P1) Resolver $\Delta U = 0$ con U en coord. cilíndricas en el disco de centro 0 y radio a, i.e.

(Ec) $\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0$

(CB) $|U(0,\theta)| < \infty \quad \forall \theta$

$U(r,-\pi) = U(r,\pi)$

$\frac{\partial U}{\partial \theta}(r,\pi) = \frac{\partial U}{\partial \theta}(r,-\pi)$

(CB) $U(a,\theta) = f(\theta) \qquad f \in C^1(-\pi,\pi)$

sol: Como el dominio es acotado, resolvemos por sep. de variables.

$U(r,\theta) = G(r)\,\varphi(\theta)$.

$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial G}{\partial r}\right) + \frac{G(r)}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$

suponiendo sols no triviales, $\varphi \neq 0$, $G \neq 0$, dividiendo por $\frac{G\varphi}{r^2}$

$\frac{r}{G}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial \theta^2} = 0$

$\frac{r}{G(r)}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = -\frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial \theta^2} = \lambda$

Separa casos: $\lambda > 0$, $\lambda < 0$, $\lambda = 0$.

$\lambda < 0$) $U = G\varphi$ con $\varphi = 0$. Se descarta!
$\Rightarrow U = 0$. (visto en clases)

$\boxed{\lambda = 0}$ $\Rightarrow$ ① $\varphi = cte$ $\quad \varphi'' = 0 \Rightarrow \varphi = c_1 + c_2\theta \quad$ y $\varphi(\pi) = \varphi(-\pi)$

$\Rightarrow c_2 = 0$

② $G = \tilde{A}_1 \ln(r) + \tilde{A}_2$

$|G(0)| < \infty \Rightarrow \tilde{A}_1 = 0 \Rightarrow G = cte$

$\Rightarrow U = cte.$

$\boxed{\lambda > 0}$ ① $\varphi'' = -\omega^2 \varphi \Rightarrow \varphi(\theta) = A\cos(\omega\theta) + B\,\mathrm{sen}(\omega\theta)$

$\lambda = \omega^2.$

$\omega > 0$

$$= \left[\tilde{A} e^{i\omega\theta} + B e^{-i\omega\theta}\right]$$

$\Rightarrow \varphi(\pi) = \varphi(-\pi) \Rightarrow A\cos(\omega\pi) + B\,\mathrm{sen}(\omega\pi) = A\cos(\omega\pi) + B\,\mathrm{sen}(-\omega\pi)$

$\Rightarrow 2B\,\mathrm{sen}(\omega\pi) = 0.$

$\Rightarrow B = 0 \quad \vee \quad \omega\pi = k\pi \quad k \in \mathbb{Z}$

$\omega = k$

$\Rightarrow \varphi'(\pi) = \varphi'(-\pi) \Rightarrow -A\omega\,\mathrm{sen}(\omega\pi) + B\omega\cos(\omega\pi) = -A\omega\,\mathrm{sen}(-\omega\pi) + B\omega\cos(-\omega\pi)$

$\Rightarrow 2A\omega\,\mathrm{sen}(\omega\pi) = 0 \qquad \omega \neq 0.$

$A = 0 \quad \vee \quad \omega\pi = k\pi \quad k \in \mathbb{Z}$

$\omega = k$

$$\Rightarrow \begin{cases} \varphi_{1k}(\theta) = \tilde{A}_1 \cos(k\theta) \\ \varphi_{2k}(\theta) = \tilde{A}_2\,\mathrm{sen}(k\theta) \end{cases}$$

Tomando la combinación lineal

$\varphi_k(\theta) = \tilde{A}_{1,k}\cos(k\theta) + \tilde{A}_{2,k}\,\mathrm{sen}(k\theta) \qquad$ cte depende de k

(2) $\dfrac{r}{G}\dfrac{d}{dr}\left(r\dfrac{dG}{dr}\right)=\omega^2 \xrightarrow{\text{por lo anterior}} = k^2$

consideremos $k\neq 0$.

$$r\left(r\frac{d^2G}{dr^2}+\frac{dG}{dr}\right)=k^2G$$

$$r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}-k^2G=0$$

Consideremos soluciones $G(r)=qr^p$ q de. y debemos encontrar p t.q. se satisfaga la ec.

$$r^2p(p-1)r^{(p-2)}+rpr^{p-1}-k^2r^p=0.$$

$$\Rightarrow p(p-1)+p-k^2=0$$

$$\Rightarrow p^2=k^2$$

$$p=\pm k$$

La solución viene dada por

$$\boxed{G_k(r)=C_kr^k+D_kr^{-k}}\ \forall k\geq 1$$

Imponiendo $|G(0)|<\infty$

$\Rightarrow D_k=0.$

La solución queda

$G_k(r)=C_kr^k \quad \forall k\geq 1$ y es válida t.b. para k

$U_k(r,\theta)=G_k(r)\varphi_k(\theta)$ y resuelve la ecuación $\forall k\geq 0.$

$$\Rightarrow \quad U = \sum_{k \geq 0} u_k(r,\theta)$$

$$= \sum_{k \geq 0} \alpha_k r^k \cos(k\theta) + \beta_k r^k \operatorname{sen}(k\theta).$$

Para encontrar α_k, β_k debemos imponer las C. exter

$$U(a,\theta) = f(\theta)$$

$$\Leftrightarrow \sum_{k \geq 0} \alpha_k a^k \cos(k\theta) + \beta_k a^k \operatorname{sen}(k\theta) = f(\theta)$$

Recordamos Series de Fourier

$$g(x) = \sum_{k \geq 0} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \operatorname{sen}\left(\frac{k\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} g(y)\,dy$$

$$a_k = \frac{1}{L} \int_{-L}^{L} \cos\left(\frac{k\pi y}{L}\right) g(y)\,dy$$

$$b_k = \frac{1}{L} \int_{-L}^{L} \operatorname{sen}\left(\frac{k\pi y}{L}\right) g(y)\,dy$$

En nuestro caso $\underline{\underline{L = \pi}}$, $a_k = \alpha_k a^k$, $b_k = \beta_k a^k$, $g = f$

$$\alpha_k a^k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ky) f(y)\,dy$$

$$\alpha_k = \frac{1}{\pi a^k} \int_{-\pi}^{\pi} \cos(ky) f(y)\,dy$$

$$\beta_k = \frac{1}{\pi a^k} \int_{-\pi}^{\pi} \operatorname{sen}(ky) f(y)\,dy$$

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\,dy$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\,dy$$
$$+ \sum_{k \geq 1} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ky)\, f(y)\, dy \right) \left(\frac{r}{a} \right)^{k} \cos(k\theta)$$
$$+ \sum_{k \geq 1} \left(\frac{1}{\pi} \int_{k}^{k} \operatorname{sen}(ky)\, f(y)\, dy \right) \left(\frac{r}{a} \right)^{k} \operatorname{sen}(k\theta)$$