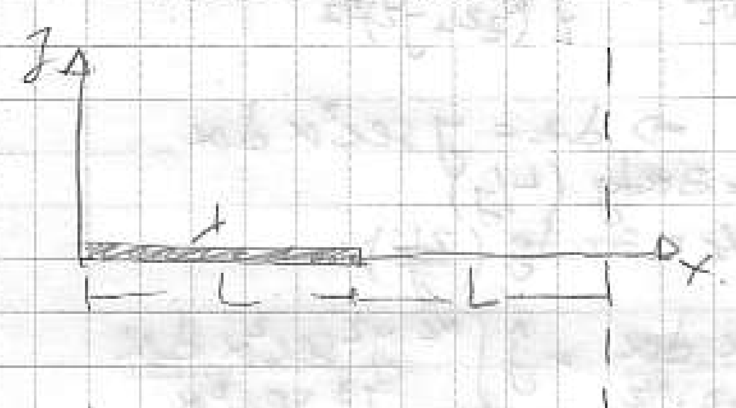
Pauta P1 Control 1 FI2A2-5

a)



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq\,(\vec{r}-\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}$$

En este caso se quiere calcular el campo en el eje "punteado". Luego, $\vec{r}$ vale $\vec{r} = 2L\,\hat{i} + y\hat{j}$.

$$\vec{r}\,' = x\,\hat{i},\quad x \in [0, L]$$
$$dq = \lambda\, dx$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda\, dx\,\left((2L-x)\hat{i} + y\hat{j}\right)}{\left((2L-x)^2 + y^2\right)^{3/2}} \qquad 1.0 \text{ pts}$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\underbrace{\int_0^L \frac{(2L-x)\,dx\,\hat{i}}{\left((2L-x)^2+y^2\right)^{3/2}}}_{I_1} + \underbrace{\int_0^L \frac{y\,dx\,\hat{j}}{\left((2L-x)^2+y^2\right)^{3/2}}}_{I_2}\right]$$

Veamos I_1

$$I_1 = \int_0^L \frac{(2L-x)\,dx\,\hat{i}}{\left((2L-x)^2+y^2\right)^{3/2}}$$ notar que es una derivada exacta, pues

$$\frac{d}{dx}\left(\frac{1}{\sqrt{(2L-x)^2+y^2}}\right) = -\frac{1}{2}\cdot\frac{1}{\left((2L-x)^2+y^2\right)^{3/2}}\cdot 2(2L-x)\cdot -1$$

$$\Rightarrow I_1 = \frac{1}{\sqrt{(2L-x)^2+y^2}}\,\hat{i}\,\Big|_0^L = \underline{\underline{\left[\frac{1}{\sqrt{L^2+y^2}} - \frac{1}{\sqrt{4L^2+y^2}}\right]\hat{i}}}$$

Veamos I_2

$$I_2 = \int_0^L \frac{y\,dx\,\hat{j}}{\left((2L-x)^2+y^2\right)^{3/2}}$$

Sea $z = (2L-x) \Rightarrow dz = -dx$

$x = 0 \Rightarrow z = 2L$

$x = L \Rightarrow z = L$

$$I_2 = \int_{2L}^{L} \frac{y(-dz)\hat{j}}{(z^2+y^2)^{3/2}} = \int_{L}^{2L} \frac{y\,dz\,\hat{j}}{(z^2+y^2)^{3/2}}$$

Sea $z = y\,tg\alpha \Rightarrow dz = y\sec^2\alpha\,d\alpha$

$z = L \Rightarrow \alpha_0 = \operatorname{arctg}(L/y)$

$z = 2L \Rightarrow \alpha_f = \operatorname{arctg}\left(\frac{2L}{y}\right)$

$$I_2 = \hat{j}\int_{\alpha_0}^{\alpha_f} \frac{y \cdot y\sec^2\alpha\,d\alpha}{(y^2 tg^2\alpha + y^2)^{3/2}} = \hat{j}\int_{\alpha_0}^{\alpha_f} \frac{y^2\sec^2\alpha\,d\alpha}{y^3\sec^3\alpha}$$

$$= \frac{\hat{j}}{y}\int_{\alpha_0}^{\alpha_f} \cos\alpha\,d\alpha = \frac{\hat{j}}{y}\,\operatorname{sen}\alpha\Big|_{\alpha_0}^{\alpha_1}$$

pero $\operatorname{sen}\alpha = \dfrac{tg\alpha}{\sqrt{1+tg^2\alpha}}$

$$\Rightarrow I_2 = \frac{1}{y}\left[\frac{2L/y}{\sqrt{1+\frac{4L^2}{y^2}}} - \frac{L/y}{\sqrt{1+\frac{L^2}{y^2}}}\right]\hat{j}$$

$$I_2 = \frac{1}{y}\left[\frac{2L}{\sqrt{y^2+4L^2}} - \frac{L}{\sqrt{y^2+L^2}}\right]\hat{j}$$

$$\Rightarrow \boxed{\vec{E} = \underbrace{\left[\frac{1}{\sqrt{L^2+y^2}} - \frac{1}{\sqrt{4L^2+y^2}}\right]\hat{\imath}}_{1{,}5\text{ pts}} + \underbrace{\frac{1}{y}\left[\frac{2L}{\sqrt{y^2+4L^2}} - \frac{L}{\sqrt{y^2+L^2}}\right]\hat{j}}_{1{,}5\text{ pts}}}$$

b) Hay que calcular el torque respecto al punto $(2L,0)$ donde está el pasador.

$\tau_0 = \int(\vec{r}-\vec{r}_0)\times d\vec{F}$, con $d\vec{F} = dQ\cdot\vec{E} = \lambda dy\cdot\vec{E}$ calculado antes

$\vec{r} = 2L\hat{\imath} + y\hat{j}$ (igual que antes, pero ahora, $y\in[0,L]$); $\vec{r}_0 = 2L\hat{\imath}$

$\Rightarrow \vec{\tau} = \int_0^L y\hat{j}\times\lambda dy\,\vec{E}$; sólo sobreviven los términos del campo que no apuntan en $\hat{\imath}$ ($\hat{j}\times\hat{j} = 0$)

$$\tau = \int_0^L y(\hat{j}\times\hat{\imath})\lambda dy\left[\frac{1}{\sqrt{L^2+y^2}} - \frac{1}{\sqrt{4L^2+y^2}}\right] \quad 1{,}5\text{ pts}$$

$\Rightarrow \hat{j}\times\hat{\imath} = -\hat{k}$

torque según $-\hat{k}$!! 0,5 pts