Capacitated Network Design - Polyhedral Structure and Computation *

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May, 1994 (Revised June, 1995)

Abstract

We study a capacity expansion problem that arises in telecommunication network design. Given a capacitated network and a traffic demand matrix, the objective is to add capacity to the edges, in modularityes of various modularities, and route traffic, so that the overall cost is minimized.

We study the polyhedral structure of a mixed-integer formulation of the problem and develop a cutting-plane algorithm using facet defining inequalities. The algorithm produces an extended formulation providing both a very good lower bound and a starting point for branch and bound. The overall algorithm appears effective when applied to problem instances using real-life data.

1 Introduction and Formulation.

In this paper we study the polyhedral structure of a mixed-integer programming formulation of a capacity expansion problem arising in telecommunications, and present computational results related with a cutting-plane algorithm which uses facet defining inequalities to strengthen the linear programming relaxation.

The generic problem that we are concerned with has been studied by many authors and is still the focus of a lot of work because of its importance. In this problem, we are given a graph and a set of multicommodity flow demands between pairs of nodes. The task is to add capacity to the edges of a graph so that the demands can be (fractionally) routed. The capacity is added in discrete units of

*This research was partially supported by a Presidential Young Investigator Award, NSF grant NCR-930751, and the Center for Telecommunications Research, Columbia University

fixed available modularities. We pay for the added capacity and possibly also for the flows. There are many possible special cases that arise from this general model. For example, the graph may be directed or undirected, there may be side constraints about the routing itself (integer valued or not, required to satisfy "survivability" constraints or not) and the number of capacity modularities may vary across models. This generic problem is strongly NP-hard [6] as it contains the set cover problem (and perhaps more to point, the fixed-charge network design problem, and thus the Steiner tree problem) as a special case. The problem has been known by names such as "network loading problem," ([10], [12]) "minimum cost capacity installation problem," ([4])and others.

We will denote the particular version of the problem that we study by CEP in the sequel, and will define it below. Our primary motivation for studying CEP is that it naturally arises as part of a much larger and complex problem concerning ATM (asynchronous transfer mode) network design that we are separately studying, as part of an ongoing study of ATM at Bellcore. This larger problem is in fact so large, complex and ill-defined that a direct polyhedral study of it would be impractical and probably not advisable. It includes "survivability" requirements, switching and concentration at nodes, and other complex features. However, the ATM problem contains several subproblems either identical or closely related to CEP. These problems have fully dense traffic matrices (i.e. every node wants to talk to every other node). The strategy we are following to solve the ATM problem is to tighten-up mixed-integer formulations involving CEP, and that is our primary concern here. Thus, our computational testing will focus on how effective our inequalities are towards obtaining a strong formulation for CEP (as opposed to developing an algorithm for solving the optimization problems). We will report on the ATM problems in a future paper.

In the model the edges are undirected, but traffic demands, and thus flows, are *directed*. This arises because the amount of traffic to be routed from node s to node t may well be different from the amount to be sent from t to s. Moreover, any fixed edge $\{i, j\}$ of the network, essentially consists of two parallel directed edges (i, j) and (j, i), and the flows on (i, j) and (j, i) do not interfere with each other. The capacity of both directed edges is the same, and thus we require that the total flow on (i, j) (and also the total flow on (j, i)) is at most the capacity of the edge $\{i, j\}$. Briefly, this constraint arises as follows. Traditional telecommunications traffic (other than video) has been bidirectional, and networks have been designed accordingly (that is, each link can carry the same amount of traffic in either direction). More important, even when unidirectional traffic is being handled (such as in our data sets) the network is *still* designed bidirectionally in the event that bidirectional traffic may also have to be carried under a future scenario. In our data sets, the transmission systems are optical and, as a result, bidirectional. While it is certainly possible to design networks with unidirectional transmission systems, that was not the case in the data sets available to us. However, it is worth noting that with purely bidirectional traffic, our model becomes equivalent to the undirected graph model.

In this paper we study CEP when there are two modularity sizes, motivated by real-life data available to us. However, the extension of our inequalities to more than two modularity sizes is straightforward. We will assume that the larger modularity size is an integral multiple of the smaller one (a realistic assumption). By rescaling demands, we may assume that the smaller modularity size is 1. We call the modularity sizes unit-batches and λ-batches, where $\lambda > 1$ is the capacity of the larger modularity size. A more precise definition of CEP is the following. Given a connected undirected graph $G = (V, E)$ with existing capacities $C_e \geq 0$ for all $e \in E$, and point-to-point traffic demand between various pairs of nodes, let $\mathcal{P}^{\mathcal{X}}$ denote the convex hull of feasible solutions to CEP. Then,

$$\mathcal{P}^{\mathcal{X}} = \text{conv}\Big\{ \quad f \in \mathcal{R}^{|K| \times 2 \times |E|} \quad , \quad x,\ y \in \mathcal{Z}^{|E|} \ :$$

$$\sum_{\{i,j\} \in E} f_{ji}^k - \sum_{\{i,j\} \in E} f_{ij}^k \;=\; d_i^k \qquad i \in V,\ k \in K \ \ i \neq k \tag{1}$$

$$\sum_{k \in K} f_{ij}^k \;\leq\; C_{i,j} + x_{i,j} + \lambda y_{i,j} \quad \{i, j\} \in E \tag{2}$$

$$\sum_{k \in K} f_{ji}^k \;\leq\; C_{i,j} + x_{i,j} + \lambda y_{i,j} \quad \{i, j\} \in E \tag{3}$$

$$x_{i,j},\ y_{i,j},\ f_{ij}^k \ \ \geq \ \ 0 \quad \Big\}$$

where K denotes the set of commodities related with the traffic demands, d_i^k is the net demand of commodity k at i and f, x and y are the variable vectors related with flow, unit-batches and λ-batches, respectively. In this formulation equation (1) is a flow conservation equation, and equations (2) and (3) indicate that total flow on directed edge (i,j) or (j,i) can not exceed total capacity of the related edge $\{i,j\}$.

We note that the dimension of $\mathcal{P}^{\mathcal{X}}$ is equal to the number of variables minus the rank of the formulation, that is, there are no additional implied equations. Although we do not prove it explicitly, this result is implied by some of the polyhedral results presented in the following sections.

Throughout the paper, we will use $x_{i,j}$ and $x_{j,i}$ interchangeably to denote the same variable x_e when $e = \{i,j\}$ and we will do the same for variables y and existing capacities C as well.

In the literature on multicommodity network flow problems, there are two main approaches related with the definition of the commodities. The first approach is to define a separate commodity for every non-zero point-to-point demand, resulting in $O(|V|^2)$ commodities in general. The second approach is to aggregate the demands with respect to their source (or destination) nodes and define a commodity for each node with positive supply (or demand). The aggregated formulation has $O(|V|)$ commodities.

For some problems similar to CEP (fixed charge network flow problem, for example) the "fine grain" disaggregated formulation results in a stronger LP-relaxation. The number of variables in this formulation is $O(|E||V|^2)$ as opposed to $O(|E||V|)$ of the aggregated formulation and, as noted in [3], when developing a cutting-plane algorithm, it can be prohibitively expensive to use the disaggregated formulation. Although it is possible to project the disaggregated formulation on the space of the aggregated formulation by using a family of inequalities, called "dicut collection inequalities" [17], the related separation problem appears to be very difficult. In this paper, we adopt the second approach (aggregated version) and define a commodity for each supply node. We also note that for CEP, the LP-relaxations for both of the formulations have the same value. If i and j are nodes, we denote by t_{ij} the amount of demand that must be routed from i to j.

The polyhedral structure of CEP (or, rather, some closely related variants) has already been previously studied. Magnanti and Mirchandani [9] have studied a special case of CEP in which there is a single commodity to be routed between two special nodes of the network and there is no existing capacity on the network. In this paper, they present some facet defining inequalities and show that this special case of CEP is closely related with the shortest path problem. We will describe the results in [9] more completely later in this paper. Another special case, which arises is the context of the lot-sizing problem with constant production capacities, has been studied by Pochet and Wolsey [16]. In this case, the network related with CEP has a special structure and there is a single modularity size. In [16], Pochet and Wolsey fully describe the convex hull of a related polyhedron by using a polynomial number of facets.

Some subproblems related with CEP have also attracted attention. Magnanti, Mirchandani and Vachani [10] study the polyhedral structure of a MIP formulation of the network loading problem (NLP) with three nodes and a single modularity size. In [10], Magnanti et al. present a complete characterization of the projection of the related polyhedron on the space of discrete variables.

In [11] these and other results are applied to extensive computational tests on the two-facility (two modularities) network loading problem on undirected graphs with bidirectional traffic and integer-valued demands. They present results with a cutting plane algorithm as well as a Lagrangean-based approach, both using the disaggregated multcommodity flow formulation. The inequalities used therein are of three types: cutset inequalities, 3-partition inequalities and a third kind, "arc residual capacity" inequalities which strengthen the capacity inequality on a single arc. Even though our model is different from that in [11] one of our inequalities is closely related to their cutset inequality, and another is somewhat related to their 3-partition inequalities. In this paper we present several facet-defining inequalities that extend these two. Our computational approach is also quite different from that in [11], in particular the separation routines.

In [15], Pochet and Wolsey study how to strengthen inequalities of the form $\sum C_j x_j \geq b$ and $\sum C_j x_j \geq y$, for $y \in R_+$ and $x_j \in Z_+^n$, essentially using the MIR (mixed-integer rounding) procedure. Inequalities of this form arise in our problem and we use some of their techniques.

Recently, Stoer and Dahl [18] studied a problem similar to ours where the flows are undirected, there are no flow costs but the capacities to be added to edges are of a more general form than those studied here. (We note that our formulation can be used to model undirected flows). One primary feature of their approach is that (in terms of our model) they would split the integral variables into sums of $0-1$ variables. As a result the inequalities they obtain have a rather combinatorial flavor and when the demands are small, this approach may be effective. Another feature of the approach in [18], again in terms of our problem, is that they study the projection of the formulation onto the space of the x and y variables, which is possible since the problem in [18] does not have flow costs. Feasibility is achieved by means of cutting planes that are generated algorithmically. A second class of models considered in [18] can in addition handle side constraints, such as survivability constraints.

Even more recently, a similar projection approach has been implemented by Barahona (see [1]) and Bienstock, Chopra, Günlük and Tsai (see [4]). This approach may well be competitive with the multicommodity flows formulation. Several facet defining inequalities for the projection are described in [12].

Next, we briefly introduce the notation used in this paper. In what follows, the set of all real numbers is denoted by R, and non-negative real numbers by R^+. Similarly Z and Z^+ denote the set of integers and non-negative integers respectively. We use "\" to denote the ordinary set difference function and when it is not ambiguous, we denote $\{i\}$ by i.

For any vector v and a subset S of its indices, we define $v(S) = \sum_{i \in S} v_i$. Similarly, for a set A of directed edges and a set Q of commodities, we define $f^Q(A) = \sum_{k \in Q} \sum_{a \in A} f_a^k$.

We define $(\alpha)^+$ to be $\max\{0, \alpha\}$ and $r(\cdot,\cdot)$ to be

$$r(\alpha,\beta) = \begin{cases} \alpha - \beta(\lceil \alpha/\beta \rceil - 1) & \text{if } \alpha, \beta > 0, \\ 0 & \text{otherwise} \end{cases}$$

so that $\alpha = \beta(\lceil \alpha/\beta \rceil - 1) + r(\alpha,\beta)$ and $\beta \geq r(\alpha,\beta) > 0$ if $\alpha, \beta > 0$. Notice that this differs from $\alpha mod(\beta)$ when α/β is integral. We will abbreviate $r(\alpha)$ for $r(\alpha, 1)$.

Let $\delta(W) = \{e = \{i,j\} \in E : i \in W, j \notin W\}$ for $W \subset V$. Given $W \subset V$, we denote the net traffic of W by $T(W)$ where

$$T(W) = \left(\max \left\{ \sum_{i \in W} \sum_{j \in V \setminus W} t_{ij}, \sum_{i \in V \setminus W} \sum_{j \in W} t_{ij} \right\} - C(\delta(W)) \right)^{+}.$$

For a feasible solution $\bar{p} = (\bar{x}, \bar{y}, \bar{f}) \in \mathcal{P}^{\mathcal{X}}$, edge $\{i,j\} \in E$ is said to be *"saturated "* if total flow on the directed edge (i,j) or (j,i) is equal to the total capacity of $\{i,j\}$, in other words if $\max\{\bar{f}^K_{ij}, \bar{f}^K_{ji}\} = \bar{x}_{i,j} + \lambda \bar{y}_{i,j} + C_{i,j}$.

2 Cut-set Facets.

There are several inequalities that make use of the fact that the capacity across a cut is at least as large as the demand across the cut. We start with a generalization of the "cut-set" inequalities studied in [9] for the single commodity problem. See [4], [1] for a similar inequality in the directed model. The inequalities are superficially similar, but from a combinatorial viewpoint behave fairly differently to reflect the different nuances of the models. From a purely technical perspective the different models lead to very different analyses and proofs.

Given a set $S \subset V$, we note that $T(S)$ gives a lower bound on the capacity to be added across the cut separating nodes in S from the rest of the network. When the value of this lower bound (implied by flow-conservation equations and capacity constraints) is fractional, the LP-relaxation can be strengthened by forcing the added capacity across the cut to be at least $\lceil T(S) \rceil$. These valid inequalities do not define facets of the the CEP polytope unless the set S satisfies certain properties. We next state these properties. Given a graph $G = (V,E)$ and a vertex subset S, we denote by $E(S)$ the set of edges with both ends in S.

Definition 2.1 *Given a connected graph $G = (V,E)$, a set S is called a* **"strong subset "** *of V with respect to G if it is a proper subset of V and both $G_S = (S, E(S))$ and $G_{\overline{S}} = (V \setminus S, E(V \setminus S))$ are connected.*

We note that given $S \subset V$, the related cut-set inequality is dominated by other cut-set inequalities whenever G_S or $G_{\overline{S}}$ is disconnected.

Theorem 2.2 *Given a strong subset S of V*

$$x(\delta(S)) + \lambda y(\delta(S)) \geq \lceil T(S) \rceil \tag{4}$$

defines a facet of $\mathcal{P}^{\mathcal{X}}$ provided $\lceil T(S) \rceil > T(S)$ and $\lceil T(S) \rceil \geq \lambda$.

Proof. Validity of (4) is obvious. To simplify notation, let $E' = \delta(S)$ and $\bar{T} = \lceil T(S) \rceil$. By construction we will show that the related face $F = \{(x, y, f) \in \mathcal{P}^{\mathcal{X}} : x(E') + \lambda y(E') = \bar{T}\}$ is not empty and then by contradiction, we will show that it is a facet.

For a fixed $e_0 \in E'$ consider $\bar{p} = (\bar{x}, \bar{y}, \bar{f})$ where

$$\bar{x}_e = \begin{cases} M & e \notin E' \\ \bar{T} & e = e_0 \\ 0 & \text{otherwise} \end{cases} \qquad \bar{y}_e = \begin{cases} M & e \notin E' \\ 0 & \text{otherwise} \end{cases}$$

(M is a large enough number) and $\bar{f}$ is such that all traffic between nodes in S ($V \setminus S$) is sent using $E(S)$ ($E(V \setminus S)$) edges and traffic crossing the cut is sent using edges with positive existing capacity and the remaining through e_0. Since both G_S and $G_{\overline{S}}$ are connected and $x(E') > T(S)$, $\bar{f}$ is feasible and thus $\bar{p} \in F$.

Notice that the edges in $E \setminus E'$ are not saturated. Therefore, without saturating them, it is possible to increase flow by a small amount for all commodities. We can do the same for e_0 as well, so without loss of generality we will assume that $\bar{f}^k_{ij}, \bar{f}^k_{ji} > 0$ for all $k \in K$ for edges with positive $\bar{x}_{i,j}$. Suppose there is an equation of the form

$$\alpha x + \beta y + \gamma f = \pi \tag{5}$$

satisfied by all points $p = (x, y, f) \in F$, where α, β and γ are vectors of appropriate dimension and π is a real number. We will show that (5) is a linear combination of (4) and flow conservation equations.

For all $e \notin E'$, it is possible to modify $\bar{p}$ by keeping $\bar{f}$ same and increasing $\bar{x}_e$ or $\bar{y}_e$ to obtain another point in F, which implies that $\alpha_e = \beta_e = 0$. We can also decrease $\bar{x}_{e_0}$ by λ and increase $\bar{y}_{e_0}$ by 1 to get a new point in F. Therefore $\alpha_{e_0} = (1/\lambda)\beta_{e_0}$ and since $e_0 \in E'$ is arbitrary, $\alpha_e = (1/\lambda)\beta_e$ for all $e \in E'$.

For any $k \in K$, it is possible to obtain new points in F by modifying $\bar{p}$ by simultaneously increasing $\bar{f}^k_{i,j}$ and $\bar{f}^k_{j,i}$ by a small amount for edges $\{i, j\}$ with positive

$\bar{x}_{i,j}$. Since e_0 is arbitrary, we can conclude that $\gamma^k_{ij} = -\gamma^k_{ji}$ for all $\{i,j\} \in E$ and $k \in K$.

To show that $\gamma = 0$, we will first choose a spanning tree $T = (V, E'')$ of G using edges in $E \setminus E'$ and edge e_0 and then arbitrarily direct its edges to obtain the directed tree $T' = (V, A)$. If necessary by subtracting a linear combination of the flow-balance equalities (1) of $\mathcal{P}^{\mathcal{X}}$ from (5) we can assume that $\gamma^k_a = 0$ for all $k \in K$ and $a \in A$. Since $\gamma^k_{ij} = -\gamma^k_{ji}$ for any $\{i,j\} \in E$, this implies that $\gamma^k_{ij} = 0$ for $\{i,j\} \in E''$ and $k \in K$.

For $\{i,j\} \in (E \setminus E') \setminus E''$ we can find the unique cycle formed by $\{i,j\}$ and the edges in E''. Notice that e_0 will not appear on this cycle since it is the only edge crossing the cut. Since flows on the tree edges are positive in both directions for all commodities, we can send small circulation flows of each commodity on this cycle and conclude that $\gamma^k_{ij} = 0$ for $\{i,j\} \in (E \setminus E') \cup e_0$ and $k \in K$.

If $|E'| = 1$, then the proof is complete. On the other hand if $|E'| \geq 2$, then we choose an edge $\{u,v\} = e_1 \in E'$ different from e_0. Next, we modify $\bar{p}$ by increasing $\bar{x}_{e_1}$ by 1 and decreasing $\bar{x}_{e_0}$ by 1 and rerouting flow so that neither e_0 or e_1 is saturated and flows on both e_0 and e_1 are positive for all commodities. Obviously this new point is on the face. Now we find the unique cycle formed by e_1 and the edges in E'' and send circulation flows to argue that $\gamma^k_{uv} = 0$ for all $k \in K$. Since e_1 is arbitrary, we can conclude that $\gamma = 0$.

Lastly, modifying $\bar{p}$ as above also implies that if $|E'| > 1$, then there is a number $\bar{\alpha} \in R$ such that $\alpha_e = \bar{\alpha} = (1/\lambda)\beta_e$ for all $e \in E'$. Therefore, (5) is a multiple of (4) (plus a linear combination of flow-balance equations). ■

Usually, inequalities of the form (4) are accompanied by other valid inequalities, obtained by means of the mixed-integer rounding, or MIR, procedure, see [13], that exploit the following fact: If no capacity is added across a cut using unit-batches, then enough capacity should be added using an integer number of λ-batches.

Example 2.3 *Consider the instance of CEP with* $V = \{1,2\}$ *and* $E = \{1,2\}$. *Let* $\lambda = 4$, $t_{12} = 7.2$, $t_{21} = 5.7$ *and* $C_{1,2} = 0.8$. *The cut-set inequality for this case is:*

$$x_{1,2} + 4y_{1,2} \geq 7 \tag{6}$$

since $\lceil \max\{7.2, 5.7\} - 0.8 \rceil = 7$. *Now assume that the flow costs are zero, the cost of a unit-batch is* $C_1 = 1$ *and the cost of a* λ*-batch is* $C_\lambda = 3$ *(so that* $C_1 > C_\lambda/\lambda$*). After including (6) to the LP-relaxation of the problem, the optimal solution has* $x_{1,2} = 0$ *and* $y_{1,2} = 7/4$*, not an integral solution. Notice that if* $y_{1,2} < 2$ *then* $y_{1,2} \leq 1$*, implying* $x_{1,2} \geq 3$*, and thus,*

$$x_{1,2} \geq 3(2 - y_{1,2})$$

is a valid inequality which cuts off the above fractional solution from the set of feasible solutions.

We next generalize this idea and introduce a new family of cut-set facets. A similar inequality, for the undirected graph model with bidirectional integer valued demands, is used in [11].

Theorem 2.4 *Given a strong subset* S *of* V *such that* $\lambda > r(\lceil T(S) \rceil, \lambda) > 0$*, then*

$$x(\delta(S)) + r(\lceil T(S) \rceil, \lambda) y(\delta(S)) \geq r(\lceil T(S) \rceil, \lambda) \lceil T(S)/\lambda \rceil \tag{7}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ *provided* $\lceil T(S) \rceil > 1$ *or* $C(\delta(S)) > 0$ *or* $|\delta(S)| = 1$*.*

Proof. To simplify notation, let $E' = \delta(S)$, $T^+ = \lceil T(S)/\lambda \rceil$ and $r^+ = r(T^+, \lambda)$. We will first rewrite (7) as

$$x(E') \geq r^+(T^+ - y(E')).$$

For any $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$, if $y(E') \geq T^+$ then it is easy to see that (7) is valid. On the other hand if $y(E') \leq T^+ - 1$ then (2) and (3) imply that

$$\begin{aligned}
x(E') &\geq \lceil T(S) \rceil - \lambda y(E') \\
&= \lambda \left\lfloor \frac{\lceil T(S) \rceil}{\lambda} \right\rfloor + r^+ - \lambda y(E') \\
&= r^+ + \lambda \left(\left\lfloor \frac{\lceil T(S) \rceil}{\lambda} \right\rfloor - y(E') \right) \\
&\geq r^+ \left(T^+ - y(E') \right).
\end{aligned}$$

For the rest of the proof, refer to [14]. ∎

We note that given a strong subset S of V, if $\lceil T(S) \rceil = 1$, $C(\delta(S)) = 0$ and $|\delta(S)| > 1$, then all of the points on the face defined by (7) satisfy the family of equations,

$$f_{ij}^k - f_{ji}^k = (x_{i,j} + y_{i,j}) \sum_{v \in V \setminus S} t_{kv}$$

for $i, k \in S$, $\{i, j\} \in \delta(S)$, and thus (7) is not facet defining. In the next section we introduce some facets of the CEP polytope that include the flow variables as well as the capacity variables, and these facets can be considered as generalizations of cut-set facets.

The model studied in [9] differs from ours primarily in that there is a single commodity (i.e. a single origin-destination node pair for which there is positive demand) and there are *three* types of capacity modularities that one can add to any edge. In [9] it is stated that the above cut-set inequalities are facet-defining, as well as a third type of cut-set inequality, which arises by applying the MIR procedure one additional time (to handle the third type of capacity variable). It is shown therein that if there are no flow costs, then under reasonable assumptions on the cost coefficients the linear program containing all cut-set inequalities has some optimal solution that is integral; and they present an efficient algorithm for computing that solution which uses the optimal dual variables.

We note that for the multicommodity case, the cut-set inequalities typically reduce the integrality gap to 30% from a much larger initial value and they are also helpful in terms of pinpointing "interesting" subset of vertices. Below we consider a large class of inequalities which include the cut-set facets as a special case.

3 Flow-cut-set Facets.

In this section we generalize the cut-set facets to include the flow variables as well. Consider a subset S of V and the cut-set facets (4) and (7) related with it. After including these facets in the LP-relaxation of CEP, there exists feasible points to the extended formulation which assign an integer amount of total capacity across the cut $\delta(S)$ but allocate this capacity fractionally among the edges in the cut. The flow-cut-set facets exclude some of these points from the feasible region.

Given a subset S of V and a non-empty partition $\{E_1, E_2\}$ of $\delta(S)$. By considering the orientation of these edges away from S, from E_i we obtain a set of arcs A_i (i.e. $A_i = \{(u,v) : u \in S, v \notin S, \{u,v\} \in E_i\}$), and similarly orienting these edges towards S we obtain a set $\bar{A}_i$.

Consider a simple instance of CEP where there is a single commodity to be routed from S to $\bar{S}$. Furthermore, assume that the cost of routing flow through A_1 is much larger than that of A_2 but cost of adding capacity on E_1 is smaller. In this case, solutions to the LP-relaxation will send all the flow using A_2, assign just enough (fractional) capacity to the E_2 edges to handle the routing, and possibly a small amount of capacity to E_1 to satisfy any cut-set facets we may have added. When combined with cut-set facets, the flow-cut-set facets force the capacity added to E_2 to be integral. These facets have the following common structure,

$$bx(E_2) \;+\; cy(E_2) \;+\; f^Q(A_1) \;\geq\; d \tag{8}$$

where Q is a subset of S and $b, c, d \in R$. (See [11] for an inequality with this general structure that strengthens the capacity inequality on one given edge).

All of the facet defining inequalities presented in this section exploit the following basic idea (see [13]). Consider the polyhedron

$$P = \text{conv}\,\{x \in Z^+, f \in R^+ : f + ax \geq b\}$$

when $a > r(b,a) > 0$ (i.e. $a, b > 0$ and b is not an integer multiple of a), and let CP denote its continuous relaxation. As described in [13], it is easy to observe that all of the points in $CP \setminus P$ violate the inequality $f \geq r(b,a)(\lceil b/a \rceil - x)$ and consequently P can also be expressed as,

$$P = \;\{x, f \in R^+ : f + ax \geq b,\; f \geq r(b,a)(\lceil b/a \rceil - x)\}\,.$$

Also notice that, for an arbitrary polyhedron, if $f + ax \geq b$ is a valid inequality for $x \in Z^+$ and $f \in R^+$ then

$$f \geq r(b,a)(\lceil b/a \rceil - x) \tag{9}$$

is a valid (MIR) inequality.

Before proceeding any further, we first state the following technical lemma, which will help us keep the facet proofs less lengthy. For a proof of this lemma, see [14].

In Lemma 3.2 we consider a facet of the form (8) and investigate some properties of the equations which are satisfied by all points of this facet.

Definition 3.1 *Given two sets S and Q such that $Q \subseteq S \subseteq V$ we define $t(W, V \setminus S) = \sum_{i \in W} \sum_{j \notin S} t_{ij}$, and we call Q a* **"commodity subset "** *of S if $t(q, V \setminus S) > 0$ for all $q \in Q$.*

Lemma 3.2 *Given a strong subset S of V, a commodity subset Q of S, a nonempty partition $\{E_1, E_2\}$ of $\delta(S)$ and a face*

$$F = \left\{ (x, y, f) \in \mathcal{P}^{\mathcal{X}} : b \sum_{e \in E_2} x_e + c \sum_{e \in E_2} y_e + \sum_{a \in A_1} \sum_{k \in Q} f_a^k = d \right\}$$

of $\mathcal{P}^{\mathcal{X}}$ where $b, c, d \in R$, assume that the equation $\alpha x + \beta y + \gamma f = \pi$ is satisfied by all points in F. Then, without loss of generality,

(i) *If F is proper (i.e. $F \neq \emptyset$), then $\alpha_e = \beta_e = 0$ for all $e \in E \setminus E_2$.*

(ii) *If there exists $\bar{p} = (\bar{x}, \bar{y}, \bar{f}) \in F$ such that $\bar{x}(E_2) + \lambda \bar{y}(E_2) + C(E_2) > \bar{f}^K(A_2)$, then $\gamma_a^k = 0$ for all $k \in K, a \notin A_1$ and $k \notin Q, a \in A_1$.*

(iii) *If $\gamma_a^k = 0$ for $k \in K, a \notin A_1$ and there is a point $\hat{p} = (\hat{x}, \hat{y}, \hat{f}) \in F$ satisfying $\hat{x}(E_2) > 0$, then there exists $\bar{\alpha} \in R$ such that $\alpha_e = \bar{\alpha}$ for all $e \in E_2$, and similarly if $\hat{y}(E_2) > 0$, then there exists $\bar{\beta} \in R$ such that $\beta_e = \bar{\beta}$ for all $e \in E_2$.*

(iv) *If $\gamma_a^k = 0$ for $k \in K, a \notin A_1$ and there is a point $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{f}) \in F$ such that $\tilde{f}^Q(A_1) > 0$, then for all $k \in Q$ there exists $\bar{\gamma}^k \in R$ such that $\gamma_a^k = \bar{\gamma}^k$ for all $a \in A_1$. Furthermore, if $\tilde{f}^Q(A_2) > 0$ as well, then, there exists $\bar{\gamma} \in R$ such that $\bar{\gamma}_a^k = \bar{\gamma}$ for all $k \in K, a \in A_1$*

∎

Given two sets S and Q such that $Q \subseteq S \subseteq V$, it is easy to see that the total flow of commodities in Q leaving S should be sufficient to satisfy the total demand in $V \setminus S$. Let $\{E_1, E_2\}$ be a partition of $\delta(S)$, and remember that A_i denotes the edges in E_i oriented from S to $V \setminus S$. Then, we can write

$$f^Q(A_1) + f^Q(A_2) \geq t(Q, V \setminus S)$$

implying

$$f^Q(A_1) + x(E_2) + \lambda y(E_2) + C(E_2) \geq t(Q, V \backslash S)$$

and

$$f^Q(A_1) + x(E_2) + \lambda y(E_2) \geq t(Q, V \backslash S) - C(E_2). \tag{10}$$

We now write an inequality of the form (9) using the fact that $f^Q(A_1) \in R^+$ and $x(E_2) + \lambda y(E_2) \in Z^+$. For a given subset Q of S, the following theorem develops a lower bound on $f^Q(A_1)$ when $x(E_2) + \lambda y(E_2)$ is less than the minimum integral capacity that can carry the total demand of Q in $V \setminus S$. We also note that (11) of Theorem 3.3 becomes the cut-set inequality (4) when $E_1 = \emptyset$.

Theorem 3.3 *Given a strong subset S of V, a commodity subset Q of S and a nonempty partition $\{E_1, E_2\}$ of $\delta(S)$, let $T' = t(Q, V \backslash S) - C(E_2)$, $r' = r(T')$ and $\bar{T} = \lceil T' \rceil$.*
(i) If $1 > r' > 0$ then

$$f^Q(A_1) \geq r' \left(\bar{T} - x(E_2) - \min\{\lambda, \bar{T}\} y(E_2)\right) \tag{11}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ provided $T' > 1$ or $C(E_2) > 0$.
(ii) If $1 > T' > 0$ and $C(E_2) = 0$ then

$$f^Q(A_1) \geq T' \left(1 - x(E_2) - y(E_2)\right) \tag{12}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ provided $|Q| = 1$.

Proof. See [14]. ■

Example 3.4 *Consider the instance of CEP with $|S| = |\bar{S}| = 1$ $E_1 = e_1$, $E_2 = e_2$ and assume that $T' = 6.8$ and $\lambda = 4$. A possible solution to this instance (that is, with appropriate cost coefficients) has $y(E_2) = 1.7$, $x(E_2) = 0$ and $f(A_1) = 0$ and this fractional solution is cut-off by the flow-cut-set inequality*

$$f(A_1) \geq 0.8 \left(7 - 4y(E_2) - x(E_2)\right) \tag{13}$$

since the right hand side is 0.16. After including (13) in the formulation the new solution has $y(E_2) = 1.75$, $x(E_2) = 0$ and $f(A_1) = 0$.

As this example demonstrates, (11) and (12) are not sufficient to force $y(E_2)$ to be integral when both $x(E_2)$ and $f(A_1)$ are zero. Next we write another inequality of the form (9) which implies that if $f^Q(A_1) = x(E_2) = 0$ then $y(E_2)$ can not be less than the minimum integral capacity that can carry $t(Q, V \backslash S) - C(E_2)$.

Theorem 3.5 *Given a strong subset S of V, a commodity subset Q of S and a nonempty partition $\{E_1, E_2\}$ of $\delta(S)$, let $T' = t(Q, V \backslash S) - C(E_2)$, $T^+ = \lceil T'/\lambda \rceil$ and $\bar{r} = r(T', \lambda)$. Then,*

$$f^Q(A_1) + \min\{1, \bar{r}\} x(E_2) \geq \bar{r}\,(T^+ - y(E_2)) \tag{14}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ provided $T' > 1$ and $\lambda > \bar{r}$.

Proof. To show that (14) is a valid for $\mathcal{P}^{\mathcal{X}}$ we first note that it is implied by non-negativity constraints whenever $y(E_2) \geq T^+$ or $\min\{1, \bar{r}\} x(E_2) \geq \bar{r}\,(T^+ - y(E_2))$. So we will concentrate on the case when $y(E_2) \leq T^+ - 1$ and $\min\{1, \bar{r}\} x(E_2) < \bar{r}\,(T^+ - y(E_2))$, and rewrite the lower bound on the total flow of Q-commodities on A_1 edges,

$$\begin{aligned} f^Q(A_1) &\geq T' - x(E_2) - \lambda y(E_2) \\ &= \lambda\,(T^+ - 1) + \bar{r} - x(E_2) - \lambda y(E_2) \\ &= \lambda\,(T^+ - 1 - y(E_2)) + \bar{r} - x(E_2). \end{aligned} \tag{15}$$

Next we consider two cases. When $\bar{r} > 1$ then using $y(E_2) \leq T^+ - 1$ and $\lambda \geq \bar{r}$, (15) can be modified as

$$\begin{aligned} f^Q(A_1) &\geq \bar{r}(T^+ - 1 - y(E_2)) + \bar{r} - x(E_2) \\ &= \bar{r}\,(T^+ - y(E_2)) - x(E_2). \end{aligned}$$

On the other hand, if $\bar{r} < 1$, then using $\lambda > 1$ and $x(E_2) < (T^+ - y(E_2))$, we can write

$$\begin{aligned} f^Q(A_1) &\geq \lambda\,(T^+ - 1 - y(E_2) - x(E_2)) + \bar{r} \\ &\geq \bar{r}\,(T^+ - y(E_2) - x(E_2))\,. \end{aligned}$$

and conclude that (14) is a valid inequality.

To show that (14) is a facet we will construct several points on the related face. Let $e_0 \in E_2$ and consider $p^1 = (x^1, y^1, f^1) \in F$, where

$$x_e^1 = \begin{cases} M & e \in E \setminus E_2 \\ 0 & \text{otherwise} \end{cases} \qquad y_e^1 = \begin{cases} M & e \in E \setminus E_2 \\ T^+ & e = e_0 \\ 0 & \text{otherwise} \end{cases}$$

and f^1 is a feasible flow vector such that it does not saturate e_0 and $(f^1)^Q(A_1) = 0$. Next we construct $p^2 = (x^1, y^2, f^2) \in F$ where y^2 is same as y^1 except $y_{e_0}^2 = T^+ - 1$ and f^2 is a feasible flow vector saturating all the edges in E_2 and satisfying $(f^2)^Q(A_1) = \bar{r}$. Lastly we construct $p^3 = (x^3, y^2, f^3) \in F$ where x_3 is same as x^1 except $x_{e_0}^2 = 1$, and f^3 saturates all the edges in E_2 and satisfies $(f^3)^Q(A_1) = \bar{r} - \min\{1, \bar{r}\}$.

Assume that (14) is not a facet and let $\alpha x + \beta y + \gamma f = \pi$ be an equation different from (11) satisfied by all points $(x, y, f) \in F$. Notice that if $t(Q, V \backslash S) > \bar{r}$ (i.e. when $C(E_2) > 0$ or $t(Q, V \backslash S) > \lambda$), then $(f^2)^Q(A_2) > 0$ and if $t(Q, V \backslash S) = \bar{r}$, then $\bar{r} > 1$ and $(f^3)^Q(A_2) > 0$. Therefore, applying Lemma 3.2 with p^1, p^2, and p^3 we can show that there exist $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in R$ satisfying;

$$\alpha_e = \begin{cases} \bar{\alpha} & e \in E_2 \\ 0 & \text{otherwise} \end{cases} \qquad \beta_e = \begin{cases} \bar{\beta} & e \in E_2 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_a^k = \begin{cases} \bar{\gamma} & a \in A_1, k \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $p^1, p^2, p^3 \in F$ also imply that, $\bar{\gamma} = \bar{\beta}/\bar{r}$, $\bar{\alpha} = \min\{1, \bar{r}\}\bar{\gamma}$, and $\pi = \bar{\beta}T^+$. ∎

Example 3.4 (continued) *Recall that, after including (13) in the formulation, the solution had $y(E_2) = 1.75$, $x(E_2) = 0$ and $f(A_1) = 0$. As $T' = 6.8$ and $\lambda = 4$, this solution does not satisfy (14) since the right hind side of*

$$f(A_1) + x(E_2) \geq 2.8\,(2 - y(E_2)) \tag{16}$$

is positive. After including (16) in the formulation, the new solution is $y(E_2) = 1.1\bar{6}$, $x(E_2) = 2.\bar{3}$ and $f(A_1) = 0$, still not an integral solution.

The last flow-cut-set facet (17) can be considered as an extension of (9) to three variables, and it states that when $y(E_2)$ is not sufficient to carry all the flow, and $x(E_2)$ is not big enough to carry the remainder, then $f(A_1)$ can not be zero. We also note that (17) of Theorem 3.6 becomes the cut-set inequality (7) when $E_1 = \emptyset$.

Theorem 3.6 *Given a strong subset S of V, a commodity subset Q of S and a nonempty partition $\{E_1, E_2\}$ of $\delta(S)$, let $T' = t(Q, V \backslash S) - C(E_2)$, $r' = r(T')$, $T^+ = \lceil T'/\lambda \rceil$ and $r^+ = r(\lceil T' \rceil, \lambda)$. Then,*

$$f^Q(A_1) \geq r' \left(r^+ \left(T^+ - y(E_2)\right) - x(E_2)\right) \tag{17}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ provided $T' > 1$ and $1 > r'$.

Proof. We first show (17) is valid. This can be shown by applying the MIR procedure twice, but we will present a direct proof. For any $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$, (17) is valid whenever $y(E_2) \geq T^+$ or $x(E_2) \geq r^+ \left(T^+ - y(E_2)\right)$. Now consider the case when $y(E_2) \leq T^+ - 1$ and $x(E_2) \leq r^+ \left(T^+ - y(E_2)\right) - 1$. We know that

$$\begin{aligned} f^Q(A_1) &\geq T' - x(E_2) - \lambda y(E_2) \\ &= \lambda \left(T^+ - 1\right) + \left(r^+ - 1\right) + r' - x(E_2) - \lambda y(E_2) \\ &= \lambda \left(T^+ - 1 - y(E_2)\right) + r^+ - 1 - x(E_2) + r'. \end{aligned}$$

Using $\lambda \geq r^+$ and $1 \geq r'$ we can write

$$\begin{aligned} f^Q(A_1) &\geq r^+ \left(T^+ - 1 - y(E_2)\right) + r^+ - 1 - x(E_2) + r' \\ &= r^+ \left(T^+ - y(E_2)\right) - 1 - x(E_2) + r' \\ &\geq r' \left(r^+ \left(T^+ - y(E_2)\right) - 1 - x(E_2)\right) + r' \\ &= r' \left(r^+ \left(T^+ - y(E_2)\right) - x(E_2)\right). \end{aligned}$$

Therefore, (17) is a valid inequality for $\mathcal{P}^{\mathcal{X}}$. To see that it is a facet, we will construct several points on the related face. Let $e_0 \in E_2$ and consider $p^1 = (x^1, y^1, f^1) \in F$, where

$$x^1_e = \begin{cases} M & e \in E \setminus E_2 \\ r^+ & e = e_0 \\ 0 & \text{otherwise} \end{cases} \qquad y^1_e = \begin{cases} M & e \in E \setminus E_2 \\ T^+ - 1 & e = e_0 \\ 0 & \text{otherwise} \end{cases}$$

and f^1 is a feasible flow vector such that it does not saturate e_0 and satisfies $(f^1)^Q(A_1) = 0$. Next we construct $p^2 = (x^2, y^2, f^1) \in F$ where x^2 and y^2 are same as x^1 and y^1 except $x^2_{e_0} = 0$ and $y^2_{e_0} = T^+$. Lastly we construct $p^3 = (x^3, y^1, f^3) \in F$ where x_3 is same as x^1 except $x^2_{e_0} = r^+ - 1$, and f^3 saturates all the edges in E_2 and satisfies $(f^3)^Q(A_1) = r'$.

Using similar arguments as in the proof of Theorem 3.5, we can use points p^1, p^2 and p^3 to show that (17) is a facet of $\mathcal{P}^{\mathcal{X}}$. ∎

Example 3.4 (continued) *When applied to the given instance, (17) becomes*

$$f(A_1) \geq 0.8\,(3\,(2 - y(E_2)) - x(E_2)) \tag{18}$$

and including (18) in the formulation finally results in the integral solution with $y(E_2) = 1$, $x(E_2) = 2$ *and* $f(A_1) = 0.8$.

4 Three-partition Facets.

When deriving the cut-set or the flow-cut-set facets, the main idea is to find an edge-cut dividing the network into two connected components, and develop lower bounds on the variables related with the edges appearing on this cut. A natural extension of this approach is to consider a multi-cut, partitioning the network into three components, and study the facets related with this multicut. This can be seen as a special case of the "metric" inequalities. Once again, in each model for capacity expansion in networks a "different" three-partition inequality arises (actually, more than one inequality may arise). See [10], [11], [4]. The inequalities are similar in flavor but differ in their right-hand side and tend to work in different ways. In general, the inequalities may be seen as a direct descendant of the so-called "partition" inequalities for the Steiner tree problem. From a technical point of view, the analysis ends up being quite different.

Let $\Delta \subset E$ be such a multicut and $\{S_1, S_2, S_3\}$ be the related partition of the node set. If each S_i is strong, then it is possible to develop a lower bound on the capacity to be added across this multicut as follows. First we add up the cut-set inequalities (4) related with each S_i and then divide both sides of the resulting inequality by two to get the valid (implied) inequality $x(\Delta) + \lambda y(\Delta) \geq (\lceil T(S_1) \rceil + \lceil T(S_2) \rceil + \lceil T(S_3) \rceil)/2$. Notice that if the right hand side is fractional (i.e. $\sum_i \lceil T(S_i) \rceil$ is odd), then it is possible to strengthen the inequality by replacing the right hand side by its ceiling to obtain,

$$x(\Delta) + \lambda y(\Delta) \geq \left\lceil \frac{\lceil T(S_1) \rceil + \lceil T(S_2) \rceil + \lceil T(S_3) \rceil}{2} \right\rceil . \tag{19}$$

Although one would expect the strengthened inequality to be a facet of the CEP polytope (similar inequalities are facet defining for NLP, see [10]), in some cases it does not even define a supporting hyperplane. The following example demonstrates one such case.

Example 4.1 *Consider the single modularity-size version of CEP when* $|V| = 3$*,* G *is the complete graph* K_3 *and there is no existing capacity. Let* $t_{12} = t_{13} = t_{23} = 5$ *and* $t_{21} = t_{31} = t_{32} = 0$*. For this case (19) becomes,* $x_{12} + x_{13} + x_{23} \geq 13$ *as* $T(1) = T(3) = 10$ *and* $T(2) = 5$*.*

Notice that before reaching its destination, each unit of t_{12}, t_{13} *or* t_{23} *has to go through the directed edges* $(1,2), (1,3)$ *or* $(2,3)$ *at least once. This observation implies that*

$$x_{12} + x_{13} + x_{23} \geq t_{12} + t_{13} + t_{23} = 15$$

is a valid inequality, dominating (19).

Next, we study the polyhedral structure of this simplified version of CEP (i.e. when there is a single modularity-size and $G = K_3 = (V_3, E_3)$). We denote the integral polyhedron related with this problem by $\mathcal{P}^{\mathcal{X}3}$and its continuous relaxation by $\mathcal{CP}^{\mathcal{X}3}$. For $\mathcal{CP}^{\mathcal{X}3}$, Lemma 4.2 establishes the necessary conditions on the capacity variables x, under which one can find a feasible flow vector.

Lemma 4.2 *Given* $\bar{x} \in \mathcal{R}^3$*, there exists a flow vector* $\bar{f}$ *such that* $(\bar{x}, \bar{f}) \in \mathcal{CP}^{\mathcal{X}3}$ *if and only if*

$$\begin{array}{llcll} (i) & \bar{x}(i) & \geq & T(i) & \text{for all } i \in V_3, \\ (ii) & \bar{x}(E_3) + C(E_3) & \geq & t_{ij} + t_{ik} + t_{kj} & \text{for all permutations } \pi = (i,j,k) \text{ of } V_3, \text{ and} \\ (iii) & \bar{x}_{i,j} & \geq & 0 & \text{for all } \{i,j\} \in E_3. \end{array}$$

Proof. The necessity of (i) - (iii) is obvious. To show that they are sufficient, we construct a feasible flow vector $\bar{f}$ which satisfies the following two conditions for every ordered pair of nodes (i, j):

- If $\bar{x}_{i,j} + C_{i,j} \geq t_{ij}$ then t_{ij} is sent directly from node i to node j.

- If $\bar{x}_{i,j} + C_{i,j} < t_{ij}$ then $\bar{x}_{i,j} + C_{i,j}$ flow is routed on (i, j) and $t_{ij} - \bar{x}_{i,j} - C_{i,j}$ via k.

It is easy to check that $\bar{f}$ satisfies the flow-balance equalities, and for all $i \neq j$

$$\sum_v \bar{f}_{ij}^v = \min\{t_{ij}, \bar{x}_{i,j} + C_{i,j}\} + (t_{ik} - \bar{x}_{i,k} - C_{i,k})^+ + (t_{kj} - \bar{x}_{k,j} - C_{k,j})^+.$$

To show that $\bar{f}$ also satisfies the capacity constraints, we consider the following two cases.

For any ordered pair (i, j), if $t_{ij} \geq \bar{x}_{i,j} + C_{i,j}$ then both $(t_{ik} - \bar{x}_{i,k} - C_{i,k})^+$ and $(t_{kj} - \bar{x}_{k,j} - C_{k,j})^+$ are zero due to (i) applied to node i and node j, respectively, and thus $\sum_v f_{ij}^v = \bar{x}_{i,j} + C_{i,j}$.

On the other hand, when $t_{ij} < \bar{x}_{i,j} + C_{i,j}$, then the total flow on (i, j) equals $t_{ij} + (t_{ik} - \bar{x}_{i,k} - C_{i,k})^+ + (t_{kj} - \bar{x}_{k,j} - C_{k,j})^+$. When either the second or the third term is zero, this is at most $\bar{x}_{i,j} + C_{i,j}$ by (i) applied to i or j, respectively. When they are both positive, this is also at most $\bar{x}_{i,j} + C_{i,j}$ by (ii). ∎

In other words, Lemma 4.2 states that $\mathcal{CP}^{\mathcal{X}3}$can be projected on the space of x variables by using (i) - (iii). We note that (i) - (ii) of Lemma 4.2 belong to a family of inequalities called *"metric inequalities"* (see [7], for example). It is known that these inequalities are sufficient to project the continuous relaxation of a capacitated multicommodity flow polyhedron on the space of the discrete variables. Metric inequalities play a fundamental role in the theory of multicommodity flows, we refer the reader to [8] for more extensive treatment. In Lemma 4.2 we identify the important metric inequalities for $\mathcal{CP}^{\mathcal{X}3}$. Also notice that if we define

$$\theta = \max_{\pi=(i,j,k)} \{t_{ij} + t_{ik} + t_{kj}\}$$

then (ii) can be replaced by a single inequality $\bar{x}(E_3) + C(E_3) \geq \theta$.

Corollary 4.3 *Given $\bar{x} \in Proj_x(\mathcal{CP}^{\mathcal{X}3})$, if $\bar{x}$ satisfies (i) with strict inequality for all nodes, and if $\bar{x}(E_3) + C(E_3) > \theta$, then it is possible to find a feasible flow vector $\bar{f}$ such that $(\bar{x}, \bar{f}) \in \mathcal{CP}^{\mathcal{X}3}$ and $\bar{f}$ does not saturate edge $e \in E_3$ if $x_e + C_e > 0$.*

Corollary 4.4 *Given an integral vector $\bar{x} \in Proj_x(\mathcal{CP}^{\mathcal{X}3})$, $\bar{x} + C > 0$, if $\bar{x}$ satisfies (i) with strict inequality for all nodes, and if $\bar{x}(E_3) + C(E_3) > \theta$, then it is possible to find a feasible flow vector $\bar{f}$ such that $(\bar{x}, \bar{f}) \in \mathcal{P}^{\mathcal{X}3}$ and $\bar{f}$ does not saturate any edge $e \in E_3$.*

Using Lemma 4.2, we next show that the projection of $\mathcal{P}^{\mathcal{X}3}$on the space of the discrete variables can be obtained by strengthening (i) and (ii). Lemma 4.5 can be considered as a generalization of the result by Magnanti, Mirchandani and Vachani. In [10] Magnanti et al. study a similar three-node network design problem (called NLP) where it is assumed that there is a single modularity size and there is no existing capacity on the edges. Furthermore, the capacity constraints are different from the ones we study here, and consequently they can assume that there are only two source nodes with positive supply nodes.

Lemma 4.5

$$
proj_x(\mathcal{P}^{\mathcal{X}3}) = \Big\{ \quad x \in \mathcal{R}^3 :
$$

$$
x(i) \geq \lceil T(i) \rceil \quad \textit{for all } i \in V_3 \tag{20}
$$

$$
\sum_{i>j} x_{i,j} \geq \max\left\{ \left\lceil \frac{\sum_i \lceil T(i) \rceil}{2} \right\rceil, \lceil \theta - C(E_3) \rceil \right\} \tag{21}
$$

$$
x_{i,j} \geq 0 \tag{22}
$$

$$
\Big\}
$$

Proof. Let Q be the polyhedron defined by (20) - (22) and notice that $proj_x(\mathcal{P}^{\mathcal{X}3}) \subseteq Q \subseteq proj_x(\mathcal{CP}^{\mathcal{X}3})$.

Consider any extreme point $\bar{x}$ of Q. If the inequalities defining $\bar{x}$ include (21) or one of (22), it is easy to see that $\bar{x}$ is integral. The remaining case occurs when $\bar{x}$ is defined by inequalities (20) alone. In this case $\bar{x}_{i,j} = (\lceil T(i) \rceil + \lceil T(j) \rceil - \lceil T(k) \rceil)/2$ implying $\bar{x}_{1,2} + \bar{x}_{1,3} + \bar{x}_{2,3} = (\lceil T(i) \rceil + \lceil T(j) \rceil + \lceil T(k) \rceil)/2$. Since $\bar{x}$ must also satisfy (21), it follows that

$$
\frac{\sum_i \lceil T(i) \rceil}{2} \geq \left\lceil \frac{\sum_i \lceil T(i) \rceil}{2} \right\rceil
$$

implying $\sum_i \lceil T(i) \rceil / 2$ (and thus $\bar{x}$) is integral. ∎

We also note that (20) - (22) provide a non-redundant description of $proj_x(\mathcal{P}^{\mathcal{X}3})$ when

$$\max\left\{\left\lceil\frac{\sum_i \lceil T(i)\rceil}{2}\right\rceil, \lceil\theta - C(E_3)\rceil\right\} > \frac{\sum_i \lceil T(i)\rceil}{2} \tag{23}$$

and (21) is redundant when (23) holds as an equality.

In the remainder of this section, we will work with three-partitions of V and using the obvious relationship between three-partitions and K_3, we describe some facets of $\mathcal{P}^{\mathcal{X}}$ using (21) of Lemma 4.5 and its extensions. Given a partition $\Pi = \{S_1, S_2, S_3\}$ of V, we use $\delta(i,j)$ to denote $\delta(S_i) \cap \delta(S_j)$ and Δ to denote $\delta(1,2) \cup \delta(1,3) \cup \delta(2,3)$. For typographical ease, we use $x(i,j), y(i,j)$ and $C(i,j)$ in place of $x(\delta(i,j)), y(\delta(i,j))$ and $C(\delta(i,j))$ respectively.

Given a three-partition of V, for the generalization of (21) of Lemma 4.5 to define a facet of $\mathcal{P}^{\mathcal{X}}$, the partition has to satisfy certain properties. We next state these properties.

Definition 4.6 *Given a capacitated network $G = (V, E)$ and related traffic demands, a three-partition $\{S_1, S_2, S_3\}$ of V is called a* **"critical partition "** *of V if every S_i is a strong subset of V, $\lceil T(S_i)\rceil > T(S_i)$ for $i = 1, 2, 3$ and*

$$\lceil T(S_i)\rceil < \lceil T(S_j)\rceil + \lceil T(S_k)\rceil$$

for any permutation (i, j, k) of $\{1, 2, 3\}$.

As in Section 3, we first consider a generic three-partition facet and investigate some properties of the equations which are satisfied by all points of this facet.

Lemma 4.7 *Given a critical partition $\Pi = \{S_1, S_2, S_3\}$ of V and a proper face*

$$F = \left\{(x, y, f) \in \mathcal{P}^{\mathcal{X}} : \sum_{j>i} a_{i,j} \sum_{e\in\delta(i,j)} x_e + \sum_{j>i} b_{i,j} \sum_{e\in\delta(i,j)} y_e = c\right\}$$

of $\mathcal{P}^{\mathcal{X}}$, where $a_{i,j}, b_{i,j} \in R$, $j > i$, and $c \in R$, assume that equation $\alpha x + \beta y + \gamma f = \pi$ is satisfied by all points in F.

If there exists $\bar{p} = (\bar{x}, \bar{y}, \bar{f}) \in F$ such that $\bar{x}(\Delta) + \lambda\bar{y}(\Delta) > \max\{\sum_i T(i)/2, \theta - C(\Delta)\}$ and $\bar{x}(i,j) + \bar{y}(i,j) + C(i,j) > 0$ for all $j > i$ then without loss of generality

,

(*i*) $\alpha_e = \beta_e = 0$ *for all* $e \in E \setminus \Delta$,

(*ii*) $\gamma = 0$,

(*iii*) *for any* $j > i$, *if* $\bar{x}(i,j) > 0$ *then there exists* $\bar{\alpha}_{i,j} \in R$ *such that* $\alpha_e = \bar{\alpha}_{i,j}$, *for all* $e \in \delta(i,j)$, *and*

(*iv*) *if* $\bar{y}(i,j) > 0$, *then there exists* $\bar{\beta}_{i,j} \in R$ *such that* $\beta_e = \bar{\beta}_{i,j}$ *for all* $e \in \delta(i,j)$.

Proof.

See [14]. ∎

Given a three-partition $\Pi = \{S_1, S_2, S_3\}$ of V, we denote $\lceil T(S_i) \rceil$ by $\bar{T}(i)$ and $\sum_{u \in S_i} \sum_{v \in S_j} t_{ij}$ by $T(i,j)$.

We use θ for $\max_\pi \{T(i,j) + T(i,k) + T(k,j)\}$ and $\bar{\theta}$ for $\lceil \theta - C(\Delta) \rceil$. Lastly we define Θ to be

$$\Theta = \max \left\{ \left\lceil \frac{\sum_i \bar{T}(i)}{2} \right\rceil, \bar{\theta} \right\}.$$

The following is a straight forward extension of Lemma 4.5 to three-partitions of V.

Theorem 4.8 *Given a critical partition* $\Pi = \{S_1, S_2, S_3\}$ *of* V, *if* $\Theta - \bar{T}(i) \geq \lambda$ *for* $i = 1, 2, 3$, *then,*

$$x(\Delta) + \lambda y(\Delta) \geq \Theta \tag{24}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ *provided* $\Theta > \max\{\sum_i \bar{T}_i/2, \theta - C(E)\}$.

Proof. The validity of (24) is due to Lemma 4.5. For the rest of the proof, refer to [14]. ∎

Next we consider the case when given a critical partition $\{S_1, S_2, S_3\}$ of V, $\Theta - \bar{T}(i) \geq \lambda$ does not hold for all S_i. Let $\bar{T}(3) \geq \bar{T}(2) \geq \bar{T}(1)$. If $\lambda > \Theta - \bar{T}(3)$, then

$$\Theta - \bar{T}(3) \geq \left(\frac{\sum_i \bar{T}(i)}{2} + \frac{1}{2} \right) - \bar{T}(3) \geq \frac{\bar{T}(1) + \bar{T}(2) - \bar{T}(3) + 1}{2}$$

implies that $2\lambda + \bar{T}(3) > \bar{T}(1) + \bar{T}(2) + 1 \geq \bar{T}(3)$. Therefore, this case arises when $2\lambda - 1 > \bar{T}(1) + \bar{T}(2) - \bar{T}(3)$, or, in other words, when the sum $\bar{T}(1) + \bar{T}(2)$ is not very big when compared to $\bar{T}(3)$.

Theorem 4.9 *Given a critical partition $\Pi = \{S_1, S_2, S_3\}$ of V, let $\bar{T}(3) \geq \bar{T}(2) \geq \bar{T}(1)$. If $\Theta - \bar{T}(3) < \lambda$ and $\bar{T}(3) \geq \lambda$, then*

$$x(\Delta) + (\Theta - \bar{T}(3))y(1,2) + \lambda y(1,3) + \lambda y(2,3) \geq \Theta \tag{25}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ provided $\Theta > \sum_i \bar{T}_i/2$.

Proof. For any point $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$ inequality (25) is clearly valid when $y(1,2) = 0$. On the other hand, if $y(1,2) \geq 1$, then notice that

$$\begin{aligned} x(\Delta) + (\Theta - \bar{T}(3))y(1,2) &+ \lambda y(1,3) + \lambda y(2,3) \\ &\geq x(1,3) + x(2,3) + (\Theta - \bar{T}(3)) + \lambda y(1,3) + \lambda y(2,3) \\ &= x(S_3) + \lambda y(S_3) + (\Theta - \bar{T}(3)) \\ &\geq \bar{T}(3) + (\Theta - \bar{T}(3)) = \Theta. \end{aligned}$$

For the rest of the proof, see [14]. ∎

Next we study facets of the CEP polytope which primarily exclude points with $y(\Delta) = \Theta/\lambda$ from the feasible region when Θ/λ is fractional. We basically consider two cases depending on which one of the two terms dominates in determining Θ. But, before proceeding any further, we need some more notation. Given a partition $\Pi = \{S_1, S_2, S_3\}$ of V, we define $r^+(i)$ to denote $r(\bar{T}(i), \lambda)$ and $T^+(i)$ to denote $\lceil \bar{T}(i)/\lambda \rceil$. Notice that $\bar{T}(i) = \lambda\,(T^+(i) - 1) + r^+(i)$ for all $S_i \in \Pi$. We further define $r_{\max} = \max\{r^+(i)\}$, $r_{\min} = \min\{r^+(i)\}$ and $r_{\text{med}} = \sum_i r^+(i) - r_{\min} - r_{\max}$.

Notice that if $\bar{T}(3) \geq \bar{T}(2) \geq \bar{T}(1)$, then $T^+(3) \geq T^+(2) \geq T^+(1)$. Furthermore, when $\bar{T}(3) < \bar{T}(1) + \bar{T}(2)$, $T^+(3)$ is no more than $T^+(2) + T^+(1)$, and

$$\left\lfloor \frac{\sum_i T^+(i)}{2} \right\rfloor - T^+(i) \geq \left\lfloor \frac{\sum_i T^+(i)}{2} \right\rfloor - T^+(3) = \left\lfloor \frac{T^+(1) + T^+(2) - T^+(3)}{2} \right\rfloor \geq 0.$$

The following theorem has the same spirit as Theorem 2.4 and (26) is a MIR inequality. Remember that

$$\Theta = \max\left\{ \left\lceil \frac{\sum_i \bar{T}(i)}{2} \right\rceil, \bar{\theta} \right\}$$

and we note that the required conditions can hold only if the second term strictly dominates the first. In other words, (26) is a facet only if $\Theta > \lceil \sum_i \bar{T}(i)/2 \rceil$.

Theorem 4.10 *Given a critical partition* $\Pi = \{S_1, S_2, S_3\}$ *of* V*, let* $\bar{T}(3) = \max\{\bar{T}(i)\}$ *and* $r^+(1) \geq r^+(2)$*. If* $\lambda > r(\Theta, \lambda)$*, then*

$$x(\Delta) + r(\Theta, \lambda) y(\Delta) \geq r(\Theta, \lambda) \lceil \Theta/\lambda \rceil \tag{26}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ *provided one of the following conditions is true.*

(*i*) $\lceil \Theta/\lambda \rceil - 1 \geq \lceil \sum_i T^+(i)/2 \rceil$.

(*ii*) $\lceil \Theta/\lambda \rceil = \lceil \sum_i T^+(i)/2 \rceil > \sum_i T^+(i)/2$, $r(\Theta, \lambda) \geq r^+(2)$ *and* $\bar{T}(2) > 1$.

(*iii*) $\lceil \Theta/\lambda \rceil = \lceil \sum_i T^+(i)/2 \rceil = \sum_i T^+(i)/2$, $r(\Theta, \lambda) \geq r_{\max}$, $r(\Theta, \lambda) > r_{\min}$ *and* $T^+(1) + T^+(2) > T^+(3)$.

Proof. Validity of (26) should be clear. To show that $F = \{(x, y, f) \in \mathcal{P}^{\mathcal{X}} : x(\Delta) + r(\Theta, \lambda) y(\Delta) = r(\Theta, \lambda) \lceil \Theta/\lambda \rceil\}$ is a facet, we first analyze cases (*i*) and (*ii*).

(*i*), (*ii*) Choose a fixed edge $e_{i,j} \in \delta(i, j)$ for all $j > i$, and consider $p^1 = (x^1, y^1, f^1)$ where

$$y^1_e = \begin{cases} M & e \notin \Delta \\ \lfloor \sum_i T^+(i)/2 \rfloor - T^+(3) & e = e_{1,2} \\ \lceil \sum_i T^+(i)/2 \rceil - T^+(2) & e = e_{1,3} \\ \lceil \Theta/\lambda \rceil - T^+(1) - 1 & e = e_{2,3} \\ 0 & \text{otherwise} \end{cases} \qquad x^1_e = \begin{cases} M & e \notin \Delta \\ 1 & e = e_{1,2} \\ 0 & e = e_{1,3} \\ r(\Theta, \lambda) - 1 & e = e_{2,3} \\ 0 & \text{otherwise} \end{cases}$$

and f^1 is a feasible flow vector. Notice that $\lceil \Theta/\lambda \rceil - T^+(1) - 1 \geq \lceil T^+(2)/2 \rceil - 1 \geq 0$, and therefore $x^1, y^1 \geq 0$. To see that it is possible to find a feasible flow vector f^1, first note that $x^1(\Delta) + \lambda y^1(\Delta) = \Theta$ and p^1 satisfies cut-set inequalities for S_1 and S_3. Next, observe that the capacity across the cut $\delta(S_2)$ is

$$\begin{aligned} x^1(\delta(S_2)) + \lambda y^1(\delta(S_2)) &\geq \lambda \left(\left\lfloor \frac{\sum_i T^+(i)}{2} \right\rfloor + \left\lceil \frac{\Theta}{\lambda} \right\rceil - 1 - T^+(1) - T^+(3) \right) + r(\Theta, \lambda) \\ &\geq \lambda \left(\left\lfloor \frac{\sum_i T^+(i)}{2} \right\rfloor + \left\lceil \frac{\sum_i T^+(i)}{2} \right\rceil - T^+(1) - T^+(3) - 1 \right) + r(\Theta, \lambda) \end{aligned}$$

$$
\begin{aligned}
&\geq \lambda\left(T^{+}(2)-1\right)+r(\Theta,\lambda) \\
&\geq T^{+}(2)
\end{aligned}
$$

(where the last inequality follows from $r(\Theta, \lambda) \geq r^{+}(2)$) so that p^1 satisfies the cut-set inequality for S_2 as well. Therefore, using Lemma 4.2, $p^1 \in \mathcal{P}^{\mathcal{X}}$ and thus $p^1 \in F$.

Assuming F is not a facet, let $\alpha x + \beta y + \gamma f = \pi$ be an equation different from (26) satisfied by all points $p = (x, y, f) \in F$. Observe that $x^1(1,2) > 0$ and $y^1(1,3) = \lceil \sum_i T^{+}(i)/2 \rceil - T^{+}(2) \geq (T^{+}(3) + T^{+}(1) - T^{+}(2))/2 \geq T^{+}(1)/2 > 0$. Lastly, if $y^1(2,3) = 0$ then,

$$
y^1(2,3) = 0 = \left\lceil \frac{\Theta}{\lambda} \right\rceil - T^{+}(1) - 1 \geq \frac{\sum_i T^{+}(i) + 1}{2} - T^{+}(1) - 1 \geq \frac{T^{+}(2) - 1}{2} \geq 0
$$

implies that $\lceil \Theta/\lambda \rceil = \lceil \sum_i T^{+}(i)/2 \rceil$ and $T^{+}(2) = 1$. In this case $x^1(2,3) = r(\Theta,\lambda) - 1 \geq r^{+}(2) - 1 = \bar{T}(2) - 1 > 0$. Therefore, we can conclude that $x^1(2,3) + \lambda y^1(2,3) > 0$.

Using Lemma 4.7, we can now argue that $\gamma = 0$ and $\alpha_e = \beta_e = 0$ for all $e \in E \backslash \Delta$. Moreover, it is possible to perturb p^1 by decreasing $x_{e_{1,3}}$ and increasing $x_{e_{1,2}}$ or $x_{e_{2,3}}$, implying that for some $\bar{\alpha} \in R$, $\alpha_e = \bar{\alpha}$ whenever $e \in \Delta$.

We next construct a point $p^2 = (x^2, y^2, f^2) \in F$ where $x^2 = 0$, y^2 is identical to y^1 with the exception that $y^2_{e_{2,3}} = \lceil \Theta/\lambda \rceil - T^{+}(1)$, and f^2 is some feasible flow vector which exists by Lemma 4.2. Perturbing p^2 by increasing $y^2_{e_{1,2}}$ and decreasing $y^2_{e_{1,3}}$ or $y^2_{e_{2,3}}$, we conclude that there exists $\bar{\beta} \in R$, such that $\beta_e = \bar{\beta}$, for all $e \in \Delta$. Furthermore, $p^1, p^2 \in F$ implies that $\bar{\beta} = r(\Theta,\lambda)\bar{\alpha}$ and thus F is indeed a facet.

(iii) Choose a fixed edge $e_{i,j} \in \delta(i,j)$ for all $j > i$, and consider $p^3 = (x^3, y^3, f^3) \in F$ where

$$
y^3_e = \begin{cases} M & e \notin \Delta \\ \left(T^{+}(1) + T^{+}(2) - T^{+}(3)\right)/2 & e = e_{1,2} \\ \left(T^{+}(1) + T^{+}(3) - T^{+}(2)\right)/2 & e = e_{1,3} \\ \left(T^{+}(2) + T^{+}(3) - T^{+}(1)\right)/2 & e = e_{2,3} \\ 0 & \text{otherwise} \end{cases}
\qquad
x^3_e = \begin{cases} M & e \notin \Delta \\ 0 & e \in \Delta \end{cases}
$$

and f^3 is a feasible flow vector. Since $y^3(i,j) > 0$ for all $j > i$, we can apply Lemma 4.7 with p^3 and show that $\gamma = 0$, $\alpha_e = \beta_e = 0$ for all $e \in E \setminus \Delta$, and for all $j > i$, there exists $\bar{\beta}_{i,j} \in R$ such that $\beta_e = \bar{\beta}_{i,j}$ for all $e \in \delta(i,j)$.

Next for each $e_{i,j}$ we perturb p^3 by decreasing $y_{e_{i,j}}$ by 1 and increasing $x_{e_{i,j}}$ by $r(\Theta, \lambda)$ to obtain new points in F. Using these points together with p^3, we conclude that for all $j > i$, if $e \in \delta(i,j)$ then, $\alpha_e = r(\Theta,\lambda)\bar{\beta}_{i,j}$.

Lastly, let $\{a,b,c\}$ be a permutation of $\{1,2,3\}$ so that $r^+(a) \geq r^+(b) \geq r^+(c)$. Since $r(\Theta,\lambda) > r_{\min} = r^+(c)$, it is possible to permute p^3 by decreasing $y_{e_{b,c}}$ by 1, increasing $x_{e_{b,c}}$ by $r(\Theta,\lambda) - 1$ and increasing $x_{e_{a,b}}$ by 1. Similarly, it is possible to permute p^3 by decreasing $y_{e_{a,c}}$ by 1, increasing $x_{e_{a,c}}$ by $r(\Theta,\lambda) - 1$ and increasing $x_{e_{a,b}}$ by 1. These new points are in F, and thus $\bar{\beta}_{a,b} = \bar{\beta}_{a,c} = \bar{\beta}_{b,c}$, implying that F is a facet of $\mathcal{P}^{\mathcal{X}}$. ∎

We also note that it is possible to relax the condition $\bar{T}(2) > 1$ from (ii) of Theorem 4.10, but in this case $C(2,3)$ has to be positive whenever $\bar{T}(2) = 1$. To avoid complicating the proof any further, we chose to skip this.

In the remainder of this section, we consider the case when for a critical partition $\{S_1, S_2, S_3\}$ of V, Θ is equal to $\lceil \sum_i \bar{T}(i)/2 \rceil$, and we identify facets of $\mathcal{P}^{\mathcal{X}}$ that exclude some of the fractional points from the feasible region when $y(\Delta)$ is less than $\lceil \Theta/\lambda \rceil$. Before that we will make an observation concerning the identity $\bar{T}(i) = \lambda\,(T^+(i) - 1) + r^+(i)$ and the cut-set inequalities. First note that

$$\sum_i \bar{T}(i) = \lambda \left(\sum_i T^+(i) - 3 \right) + \sum_i r^+(i)$$

implying

$$\frac{\sum_i \bar{T}(i)}{2} = \lambda \left(\frac{\sum_i T^+(i)}{2} - \frac{3}{2} \right) + \frac{\sum_i r^+(i)}{2}.$$

Therefore, depending on $\sum_i T^+(i)$, we can write

$$\left\lceil \frac{\sum_i \bar{T}(i)}{2} \right\rceil = \begin{cases} \lambda \left(\left\lceil \frac{\sum_i T^+(i)}{2} \right\rceil - 2 \right) + \left\lceil \frac{\sum_i r^+(i)}{2} \right\rceil & \text{if } \sum_i T^+(i) \text{ is odd} \\ \lambda \left(\left\lceil \frac{\sum_i T^+(i)}{2} \right\rceil - 2 \right) + \left\lceil \frac{\lambda + \sum_i r^+(i)}{2} \right\rceil & \text{if } \sum_i T^+(i) \text{ is even.} \end{cases}$$

Next, note that when $x(\Delta) = 0$, the cut set inequalities imply that $y(\Delta) \geq$

$\lceil \sum_i T^+(i)/2 \rceil$ and when $y(\Delta) = \lceil \sum_i T^+(i)/2 \rceil - 1$, then

$$x(\Delta) \geq \begin{cases} r_{\min} & \text{if } \sum_i T^+(i) \text{ is odd} \\ r_{\text{med}} & \text{if } \sum_i T^+(i) \text{ is even.} \end{cases}$$

This is easy to see as $y(\Delta) = \lceil \sum_i T^+(i)/2 \rceil - 1$ implies that $y(\delta(S_i)) \leq T^+(i) - 1$ holds for some $i \in \{1,2,3\}$ and using the cut-set inequality (7), $x(\delta(S_i)) \geq r^+(i)$. Furthermore if $\sum_i T^+(i)$ is even, then either $y(\delta(S_i)) \leq T^+(i) - 2$ for some $i \in \{1,2,3\}$ and $x(\delta(S_i)) \geq r^+(i) + \lambda$, or $y(\delta(S_i)) \leq T^+(i) - 1$ and thus $x(\delta(S_i)) \geq r^+(i)$ holds for two separate subsets.

Next, we study the case when $\sum_i T^+(i)$ is odd more closely. For a given $p = (x,y,f) \in \mathcal{P}^{\mathcal{X}}$, let k denote $(\lceil \sum_i T^+(i)/2 \rceil - y(\Delta))^+$. Using the previous observations and the three-partition inequality (24) we can write,

$$x(\Delta) \geq \begin{cases} 0 & \text{if } k = 0 \\ r_{\min} & \text{if } k = 1 \\ \lceil \sum_i r^+(i)/2 \rceil + \lambda(k-2) & \text{if } k \geq 2. \end{cases} \tag{27}$$

As seen in Figure 1, it is possible to write valid inequalities stronger than $x(\Delta) + \lambda y(\Delta) \geq \lceil \sum_i \bar{T}(i)/2 \rceil$ when $x(\Delta) < \lceil \sum_i r^+(i)/2 \rceil$. We note that $(1/\lambda) \lceil \sum_i \bar{T}(i)/2 \rceil$, the value $y(\Delta)$ assumes when $x(\Delta) + \lambda y(\Delta) = \lceil \sum_i \bar{T}(i)/2 \rceil$ and $x(\Delta) = 0$ is not necessarily integral and it is strictly less than $\lceil \sum_i T^+(i)/2 \rceil$. Depending on the value of $\lceil \sum_i r^+(i)/2 \rceil$, $(1/\lambda) \lceil \sum_i \bar{T}(i)/2 \rceil$ can be larger or smaller than $\lceil \sum_i T^+(i)/2 \rceil - 1$, but in either case point p_2 lies above the line $x(\Delta) + \lambda y(\Delta) = \lceil \sum_i \bar{T}(i)/2 \rceil$. In other words, $r_{\min} + \lambda (\lceil \sum_i T^+(i)/2 \rceil - 1) \geq \lceil \sum_i \bar{T}(i)/2 \rceil$.

We first consider the case when p_2 lies above the line joining p_1 and p_3.

Theorem 4.11 *Given a critical partition* $\Pi = \{S_1, S_2, S_3\}$ *of* V, *if* $\Theta = \left\lceil \frac{\sum_i \bar{T}(i)}{2} \right\rceil$ *and* $r_{\min} \geq \frac{1}{2} \left\lceil \frac{\sum_i r^+(i)}{2} \right\rceil$, *then,*

$$x(\Delta) \geq \frac{1}{2} \left\lceil \frac{\sum_i r^+(i)}{2} \right\rceil \left(\left\lceil \frac{\sum_i T^+(i)}{2} \right\rceil - y(\Delta) \right) \tag{28}$$

is a facet of $\mathcal{P}^{\mathcal{X}}$ *provided* $\Theta > \max\{2, \theta - C(\Delta)\}$ *and both* $\sum_i T^+(i)$ *and* $\sum_i r^+(i)$ *are odd.*

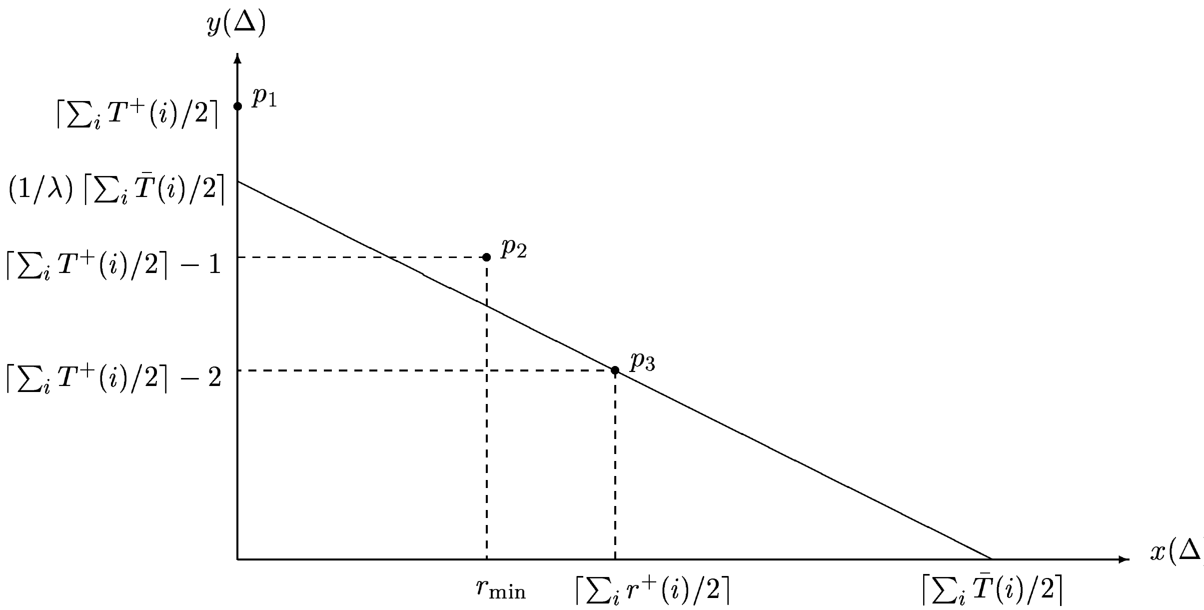

Figure 1: Finding new cuts using cut-set and 3-partition inequalities ($\sum_i T^+(i)$ is odd).

Proof. Validity of (28) is due to (27). To see that it is a facet, let $\bar{T}(3) \geq \bar{T}(2) \geq \bar{T}(1)$, and F be the face of $\mathcal{P}^{\mathcal{X}}$ implied by (28). Choose a fixed edge $e_{i,j} \in \delta(i,j)$ for all $j > i$, and consider $p^1 = (x^1, y^1, f^1)$ where

$$y_e^1 = \begin{cases} M & e \notin \Delta \\ \lfloor \sum_i T^+(i)/2 \rfloor - T^+(3) & e = e_{1,2} \\ \lfloor \sum_i T^+(i)/2 \rfloor - T^+(2) & e = e_{1,3} \\ \lfloor \sum_i T^+(i)/2 \rfloor - T^+(1) & e = e_{2,3} \\ 0 & \text{otherwise} \end{cases} \qquad x_e^1 = \begin{cases} M & e \notin \Delta \\ \lceil \sum_i r^+(i)/2 \rceil - r^+(3) & e = e_{1,2} \\ \lceil \sum_i r^+(i)/2 \rceil - r^+(2) & e = e_{1,3} \\ \lfloor \sum_i r^+(i)/2 \rfloor - r^+(1) & e = e_{2,3} \\ 0 & \text{otherwise} \end{cases}$$

and f^1 is a feasible flow vector. Clearly $y^1 \geq 0$, $\lambda y^1(\Delta) + x^1(\Delta) = \Theta$ and p^1 satisfies the cut-set inequalities for all $S_i \in \Pi$. As we show next, $x^1 \geq 0$ and thus $p^1 \in F$.

To see that $x^1 \geq 0$, note that when $\sum_i r^+(i)$ is odd, $r_{\min} \geq \frac{1}{2} \left\lceil \frac{\sum_i r^+(i)}{2} \right\rceil$ implies

$$r_{\text{med}} + r_{\min} \quad \geq \quad \frac{\sum_i r^+(i)}{2} + \frac{1}{2}$$

$$\begin{aligned}\frac{1}{2}(r_{\text{med}} + r_{\text{min}}) &\geq \frac{r_{\text{max}}}{2} + \frac{1}{2} \\ r_{\text{med}} + r_{\text{min}} &\geq r_{\text{max}} + 1\end{aligned}$$

so that $x^1 \geq 0$ and $x^1(1,2), x^1(1,3) > 0$. To see that $x^1(2,3) + \lambda y^1(2,3) > 0$, notice that if $y^1(2,3) = 0$ then,

$$0 = \frac{T^+(3) + T^+(2) - T^+(1) - 1}{2} \geq \frac{T^+(3) - 1}{2}$$

implies that $T^+(3) = T^+(2) = T^+(1) = 1$. If at the same time $x^1(2,3) = 0$ then

$$0 = \frac{r^+(3) + r^+(2) - r^+(1) - 1}{2} = \frac{\bar{T}(3) + \bar{T}(2) - \bar{T}(1) - 1}{2} \geq \frac{\bar{T}(3) - 1}{2}$$

implying $\lceil \sum_i \bar{T}(i)/2 \rceil = 2$, a contradiction.

If we let $\alpha x + \beta y + \gamma f = \pi$ be an equation satisfied by all $p = (x, y, f) \in F$, then by applying Lemma 4.7 with p^1, we can show that $\gamma = 0$ and $\alpha_e = \beta_e = 0$ for all $e \in E \setminus \Delta$. It is possible to modify p^1 by decreasing $x_{e_{1,2}}$ or $x_{e_{1,3}}$ by 1 and increasing $x_{e_{2,3}}$ by 1 (and modifying flow) to obtain new points in F, which implies that there is an $\bar{\alpha} \in R$ such that $\alpha_e = \bar{\alpha}$ for all $e \in \Delta$.

Lastly, we construct $p^2 = (x^2, y^2, f^2) \in F$ where $x^2 = 0$, f^2 is a feasible flow vector and y^2 is identical to y^1 with the exception that $y^2_{e_{1,2}} = y^1_{e_{1,2}} + 1$ and $y^2_{e_{1,3}} = y^1_{e_{1,3}} + 1$. Since it is possible to find new points by modifying p^2 by decreasing $y_{e_{1,2}}$ or $y_{e_{1,3}}$ by 1 and increasing $y_{e_{2,3}}$ by 1, we can argue that for some $\bar{\beta} \in R$, $\beta_e = \bar{\beta}$ for all $e \in \Delta$. Finally $p^1, p^2 \in F$ implies that $\bar{\beta} = \frac{1}{2}\left\lceil \frac{\sum_i r^+(i)}{2} \right\rceil \bar{\alpha}$, which completes the proof. ∎

We next consider the case when p_2 lies below the line joining p_1 and p_3. Notice that when $r_{\text{min}} < (1/2)\lceil \sum_i r^+(i)/2 \rceil$, (28) of Theorem 4.11 is not valid, but in this case we can write two new valid inequalities using p_1 and p_2 or p_2 and p_3. These inequalities are,

$$x(\Delta) \geq r_{\text{min}} \left(\left\lceil \sum_i T^+(i)/2 \right\rceil - y(\Delta) \right) \tag{29}$$

and

$$x(\Delta) \geq r_{\text{min}} + \left(\left\lceil \sum_i r^+(i)/2 \right\rceil - r_{\text{min}} \right) \left(\left\lceil \sum_i T^+(i)/2 \right\rceil - y(\Delta) - 1 \right) \tag{30}$$

Unfortunately, these inequalities do not define facets of $\mathcal{P}^{\mathcal{X}}$ since any point on the faces related with (29) and (30) also satisfies $y(a) = T^+(a) - 1$ where $a = \text{argmin}\{\bar{T}(i)\}$. We next present a facet of $\mathcal{P}^{\mathcal{X}}$ which combines (29) and (30).

Theorem 4.12 *Given a critical partition $\Pi = \{S_1, S_2, S_3\}$ of V, let $r^+(3) \geq r^+(2) \geq r^+(1)$. If $\Theta = \lceil \sum_i \bar{T}(i)/2 \rceil$, and $r^+(1) \leq \frac{1}{2} \lceil \sum_i r^+(i)/2 \rceil$, then*

$$\frac{x(1,2)+x(1,3)}{r^+(1)} + \frac{x(2,3)}{\min\{r^+(2), \lceil \sum_i r^+(i)/2 \rceil - r^+(1)\}} \geq \left(\left\lceil \frac{\sum_i T^+(i)}{2} \right\rceil - y(\Delta) \right) \tag{31}$$

defines a facet of $\mathcal{P}^{\mathcal{X}}$ provided $\max_i\{T^+(i)\} > 1$ and $\sum_i T^+(i)$ is odd.

Proof. Before showing that (31) is a valid inequality, we first define $x(1)$ to denote $x(1,2) + x(2,3)$, α to denote $1/r^+(1)$ and β to denote the coefficient of $x(2,3)$ in (31). We further let $g(x)$ to denote the left hand side of (31) so that $g(x) = \alpha x(1) + \beta x(2,3)$. Notice that $\lceil \sum_i r^+(i)/2 \rceil \geq 2r^+(1)$ together with $r^+(2) \geq r^+(1)$ implies that $\alpha \geq \beta$ and $\beta r^+(2) \geq 1$.

First note that (31) is valid for any $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$ whenever $y(\Delta) \geq \lceil \sum_i T^+(i)/2 \rceil$. Next, consider the case when $y(\Delta) \geq \lceil \sum_i T^+(i)/2 \rceil - 1$, so that, there exists an index $i \in \{1,2,3\}$ with $y(\delta(S_i)) < T^+(i)$. If $y(\delta(S_1)) < T^+(1)$ then, the cut-set inequality for S_1 implies $x(1) \geq r^+(1)$ and thus $g(x) \geq 1$. On the other hand, if $y(\delta(S_1)) \geq T^+(1)$ then, using the cut-set inequalities for S_2 or S_3 we have $x(\Delta) \geq r^+(2)$ and $g(x) \geq \beta x(\Delta) \geq \beta r^+(2) \geq 1$.

The last case we consider is when $y(\Delta)$ is at most $\lceil \sum_i T^+(i)/2 \rceil - 2$. Let $k(i) = (T^+(i) - y(\delta(S_i)))^+$ and $K = \lceil \sum_i T^+(i)/2 \rceil - y(\Delta) \geq 2$ and note that $\sum_i k(i) \geq 2K - 1$. If $k(1) \geq 1$ then, to find a lower bound on $g(x)$, we look at the optimal value of the following linear program.

$$\begin{aligned} \min \quad z = \quad & \alpha x(1) + \beta x(2,3) \\ \text{st.} \quad\quad x(1) \quad & \geq \quad r^+(1)k(1) \\ x(1) + x(2,3) \quad & \geq \quad \lambda(K-2) + \lceil \textstyle\sum_i r^+(i)/2 \rceil \\ x(1), x(2,3) \quad & \geq \quad 0 \end{aligned}$$

where we minimize $g(x)$ subject to some valid inequalities. It is easy to see that the optimal solution has $x(1) = r^+(1)k(1)$ and $x(2,3) = \lambda(K-2) + \lceil \sum_i r^+(i)/2 \rceil -$

$r^+(1)k(1)$ yielding,

$$\begin{aligned} z &= \frac{r^+(1)k(1)}{r^+(1)} + \frac{\lambda(K-2) + \lceil \sum_i r^+(i)/2 \rceil - r^+(1)k(1)}{\min\{r^+(2), \lceil \sum_i r^+(i)/2 \rceil - r^+(1)\}} \\ &\geq k(1) + (K-2) + \frac{\lceil \sum_i r^+(i)/2 \rceil - r^+(1)k(1)}{\min\{r^+(2), \lceil \sum_i r^+(i)/2 \rceil - r^+(1)\}} \\ &\geq k(1) + (K-2) + 1 - \frac{r^+(1)\,(k(1)-1)}{\min\{r^+(2), \lceil \sum_i r^+(i)/2 \rceil - r^+(1)\}} \\ &\geq k(1) + 1 + (K-2) - k(1) + 1 \end{aligned}$$

so that $g(x) \geq z \geq K$.

On the other hand, if $k(1) = 0$, then $k(2) + k(3) \geq 2K - 1$ and $\max\{k(1), k(2)\} \geq K$. Writing the cut-set inequalities for S_2 and S_3 we have, $x(\Delta) \geq \max\{x(1,2) + x(2,3), x(1,2) + x(2,3)\} \geq \max\{r^+(2)k(2), r^+(3)k(3)\} \geq r^+(2)K$. Therefore $g(x) \geq \beta x(\Delta) \geq \beta r^+(2)K \geq K$ and (31) is satisfied by all $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$.

For the rest of the proof, refer to [14]. ■

In Theorems 4.11 and 4.12, we considered the case when $\Theta = \lceil \sum_i \bar{T}(i)/2 \rceil$ and $\sum_i T^+(i)$ is odd. If $\sum_i T^+(i)$ is even, then for any $p = (x, y, f) \in \mathcal{P}^{\mathcal{X}}$ we can bound $x(\Delta)$ from below by

$$x(\Delta) \geq \begin{cases} 0 & \text{if } k = 0 \\ r_{\text{med}} & \text{if } k = 1 \\ \left\lceil \frac{\lambda + \sum_i r^+(i)}{2} \right\rceil + \lambda(k-2) & \text{if } k \geq 2. \end{cases} \tag{32}$$

where $k = (\lceil \sum_i T^+(i)/2 \rceil - y(\Delta))^+$. Using (32), it is possible to develop valid inequalities of the form (28) , (29) or (30), but these inequalities are not facet defining.

5 Computational Results.

In this section, we present the results of our computational experience with a cutting-plane algorithm. The approach described below has been elaborated and modified so as to work in the *one*-facility network loading problem on directed graph, with very good results, see [4].

We developed an iterative algorithm which uses the facet defining inequalities as cutting-planes and includes them in the formulation whenever they are valid (and violated but not necessarily facet defining). The algorithm has three modules, one for each class of facets we presented in Sections 2 - 4. We used these modules in a hierarchical manner, and for a given iteration, executed a module only if no violated cuts are found by the previous modules. For each module, there is an upper limit on the number of cuts that can be introduced to the extended formulation in a single iteration. During the course of our study, we observed that it is better to use these modules in the following order: the cut-set module, the three-partition module and the flow-cut-set module. After obtaining the last formulation, we ran branch-and-bound. CPLEX, Version 2.1, was used throughout. The cutting plane algorithm was run on a SPARC 10-40, and branch-and-bound on a SPARC 10-51.

We used two sets of real-life data, which arise, as described before, as part of ATM network design problems. The traffic demand matrices are fully dense and it is not practical to use the disaggregated formulation (i.e. defining a commodity for every source-destination pair) for these problems. As we explain later, we also made some modifications on the data to generate additional test problems while disturbing the underlying structure in a minimal way. The first data set is of a network with 15 nodes and 22 edges (see Figure 2). The traffic demands are fairly large when compared with the existing capacity on the edges and there is a cost related with flow variables as well as the capacity variables. The second network (see Figure 3) is much denser when compared with the first one and it has 16 nodes and 49 edges. In this data set, traffic demands are quite small and there is no existing capacity. Further, there is no cost related with the flow variables. In both of the test problems the cost of adding capacity on an edge has a fixed component (related with the switches on both ends of the edge) and a variable component proportional to the actual length of the edge. The unit-batches correspond to so-called OC-3 facilities and and λ-batches correspond to OC-12 facilities and thus, λ is 4. The cost related with these facilities is such that cost of an OC-12 facility is more than the cost of one OC-3 facility but it is less than that of two OC-3 facilities and, therefore, in the optimal solution x variables are either 0 or 1. We included these bounds for the x variables in the original formulation but did not modify the valid inequalities using this information.

For each of the three modules of the algorithm, there is an exponential number of related facets and to implement the algorithm we need to find a practical way to choose violated inequalities, or, in other words, we need to find a way to solve the separation problem. Little effort was spent on the separation problem and it is likely that our cutting plane algorithm can be substantially speeded up by developing more efficient separation modules. The networks related with the data sets are quite different and this was reflected in the type of valid inequalities that became active. We will postpone addressing the separation problem and look at the data sets more closely.

5.1 Data Set 1

For every strong subset of the node set, there are two related cut-set facets and even when the number of nodes is small (15 in this case), there are potentially $2^{|V|}$ subsets to be considered. The number of strong subsets of a graph is closely related with the density of the graph and as seen in Figure 2, the network related with this instance is fairly sparse.

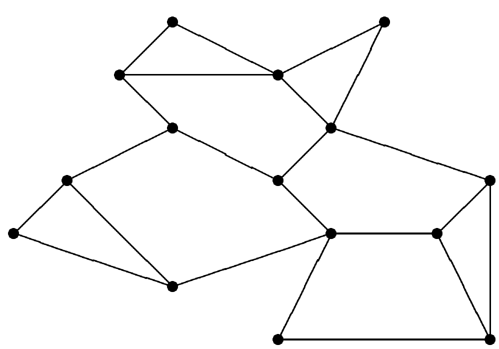

Figure 2: Network 1, $n = 15$ and $m = 22$

In this example, there are only 190 strong subsets and it is feasible to check all of them to see if the related cut-set facets are violated or not. Recall that for a subset to qualify as a strong subset, both the subset and its complement have to be connected.

Similarly, the number of critical partitions of Network 1 is not very large (close to one thousand) and it is possible to check all of them in each iteration to see if they are violated or not. Lastly, we need to consider a number of flow-cut-set facets for each strong subset S, each commodity subset Q of S and each nonempty partition $\{E_1, E_2\}$ of $\delta(S)$. In our experiments we noticed that these cuts are more effective when (i) Q is "compact", i.e. small and connected, (ii) $|E_2|$ is small and (iii) edges in E_2 are "close" to Q, mostly when they are incident with nodes in Q. Using these observations, for each strong subset S, we generated sets Q such that $Q = \{v\}$ for all $v \in S$ and $Q = \{u, v\}$ for all $u, v \in S$ and $\{u, v\} \in E$. For choosing E_1 and E_2, we looked at the partitions that consist of no more than three edges in E_2.

We first ran the algorithm without a time limit and generated an extended formulation by including all of the violated cuts in the original LP-relaxation. The optimal integral solution to this problem has a cost of 2231 and the lower bound generated by the extended LP was 2222, only 0.4% away from the optimum. This run took approximately 30 minutes on a SPARC10 - 40 machine and the statistics of this run are presented in Table 1. We define the "scaled gap" to be the difference between the value of the extended formulation and the optimal (integral) solution divided by the difference between the value of the LP-relaxation and the optimal (integral) solution.

As seen in Table 1, the algorithm very quickly narrowed the gap: after approximately 21 seconds the scaled gap was less than 3% (i.e., the true gap was under 1%) and after one minute of run time the scaled gap was less than 2%. After iteration 9 it takes almost half an hour to cut the scaled gap from 1.9% to 1.3%.

When we applied branch-and-bound using the resulting extended formulation, the (integral) optimum was found in a few seconds. To balance the run-time between the cutting-plane algorithm and branch-and-bound, we next limited the use of flow-cut-set facets and stopped the algorithm after 70 seconds. After this modification, total run-time (i.e., generating the extended formulation and running branch-and-bound) was reduced to under two minutes. In contrast, running branch-and-bound without any cuts required more than an hour to find the integral optimal solution.

iteration number	number of cuts	cut type	LP value	gap (%)	scaled gap(%)	time (sec)
0	0	-	1534	31.0	100	.42
1	45	c-s	1971	11.4	36.8	1.62
2	7	c-s	1998	10.0	32.3	2.08
3	37	3-p	2156	3.4	11.0	2.95
4	3	c-s	2156	3.4	11.0	4.17
5	7	3-p	2203	1.2	3.9	4.67
6	2	3-p	2204	1.2	3.9	5.45
7	57	f-c	2210	0.9	2.9	21.13
8	58	f-c	2215	0.7	2.3	51.32
9	56	f-c	2218	0.6	1.9	93.58
10	57	f-c	2220	0.5	1.6	239.93
11	30	f-c	2221	0.4	1.3	488.80
12	1	3-p	2221	0.4	1.3	490.62
13	24	f-c	2222	0.4	1.3	746.68
14	12	f-c	2222	0.4	1.3	1000.73
15	2	f-c	2222	0.4	1.3	1251.58
16	1	f-c	2222	0.4	1.3	1502.82
17	0	f-c	2222	0.4	1.3	1755.48

Table 1: Example run of the algorithm on Data Set 1 (no time limit).

Next, we modified the original data ('Cap1') to generate new problem instances and to test the performance of the algorithm when applied to instances with different nature. Keeping the underlying network the same, we generated four more instances by changing the data as follows: The second data set is same as the first one, except that the existing capacities are assumed to be zero; the third set is obtained by doubling the traffic demands and the last two sets are generated by respectively increasing and decreasing the flow costs. Tables 2 and 3 summarize the results of these runs.

The run-times are presented in Tables 2 and 3. We note that for all of the test problems, the total CPU-time needed to find the optimal solution was under two

problem	z(LP)	z(ELP)	z(IP)	gap(%)	sc. gap(%)	time(sec)
1: Cap1	1534	2218	2231	0.58	1.87	74
2: NoXcap1	4075	4576	4607	0.67	5.83	89
3: 2traf1	5608	6339	6354	0.23	2.01	74
4: NoFC1	949	1623	1631	0.49	1.17	83
5: 5FC1	2411	3105	3132	0.86	3.74	81

Table 2: Example problems generated using Data Set 1

problem	# of cuts	B&B time	Pure B&B time
1: Cap1	299	10sec	1hour
2: NoXcap1	336	15sec	12mins
3: 2traf1	124	11sec	unfinished
4: NoFC1	282	8sec	unfinished
5: 5FC1	307	9sec	30mins

Table 3: B&B times for Set 1

minutes and the algorithm is not effected by the changes in the input as long as the underlying network stays the same. As seen in the Tables 2 and 3, when we apply branch-and-bound without any cutting-planes, the run-times vary from 12 minutes to several hours. For Problems 3 and 4 when we terminated the run after more than 3 hours of CPU-time, the branch-and-bound tree had more than 20,000 nodes and the gap between the upper and lower bounds was still large.

5.2 Data Set 2

As seen in Figure 3, the network related with this data set is dense and consequently the number of strong subsets is quite large. There are more than 25,000 strong subsets related with this network and although it is still feasible to consider all of the cut-set facets, it is not possible to do the same for all of the three-partition or flow-cut-set facets. We stress, however, that it is quite practical to *enumerate* all strong subsets at the start of the algorithm: this only requires a very small amount of computation.

For this instance, we modified the algorithm and defined flags related with each strong subset . When executing the cut-set module, we marked a strong subset if the related cut-set inequalities are violated or when the slacks related with the cuts are less than 10% of the right hand side. Using these flags, we only considered the three-partitions which are formed by these subsets. Similarly, we only used the flow-cut-set facets related with the chosen subsets. The number of "important" strong subsets, selected as above, was under 100 and in terms of finding a good lower bound, they were as effective as the whole list. We also note that, in this case the flow-cut-set facets were not very effective as traffic demands are small and the flow costs are zero.

A word about this approach. Although seemingly inelegant and inefficient (it reeks of enumeration), it turns out that with proper data management techniques in fact it leads to an extremely effective and efficient (and quick) algorithm for network loading problems; see [4]. It is worth noting that the problem of detecting whether a cut inequality is violated is NP-complete ([2] has proved a related result).

The LP-relaxation related with this data set has a value of 1,950 and the corresponding optimal integral solution has a value of 10,704 (as we learned later). The best lower bound we obtained by applying the cutting plane algorithm, with the modifications describes above, was 8,491. In other words, this lower bound is 20% was away from the optimal value and the scaled gap is more than 25%. When we applied branch-and-bound using the resulting extended formulation, the gap between the upper and lower bounds generated by CPLEX was more than 10% after a few hours. After many hours of CPU time, and before the problem was solve, CPLEX exhausted the system memory (consuming in the process approximately 48 megabytes).

We next studied the fractional optimal solution to the extended formulation and realized that the overall capacity added to the whole network (that is, $x(E) + y(E)$) was quite small. The main reason for this was that the traffic demands are small. Even though the cut-set facets and the three-partition facets force the degree (i.e. $x(\delta(S)) + y(\delta(S))$) of a strong subset to be at least one, these cuts are myopic, and

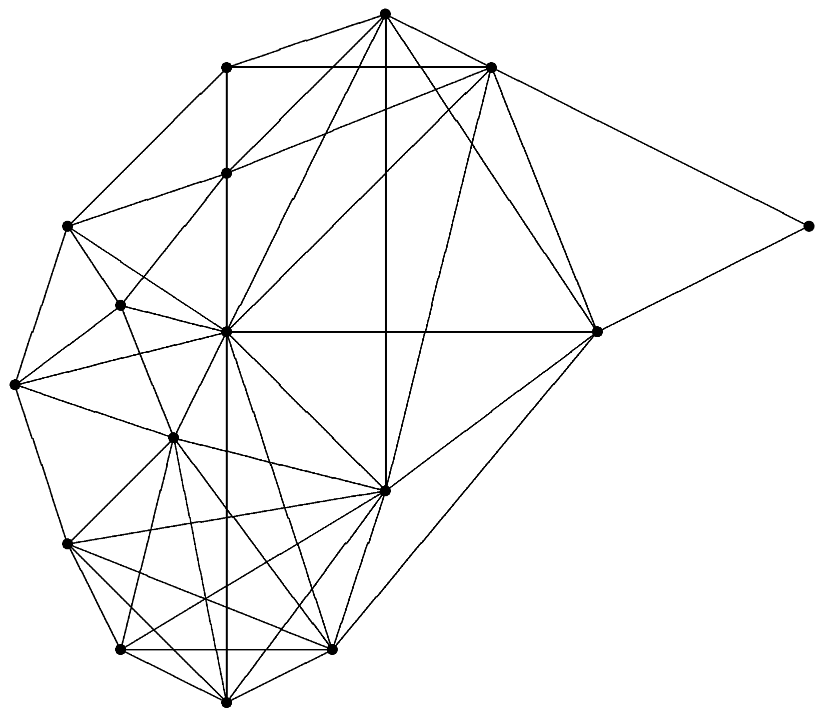

Figure 3: Network 2, $n = 16$ and $m = 49$

they do not force a lower bound on the overall capacity. Since there is no existing capacity for this problem and as the resulting network has to be connected, the optimal solution should add capacity on at least 15 (= # of nodes−1) edges so that the optimal solution would contain enough edges to form a connected network.

Using this observation, we then added a new module to the algorithm that checks whether or not some simple valid spanning tree cuts are satisfied by fractional solutions. In this module we have two kinds of valid inequalities. The first one of these can be obtained by shrinking a subset of nodes and requiring the resulting network to have enough edges to form a tree. The second one basically states that after deleting some edges (that is shrinking pairs of nodes) the solution to the design problem should have enough number of edges to form a spanning tree together with the deleted edges. We used the list of strong subsets for the first family of the spanning tree cuts and shrank the edges with $x_e + y_e > 1$ for the second one.

After including this module in the cutting plane algorithm, the lower bound generated by the extended formulation went up to 10,339 or, only 3.4% off the optimal solution. Using the resulting formulation, branch-and-bound was able to find an optimal solution, with the entire procedure taking under half an hour. To study the effect of this new module more closely, we ran the algorithm by disabling all other modules and the resulting lower bound was 9,071, or less than 10% away from the optimum. However, the resulting extended formulation was very ineffective for branch-and-bound.

Lastly, we generated a larger extended formulation by first applying our cutting-plane algorithm and then setting some of the design variables to zero, and repeating this procedure iteratively until an integral solution was generated. This way we generated many valid inequalities and using this formulation, branch-and-bound was able to find an optimal solution in about 15 minutes, that is the run-time was reduced by a factor of two.

We also generated two more problems related with this data set by increasing the traffic demands and by changing the objective function coefficients of the flow variables. In Tables 4 and 5, we summarize the statistics related with these of the problems.

problem	z(LP)	z(ELP)	z(IP)	gap(%)	sc. gap(%)	time(sec)
1: Cap2	1950	10339	10704	3.4	4.2	160
2: 2traf2	3901	10792	11789	8.5	12.7	175
3: 1FC2 (a)	4092	12779	14384	10.9	15.3	177
4: 1FC2 (b)	4092	13379	14384	7.0	9.8	5hrs

Table 4: Example problems generated using Data Set 2

In Table 4, problems 3 and 4 correspond to the same problem for different lengths of run-time.

As it can be seen in Tables 4 and 5, the algorithm is not as successful for the

problem	# of cuts	B&B time	Pure B&B time
1: Cap2	247	15min	unsolved
2: 2traf2	258	3hrs	unsolved
3: 1FC2 (a)	508	10hrs*	unsolved

Table 5: B&B times for Set 2

second and third problems (these are the ones we generated by modifying the original data).

For the second problem, the scaled gap was more than 10% after the first phase and branch-and-bound takes just under three hours. When applying the algorithm to this data set, we limited the use of flow-cut-set facets (to keep the size of the extended-LP small). Since these inequalities play a more important role when the volume of traffic is high, this change results in a larger gap and thus a much longer branch-and-bound time. Nonetheless, in terms of application, we want to note that the solution time for this problem is reasonable.

For the third problem (1FC2 (a)), we should say that the extended formulation generated by the cutting-plane algorithm was not strong enough and we could not solve the problem to optimality using CPLEX (sequential) branch-and-bound. The run time and the optimal value reported in Table 5 were obtained by J. Eckstein by running his parallel branch-and-bound code CMMIP on a 64 processor CM-5 machine [5]. Starting with the extended formulation, the code took approximately 10 hours to solve this problem to optimality, generating a B-B tree with 2.4 million nodes. This negative result shows that the facet defining inequalities that we have presented in this paper are not sufficient to solve hard problems (i.e., resulting from dense graphs, with dense traffic matrices with flow costs) and more work needs to be done on the polyhedral structure of CEP.

As a further test of the strength of our inequalities, we performed the following experiments. Suppose we have generated valid inequalities for a problem instance, and the demand data were to change in a small way. Then the inequalities would generally become invalid. However, we can recompute the coefficients in the inequalities so that they become valid once again, in a small fraction of the time it took to compute the original inequalities. Note that the resulting inequalities are probably not facet-defining. Nevertheless, how strong are they? This question has great practical significance, since we will usually solve many problems that differ slightly from one another in the demand amounts. To test this, for selected problems we (a) generated an extended formulation as described above, and then (b) randomly perturbed each demand by 10 % and 20 %. Table 6 describes the results of these tests. Here LP is the LP-relaxation of the perturbed problem, RELP is the *reconstructed* extended formulation, ELP is the extended formulation for the perturbed problem (obtained in the normal way) and IP is the perturbed mixed-integer program. As we can see, the strategy of recomputing cuts appears quite effective. In a certain sense, this shows that our inequalities are "stable" and more "combinatorial" than driven by the demand amounts.

	Data Set 2		Data Set 1	
Perturbation	10%	20%	10%	20%
z(LP)	1955.83	1967.58	1423.44	1401.54
z(RELP)	10315.51	10316.61	2118.73	2112.92
z(ELP)	10315.51	10321.59	2160.97	2157.63
z(IP)	10704.00	10704.00	2182.37	2164.57
B&B time	430sec	382sec	9sec	9sec
Gap	3.6%	3.6%	2.9%	2.4%
Sc. Gap	4.4%	4.4%	8.3%	6.8%

Table 6: Perturb & Reconstruct

6 Extensions.

There are several areas that we plan to explore in the future. The cutsets we described above involve families of subsets of nodes. Roughly speaking, our algorithms maintained a list of "active" subsets. It is easy to decide when a subset is no longer active, but all the approaches we can think of for generating new active sets involve problems similar to the maximum-cut problem.

Another issue is that of generating strong inequalities involving partitions of the node set into more than three classes. Early work on our part appears to show that the structure of the "better" facets is quite complex (they strongly depend on the demand amounts – one can easily generate interesting-looking combinatorial facets that never come into play). Instead, we are developing an approach for automatically computing $face$-defining violated inequalities. Roughly, this approach involves recursively solving problems of type CEP that have a simpler structure.

A simple change to our formulation is that of replacing each edge by three parallel edges, one for existing capacity, one for x-capacity and one for y-capacity, and similarly splitting the flows in the edge into a sum of three values. This will merely increase the number of continuous variables by a factor of three, but the benefit is that we will have a richer family of "flow cut-set" inequalities. As a preliminary step in this direction, we are improving our separation procedure for these inequalities. We note that there are other ways of tightening the split formulation.

A different kind of reformulation involves using path variables instead of flow variables. However, the integral variables remain the same, and potentially the resulting problem is just as difficult as the original one (although there are more ways of strengthening the path formulation). We will test all of these ideas in a future paper.

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