Matroids and the Greedy Algorithm

Dpto. Ingeniería Industrial, Universidad de Chile

IN73P, Tópicos de Optimización Discreta

October 8, 2007

Outline

1. Introduction
2. Matroids
3. Matroid Intersection

Kruskal's Algorithm

The Problem (MST):

Given a connected undirected graph $G(V, E)$, and $c \in \mathbb{R}^E$, find a spanning tree $T \subseteq E$ of maximum weight $c(T)$.

The Algorithm:

```
Set $T \leftarrow \emptyset$.
while $\exists e \in E \setminus T : \{e\} \cup T$ is a forest do
    Choose such $e$ with $c_e$ maximum
    $T \leftarrow T \cup \{e\}$.
end while
```

Some cosmetic changes:

- Let's define $\mathcal{I} := \{J : J \subset E, J \text{ is a forest in } G\}$, we will call $\mathcal{I}$the set of *independent sets*.
- Then we can re-write our algorithm as:

The Greedy Algorithm (GA):

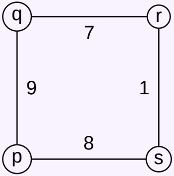
```
Set $T \leftarrow \emptyset$.
while $\exists e \notin T, c_e > 0$ and $T \cup \{e\} \in \mathcal{I}$ do
    Choose such $e$ with $c_e$ maximum
    $T \leftarrow T \cup \{e\}$.
end while
```

Is there anything else?

- We will see that the greedy algorithm solves MST.
- Are there other families $\mathcal{I}$ for which the greedy algorithm solves the asociated problem?

An Example:

- Consider $\mathcal{I}$ the set of all matchings in $G(V, E)$.
- The example shows that it does not solve the problem.



- Families for which GA return the optimal solution are called matroids.
- Can we characterize them?

Basic Definitions

Matroid:

Given a ground set S and $\mathcal{I} \subseteq \mathcal{P}(S)$ (called the set of independet sets), we say that $M = (S, \mathcal{I})$ is a matroid if:

M0 $\emptyset \in \mathcal{I}$.

M1 If $J' \subseteq J \in \mathcal{I}$, then $J' \in \mathcal{I}$.

M2 For every $A \subset S$, every maximal independet set contained in A has the same cardinality.

Understanding the Axioms:

- It's clear that M0 and M1 are necesary for the correctness of the GA.
- To see necesity of M2 note that:
  - Let $c_e \in \{0, 1\}$, $A = \{e \in S : c_e = 1\}$, $\Rightarrow$ $c(J) = |A \cap J|$.
  - The GA will begin with $J = \emptyset$, the GA will finish with a maximal independent set contained in A, but M2 ensure that it will be of maximum weight.

Some more Terminology

- Given $A \subseteq S$, any maximal $J \subseteq A, J \in \mathcal{I}$ is called a basis of A; and define rank of $A \subseteq S$ as $r(A) = \max\{|J| : J \in \mathcal{I}, J \subseteq A\}$.

Some Examples:

The forest of a graph define a matroid:

M0 The empty set is a forest.

M1 If J is a forest, any subset of it is a forest.

M2 Let $A \subseteq S$, and J a basis of A.

- $\Rightarrow J$ is a maximal forest in $G' = (V, A)$.
- $\Rightarrow J$ is a spanning tree in every componen of G'.
- Let $\{V_i\}_{i=1}^{k}$ be each connected component in G'.
- $\Rightarrow |J| = \sum_{i=1}^{k}(|V_i| - 1) = |V| - k$, which is independent of J.

Some Examples:

Linear Matroids:

Let K be a field, and $N \in K^{n \times S}$, for some set S. Let $\mathcal{I} = \{J \subseteq S$: columns indexed by J are linearly independent (l.i.)$\}$.

- M0 By convention the empty set of columns is l.i.
- M1 If $J \in \mathcal{I}$, then any subset of columns of J is l.i.
- M2 Given $A \subseteq S$, for any basis J of A, $F := \langle N_i : i \in A \rangle$ = $\langle N_i : i \in J \rangle$, and all basis of F have the same cardinality (basic linear algebra theorem).

Some Examples:

Uniform Matroids:

Given a set S and $k \in \mathbb{N}$, we define $\mathcal{I} = \{J : J \subseteq S, |J| \leq k\}$.

- M0 Clearly $\emptyset \in \mathcal{I}$.
- M1 If $J \in \mathcal{I}$, and $J' \subseteq J$, then $|J'| \leq |J| \leq k$.
- M2 Given $A \subseteq S$, and J a basis of A, then $|J| = |J \cap A| = \min\{k, |A|\}$, which is independent of J.

Some Examples:

We saw that given $G = (V, S)$, the GA fail to optimize over the set $\mathcal{I} = \{J : J \subseteq S, J \text{ is a matching }\}$. However we still can define a matroid related to Matchings:

A matching-related matroid:

Let $G = (V, E)$ be a graph, $S = V$, and $\mathcal{I} = \{J \subseteq S :$ there is a matching M in G covering all elements in $J\}$.
Proof: M0 and M1 trivially holds. Let $A \subseteq S$ and J_1, J_2 two basis with $|J_1| < |J_2|$; let M_1, M_2 be the related matchings in G. Decompose $G' = (V, M_1 \Delta M_2)$ in cycles and paths. $|J_1| < |J_2|$ implies that $\exists$ a path starting in $v \in J_2 \setminus J_1$ and ending in $w \notin J_1$. By swapping edges in this path, we obtain a matching M_1' covering $J_1 \cup \{v\}$, $\Rightarrow\Leftarrow$.

Correctness of the Greedy Algorithm:

The results is due to Rado [1957], and rediscovered by Edmonds [1970]

Theorem

For any matroid $M = (S, \mathcal{I})$*, and any* $c \in \mathbb{R}^S$*, the GA finds a maximum-weight independent set.*

Proof.

By contradiction.

- Let $J = \{e_1, \ldots, e_m\}$ be the solution reported by the GA, where the order is given by the order in which the algorithm choose the elements. We can see that $c_{e_i} \geq c_{e_{i+1}}, i = 1, \ldots, m-1$.

☐

Correctness of the Greedy Algorithm:

Proof.

- Let $J' = \{q_1, \ldots, q_l\}$ be a maximum-weight independent set, where $c_{q_i} \geq c_{q_{i+1}}, i = 1, \ldots, l+1$.
- Let $k = \min\{i : c_{e_i} < c_{q_i}\}$ (use $c_{e_{m+1}} = -\infty$).
- Then the GA did not choose any of $\{q_i\}_{i=1}^{k}$ in step k.
- Whatever GA choose at step k has weight $< c_{q_k}$.
- Then $\forall i = 1, \ldots, k$, $q_i \in \{e_j\}_{j=1}^{k-1}$ or $\{e_j, q_i\}_{j=1}^{k-1} \notin \mathcal{I}$.
- Then $J_{k-1} := \{e_j\}_{j=1}^{k-1}$ is a basis of $Q := J_{k-1} \cup J'_k$.
- But this contradicts M2, since J'_k is also a basis of Q.

☐

Some Consecuences:

- We say that $M = (S, \mathcal{I})$ is an independent system (IS) if it satisfies M0 and M1.
- We could apply the GA to $M = (S, \mathcal{I})$ an IS.
- Let M be an IS that does not satisfies M2, let $A \subseteq S$ violating M2.
- Let J_1, J_2 be two basis of A with $|J_1| < |J_2|$ and set $c(e) = \mathbb{1}_A(e) + \varepsilon \mathbb{1}_{J_1}(e)$, where $\varepsilon < \frac{1}{|J_1|}$.
- Then the GA will return J_1, which is not a maximal weight independent set in M.

Theorem

Let $(S, \mathcal{I})$ be an IS. Then the GA finds an optimal independent set $\forall c \in \mathbb{R}^S \Leftrightarrow (S, \mathcal{I})$ is a matroid.

Complexity of the GA

- We would like to claim that the GA is efficient.
  - How we estimate the work involved in deciding $Q : J \cup \{e\} \in \mathcal{I}$?
  - If Q can be answered in polynomial time (PT), then the GA can be implemented in PT.
  - The GA can be implemented in polynomial time if and only if Q can be answered in PT.
  - Each matroid is given by an oracle that answers Q.
  - We say that a matroid algorithm is PT, if the number of oracle questions is bounded by a polynomial in $|S|$ and all other work is polynomially bounded on $|S|$ and the size of the rest of the input.
  - The GA is PT.

Do we need all this?

- Could we measure the complexity of matroid algorithms in a different way?
- We would need a general way to describe matroids.
- If we describe $\mathcal{I}$ as a list, can be exponential in $|S|$.
- Given $n = |S|$, there is $g(n) = \Theta(e^n)$, such that the number of matroids in $S \geq 2^{g(n)}$ (Welsh [1976]). Then would need exponential encodigs for each matroid!

Matroid Polytopes

Observation

Given a matroid $M = (S\mathcal{I})$ with rank function r, $J \in \mathcal{I}$ and x^o its characteristic vector, then

$$x^o(A) = |J \cap A| \leq r(A)$$

A valid LP bound

Consider the following LP:

$$\begin{array}{lll} (\mathrm{P}) \ \max & cx & \\ s.t. & x(A) \leq r(A) & \forall A \subseteq S \\ & x_e \geq 0 & \forall e \in S \end{array}$$

Another proof of the correctness of GA

Theorem (Edmonds, 1970)

Let $M = (S, \mathcal{I})$ be a matroid with rank function r, $c \in \mathbb{R}^S$, and x^o be the characteristic vector of J found by the GA. Then x^o is an optimal solution to (P).

Proof.

Note that the dual of (P) is

$$
\begin{array}{lll}
\text{(D)} \ \min & \sum(r(A)y_A : A \subseteq S) & \\
s.t. & \sum(y_A : e \in A \subseteq S) \geq c_e & \forall e \in S \\
 & y_A \geq 0 \quad \forall A \subseteq S &
\end{array}
$$

Another proof of the correctness of GA

(continued).

The complementary slackness conditions are:

C1 $x_e > 0 \Rightarrow \sum(y_A : e \in A \subseteq S) = c_e, \forall e \in S$

C2 $y_A > 0 \Rightarrow x(A) = r(A), \forall A \subseteq S$

We build y as follows:

- Order $\{e_i\}_{i \in S}$ such that $c_{e_i} \geq c_{e_{i+1}}, i = 1, \ldots, n$
- Define $T_i = \{e_j\}_{j=1}^{i}$, and let m such that $c_{e_m} > 0 \geq c_{e_{m+1}}$.
- Let $y_A^o = \begin{cases} c_{e_i} - c_{e_{i+1}} & A = T_i, i = 1, \ldots, m-1 \\ c_{e_m} & A = T_m \\ 0 & \text{otherwise} \end{cases}$

Another proof of the correctness of GA

(continued).

- Note that $\sum(y^o_A : e_j \in A \subseteq S) = 0, \forall j > m$.
- Also $\sum(y^o_A : e_j \in A \subseteq S) = \sum_{i=j}^{m} y^o_{T_i}$
  $= \sum_{i=j}^{m-1} c_{e_i} - c_{e_{i+1}} + c_{e_m} = c_{e_j}, \forall j \leq m$.
- Then y^o is feasible for (D) and satisfy C1.
- Finally, if $y^o_A > 0 \Rightarrow A = T_i$. Then is enough to proof that $x(T_i) = r(T_i)$.
- By contradiction, if not, then $\exists e_k \in T_i \setminus J$ such that $(J \cap T_i) \cup \{e_k\} \in \mathcal{I}$. But e_k was not added to J during the GA, contradiction.

□

Some consecuences:

Theorem (Convex hull of independent set)

Let $M = (S, \mathcal{I})$ *be a matroid with rank function* r. *The convex hull of all independent sets is*

$$\{x \in \mathbb{R}^s : x \geq 0, x(A) \leq r(A), \forall A \subseteq S\}$$

Theorem

Let $M = (S, \mathcal{I})$ *be a matroid, let* $c \in \mathbb{R}^S$, *and let* $J \in \mathcal{I}$. *Then* J *is a maximum-weight independent set w.r.t.* c *if and only if*

- $e \in J$ *implies* $c_e \geq 0$.
- $e \notin J, J \cup \{e\} \in \mathcal{I}$ *implies* $c_e \leq 0$.
- $e \notin J, f \in J, (J \cup \{e\}) \setminus \{f\} \in \mathcal{I}$ *implies* $c_e \leq c_f$.

Properties, Axioms, Constructions

Definition (Circuits):

Given an independent system $(S, \mathcal{I})$, $C \subseteq S$ is a circuit if $\forall e \in C$, $C \setminus \{e\} \in \mathcal{I}$ (i.e. C is a minimal dependent set).

Theorem (Unicity of circuits)

Let $(S.\mathcal{I})$ be a matroid, $J \in \mathcal{I}$, $e \in S$. Then $J \cup \{e\}$ contains at most one circuit.

Proof.

- Assume not, let $C_1, C_2 \subset J \cup \{e\}$ two circuits, and J minimal ($C_1 \cup C_2 = J \cup \{e\}$).
- $\exists a \in C_1 \setminus C_2$, $b \in C_2 \setminus C_1 \Rightarrow$ (by minimality of J) $J' = C_1 \cup C_2 \setminus \{a, b\} \in \mathcal{I}$.
- $\Rightarrow J', J$ basis of $C_1 \cup C_2$ contradiction.

Properties, Axioms, Constructions

Posible characterizations of Matroids:

We can characterize a matroid through its Independet Sets, Rank function, Set of Circuits, or its Basis.

Matroids by Circuits:

A set $\mathcal{C} \subseteq \mathcal{P}(S)$ is the set of circuits of a matroid iff:

- C0 $\emptyset \notin \mathcal{C}$.
- C1 If $C_1, C_2 \in \mathcal{C}$, and $C_1 \subseteq C_2$, then $C_1 = C_2$.
- C2 If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and $e \in C_1 \cup C_2$, then $\exists C \in \mathcal{C}, C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Properties, Axioms, Constructions

Proof.

- Necesity: let $\mathcal{C}$ the circuit family of matroid $(S, \mathcal{I})$.
  - Then C0 and C1 are obvious.
  - If C2 does not hold, then $J = (C_1 \cup C_2) \setminus \{e\} \in \mathcal{I}$, but then J contains two circuits!
- Suficiency: let $\mathcal{I} = \{J \subseteq S : C \nsubseteq J, \forall C \in \mathcal{C}\}$, we will show that $M = (S, \mathcal{I})$ is a matroid:
  - Clearly M0 and M1 holds.
  - If M2 does not hold, let J_1, J_2 be basis of $A \subseteq S$ with $|J_1| < |J_2|$ and with $|J_1 \cap J_2|$ maximal.
  - Note that $\exists e \in J_1 \setminus J_2$ (if not J_1 is not maximal).
  - Now $\exists ! C \in \mathcal{C} : C \subseteq J_2 \cup \{e\}$ (if not, contradict C2).
  - $C \nsubseteq J_1, \Rightarrow \exists f \in C \setminus J_1, \Rightarrow J_3 = (J_2 \cup \{e\}) \setminus \{f\} \in \mathcal{I}$.
  - But $|J_3| > |J_1|$ and $|J_3 \cap J_1| > |J_2 \cap J_1|$, contradiction!

On oracles and non-equivalences:

- We have seen that there are diferent characterization of matroids.
- Could we use different oracles for matroids?
    - We have the independent set oracle ($\mathcal{O}_I$).
    - Consider the cycle oracle ($\mathcal{O}_C$).
    - The problem of decidyng if $S \in \mathcal{I}$ is in $\mathcal{P}$ if we use $\mathcal{O}_I$, but is exponential if we use $\mathcal{O}_C$.
- Conclusions:
    - Not all oracles for matroids are as powerfull.
    - There are stronger oracles than $\mathcal{O}_I$ (for example: given $A \subseteq S$, return largest cycle in A).
    - We choose $\mathcal{O}_I$ because of aplications.

Constructions:

- If $(S, \mathcal{I})$ is a matroid, and $B \subset S$, then $M \setminus B := M' = (S', \mathcal{I}')$, where $S' = S \setminus B$ and $\mathcal{I}' = \{J \in \mathcal{I} : J \subseteq J'\}$, is a matroid.
- If $(S, \mathcal{I})$ is a matroid, and $k \in \mathbb{N}_+$, then $(S, \mathcal{I}')$, where $\mathcal{I}' = \{J \in \mathcal{I} : |J| \leq k\}$, is a matroid.
- If $M_i = (S_i, \mathcal{I}_i)$ $i = 1, 2$ are matroids, $S_1 \cap S_2 = \emptyset$, then $M_1 \oplus M_2 := M' = (S', \mathcal{I}')$, where $S' = S_1 \cup S_2$ and $\mathcal{I}' = \{J_1 \cup J_2 : J_1 \in \mathcal{I}_1, J_2 \in \mathcal{I}_2\}$, is a matroid.
- If $(S, \mathcal{I})$ is a matroid, $B \subseteq S$, J basis of B, then $M/B := M' = (S', \mathcal{I}')$, where $S' = S \setminus B$ and $\mathcal{I}' = \{J' \subseteq S' : J' \cup J \in \mathcal{I}\}$, is a matroid.

Constructions:

An Aplication:

If $M = (S, \mathcal{I})$ is a matroid, $B \subseteq S$, then $M' = M/B \oplus M \setminus \overline{B}$ is a matroid on S, and its bases are the bases of M that intersect B in a basis of B.

Theorem (Nested Bases)

Let $\{T_i\}_{i=o}^{l} + 1 \subseteq \mathcal{P}(S)$ *such that* $T_o = \emptyset$, $T_{l+1} = S$ *and* $T_i \subseteq T_{i+1} : i = 0, \ldots, l$. *The bases of* T_l *in* M *that intersect* T_i *in a basis of* T_i *for* $i = 1, \ldots, l$ *are the bases of* T_l *in* $N = N_o \oplus N_1 \oplus \ldots \oplus N_l$, *where* $N_i = (M/T_i) \setminus \overline{T_{i+1}}$.

The Problem

Given M_1, M_2 two matroids defined on the same set S, we want to find a maximum weight common independent set.

An Example

Maximum weight matching in bipartite graphs:
Let $G = (S_1 \cup S_2, E)$ be a bipartite graph ($E \subseteq S_1 \times S_2$).
Define $M_v = (S_v, \mathcal{I}_v)$ where $S_v = \delta v$ and $\mathcal{I} = \{J \subseteq S_v : |J| = 1\}$. Then M_v is a matroid, and $M_{S_i} = \bigoplus_{v \in S_i} M_v$ is also a matroid. Thus a maximum weight matching in G is the maximum weight of a common independent set in M_{S_i}.

Looking for a min-max relation

- Let $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $A \subseteq S$.
- Then $J \cap A \in \mathcal{I}_1$ and $J \cap \overline{A} \in \mathcal{I}_2$
- Then $|J| = |J \cap A| + |J \cap \overline{A}| \leq r_1(A) + r_2(\overline{A})$.

Theorem (Matroid Intersection Theorem)

For matroids M_1, M_2 on S

$$\overline{r} = \max\{|J| : J \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(A) + r_2(\overline{A}) : A \subseteq S\} = \underline{r}$$

- Note that Köning's theorem follows directly from the matroid intersection theorem.
- We denote $r_{12}(A) = r_1(A) + r_2(\overline{A})$.

A Proof of the matroid intersection theorem:

Proof.

- The $\leq$ part is done. By induction on $|S|$.
  - If $\nexists e \in S : \{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$, then $\bar{r} = 0$, and $\forall e \in S$ $\{e\} \notin \mathcal{I}_1$ or $\{e\} \notin \mathcal{I}_2$
  - Let $A = \{e : r_1(\{e\}) = 0\}$, $\Rightarrow \underline{r} \leq r_{12}(A) = \bar{r}$.
  - Let $k = \underline{r}$, and $\{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. If $\exists A \subseteq S' = S \setminus \{e\}$ such that $k = r_1(A) + r_2(S' \setminus A)$ we are done.
  - If $M_i' = M_i/\{e\}$ and $k - 1 \leq \min\{r_{12}'(B) : B \subseteq S'\}$, then $\exists J' \in M_1' \cap M_2', |J'| \geq k - 1$, then $J' \cup \{e\} \in \mathcal{I}_i$ and we are done.
  - Note that if $\exists A \subseteq S' : r_1(A) + r_2(S' \setminus A) \leq k - 1$ and $B \subseteq S' : r_1(B \cup \{e\}) - 1 + r_2((S' \setminus B) \cup \{e\}) - 1 \leq k - 2$.
  - By subaditivity of r_i $r_{12}(A \cup B \cup \{e\}) + r_{12}(A \cap B) \leq 2k - 1$
  - But then $k = \underline{r} \leq k - 1$ contradiction!

□

The Idea

We will generalize the alternating path algorithm for bipartite matchings:

- We will have $J \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- The algorithm will look for larger J or for A such that $r_{12}(A) = |J|$.
- In the case of a bipartite matching, an augmenting path $e_1, f_1, \ldots, e_m, f_m, e_{m+1}$ satisfies:
    - $e_i \notin J$, $f_i \in J$.
    - $J \cup \{e_1\} \in \mathcal{I}_2$, $J \cup \{e_{m+1}\} \in \mathcal{I}_1$.
    - $(J \cup \{e_i\}) \setminus \{f_i\} \in \mathcal{I}_1$, $(J \cup \{e_{i+1}\}) \setminus \{f_i\} \in \mathcal{I}_2$.

Some previous definitions:

- Let $G = G(M_1, M_2, J)$ be a directed graph where:
    - $V(G) = S \cup \{r, s\}$.
    - $es \in E \forall e \in S \setminus J : J \cup \{e\} \in \mathcal{I}_1$.
    - $re \in E \forall e \in S \setminus J : J \cup \{e\} \in \mathcal{I}_2$.
    - $ef \in E \forall e \in S \setminus J, f \in J : J \cup \{e\} \notin \mathcal{I}_1$, $(J \cup \{e\}) \setminus \{f\} \in \mathcal{I}_1$.
    - $fe \in E \forall e \in S \setminus J, f \in J : J \cup \{e\} \notin \mathcal{I}_2$, $(J \cup \{e\}) \setminus \{f\} \in \mathcal{I}_2$.

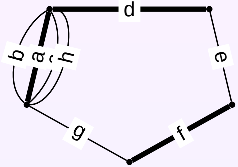
The Augmenting Path Theorem:

Theorem

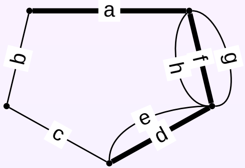
- *If there is no* (r, s)*-dipath in* G*, then* J *is maximum; in fact, if* $A \subseteq S$ *and* $\delta(A \cup \{r\}) = \emptyset$*, then* $|J| = r_{12}(A)$*.*
- *If there exists an* (r, s)*-dipath in* G*, then* J *is not maximum; in fact, if* $r, e_1, f_1, \ldots, e_m, f_m, e_{m+1}, s$ *is the node-sequence of a chordless* (r, s)*-diapth, then* $J \Delta \{e_1, f_1, \ldots, e_m, f_m, e_{m+1}\} \in \mathcal{I}_1 \cap \mathcal{I}_2$*.*

An Example:

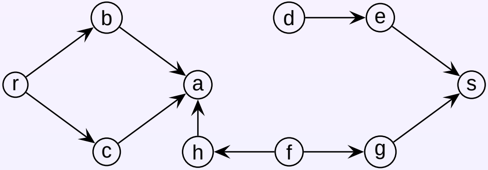
Consider G_1, G_2 two forest matroids, J is the set of black edges, and we present $G(M_1, M_2, J)$, J is maximum.



G_1



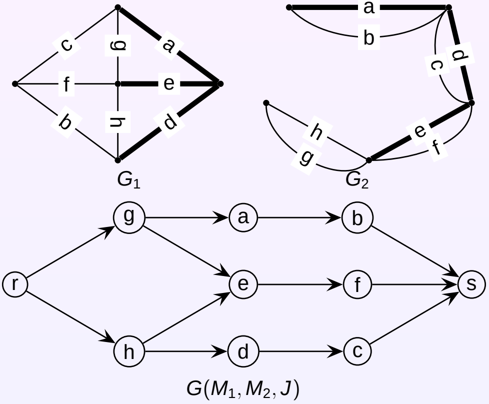
G_2



$G(M_1, M_2, J)$

Another Example

Consider G_1, G_2 two forest matroids, J is the set of black edges, and we present $G(M_1, M_2, J)$, J is not maximum.



Matroid Intersection Algorithm (MIA):

```
Set $J = \emptyset$.
loop
    Construct $G = G(M_1, M_2, J)$.
    if $\exists$ $(r, s)$-dipath $P$ in $G$ then
        Let $r, e_1, f_1, \ldots, e_m, f_m, e_{m+1}, s = P$ a chordless
        $(r, s)$-dipath.
        Let $J \leftarrow J \Delta \{e_1, f_1, \ldots, e_m, f_m, e_{m+1}\}$.
    else
        Let $A = \{e \in S : \exists P,\ (r, e) - \text{dipath in } G\}$.
        stop
    end if
end loop
```

Some Notes on the Algorithm:

- Note that if there is no (r,s)-dipath, A satisfies the condition of $|J| = r_1(J \cap A) + r_2(J \cap \overline{A})$.
- the proof comes from the fact that $J \cap A$ is an M_1-basis for A; also $J \cap \overline{A}$ is an M_2-basis for $\overline{A}$.
- The condition of *chordless* path is essential.
- If there exists a path, there exists a chordless path.
- Note we will need at most $n := |S|$ aughmentations.
- $G(M_1, M_2, J)$ can be constructied in $\mathcal{O}(n^2)$ oracle calls.
- Finding an (r,s)-dipath is polynomial.
- MIA is a polynomial-time matroid algorithm.

Weighted matroid interesction problem:

The Problem:

Given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and $c \in \mathbb{R}^S$, find

$$\begin{array}{lll} (WMIP) & \max & c(J) \\ & s.t. & J \in \mathcal{I}_1 \cap \mathcal{I}_2 \end{array}$$

- We have solved this problem in two special cases:
  - $c_e = 1\ \forall e \in S$.
  - $\mathcal{I}_1 = \mathcal{I}_2$.

Matroid Intersection Polyhedra

- Is clear that the following gives a valid upper bound for WMIP:

$$(MIP) \qquad \begin{aligned} &\max cx \\ &x(A) \leq r_1(A) \ \forall A \subseteq S \\ &x(A) \leq r_2(A) \ \forall A \subseteq S \\ &x_e \geq 0 \ \forall e \in S \end{aligned}$$

Theorem (Matroid Intersection Polytope Theorem)

The convex hull of all common independent sets is the set described by MIP.

Some Notes:

- Note that if P_i is the convex hull of all independent sets in $\mathcal{I}_i$, then MIP=$P_1 \cap P_2$, i.e. the common vertices of P_i are the vertices of the intersection of P_i.
  - This is quite surprising!
  - In general, intersection generate many new vertices.
- How does the polytope look for the case of bipartite matching?

The Weighted Case:

Towards a proof:

- Consider the dual of MIP.

$$\begin{array}{lll} (DMIP) & \min \sum (r_1(A)y_A^1 + r_2(A)y_A^2 : A \subseteq S) \\ s.t. & \sum (y_A^1 + y_A^2 : A \subseteq S, e \in A) \geq c_e \quad \forall e \in S \\ & y_A^1, y_A^2 \geq 0 \quad \forall A \subseteq S \end{array}$$

- Let $(\overline{y}^1, \overline{y}^2)$ an optimal solution to DMIP.
- Let $c_e^1 := \sum (y_A^1 : A \subseteq S, e \in A)$ and $c^2 = c - c^1$.
- Note that $\overline{y}^i$ is an optimal dual solution to P_i with objective c^i (i.e. to $\max c^i x : x(A) \leq r_i(A), \quad A \subseteq S$).
- The converse is also true (i.e. if $\overline{y}_i$ is dual optimal to P_i, then $(\overline{y}_1, \overline{y}_2)$ is optimal for DMIP).

Towards a proof:

- $c^1 + c^2 = c$ is call a weight splitting.
- If exists $J \in \mathcal{I}_i$ such that J is optimal for P_i, then J is optimal for MIP ($c(J) = c^1(J) + c^2(J)$ $\geq c^1(J') + c^2(J') = c(J')$ for all $J' \in \mathcal{I}_1 \cap \mathcal{I}_2$).
- In fact such a weight splitting and J always exists (consecuence of the matroid intersection theorem).
- The proof comes from total dual integrality of MIP
  - Idea of Proof:
  - given $c \in \mathbb{Z}^S$, an optimal solution $(\overline{y}_1, \overline{y}_2)$, define c^i.
  - get optimal dual solution defined by GA.
  - Then prove that constraint matrix restricted to non-zero dual variables is TDI (structure is triangular).
  - from here we conclude that MIP is integral

Matroid Intersection Algorithms:

A Primal Dual Algorithm:

```
$k = 0, J_k = \emptyset$.
**loop**
    Construct $G = G(M_1, M_2, J_k, c)$
    **if** $\exists$ $(r, s)$-dipath in $G$ **then**
        Find least cost $(r, s)$-dipath $P$ of minimal cardinality.
        Aument $J_k$ using $P$ to obtain $J_{k+1}$.
        $k \leftarrow k + 1$.
    **else**
        Choose $J = J_p$ such that $c(J) \geq c(J_i), \forall i = 1, \ldots, k$.
        stop.
    **end if**
**end loop**
```