A MODEL FOR RESIDENTIAL SUPPLY

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Abstract

The release of agricultural land for urban use by landowners, the development of this land into urbanized land lots by land developers and the construction of buildings by housing developers is a complex multistage process which we study and model as a chain of sub-markets. The theoretical complexity comes from the interaction between supply agents at each stage, the existence of a natural monopoly associated with a location externality such as agglomeration, access and neighborhood advantages and the spatial context of this linked market.

Analytical models are derived for each supplier agent type, assuming they behave stochastically. We apply the maximum entropy framework to generate models consistent with discrete choice theory. The cases of perfect and imperfect competition are considered, along with constrained and unconstrained supply. The usual economic properties required for deterministic models are demonstrated to hold here. The calibration of parameters is discussed and equilibrium equations for quantity and prices are derived.

Thus, this new model can be conceived as a key tool in an overall land use model, providing a detailed and economically consistent description of the supply side. With this enhancement, a land use equilibrium is established on more solid ground.

JEL classification codes: R0 - Urban, Rural, and Regional Economics: General

1. INTRODUCTION

In the context of urban economics, demand for and supply of locational choices can be described as two optimization processes that provide the basic inputs to produce market equilibrium. Under the paradigm of rationality of consumers and suppliers, each optimization process describes the expected behavior of agents in the market.

The commodity traded in the market analyzed here is a residential option, which is described by several attributes. Some are associated with the location of the land lots, and include the building and natural environment, as well as accessibility and attractiveness. Other attributes describe the land lot itself, including size and view, plus the dwelling type, i.e. size, number of rooms, etc. Urban economists have recognized that locational attributes are not only associated with transport costs, but also with access to agglomeration economies related to the built environment. This property makes such a good very peculiar - it becomes quasi-unique. Attributes associated with the dwelling, on the other hand, do not introduce any peculiarity; they are produced and traded in the competitive building market.

Alonso's (1964) bid-auction model is consistent with this economic condition of quasi-uniqueness by assuming that location options are traded in auctions; we can make the softer, but perhaps also more realistic, assumption that agents have information on a *common value* at each location (see McAffe and McMillan 1987). This means that each location has a widely known estimate of the value before the auction, but the final price is still defined by the rule of the highest bidder, who gets the property or right to use this location. The theoretical discussion of this approach produced the bid-choice model (Martínez, 1992). In contrast with normal competitive goods, where prices are the result of a competitive equilibrium between demand and supply, in this framework the price formation process has two components. The dwelling cost is a competitive component with a production function, whereas the location value is a monopolistic component resulting from capitalizing the consumers preferences expressed by their willingness-to-pay.

In this market context, suppliers produce location-dwelling options subject to the conditions that the final price is defined by the best bid. Thus, their production problem can be described as choosing the number of location options supplied for each *combination of location, land lots and dwellings that yields them the maximum profit for a given dwelling prices*. Here we model this problem for several assumptions of market suppliers' behavior and market conditions.

2 THE SUPPLY MODEL'S ECONOMIC STRUCTURE

2.1 THE MARKET STRUCTURE

The first main element of the model is that we decompose the housing supply system into a three-step production process, where each step involves different producers or agents; this approach extends the Ueda et. al. (1996) model. These steps describe the residential supply market as a production chain, starting with a landowner, followed by a land developer and finally a housing developer, as shown in Figure 1.

LANDOWNERS (L):
Agricultural land is made available for urban development. Agriculture landowners release a total of Q_i^0 broadacres of land in each zone i to maximize their profit obtained from the price differential between the agricultural land price $(\overline{p}_i)$[1] and the release price (p_i), paying a marketing cost $\overline{c}_i$ and subject to urban regulation policies.

LAND DEVELOPERS (LD)
The land developer agent divides the released land in each zone into a number of land lots (L_{ki}) of different sizes (q_k). This agent maximizes the profit obtained from buying the land at a price (p_i) and selling an optimum number of lots of each size at a price p_{ki}, while paying out an exogenous marketing-management-development cost $\overline{c}_{ki}$. For computational simplicity, the land size is defined here as a discrete variable, with index k identifying the range of land lot sizes.

HOUSING DEVELOPERS (HD)
These agents build houses on the land lots. They decide the number of houses of type v to be built on each land lot(X_{vki}) to maximize the profit obtained from the price difference between buying land at prices(p_{ki}) and selling houses at prices (r_{vki}), incurring exogenous building costs $(\overline{c}_{vki})$. Housing types

[1] Overlining denotes exogenous variables (model parameters).

are defined with regard to the building attributes, for example, detached, dual occupancy, etc., and may be extended to consider flats.

2.2 THE BEHAVIORAL MODEL

We assume, for each step in the chain, a free market where producers *maximize the profits* (p) obtained from buying partially developed land, investing in some extra development, such as managing and marketing, at a cost per unit area, and selling at a profit.

The model is specified within a static equilibrium framework, where supply and demand adjust to achieve an equilibrium condition that defines the static set of prices in the system which are instantaneously informed to all actors. A similar model structure may be studied in future research within a dynamic context, where producers in the chain make decisions based on information lagged over time.

The producers' behavior is assumed to be represented by the maximization of their stochastic profit function in choosing among deterministic supply options. This means that we assume actors as having probabilistic behavior, which may be interpreted as either homogeneous actors applying mixed strategies and/or a degree of heterogeneity in actors' behavior and attributes. We maintain that this framework is highly natural for the choice problem in this market. An alternative probabilistic framework would be to assume deterministic behavior upon facing probabilistically available options, with actors choosing the optimal stochastic outcome. The analysis of this second option is not handled in this paper.

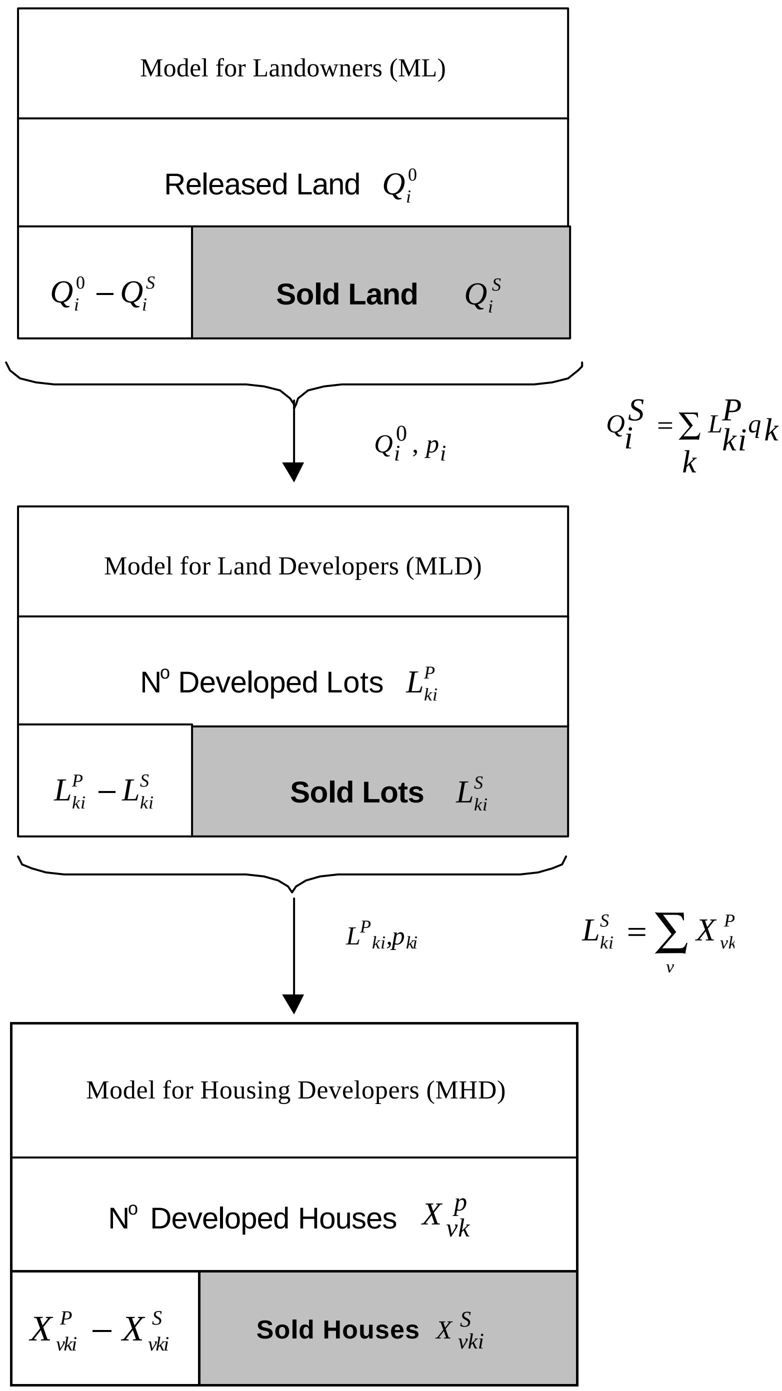
This stochastic behavior is modeled applying a *maximum entropy* (or probabilistic) framework, extending the concepts introduced in Wilson (1970) and Roy and Lesse (1981). Additionally, observed information on total supply and demand is introduced to improve the fit of the model. This defines optimization problems yielding demand and supply functions for each agent that we prove to be consistent with standard micro-economic theory and also with random choice theory.

2.3 ECONOMIC EQUILIBRIUM

The important question about whether residential supply represents a perfect or imperfect competitive market is discussed here, deriving models for these two hypothetical cases. In the first case, each agent behaves as a price taker, whilst in the second case agents can anticipate the consumers' demand function and extract their maximum willingness-to-pay. For this latter case, oligopolistic demand functions are obtained by inverting the competitive demand function in terms of supply price, which is then introduced into the supply function at the previous step in the market chain.

There is a strong theoretical argument, developed in the tradition of Alonso's (1964) urban economics, to support the thesis that landowners enjoy a non-competitive or oligopolistic land market. It is also plausible to assume that land and housing developers normally operate in a competitive market, because firms produce lots and houses anywhere, without having natural locational advantages. This is valid unless we consider that imperfect competition in the land market induces speculative behavior in developers, who manage to partially capitalize locational advantages. Although there is the cost of extra complexity associated with modeling imperfect competition, the importance of this model is that it is thoroughly consistent with the bid-auction model of land-use. This consistency is theoretically sound, because the argument that locational advantages justify the best bid rule in the land-use model is valid across the supply hierarchy, each one capturing a portion of the total monopoly power as land is developed.

Figure 1. Diagram of Model Chained Structure



We study the *long run* case, where supply is adjusted to find static equilibrium. The classical long run competitive equilibrium model assumes that all firms are identical –homogeneous - with the same technology and capital, which generates equi-profit equilibrium. However, in the urban land market, goods are differentiated due to the quasi - unique characteristics of the land; thus landowners are not homogeneous as suppliers because they hold monopoly power on spatially differentiated locational externalities, which yields differentiated profits. In this case, for the equi-profit condition to hold, it would be required to assume strong state intervention, a case not considered in this model. Since profit differentials are generated by locational externalities, then it sounds reasonable to consider that profit is maximized in each homogenous zone, independently of other zones.

Firstly, we analyze the more classical assumption that supply exactly equals demand, which is modeled by specifying an *unconstrained* supply model. Then, we relax this condition allowing for some excess supply, but it asymptotically diminishes towards the classical Walras price equilibrium; this is modeled by *non-linear models with capacity constraints*, which generate production functions (denoted by superscript P) as well as functions for sold stock (S). We derive these functions for landowners $\left(Q_i^0, Q_i^S\right)$, land developers $\left(L_{ki}^P, L_{ki}^S\right)$ and housing developers $\left(X_{vki}^P, X_{vki}^S\right)$.

2.4 THE ENTROPY MODELING APPROACH

Finally, it is important to discuss some wider developments in entropy-type models, which are making them relevant in the context of economics and thus particularly useful in our study. These concepts are expressed in more detail in Roy (1997). Typically, the only cost-related data present in the spatial entropy models of Wilson (1970) were transport costs. Thus, entropy models for determination of transport flows were interpreted as minimizing the expected total cost of travel. However, in an important statement, Smith (1990) proposed that this 'behavioral' data in an entropy model should be encapsulated in constraints that are consistent with an appropriate model theory. Roy accepted this advice in using entropy to form *probabilistic* models demonstrating asymptotic convergence properties to the classical *deterministic* models of microeconomic theory, making the classical models special cases of the more general entropy models. For instance, in the supply behavior of the competitive firm (or an aggregate of ‘identical’ firms), the profit *objective* of the deterministic theory of the firm is effectively enhanced in the probabilistic analogue by addition of a weighted entropy term, in which the enhanced objective can be interpreted as *expected* profits. Secondly, adapting ideas from Lesse (1982), it is possible to treat *Lagrange multipliers* of the estimated models as *parameters* in transformed versions for projection, allowing projected quantities to be endogenous and price-responsive. As these parameters are directly related to *elasticities* in microeconomics, their stability is only guaranteed over a relatively short time period.

In addition, in examining the issue of *available capacities* in modeling Walrasian equilibrium, advice was obtained from an early paper of Hotelling (1932). In a *regional* context, Hotelling demonstrated the rationale for generic *logistic* forms of regional supply and demand functions, which demonstrate *spill-over* effects into adjacent regions when capacity in any region is hard-pressed. Such *non-separable* functions can be obtained via entropy using a parallel argument, where units of available capacity are taken as *distinguishable* (or heterogeneous) in the entropy objective of the *constrained* model, acting somewhat like a compressing spring (repulsion effect) as the capacity limit in any zone is approached.

Last, but not least, most probabilistic demand (and supply) functions arising from the entropy framework are analytically invertible. This opens the door to development of tractable model frameworks for imperfect competition.

The aim of the paper is to present several models. In the following section, models of competitive markets are derived both for the constrained and unconstrained cases. The extension of these models to imperfect competition is presented in Section Four. Several models are formally presented in these two Sections, leaving comments, interpretations and further analysis to Section Five.

3. COMPETITIVE LAND MARKET

Although we have argued that the urban land market is normally imperfect, we shall start with models of a hypothetical competitive market for the sake of completeness and simplicity in the presentation and to obtain a comparison with the case of imperfect competition. For similar reasons, we first present the unconstrained model, so as to introduce later the extra complexity associated with the constrained model.

3.1 UNCONSTRAINED MODELS FOR COMPETITIVE MARKETS

The assumption in the unconstrained model is that supply from one step equals demand for the next agent in the chain.

<u>Landowner Model (L/C/UNCNST)</u>[2]

The entropy represents the number of ways that distinguishable units of land can be allocated into the different housing zones. When this quantity is maximized under the necessary constraints, enforcing compliance with base period observations, we obtain the most likely quantities of land released. The entropy, given by

$$S^L = -\sum_i \left\{ Q_i \left(ln\, Q_i - 1 \right) \right\}$$

is maximized under the landowners' profit constraints, given by

$$p^L = \sum_i Q_i \left(p_i - \overline{p}_i - \overline{c}_i \right),$$

which assumes landowning as an homogeneous industry. Additionally, total agricultural land potentially available for release for residential use is constrained by:

$$\sum_i Q_i = Q^0$$

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[2] Notation for models is step or agent by L, LD, and HD respectively; C and IC for perfect and imperfect competition; CNST and UNCNST for constrained and unconstrained models.

With a_1 the Lagrange multiplier on the profit constraint and g_1 that on the total quantity constraint, we obtain the Lagrangian function for each profit assumption:

$$F^L = -\sum_i \{Q_i(ln\,Q_i - 1)\} + a_1\left[\sum_i Q_i(p_i - \overline{p}_i - \overline{c}_i) - p_L\right] + g_1\left(\sum_i Q_i - Q^0\right)$$

with first order conditions given by:

$$\frac{\partial F^L}{\partial Q_i} = -ln\,Q_i + a_1(p_i - \overline{p}_i - \overline{c}_i) + g_1 = 0$$

This yields the landowners' maximum profit supply function:

$$Q_i = \exp[a_1(p_i - \overline{p}_i - \overline{c}_i) + g_1] \qquad (1)$$

In this model, α_1 is calibrated by linear extrapolation to reproduce observed profit and/or by using equation (1) with the more readily available supply data. It represents the price sensitivity of supply to profit variations. γ_1 is calculated by substituting equation (1) into the total land constraint, yielding

$$g_1 = ln\left[\frac{Q^0}{\sum_i exp[a_1(p_i - \overline{p}_i - \overline{c}_i)]}\right]$$

which depends on prices and other exogenous parameters. Note that, upon substituting this into the supply function we obtain the multinomial logit formula, which makes our supply function alternatively interpretable as a random choice model with a Gumbel distribution of suppliers' profit.

In order to use (1) in projection, we need to transform the Lagrangian function F^L above to $F^{L'}$ such that the Lagrange multipliers a_1 and g_1 can be interpreted as parameters via the following Legèndre transform (Lesse, 1982)

$$F^{L'} = F^L - p_L \text{ ¶ } F^L / \text{¶ } p_L - Q^0 \text{ ¶ } F^L / \text{¶ } Q^0$$

which comes out as

$$Z^{L'} = -\text{å}_i\, Q_i\, [ln\, Q_i - 1] + a_1\, \text{å}_i\, Q_i\, (p_i - \overline{p}_i - \overline{c}_i) + g_1\, \text{å}_i\, Q_i$$

If this expression is maximized in terms of Q_i with a_1 and g_1 as given parameters, relation (1) is produced, which upon substitution into the constraints yields their observed right-hand sides. Thus, taking a_1 and g_1 as given, relation (1) can be used for projection when prices or costs change. The above formalism applies in principle for all the following cases, both for perfect and imperfect competition.

Note that, one could assume that landowners are fully differentiable by zone due to location externalities, maximizing local profit given by

$$p_i^L = Q_i(p_i - \overline{p}_i - \overline{c}_i)$$

which leads to

$$F^{L} = -\sum_{i} Q_{i}\left(lnQ_{i} - 1\right) + \sum_{i} a_{1i}\left(Q_{i}\left(p_{i} - \overline{p}_{i} - \overline{c}_{i}\right) - p_{i}^{L}\right) + g_{1}\left(\sum_{i} Q_{i} - Q^{0}\right)$$

yielding

$$Q_{i}^{'} = \exp\left[a_{1i}\left(p_{i} - \overline{p}_{i} - \overline{c}_{i}\right) + g_{1}\right] \qquad (2)$$

In this case, in order to calibrate the a_{1i} parameters from profit levels, observations of profits for each zone are required, which might not be easy to obtain.

As this approach to obtain supply models is also used in deriving the following models in this paper, it is therefore worth commenting on an optional interpretation of the model. Let us define the suppliers' objective function as the maximization of profit plus a weighted entropy term. The Lagrangian of this problem is equal to $\overline{F}^{L} = (1/a_{1})F^{L}$ which has the same solution as F^{L} for a given domain and positive a_{1}. With this formulation the parameter $1/a_{1}$ becomes proportional to the standard deviation of the stochastic distribution introduced by the entropy term, so that as it tends to zero, the models tend to reproduce the deterministic profit maximization case. This statistical interpretation of the a parameters is also consistent with the logit model, where a_{1} represents the scale parameter of the Gumbel distribution that is proportional to the inverse of the standard deviation. It is also consistent with the interpretation of $\overline{F}^{L}$ as *expected* profits.

<u>Land Developer Model (LD/C/UNCNST)</u>

The total number of L_{i}^{0} lots is split into L_{ki} lots of size q_{k}, yielding the following entropy:

$$S^{LD} = -\sum_{ki} L_{ki}\left(ln\, L_{ki} - 1\right)$$

which is maximized subject to the aggregated profit constraint:

$$p^{LD} = \sum_{ki} L_{ki}\left(p_{ki} - p_{i} q_{k} - \overline{c}_{ki}\right) \ ,$$

and complementary quantity sum information from the observed land market introduced to the model :

$$L_{i}^{0} = \sum_{k} L_{ki} \qquad L_{k}^{0} = \sum_{i} L_{ki}$$

Thus, the first order conditions with Lagrange multipliers a_{2}, g_{i2} and h_{k}, respectively, are:

$$\frac{\partial F^{LD}}{\partial L_{ki}} = -ln\left(L_{ki}\right) + a_{2}\left(p_{ki} - p_{i} q_{k} - \overline{c}_{ki}\right) + g_{i2} + h_{k} = 0$$

which yield the optimum values:

$$L_{ki} = \exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i q_k - \bar{c}_{ki}\right)\right] \quad (3)$$

As before, a_2 is calibrated from profit or supply data. In this case, the model is doubly constrained, so that parameters g_{i2} and h_k are interdependent. However, they are easy to evaluate by the usual iterative algorithm to obtain balancing factors (see for example Wilson 1970). As shown by Anas (1983), this model can also be interpreted as multinomial logit.

Introducing the equilibrium condition between supplied and developed land:

$$Q_i = \sum_k L_{ki} q_k$$

we can obtain the expression of the land price for a unit of land p_i, given by:

$$p_i^e = \bar{p}_i + \bar{c}_i + \frac{1}{a_1}\left\{-g_1 + ln \sum_k q_k \; exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i^e q_k - \bar{c}_{ki}\right)\right]\right\} \quad (4)$$

This equation expresses the expected price at equilibrium in a fixed-point format, called the logsum fixed-point, that is $p_i^e = f\left(p_i^e\right)$, which cannot be solved analytically for p_i^e but has a numerical unique solution for a given set of p_{ki} and g_1. Observe that numerical complexity is introduced by the change of units between area land units and numbers of lots, which requires the transformation factor q_k.

Housing Developers (HD/C/UNCNST)

In this case, land lots are further developed and the entropy expresses the number of ways in which lots can be enhanced with houses of different types:

$$S^{HD} = -\sum_{vki} X_{vki}\left(ln X_{vki} - 1\right)$$

which is maximized subject to the aggregated profit:

$$p^{HD} = \sum_{vki} X_{vki}\left(r_{vki} - p_{ki} - \bar{c}_{vki}\right)$$

and the conditions from observed quantity information:

$$\sum_{vk} X_{vki} = X_i^0 \qquad \sum_i X_{vki} = X_{vk}^0$$

It follows that:

$$\frac{\partial F^{HD}}{\partial X_{vki}} = -ln\, X_{vki} + a_3\left(r_{vki} - p_{ki} - \bar{c}_{vki}\right) + g_{i3} + h_{vk} = 0$$

yielding the housing supply function as:

$$X_{vki} = \exp\left[a_3\left(r_{vki} - p_{ki} - \bar{c}_{vki}\right) + g_{i3} + h_{vk}\right] \qquad (5)$$

As before, this is a doubly-constrained model and its parameters can be obtained using the same technique.

Introducing the supply demand consistency equation:

$$L_{ki} = \sum_v X_{vki}$$

we can obtain the expression for lot prices at equilibrium p^e_{ki}, given by:

$$p^e_{ki} = \frac{1}{a_2 + a_3}\left\{a_2\left(p_i q_k + \bar{c}_{ki}\right) - g_{i2} - h_k + ln\sum_v exp\left[a_3\left(r_{vki} - \bar{c}_{vki}\right) + g_{i3} + h_{vk}\right]\right\} \qquad (6)$$

In this case, the price function is explicit and depends on land prices as the production factor, and on selling prices of house options using the lot size q_k.

3.2 Constrained Models For Competitive Markets

These models assume that the total supplied units may be in excess of the actual demand, which requires some extensions of the previous unconstrained models in this direction. In the specification of excess supply one may assume a linear or non-linear decrease of stock. In this latter and perhaps more interesting case, the entropy function may be specified by an extended model recognizing distinguishable capacity units, where the *slope* of the supply function decreases in a *logistic* form with price as production approaches capacity (Hotelling, 1932).

<u>Landowner Model (L/C/CNST)</u>

From a total of Q^0_i broadacres of building land released for development in the base year in zone *i*, Q^S_i is sold. The entropy now represents the number of ways that the units of supplied land and unsold land can be allocated from within the distinguishable units of capacity, written as:

$$S^L = ln\left[Q^0_i!/\left(Q^0_i - Q^S_i\right)!\,Q^S_i!\right]$$

which yields the following objective:

$$S^L = -\sum_i \left(Q^0_i - Q^S_i\right)\left[ln\left(Q^0_i - Q^S_i\right) - 1\right] - \sum_i Q^S_i\left[log\, Q^S_i - 1\right]$$

The entropy is maximized under the following constraints:

$$\sum_i Q^S_i = Q^0 \qquad\qquad p^L = \sum_i \left[Q^S_i\left(p_i - \bar{c}_i\right) - Q^0_i\,\bar{p}_i - \left(Q^0_i - Q^S_i\right)I\,\bar{p}_i\right]$$

where *I* represents the interest rate which generates the cost of holding stock.

With γ_1 and a_1 as the respective Lagrange multipliers, we obtain

$$Q_i^S = Q_i^o \frac{\exp\left[g_1 + a_1\left(p_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]}{\left\{1 + \exp\left[g_1 + a_1\left(p_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]\right\}} \qquad (7)$$

which is the classical logistic functional form. In this case, parameters may be calibrated by linear extrapolation for α_1 and solving a fixed-point problem for γ_1, because in this case, g_1 cannot be directly expressed analytically.

Land Developer Model (LD/C/CNST)

The total number of lots L_i^0 is subdivided by lot size into the number of produced lots L_{ik}^P and sold lots L_{ik}^S. The entropy is:

$$S^{LD} = \ln\left[\prod_i \frac{L_i^0!}{\Pi_k L_{ik}^P!} \prod_{ik} \frac{L_{ik}^P}{\left(L_{ik}^P - L_{ik}^S\right)! L_{ki}^S!}\right]$$

This reduces to

$$S^{LD} = -\sum_{ki} L_{ki}^S \left[ln\, L_{ki}^S - 1\right] - \sum_{ik}\left(L_{ki}^p - L_{ki}^S\right)\left[ln\left(L_{ki}^p - L_{ki}^S\right) - 1\right]$$

being maximized under the following total quantity constraints:

$$\sum_k L_{ki}^P = L_i^{Po} \qquad \sum_i L_{ki}^P = L_k^{Po}$$

$$\sum_k L_{ki}^S = L_i^{S0} \qquad \sum_i L_{ki}^S = L_k^{S0}$$

plus the usual profit constraint:

$$p^{LD} = \sum_{ki}\left(L_{ki}^S p_{ki} - L_{ki}^P p_i q_k - L_{ki}^P \overline{c}_{ki}\right) - \sum_{ki}\left(L_{ki}^P - L_{ki}^S\right) I p_i q_k$$

If these are associated with Lagrange multipliers f_i, c_k, g_{i2}, h_k and α_2, we can finally write:

$$L_{ki}^P = \exp\left[f_i + c_k + a_2\left(-p_i q_k - \overline{c}_{ki} - I p_i q_k\right)\right]\left[1 + \exp\left(g_{i2} + h_k + a_2\left(p_{ki} + I p_i q_k\right)\right)\right] \qquad (8)$$

$$L_{ki}^S = \exp\left[f_i + c_k + a_2\left(-p_i q_k - \overline{c}_{ki} - I p_i q_k\right)\right]\exp\left[g_{i2} + h_k + a_2\left(p_{ki} + I p_i q_k\right)\right] \qquad (9)$$

If we introduce the consistency condition $\sum_k q_k L^P_{ik} = Q^S_i$, using (7) and (8) we obtain the following equation for the equilibrium land price:

$$Q^o_i \frac{\exp\left[g_1 + a_1\left(p^e_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]}{1+\exp\left[g_1 + a_1\left(p^e_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]} = \sum_k q_k \frac{1+\exp\left[g_2 + h_k + a_2\left(p_{ki} + Ip^e_i q_k\right)\right]}{\exp\left[-f_i - c_k + a_2\left(p^e_i q_k + \overline{c}_{ki} + Ip^e_i q_k\right)\right]} \qquad (10)$$

which represents a fixed-point equation for p^e_i, even if we assume no holding stock cost (I=0). As for equation (4), this equation cannot be analytically solved for p^e_i. As demonstrated for related models for commodity flows (Roy and Johansson, 1993), both existence and uniqueness of the equilibrium is guaranteed.

Housing Developer Model (HD/C/CNST)

Agents allocate from supply L^p_{ki} at each zone *i*, distinguishable lots for housing development, building X^p_{vki} houses of type *v* on them, of which X^s_{vki} get sold. Then, the system entropy becomes

$$S^{HD} = ln \prod_{ki} \frac{L^p_{ki}!}{\left(L^p_{ki} - \sum_v X^p_{vki}\right)! \prod_v X^p_{vki}!} \left\{ \prod_{vki} \frac{X^p_{vki}!}{\left(X^p_{vki} - X^S_{vki}\right)! X^S_{vki}!} \right\}$$

With the usual approximations, S is written after some cancellation as:

$$S^{HD} = -\sum_{ki}\left(L^P_{ki} - \sum_v X^P_{vki}\right)\left[ln\left(L^P_{ki} - \sum_v X^P_{vki}\right) - 1\right] - \sum_{vki}\left(X^P_{vki} - X^S_{vki}\right)\left[ln\left(X^P_{vki} - X^S_{vki}\right) - 1\right] - \sum_{vki} X^S_{vki}\left[ln\, X^S_{vki} - 1\right]$$

This is maximized under the following constraints

$$\sum_i X^S_{vki} = X^{S0}_{vk} \qquad \sum_k X^S_{vki} = X^{S0}_{vi} \qquad \sum_v X^S_{vki} = X^{S0}_{ki}$$

$$\sum_{vk} X^P_{vki} = X^0_i \quad ; \quad \sum_i X^P_{vki} = X^0_{vk}$$

$$p^{HD} = \sum_{vki}\left(X^S_{vki} r_{vki} - X^P_{vki}\left(p_{ki} + \overline{c}_{vki}\right) - \left(X^P_{vki} - X^S_{vki}\right) Ip_{ki}\right)$$

attached to the objective with Lagrange multipliers $\lambda_{vk}, \phi_{vi}, \psi_{ki}, \gamma_{i3}, \eta_{vk}$ and α_3 respectively. Upon differentiation with respect to X^P_{vki} and X^S_{vki}, equating to zero and performing summations to remove X_{vki} from the right-hand side, we finally obtain the production of housing:

$$X^P_{vki} = L^P_{ki} \frac{\exp\left(-r^P_{vki}\right) + \exp\left(-r^P_{vki} - r^S_{vki}\right)}{1 + \sum_v \left[\exp\left(-r^P_{vki}\right) + \exp\left(-r^P_{vki} - r^S_{vki}\right)\right]} \qquad (11)$$

with the amount sold of each type on each lot category being:

$$X^{S}_{vki} = L^{P}_{ki} \quad \frac{\exp\left(-r^{P}_{vki} - r^{S}_{vki}\right)}{1+\sum_{v}\left[\exp\left(-r^{P}_{vki}\right)+\exp\left(-r^{P}_{vki} - r^{S}_{vki}\right)\right]} \tag{12}$$

with

$$-r^{P}_{vki} = a_{3}\left(p_{ki} - \bar{c}_{vki} + Ip_{ki}\right) - g_{i3} - h_{vk}$$

$$-r^{S}_{vki} = a_{3}\left(r_{vki} + Ip_{ki}\right) + l_{vk} + f_{vi} + y_{ki} \qquad \text{and}$$

Introducing the equilibrium condition between sold lots and those transformed to housing lots, given by $L^{S}_{ki} = \sum_{v} X^{P}_{vki}$, we obtain:

$$L^{S}_{ki} = \sum_{v} X^{P}_{vki} = L^{P}_{ki}\left(\frac{A_{ki}}{1+A_{ki}}\right) \tag{13}$$

with $A_{ki} = \sum_{v}\left[\exp\left(-r^{P}_{vki}\right) + \exp(-r^{P}_{vki} - r^{S}_{vki})\right]$, and using equation (8):

$$L^{P}_{ki} = B_{ki}\left(1+C_{ki}\right) \qquad \text{and} \qquad L^{S}_{ki} = B_{ki} \cdot C_{ki} \tag{14}$$

with $B_{ki} = \exp\left[f_{i} + c_{k} + a_{2}\left(-p_{i}q_{k} - \bar{c}_{ki} - Ip_{i}q_{k}\right)\right]$ and $C_{ki} = \exp\left(g_{i} + h_{k} + a_{2}(p_{ki} + Ip_{i}q_{k})\right)$.

Substituting (14) into (13), we obtain a market equilibrium condition $A_{ki} = C_{ki}$. To extract p_{ki} it is necessary to assume no holding costs (I=0), which yields:

$$p^{e}_{ki} = \frac{1}{a_{2} - a_{3}}\left\{-g_{i2} - h_{k} + ln\sum_{v} exp\left[-g_{i3} - h_{vk} + a_{3}\bar{c}_{vki}\right]\left(1+exp\left(a_{3}(r_{vki}) + l_{vk} + f_{vi} + j_{ki}\right)\right)\right\} \tag{15}$$

which is the expected equilibrium price for lots *k* in zone *i*. It is worth noting that this price is expressed in terms of model parameters only, except for the house price or rent r_{vki}. This makes lot prices dependent on the consumer/supply equilibrium in the residential housing market.

In this family of constrained models, supply functions do not reproduce the multinomial logit formula; hence the underlying distribution is different from IID Gumbel.

3.3 HOTELLING'S LEMMA

This is a classical lemma in deterministic micro-economics, also called the derivative property of profit functions, which states that the firm's net supply function is equal to the price derivative of the profit function. In other words, it should be possible to obtain the supply function from the profit function; in fact this is the approach used by Ueda et. al (1996) to derive supply and demand functions.

<u>The landowners case</u>

In this section, we shall prove the lemma for the probabilistic landowners supply model of the competitive and constrained case. Let us establish $(1/\alpha_2)$ times the Legèndre transform (Lesse, 1982) of the total original entropy objective F, converting it into the 'equivalent' <u>unconstrained</u> objective, identified as an *expected* profit.

$$Z^L = \frac{1}{a_1}\begin{bmatrix} -\sum_i \left(Q_i^0 - Q_i^S\right)\left[ln\left(Q_i^0 - Q_i^S\right) - 1\right] - \sum_i Q_i^S\left[ln\, Q_i^S - 1\right] + g_1 \sum_i Q_i^S \\ + a_1\left(\sum_i Q_i^S\left(p_i - \overline{c_i}\right) - Q_i^0\,\overline{p_i} - \left(Q_i^0 - Q_i^S\right)Ip_i\right) \end{bmatrix}$$

after considerable cancellation, we finally obtain:

$$Z^L = \frac{1}{a_1}\left[\sum_i \left(Q_i^0\right) ln\left(1 + exp\left(g_1 + a_1\left(p_i + \overline{p_i}I - \overline{c_i}\right)\right)\right) + K\right]$$

with K being a constant. Thus:

$$\frac{\partial Z^L}{\partial p_{ki}} = Q_i^0\, \frac{\exp\left(g_1 + a_1\left(p_i + \overline{p_i}I - \overline{c_i}\right)\right)}{1 + \exp\left(g_1 + a_1\left(p_i + \overline{p_i}I - \overline{c_i}\right)\right)} = Q_i^S$$

<u>The land developers case.</u>

Similarly, for the probabilistic land developer supply function of the constrained model in (8), the transform yields:

$$\begin{aligned} Z = \left(\frac{1}{a_2}\right)\{ &\sum_i L_i^0\left[ln\, L_i^0 - 1\right] - \sum_{ik} L_{ki}^S\left[-f_i - c_k - a_2\bar{c}_{ki} - a_2 p_i q_k - a_2 Ip_i q_k - g_{i2} - h_k + a_2 p_{ki} + a_2 Ip_i q_k - 1\right] \\ &+ \sum_{ik}\left[L_{ki}^P - L_{ki}^S\right] - \sum_{ik} L_{ki}^P\left[-f_i - c_k - a_2\bar{c}_{ki} - a_2 p_i q_k - a_2 Ip_i q_k\right] + \sum_{ik} L_{ki}^S\left[-f_i - c_k - a_2\bar{c}_{ki} - a_2 p_i q_k - a_2 Ip_i q_k\right] \\ &+ \sum_{ik}\left[-f_i L_{ki}^P - c_k L_{ki}^P\right] - \sum_{ik}\left[g_{i2} L_{ki}^S + h_k L_{ki}^S\right] + a_2 \sum_{ik}\left[L_{ki}^S p_{ki} - L_{ki}^P \bar{c}_{ki} - L_{ki}^P p_i q_k - \left(L_{ki}^P - L_{ki}^S\right) q_k Ip_i\right] + K \} \end{aligned}$$

with K a constant. After considerable cancellation, we finally obtain:

$$Z = \left(\frac{1}{a_2}\right)\{ \sum_i L_i^0\left[ln\, L_i^0 - 1\right] + \sum_{ik} L_{ki}^P + K \} \qquad (16)$$

Now, substituting from (8), we can prove Hotelling’s Lemma as follows:

$$\frac{\partial Z}{\partial p_{ki}} = \left(\frac{1}{a_2}\right)\{a_2 L_{ki}^P\} \qquad (17)$$

where an increase in the output price increases profit. Thus, our probabilistic result maintains this fundamental property of the classical deterministic theory. An analogous result, but with a minus sign, occurs deriving Z with respect to the input prices p_i. Similar results may be demonstrated via the Legèndre transform for the housing developers.

3.4 SUMMARY OF MODELS FOR COMPETITIVE MARKETS.

In this section we presented a family of supply models combining the assumption of a competitive market and a stochastic profit maximizing setting within an entropy framework. The well-known balancing factor technique is applied to evaluate the model parameters that adjust to prices and to constraints. It is worth noting that some of these constraints represent the natural consistency link between supply and demand in the production chain; hence they are endogenous in the complete chained model.

Regarding the functional form of the models, we derived explicit analytical expressions for demand and for equilibrium prices (see Table 1), except for the landowner model where equilibrium prices remain as fixed-point expressions. The unconstrained demand models reproduce the multinomial logit formula, while for the constrained case we obtained, in most cases, the logistic functional form.

These results show that the unconstrained set of models is consistent with the random choice theory as each one represents a multinomial logit model, hence the underlying distribution of profit functions is identical and independent (IID) Gumbel. Moreover, noting that equilibrium prices are given by the known logsum formula, one can replace prices in the demand formula by equilibrium prices to obtain a nested logit model. An additional result is that the parameter associated to prices in the supply models, denoted by a, has a statistical interpretation as the estimate (except for an exogenous and known factor) of the standard deviation of the underlying distribution of profits. This provides a method to compare our model with the deterministic model since the larger the value of the a estimated, the smaller the standard deviation of profit and the closer to a deterministic case.

TABLE 1: SUMMARY OF EQUATIONS FOR COMPETITIVE MARKETS

MODELS	EQUATION.		FORMULA	MODEL
COMPETITIVE UNCONSTRAINED	Q_i	1	$Q_i = \exp\left[a_1\left(p_i - \overline{p}_i - \overline{c}_i\right) + g_1\right]$	MNL
	L_{ki}	3	$L_{ki} = \exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i q_k - \overline{c}_{ki}\right)\right]$	MNL
	X_{vki}	5	$X_{vki} = \exp\left[a_3\left(r_{vki} - p_{ki} - \overline{c}_{vki}\right) + g_{i3} + h_{vk}\right]$	MNL
	p_i^e	4	$p_i^e = \overline{p}_i + \overline{c}_i + \frac{1}{a_1}\left\{-g_1 + \ln \sum_k q_k \exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i^e q_k - \overline{c}_{ki}\right)\right]\right\}$	Fixed point
	p_{ki}^e	6	$p_{ki}^e = \frac{1}{a_2 + a_3}\left\{a_2\left(p_i q_k + \overline{c}_{ki}\right) - g_{i2} - h_k + \ln \sum_v \exp\left[a_3\left(r_{vki} - \overline{c}_{vki}\right) + g_{i3} + h_{vk}\right]\right\}$	Logsum
COMPETITIVE CONSTRAINED (assumption I=0)	Q_i^S	7	$Q_i^S = Q_i^o \frac{\exp\left[g_1 + a_1\left(p_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]}{\left\{1 + \exp\left[g_1 + a_1\left(p_i + \overline{p}_i \mathrm{I} - \overline{c}_i\right)\right]\right\}}$	Logistic
	L_{ki}^P	8	$L_{ki}^P = \exp\left[f_i + c_k + a_2\left(-p_i q_k - \overline{c}_{ki}\right)\right]\left[1 + \exp\left(g_{i2} + h_k + a_2 p_{ki}\right)\right]$	
	L_{ki}^S	9	$L_{ki}^S = \exp\left[f_i + c_k + a_2\left(-p_i q_k - \overline{c}_{ki}\right)\right]\exp\left[g_{i2} + h_k + a_2\left(p_{ki}\right)\right]$	MNL
	X_{vki}^P	11	$X_{vki}^P = L_{ki}^P \frac{\exp\left(-r_{vki}^P\right) + \exp\left(-r_{vki}^P - r_{vki}^S\right)}{1 + \sum_v \left[\exp\left(-r_{vki}^P\right) + \exp\left(-r_{vki}^P - r_{vki}^S\right)\right]}$	Logistic
	X_{vki}^S	12	$X_{vki}^S = L_{ki}^P \frac{\exp\left(-r_{vki}^P - r_{vki}^S\right)}{1 + \sum_v \left[\exp\left(-r_{vki}^P\right) + \exp\left(-r_{vki}^P - r_{vki}^S\right)\right]}$	Logistic
	p_i^e	10	$Q_i^o \frac{exp\left[g_1 + a_1\left(p_i^e - \overline{c}_i\right)\right]}{1 + exp\left[g_1 + a_1\left(p_i^e - \overline{c}_i\right)\right]} = \sum_k q_k \frac{1 + exp\left[g_2 + h_k + a_2 p_{ki\,k}\right]}{exp\left[-f_i - c_k + a_2\left(p_i^e q_k + \overline{c}_{ki}\right)\right]}$	Fixed Point
	p_{ki}^e	15	$p_{ki}^e = \frac{1}{a_2 - a_3}\left\{-g_{i2} - h_k + ln \sum_v exp\left[-g_{i3} - h_{vk} + a_3 \overline{c}_{vki}\right]\left(1 + exp\left(a_3\left(r_{vki}\right) + l_{vk} + f_{vi} + j_{ki}\right)\right)\right\}$	Logsum

4. IMPERFECT COMPETITION

Land developers may supply lots, and landowners land, anticipating the consumers' willingness-to-pay for their outputs. In this case, their expected behavior is to maximize their oligopolistic profit. In the case of the housing developers, this is implicit in the above model when rents (r_{vki}) represent the expected maximum bid value, which is anticipated by these agents. Analytically, this means that in the calculation of first order conditions, prices should be differentiated as appropriate functions of quantities.

4.1 UNCONSTRAINED MODELS FOR IMPERFECT COMPETITION

Landowners (L/IC/UNCNST)

In contrast to the competitive case, here land prices are assumed as dependent on supply Q_i, then:

$$\frac{\partial F}{\partial Q_i} = -\ln Q_i + a_1\left(-\overline{p}_i - \overline{c}_i\right) + g_1 + a_1 Q_i \frac{\partial\left(p_i^*\right)}{\partial Q_i} + a_1 p_i^*$$

To derive the oligopolistic price p_i^*, we should invert the aggregated demand function L_{ki} in p_i (using equation 3). Take $Q_i = \sum_k q_k L_{ki}$ given by:

$$Q_i = \sum_k q_k \exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i q_k - \overline{c}_{ki}\right)\right]$$

Differentiating implicitly, we obtain:

$$\frac{\partial Q_i}{\partial p_i} = -a_2 \sum_k q_k^2 \exp\left[g_{i2} + h_k + a_2\left(p_{ki} - p_i q_k - \overline{c}_{ki}\right)\right] = -a_2 \sum_k q_k^2 L_{ki}$$

which yields:

$$Q_i \frac{\partial p_i}{\partial Q_i} = -\frac{\sum_k q_k L_{ki}}{a_2 \sum_k q_k^2 L_{ki}}$$

Also, we have from equation (3):

$$p_i^* = \frac{1}{a_2 q_k}\left(g_{i2} + h_k + a_2\left(p_{ki} - \overline{c}_{ki}\right) - \ln L_{ki}\right) \quad \forall k \qquad (18)$$

Then replacing the above expressions in the landowner's profit maximizing equation $\frac{\partial F}{\partial Q_i} = 0$, we obtain a first fixed-point equation for L_{ki}, denoted as $L^1_{ki} = f^1(L^1_{ki})$. Solving L^1_{ki}, we can obtain the profit maximizing Q_i which is the land supply model for imperfect competition.

Land developers (LD/IC/UNCNST)

Similarly, we assume lot prices as dependent on supply, then:

$$\frac{\partial F}{\partial L_{ki}} = -ln(L_{ki}) + g_{i2} + h_k - a_2 p_i q_k - a_2 \bar{c}_{ki} + a_2 L_{ki} \frac{\partial p^*_{ki}}{\partial L_{ki}} + a_2 p^*_{ki}$$

To derive the oligopolistic price p^*_{ki}, we invert the aggregate demand function $\sum_v X_{vki}$ in p_{ki} (using equation 5) and taking $L_{ki} = \sum_v X_{vki}$ we obtain

$$p^*_{ki} = \left(\frac{1}{a_3}\right) ln\left[\frac{1}{L_{ki}} \sum_v exp\left[a_3 (r_{vki} - \bar{c}_{vki}) + g_{i3} + h_{vk}\right]\right] \quad (19)$$

which yields $L_{ki} \frac{\partial p^*_{ki}}{\partial L_{ki}} = -\frac{1}{a_3}$

Then the oligopolistic supply function L_{ki} that complies with the land developer profit maximizing condition $\frac{\partial F}{\partial L_{ki}} = 0$ is

$$L_{ki} = \exp\left[g_{i2} + h_k + a_2\left(p^*_{ki} - p_i q_k - \bar{c}_{ki} - \frac{1}{a_3}\right)\right]$$

and replacing p^*_{ki} we obtain :

$$L^2_{ki} = \left[\exp\left(g_{i2} + h_k + a_2\left(-p_i q_k - \bar{c}_{ki} - \frac{1}{a_3}\right)\right)\right]^{\frac{a_3}{a_2 + a_3}} r_{ki}^{\frac{a_2}{a_2 + a_3}} \quad (20)$$

where $r_{ki} = \sum_v \exp\left[a_3 (r_{vki} - \bar{c}_{vki}) + g_{i3} + h_{vk}\right]$.

It is worth noting that, in order to assure equilibrium, equations (18) to (20) should be solved for p^*_i, p^*_{ki} and L_{ki} simultaneously, and $L_{ki} = L^1_{ki} = L^2_{ki}, \forall k, i$. To reduce the set of equations replace equations (19) into (18) obtaining $p^*_i = p^*_i(L_{ki})$, then replace $L^2_{ki}(p_i)$ from equation (20) into the right-hand side of $L^1_{ki} = f(L^1_{ki})$ yielding a solution fixed point $(L^*_{ki}, p^*_i) = F(L^*_{ki}, p^*_i)$. This complex fixed-point problem is generated by the difficulty of complying simultaneously with the profit maximizing conditions and prices being the inverse of the demand functions. In addition, there is the requirement of consistency in outputs at two levels of the supply chain.

Finally, it is of interest to use the land developers case as a means of comparing the competitive solution (3) with the imperfect competition solution (20). The most obvious difference is the occurrence of the r_{ki} term in (20), which is a non-linear composite of the influences of the different housing types n. On the other hand, in the simple sequential model (3), there is no *direct* influence at the land developer level of housing development rents and costs. In fact the r_{ki} terms resemble 'inclusive values' summed over the options at the next lower level, similar to those associated with nested logit models. This feedback effect is a strength of the imperfect competition models.

<u>Housing Developers (HD/IC/UNCNST)</u>

Additionally, equations (11) and (12) naturally give the supply model for the housing developer under imperfect competition, because for the whole three steps of the supply model, residential demand prices are assumed exogenous.

4.2 Constrained Models For Imperfect Competition

<u>Landowners (L/IC/CNST)</u>

As before, in contrast to the competitive case, here land prices are assumed as dependent on supply Q_i, then:

$$\frac{\partial F}{\partial Q_i^S} = ln\left(Q_i^0 - Q_i^S\right) - ln\,Q_i^S + a_1\left(-\bar{p}_i + I\,\overline{p_i}\right) + g_1 + a_1 Q_i^S\,\frac{\partial\left(p_i^*\right)}{\partial Q_i^S} + a_1 p_i^* \qquad (21)$$

To derive the oligopolistic price p_i^*, we should invert the aggregated demand function L_{ki}^P in p_i^* (using equation 8). Take $Q_i^S = \sum_k q_k L_{ki}^P$ given by:

$$Q_i^S = \sum_k q_k\, exp\left[f_i + c_k + a_2\left(-p_i q_k - \bar{c}_{ki} - Ip_i q_k\right)\right]\left[1 + exp\left(g_{i2} + h_k + a_2\left(p_{ki} + Ip_i q_k\right)\right)\right]$$

Differentiating implicitly, we obtain:

$$\frac{\partial Q_i^S}{\partial p_i^*} = -a_2(1-I)\sum_k q_k^2 \exp\left[f_i + c_k + a_2\left(-p_i q_k - \bar{c}_{ki} - Ip_i q_k\right)\right]\left[1 + \exp\left(g_{i2} + h_k + a_2\left(p_{ki} + Ip_i q_k\right)\right)\right]$$

$$= -a_2(1-I)\sum_k q_k^2 L_{ki}$$

which yields:

$$Q_i^S\,\frac{\partial p_i}{\partial Q_i^S} = -\frac{\sum_k q_k L_{ki}^P}{a_2(1-I)\sum_k q_k^2 L_{ki}^P}$$

Also, we have from equation (8), assuming no holding stock cost (I=0):

$$L_{ki}^{P} = \exp\left[f_i + c_k + a_2\left(-p_i^{*}q_k - \bar{c}_{ki}\right)\right]\left[1+\exp\left(g_{i2} + h_k + a_2(p_{ki})\right)\right]$$

hence
$$p_i^{*} = \frac{1}{a_2 q_k}\left(f_i + c_k - a_2\bar{c}_{ki} - ln\left[\frac{L_{ki}^{P}}{1+exp(g_{i2} + h_k + a_2(p_{ki}))}\right]\right) \quad \forall\, k \tag{22}$$

Then replacing the above expressions in equation (21) for $\frac{\partial F}{\partial Q_i^S} = 0$, we obtain a fixed-point equation for L_{ki}^{P}. With the solution for L_{ki}^{P}, we can obtain the value for Q_i^S which is the land supply model for imperfect competition.

<u>Land Developers (LD/IC/CNST)</u>

First order conditions of the optimization problem LD/C/CNST with variable prices are:

$$\frac{\partial F}{\partial L_{ki}^{S}} = -ln\left(L_{ki}^{S}\right) + ln(L_{ki}^{P} - L_{ki}^{S}) + g_{i2} + h_k + a_2\left(Ip_i q_k\right) + a_2\frac{\partial}{\partial L_{ki}}\left(L_{ki}\, p_{ki}^{*}\right)$$

Analogously to the unconstrained case, here we obtain the aggregate housing developer demand function for land lots $L_{ki}^{S}(p_{ki}) = \sum_{v} X_{vki}^{P}$ from equation (11). This can be seen as non-analytically invertible if we retain the holding cost term (Ip_{ki}). However, this will be quite small for cases of fairly low interest rates and low unused capacity. Thus neglecting this term, the inversion comes directly from (11) as:

$$p_{ki}^{*} = \left(\frac{1}{a_3}\right)\left\{ln\left(\frac{L_{ki}^{P} - L_{ki}^{S}}{L_{ki}^{S} G_{ki}}\right)\right\} \tag{23}$$

with $G_{ki} = \sum_{v}\left[\exp(-g_{i3} - h_{vk} + a_3\bar{c}_{vki})\left(1+\exp(-r_{vki}^{S})\right)\right]$, which does not depend on p_{ki}^{*}. Then, differentiating with respect to L_{ki}^{S} yields:

$$L_{ki}^{S}\left(\frac{\partial p_{ki}^{*}}{\partial L_{ki}^{S}}\right) = \frac{L_{ki}^{P}}{a_3(L_{ki}^{P} - L_{ki}^{S})}$$

and the supply function for land developers (for I=0) is:

$$L_{ki}^{S*} = \left(L_{ki}^{P} - L_{ki}^{S}\right)\exp\left[g_{i2} + h_k + a_2 p_{ki}^{*} - \frac{a_2 L_{ki}^{P}}{a_3(L_{ki}^{P} - L_{ki}^{S})}\right] \tag{24}$$

where $p^*_{ki}(L^P_{ki}, L^S_{ki})$ is given by equation (23). If we substitute for p^*_{ki} and L^P_{ki} this equation becomes a special type of fixed-point problem whose convergence needs also further investigation.

5. *Final Comments*

In this paper we have presented a set of microeconomic supply models applying the entropy framework under the common assumption that producers behave probabilistically . This yields expressions for expected profits. We have shown that the models comply in the limit with micro-economic conditions which are standard for a deterministic model. We have also shown that the known logit multinomial and nested models reproduce the competitive and unconstrained market conditions.

Each model may potentially be enriched with other informational constraints, either to enhance those introduced above or to incorporate planning regulations. This 'natural' way of handling constraints is a virtue of the probabilistic entropy approach.

A more complete analysis of parameter calibration and parameter stability is required, as we have only mentioned methods that have proved to be successfully applied in similar models. However, we foresee that the more complex expressions presented here may require an extension of the usual linear extrapolation and fixed-point methods, such as Newton- Raphson or even the interior point method.

Imperfect competition models generate supply functions that represent fixed-point problems. The question of whether or not the various solutions exist or are unique requires further research. Each model of imperfect competition may be interpreted as if the agents maximize their profit in a stochastic framework, which follows directly from the invariant Legèndre transform, where the transformed expected profits expression substitutes for the optimizing profit objective of the corresponding deterministic model. Based on this interpretation, the supply model has an orthodox economic interpretation.

The supply model developed here contributes to land-use models describing the behavior of the supply side, which can be used in a demand-supply equilibrium process. For example, it can be directly used as the supply function of the bid-choice model applied to Santiago, called MUSSA (Martínez and Donoso, 1995). Under this consistent economic model, the land-use market equilibrium can be properly characterized. Nevertheless, for applications in land use models, this supply model should be extended to include supply based on existent building stock and its regeneration.

This supply model describes the transmission mechanism of the landowner's monopolist power, associated with locational advantages, that is, access and locational externalities, through the chain from land developers to the final consumer. Conversely, it describes how the consumer's valuation of these locational attributes is capitalized by landowners and intermediate developers. Appropriate calibration of parameters defines the proportion of the power captured by each agent. In a wider context, this mechanism applies to the supply process of any goods with a quasi-unique characteristic.

We have developed a set of models under competitive and constrained conditions. Upon their application to real cities, specific market conditions should be assessed. From general analysis and experience it seems plausible to consider imperfect competition and test whether the constrained or unconstrained models perform best. The new models will then represent theoretically well-founded ways for further understanding of the complex links between urban land development and transport investment.

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