdot \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cdot \operatorname{tg}\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

$$P_p = \frac{1}{2}\gamma L^2 \cdot \underbrace{\operatorname{tg}^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}_{K_p} \longrightarrow \boxed{P_p = \frac{1}{2}\gamma L^2 \cdot K_p}$$

$P_p > P_a$ siempre ya que. $K_p > K_a$

P2| Para un muro de contención como se muestra en la figura, calcule el empuje activo provocado por el suelo que éste contiene. ¿Se mantiene la condición que $\alpha = \frac{\pi}{4} + \frac{\phi}{2}$ en este caso?

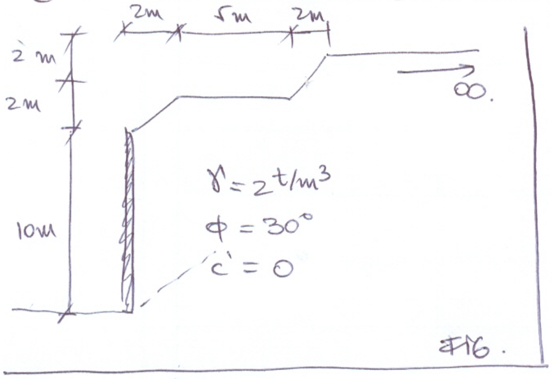
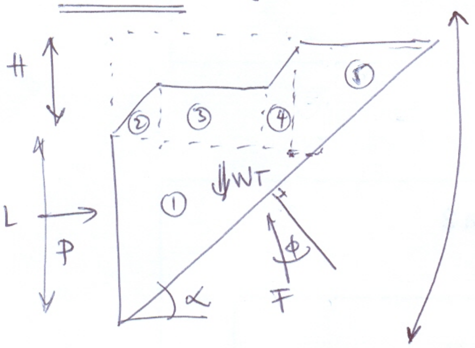


FIG.

Para resolver este problema, se deben suponer 3 cuñas genéricas y ver si los valores de α provocan que P sea el máximo:

Del P1 : $\boxed{P = W \cdot tg(\alpha - 30°)}$

* Cuña 1: $(\forall \alpha \leq 48{,}01°)$



$\boxed{\left(\alpha_{máx} \Leftrightarrow tg\alpha = \frac{10}{2+5+2} = \frac{10}{9}\right)}$

Área ①: $A_1 = \frac{L}{2} \cdot \frac{L}{tg\alpha} = \frac{L^2}{2} ctg\alpha = \boxed{50\, ctg\alpha}$

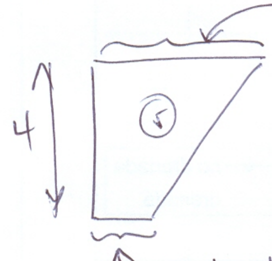
Área ②: $A_2 = \frac{2 \cdot 2}{2} = \boxed{2}$

Área ③: $A_3 = 5 \cdot 2 = \boxed{10}$

Área ④: $\frac{(4+2)}{2} \cdot 2 = \boxed{6} = A_4$

Área ⑤: $A_5 = ?$

$(L+H)\, ctg(\alpha) - (2+2+5) = 14 \cdot ctg(\alpha) - 9$



$L \cdot ctg(\alpha) - (2+2+5) = 10\, ctg(\alpha) - 9$

$\Rightarrow$ Área ⑤ $\Rightarrow A_5 = \frac{4}{2} \cdot (14\, ctg(\alpha) - 9 + 10\, ctg(\alpha) - 9) = 2 \cdot (24\, ctg(\alpha) - 18)$

$$\boxed{A_5 = 48\, ctg(\alpha) - 36}$$

Luego, el peso será:

$$W = \gamma \cdot \sum_i A_i = 2 \cdot \left[50\,ctg\alpha + 2 + 10 + 6 + (48\,ctg\alpha - 36) \right]$$

$$= 2 \cdot (98\,ctg\alpha - 18) = 4 \cdot (49 \cdot ctg\alpha - 9).$$

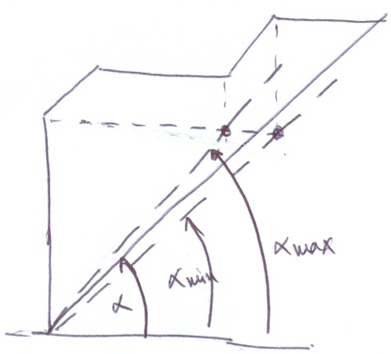
$\therefore$ $\boxed{P = 4 \cdot (49\,ctg\alpha - 9) \cdot tg(\alpha - 30)}$ Si damos valores a α y vemos cómo varía P:

α	48,01°	47°	46°	45°	...
P	45,7	44,9	44,0.	42,9.	

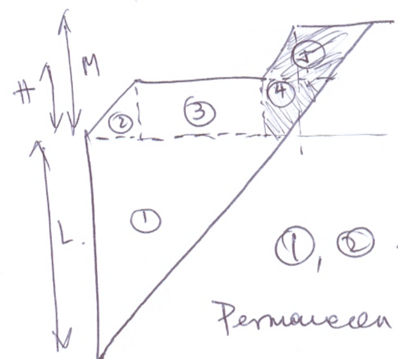
A medida que disminuye α P va disminuyendo.

Es necesario entonces analizar una 2ª cuña:

* <u>Cuña 2</u>: $\alpha \in [48{,}01°;\ 55{,}01°]$.



Áreas a considerar



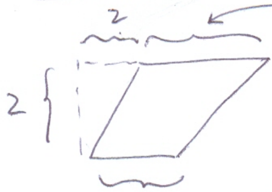
①, ② y ③ Permanecen iguales que antes. falta calcular ④ y ⑤:

<u>Área ④</u>: $(L+H)\,ctg\alpha - (2+5) = 12\,ctg\alpha - 7.$

$L\,ctg\alpha - (2+5) = 10\,ctg\alpha - 7.$

$\Rightarrow$ Área ④ : $A_4 = \frac{2}{2} \cdot (10\,ctg\alpha - 7 + 12\,ctg\alpha - 7) = \boxed{22\,ctg\alpha - 14. = A_4}$

Área 5



$(L+M)\,ctg\,\alpha - (2+5+2) = 14\,ctg\,\alpha - 9.$

$(L+H)\,ctg\,\alpha - (2+5) = 12\,ctg\,\alpha - 7.$

$$\Rightarrow A_5 = \left((2+14\,ctg\,\alpha-9)+(12\,ctg\,\alpha-7)\right)\cdot\frac{\cancel{2}}{\cancel{2}} - 2\cdot\frac{\cancel{2}}{\cancel{2}} = 26\,ctg\,\alpha - 16 = \boxed{2(13\,ctg\,\alpha-8)}$$

$$\therefore W = 2\cdot\left(50\,ctg\,\alpha + 2 + 10 + (22\,ctg\,\alpha - 14) + (26\,ctg\,\alpha - 16)\right)$$

$$\boxed{W = 2\cdot(98\,ctg\,\alpha - 18)}$$

(ja ja ja ... me quedó igual, pero no importa ... Pensé que daba un peso distinto).

$$\therefore \boxed{P = 4(49\,ctg\,\alpha - 9)\cdot tg(\alpha-30)}$$

α	49°	50°	51°	52°	53°	54°	55°
P	46,3	46,8	47,1	47,32	47,41	47,37	47,21

Por aquí está el máximo, entonces podemos encontrarlo analíticamente:

$$P = 196\cdot ctg(\alpha)\cdot tg(\alpha-30°) - 36\cdot tg(\alpha-30°).$$

De P1):
$$\frac{\partial(ctg(\alpha)\cdot tg(\alpha-30))}{\partial\alpha} = \frac{sen\,\phi\cdot\cos(2\alpha-\phi)}{sen^2\alpha\cdot\cos^2(\alpha-\phi)} = \frac{\overbrace{sen(30)}^{=\frac{1}{2}}\cdot\cos(2\alpha-30)}{sen^2(\alpha)\cdot\cos^2(\alpha-30)}.$$

$$\Rightarrow \frac{\partial P}{\partial\alpha} = \frac{\cancel{196}}{\cancel{2}}\cdot\frac{\cos(2\alpha-30)}{sen^2\alpha\cdot\cos^2(\alpha-30)} - 36\cdot\sec^2(\alpha-30)$$

$$\frac{\partial P}{\partial\alpha} = \frac{98\,\cos(2\alpha-30)}{sen^2\alpha\cdot\cos^2(\alpha-30)} - \frac{36}{\cos^2(\alpha-30)}$$

$$\frac{\partial P}{\partial \alpha} = \frac{98\cos(2\alpha-30) - 36\,\text{sen}^2\alpha}{\text{sen}^2\alpha \cdot \cos^2(\alpha-30)}$$

Se anula cuando: $98\cdot\cos(2\alpha-30) = 36\cdot\text{sen}^2\alpha$.

$$\Leftrightarrow \cos(2\alpha-30) = \frac{18}{49}\cdot\text{sen}^2\alpha.$$

$$\Rightarrow \boxed{\alpha^* = 53{,}19^\circ}$$

Veamos si se cumple:

$$\frac{\pi}{4}+\frac{\phi}{2} = \frac{\pi}{4}+\frac{\pi/6}{2} = \frac{\pi}{3} = 60^\circ$$

NO SE CUMPLE QUE EL ÁNGULO DE LA CUÑA DE FALLA SEA $45+\frac{\phi}{2}$ (ES MENOR) !

Calculemos finalmente el empuje:

$$P_a = 4\cdot(49\,\text{ctg}(53{,}19) - 9)\cdot\text{tg}(53{,}19-30)$$

$$\boxed{P_a = 47{,}41\ t/m.}$$