Everything should be made as simple as possible, but not simpler.

Albert Einstein

When a mathematician has no more ideas, he pursues axiomatics.

Felix Klein

I hope, good luck lies in odd numbers ...
They say, there is divinity in odd numbers,
either in nativity, chance, or death.

William Shakespeare

Chapter 1. Topological Degree in Finite Dimensions

In this basic chapter we shall study some basic problems concerning equations of the form $f(x) = y$, where f is a continuous map from a subset $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and y is a given point in $\mathbb{R}^n$. First of all we want to know whether such an equation has at least one solution $x \in \Omega$. If this is the case for some equation, we are then interested in the question of whether this solution is unique or not. We then also want to decide how the solutions are distributed in Ω. Once we have some answers for a particular equation, we need also to study whether these answers remain the same or change drastically if we change f and y in some way. It is most probable that you have already been confronted, more or less explicitly, by all these questions at this stage in your mathematical development.

Let us review, for example, the problem of finding the zeros of a polynomial. First we learn that a real polynomial need not have a real zero. Then we are taught that a real polynomial of odd degree, say $p_{2m+1}(t) = t^{2m+1} + p_{2m}(t)$, has a real zero, and you will recall the simple proof which exploits the fact that $p_{2m}(t)$ is 'negligible' relative to t^{2m+1} for large t, and therefore $p_{2m+1}(t) > 0$ for $t \geqq r$ and $p_{2m+1}(t) < 0$ for $t \leqq -r$ with r sufficiently large, which in turn implies that p_{2m+1} has a zero in $(-r, r)$, by Bolzano's intermediate value theorem. Next we learn that every polynomial of degree $m \geqq 1$ has at least one zero in the complex plane $\mathbb{C}$. Then we introduce the multiplicity of a zero z_0. If this is k, then z_0 is counted k times, and by means of this concept the more precise statement is arrived at that every polynomial of degree $m \geqq 1$ has exactly m zeros in $\mathbb{C}$. At this stage the problem of finding the zeros of a polynomial over $\mathbb{C}$ is solved for the pure algebraist and he will turn to the same question for more general functions over more general structures. The 'practical' man, if he is fair, will appreciate that the 'pure' fellows have proved a nice theorem, but it does not satisfy his needs. Suppose that he is led to investigate the behaviour as $t \to \infty$ of solutions of a linear system $x' = Ax$ of ordinary differential equations, where A is an $n \times n$ matrix. Then the information that the characteristic polynomial of A has exactly n zeros in $\mathbb{C}$, the eigenvalues of A, is not enough for him since he has to know whether they are in the left or right half plane or on the imaginary axis. In another situation he may have obtained his polynomial by interpolation of certain experimental data which usually contain some hopefully small errors. Then he may need to know that the zeros of polynomials close to p are close to the zeros of p.

Now, we want to construct a tool, the topological degree of f with respect to Ω and y, which is very useful in the investigation of the problems mentioned at the beginning. To motivate the process, let us recall the winding number of plane curves and its connection with theorems on zeros of analytic functions. If you missed this topic in an elementary course in complex analysis, you may either consult Ahlfors [1], Dieudonné [1], Krasnoselskii et al. [1], or believe in what we are going to mention in the sequel, since we shall indicate in § 6.6 how the winding number is related to the degree in the case of $\mathbb{R}^2$.

Let $\Gamma \subset \mathbb{C}$ be an oriented closed curve with the continuously differentiable (C^1 for short) representation $z(t)$ ($t \in [0,1]$, $z(0) = z(1)$) and let $a \in \mathbb{C}\backslash\Gamma$. Then, the integer

$$(1) \qquad w(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-a} = \frac{1}{2\pi} \int_0^1 \frac{x(t)\,y'(t) - x'(t)\,y(t)}{x^2(t) + y^2(t)}\,dt$$
$$\text{for } z(t) = x(t) + i y(t) + a$$

is called the *winding number* (or *index*) of Γ with respect to $a \in \mathbb{C}\backslash\Gamma$, since it tells us how many times Γ winds around a, roughly speaking. If Γ is only continuous then we can approximate Γ as closely as we wish by C^1-curves, and it is easy to see that all these approximations have the same winding number provided that they are sufficiently close to Γ. More precisely, if $z_1(t)$ and $z_2(t)$ are C^1-representations of the closed curves Γ_1 and Γ_2 with the same orientation as Γ and are such that

$$\max\{|z_j(t) - z(t)| : t \in [0,1]\} < \min\{|a - z(t)| : t \in [0,1]\} \qquad \text{for } j = 1, 2$$

then $w(\Gamma_1, a) = w(\Gamma_2, a)$. Therefore, we can define $w(\Gamma, a)$ to be $w(\Gamma_1, a)$ for any such Γ_1. Then we have defined

$$w\colon \{(\Gamma, a) : \Gamma \text{ closed continuous, } a \in \mathbb{C}\backslash\Gamma\} \to \mathbb{Z}$$

and it is not hard to see that this function w has the following properties:

(a) w is continuous in (Γ, a), i.e. constant in some neighbourhood of (Γ, a).
(b) $w(\Gamma, \cdot)$ is constant on every connected component of $\mathbb{C}\backslash\Gamma$ – in particular, equal to zero on the unbounded component.
(c) If the curves Γ_0 and Γ_1 are homotopic in $\mathbb{C}\backslash\{a\}$, then $w(\Gamma_0, a) = w(\Gamma_1, a)$. More explicitly, let $z_0(t)$ and $z_1(t)$ be representations of Γ_0 and Γ_1 such that there exists a continuous $h\colon [0,1] \times [0,1] \to \mathbb{C}\backslash\{a\}$ satisfying $h(0,t) = z_0(t)$ and $h(1,t) = z_1(t)$ in $[0,1]$ and $h(s,0) = h(s,1)$ for every $s \in [0,1]$; then $w(\Gamma_s, a)$ is the same integer for all $s \in [0,1]$, where Γ_s is the closed curve represented by $h(s, \cdot)$.
(d) If Γ^- denotes the curve Γ with its orientation reversed, then $w(\Gamma^-, a) = -w(\Gamma, a)$.

Property (c) is the most important one, since it allows us for example to calculate the winding number of a complicated curve by means of the winding number of a possibly simpler homotopic curve. Furthermore, (a) and (b) are simple consequences of (c).

Now, let $G \subset \mathbb{C}$ be a simply connected region, $f\colon G \to \mathbb{C}$ be analytic and $\Gamma \subset G$ be a closed C^1-curve such that $f(z) \neq 0$ on Γ. Then the 'argument principle' tells us that

$$w(f(\Gamma), 0) = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = \sum_k w(\Gamma, z_k)\, \alpha_k, \tag{2}$$

where the z_k are the zeros of f in the regions enclosed by Γ and the α_k are the corresponding multiplicities. If we assume in addition that Γ has positive orientation and no intersection points, then we know from Jordan's curve theorem, which will be proved in this chapter, that there is exactly one region $G_0 \subset G$ enclosed by Γ, and $w(\Gamma, z_0) = 1$ for every $z_0 \in G_0$. Thus, (2) becomes

$$w(f(\Gamma), 0) = \sum_k \alpha_k,$$

i.e. the total number of zeros of f in G_0 is obtained by calculating the winding number of the image curve $f(\Gamma)$ with respect to 0. In general, $w(\Gamma, z_k)$ can also be negative and then we can only conclude that f has at least $|w(f(\Gamma), 0)|$ zeros in the regions enclosed by Γ.

In the more general case of continuous maps from subsets of $\mathbb{R}^n$ into $\mathbb{R}^n$ we shall imitate these ideas. We consider open bounded subsets $\Omega \subset \mathbb{R}^n$ instead of the regions enclosed by Γ, continuous maps $f\colon \bar{\Omega} \to \mathbb{R}^n$ and points $y \in \mathbb{R}^n$ which do not belong to the image $f(\partial\Omega)$ of the boundary of Ω. With each such 'admissible' triple (f, Ω, y) we associate an integer $d(f, \Omega, y)$ such that the properties of the function d allow us to give significant answers to the questions raised at the beginning. Of course, as in daily life, we cannot achieve everything, but the following minimal requirements and their useful consequences turn out to be a good compromise.

The first condition is simply a normalization. If $f = \mathrm{id}$, the identity map of $\mathbb{R}^n$ defined by $\mathrm{id}(x) = x$, then $f(x) = y \in \Omega$ has the unique solution $x = y$, and therefore we require

(d1) $$d(\mathrm{id}, \Omega, y) = 1 \quad \text{for } y \in \Omega.$$

The second condition is a natural formulation of the desire that d should yield information on the location of solutions. Suppose that Ω_1 and Ω_2 are disjoint open subsets of Ω and suppose that $f(x) = y$ has finitely many solutions in $\Omega_1 \cup \Omega_2$ but no solution in $\bar{\Omega}\backslash(\Omega_1 \cup \Omega_2)$. Then the number of solutions in Ω is the sum of the numbers of solutions in Ω_1 and Ω_2, and this suggests that d should be additive in its argument Ω, that is

(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega}\backslash(\Omega_1 \cup \Omega_2))$.

The third and last condition reflects the desire that for complicated f the number $d(f, \Omega, y)$ can be calculated by $d(g, \Omega, y)$ with simpler g, at least if f can be continuously deformed into g such that at no stage of the deformation we get solutions on the boundary. This leads to

(d 3) $d(h(t,\cdot),\Omega,y(t))$ is independent of $t\in J=[0,1]$ whenever $h\colon J\times\bar{\Omega}\to\mathbb{R}^n$ and $y\colon J\to\mathbb{R}^n$ are continuous and $y(t)\notin h(t,\partial\Omega)$ for all $t\in J$.

There are essentially two different approaches to the construction of such a function d. The older one uses only concepts from algebraic topology, which is quite natural, since (d 1)–(d 3) involve only topological concepts such as open sets and continuous maps and a 'little bit' theory of groups like $\mathbb{Z}$; see, for example, Alexandroff and Hopf [1], Cronin [2], Dold [2], Dugundji and Granas [1].

We shall present the more recent second approach which is simpler for 'true' analysts, not worrying much about topology and algebra, since it uses only some basic analytical tools such as the approximation theorem of K. Weierstraß, the implicit function theorem and the so-called lemma of Sard (see § 2). Presentations still using topological arguments can be found in books on differential topology, for example, in Guillemin and Pollack [1], Hirsch [1] and Milnor [2], while purely analytical versions have been given by Nagumo [1] and Heinz [1] in the 1950s. An interesting mixture of the two methods has been given in Peitgen and Siegberg [1] – an outgrowth of recent efforts in finding numerical approximations to degrees and fixed points, based on the observation that the essential steps of the old method can be put into the form of algorithms.

In principle, it is an inessential question how we introduce degree theory, since there is only one $\mathbb{Z}$-valued function d satisfying (d 1)–(d 3), as you will see in § 1, and since it are the properties of d which count, as you will see throughout this chapter. Starting with the uniqueness of d, by exploiting (d 1)–(d 3) until we end up with the simplest case $f(x)=Ax$ with $\det A\neq 0$, has the advantage that the basic formula, which a purely analytical definition has to start with, does not fall from heaven – it is enough that the natural numbers do (according to L. Kronecker) – and that we are already motivated to introduce some prerequisites which we need anyway later on. However, you will keep in mind that choosing the analytical approach we lose topological insight to a considerable extent, while going through the mill of the elements of combinatorial topology you will hardly become aware of the fact that the same goal can be arrived at so simply by an analytical procedure. Thus, the essential question is why we introduce degree theory, but this has already been answered by the general remarks given in the foreword and the more special ones in this introduction which we are going to close by a few historical remarks.

The winding number is a very old concept. Its essentials can already be found in papers of C. F. Gauß and A. L. Cauchy at the beginning of the 19th century. Later on L. Kronecker, J. Hadamard, H. Poincaré and others extended formula (1) by consideration of integrals of differentiable maps over $\{x\in\mathbb{R}^n\colon |x|=1\}$. Finally, L. E. J. Brouwer established the degree for continuous maps in 1912. It is now tradition to speak of the Brouwer degree. The way towards an analytical definition was paved by A. Sard's investigation of the measure of the critical values of differentiable maps in 1942. You will find much more in the interesting papers of Siegberg [1], [2].

§ 1. Uniqueness of the Degree

In this section we shall show that there is only one function

$$d: \{(f, \Omega, y): \Omega \subset \mathbb{R}^n \text{ open and bounded, } f: \bar{\Omega} \to \mathbb{R}^n \text{ continuous, } y \in \mathbb{R}^n \setminus f(\partial\Omega)\} \to \mathbb{Z}$$

satisfying

(d1) $d(\mathrm{id}, \Omega, y) = 1$ for $y \in \Omega$

(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1, Ω_2 are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

(d3) $d(h(t, \cdot), \Omega, y(t))$ is independent of $t \in J = [0, 1]$ whenever $h: J \times \bar{\Omega} \to \mathbb{R}^n$ is continuous, $y: J \to \mathbb{R}^n$ is continuous and $y(t) \notin h(t, \partial\Omega)$ for all $t \in J$.

This will be done by reduction to more agreeable conditions, the final one being the case where f is linear, i.e. $f(x) = Ax$ with $\det A \neq 0$. During the simplifying process we introduce basic tools which are also needed for the construction of the function d in § 2, and you will see already here that the homotopy invariance (d3) of d is a very powerful property.

Let us start with some notation for the whole chapter.

1.1 Notation. We let $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$ with $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. For subsets $A \subset \mathbb{R}^n$ we use the usual symbols $\bar{A}$, ∂A to denote the closure and the boundary of A, respectively. If also $B \subset \mathbb{R}^n$ then $B \setminus A = \{x \in B : x \notin A\}$, which may be the empty set $\emptyset$. The open and the closed ball of centre x_0 and radius $r > 0$ will be denoted by

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\} = x_0 + B_r(0) \quad \text{and} \quad \bar{B}_r(x_0) = \overline{B_r(x_0)}.$$

Unless otherwise stated, Ω is always an open bounded subset of $\mathbb{R}^n$.

For maps $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ we let $f(A) = \{f(x) : x \in A\}$ and $f^{-1}(y) = \{x \in A : f(x) = y\}$. The identity of $\mathbb{R}^n$ is denoted by id, i.e. $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$. Linear maps will be identified with their matrix $A = (a_{ij})$ and we write $\det A$ for the determinant of A. We shall also use *L. Kronecker's symbol* δ_{ij}, defined by $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, so that $\mathrm{id} = (\delta_{ij})$. If $B \subset \mathbb{R}^n$ is compact, i.e. closed and bounded, then $C(B)$ is the space of continuous $f: B \to \mathbb{R}^n$, and we let $|f|_0 = \max_B |f(x)|$ for $f \in C(B)$. We shall write $f \in C(B; \mathbb{R}^m)$ to emphasize $f(B) \subset \mathbb{R}^m$, if necessary.

You will recall that $f: \Omega \to \mathbb{R}^n$ is said to be differentiable at x_0 if there is a matrix $f'(x_0)$ such that

$$f(x_0 + h) = f(x_0) + f'(x_0)\,h + \omega(h) \qquad \text{for } h \in \Omega - x_0 = \{x - x_0 : x \in \Omega\}$$

where the remainder $\omega(h)$ satisfies $|\omega(h)| \leqq \varepsilon\,|h|$ for $|h| \leqq \delta = \delta(\varepsilon, x_0)$. In this case $f'(x_0)_{ij} = \partial_j f_i(x_0) = \partial f_i(x_0)/\partial x_j$, the partial derivative of the ith component f_i with respect to x_j.

At several points in this book it will be more convenient to use *E. Landau's symbol* instead of the $\varepsilon - \delta$ formulation of the condition for the remainder, i.e. we shall say that '$\omega(h) = o(|h|)$ as $h \to 0$' iff $|\omega(h)|/|h| \to 0$ as $|h| \to 0$. Thus differentiability of f at x_0 means $f(x_0 + h) - f(x_0) - f'(x_0)h = o(|h|)$ as $h \to 0$. The formal advantage consists in the freedom to write things like $\alpha o(|h|) = o(|h|)$ if α is constant, or $\omega_1(h) + \omega_2(h) = o(|h|)$ if $\omega_i(h) = o(|h|)$ for $i = 1, 2$, etc. We denote by

$C^k(\Omega)$ the set of $f: \Omega \to \mathbb{R}^n$ which are k-times continuously differentiable in Ω, while $\bar{C}^k(\Omega) = C^k(\Omega) \cap C(\bar{\Omega})$ and $\bar{C}^\infty(\Omega) = \bigcap_{k \geqq 1} \bar{C}^k(\Omega)$. If $f'(x_0)$ exists then $J_f(x_0) = \det f'(x_0)$ is the *Jacobian* of f at x_0, and x_0 is called a *critical point* of f if $J_f(x_0) = 0$. Since these points play an important role we also introduce $S_f(\Omega) = \{x \in \Omega: J_f(x_0) = 0\}$ and write S_f for brevity whenever Ω is clear from the context. Furthermore, a point $y \in \mathbb{R}^n$ will be called a *regular value* of $f: \Omega \to \mathbb{R}^n$ if $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, and a *singular value* otherwise.

In general $\mathbb{R}$-valued maps will be denoted by Greek letters while we shall use Latin letters for vector-valued functions.

1.2 From $C(\bar{\Omega})$ to $\bar{C}^\infty(\Omega)$. The first step in the reduction is to show that d is already uniquely determined by its values on $\bar{C}^\infty$-functions. To this end let us mention the following two facts.

Proposition 1.1. *Let $A \subset \mathbb{R}^n$ be compact and $f: A \to \mathbb{R}^n$ continuous. Then f can be extended continuously to $\mathbb{R}^n$, i.e. there exists a continuous $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{f}(x) = f(x)$ for $x \in A$.*

Proof. Since A is compact, there exists a dense and at most denumerable subset $\{a^1, a^2, \dots\}$ of A. Let $\varrho(x, A)$ be the distance of the point x to A, i.e. $\varrho(x, A) = \inf\{|x - a|: a \in A\}$, and

$$\varphi_i(x) = \max\left\{2 - \frac{|x - a^i|}{\varrho(x, A)}, 0\right\} \quad \text{for } x \notin A.$$

Then

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in A \\ \left(\sum_{i \geqq 1} 2^{-i} \varphi_i(x)\right)^{-1} \sum_{i \geqq 1} 2^{-i} \varphi_i(x) f(a^i) & \text{for } x \notin A \end{cases}$$

defines a continuous extension of f. If you find this difficult, it does not matter, since we shall give a detailed proof of a much more general extension theorem later on. □

Proposition 1.2. (a) *Let $A \subset \mathbb{R}^n$ be compact, $f \in C(A)$ and $\varepsilon > 0$. Then there exists a function $g \in C^\infty(\mathbb{R}^n)$ such that $|f(x) - g(x)| \leqq \varepsilon$ on A.*
(b) *Given $f \in \bar{C}^1(\Omega)$, $\varepsilon > 0$ and $\delta > 0$ such that $\Omega_\delta = \{x \in \Omega: \varrho(x, \partial\Omega) \geqq \delta\} \neq \emptyset$, there exists $g \in C^\infty(\mathbb{R}^n)$ such that $|f - g|_0 + \max_{\Omega_\delta} |f'(x) - g'(x)| \leqq \varepsilon$.*

Proof. Let $\tilde{f}$ be a continuous extension of f to $\mathbb{R}^n$ and let

$$f_\alpha(x) = \int_{\mathbb{R}^n} \tilde{f}(\xi)\, \varphi_\alpha(\xi - x)\, d\xi \quad \text{for } x \in \mathbb{R}^n \quad \text{and} \quad \alpha > 0,$$

where $(\varphi_\alpha)_{\alpha>0}$ is the family of 'mollifiers' $\varphi_\alpha\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi_1(x) = \begin{cases} c \cdot \exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

with $c > 0$ such that $\int\limits_{\mathbb{R}^n} \varphi_1(x)\,dx = 1$, and $\varphi_\alpha(x) = \alpha^{-n}\varphi_1(x/\alpha)$. We have $\varphi_\alpha \in C^\infty(\mathbb{R}^n)$, $\int\limits_{\mathbb{R}^n} \varphi_\alpha(x)\,dx = 1$ and $\bar{B}_\alpha(0)$ is the support of φ_α, i.e.

$$\operatorname{supp}\varphi_\alpha = \overline{\{x \in \mathbb{R}^n : \varphi_\alpha(x) \neq 0\}} = \bar{B}_\alpha(0),$$

for every $\alpha > 0$. Therefore $f_\alpha \in C^\infty(\mathbb{R}^n)$ and $f_\alpha(x) \to f(x)$ as $\alpha \to 0$ uniformly on A. Hence $g = f_\alpha$ with α sufficiently small satisfies part (a). The second part follows by differentiation of $f_\alpha(x) = \int\limits_{\mathbb{R}^n} f(\xi + x)\,\varphi_\alpha(\xi)\,d\xi$ for $x \in \Omega_\delta$ and $\alpha < \delta$. □

Now, consider $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)$. Then $\alpha = \varrho(y, f(\partial\Omega)) > 0$ and we find $g \in \bar{C}^\infty(\Omega)$ such that $|f - g|_0 < \alpha$. The function $h\colon [0,1] \times \bar{\Omega} \to \mathbb{R}^n$, defined by $h(t,x) = (1-t)\,f(x) + t\,g(x)$, is continuous and we have $|h(t,x) - y| \geqq |f(x) - y| - |f - g|_0 > 0$ on $\partial\Omega$. Therefore, (d3) with $y(t) \equiv y$ implies $d(f, \Omega, y) = d(g, \Omega, y)$. This concludes the first step.

1.3 From Singular to Regular Values. Let $f \in \bar{C}^\infty(\Omega)$ and $y \notin f(\partial\Omega)$. If y is a regular value of f then $f(x) = y$ has at most finitely many solutions. To see this, let us recall

Proposition 1.3 (Inverse Function Theorem). *Let $f \in C^1(\Omega)$ and $J_f(x_0) \neq 0$ for some $x_0 \in \Omega$. Then there exists a neighbourhood U of x_0 such that $f_{|U}$ is a homeomorphism onto a neighbourhood of $f(x_0)$.*

If you do not remember the standard proof by means of Banach's fixed point theorem, you should not be frustrated since we shall prove the theorem in a more general setting later on.

Thus, if y is regular then we have $J_f(x) \neq 0$ whenever $f(x) = y$, and Proposition 1.3 implies that these solutions are isolated, i.e. to $x_0 \in f^{-1}(y)$ there exists $U(x_0)$ such that $f^{-1}(y) \cap U(x_0) = \{x_0\}$. Consequently, $f^{-1}(y)$ must be finite. Otherwise there would be an accumulation point $x_0 \in \bar{\Omega}$ of solutions, by the compactness of $\bar{\Omega}$. Since f is continuous this would imply $f(x_0) = y$ and therefore $x_0 \in \Omega$ since $y \notin f(\partial\Omega)$. Hence, x_0 is an isolated solution, a contradiction.

Now, let $y_0 \notin f(\partial\Omega)$ be any point. Then $B_\alpha(y_0) \cap f(\partial\Omega) = \emptyset$ for $\alpha = \varrho(y_0, f(\partial\Omega))$. Therefore, (d3) with $h(t,x) = f(x)$, $y(t) = t\,y_0 + (1-t)\,y$ and $y \in B_\alpha(y_0)$ implies

(1) $$d(f, \Omega, y) = d(f, \Omega, y_0) \qquad \text{for every } y \in B_\alpha(y_0).$$

Since our next proposition guarantees in particular that $B_\alpha(y_0)$ contains regular values of f, it will then be enough to consider such values.

Proposition 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. Then $\mu_n(f(S_f)) = 0$, where μ_n denotes the n-dimensional Lebesgue measure.*

Proof. All you need to know here about μ_n is that $\mu_n(J) = \prod_{i=1}^{n} (b_i - a_i)$ for the interval $J = [a, b] \subset \mathbb{R}^n$ and that $M \subset \mathbb{R}^n$ has measure zero (i.e. $\mu_n(M) = 0$) iff to every $\varepsilon > 0$ there exist at most countably many intervals J_i such that $M \subset \bigcup_i J_i$ and $\sum_i \mu_n(J_i) \leqq \varepsilon$. Then it is easy to see that an at most countable union of sets of measure zero also has measure zero.

Since an open set Ω in $\mathbb{R}^n$ may be written as a countable union of cubes, say $\Omega = \bigcup_i Q_i$, it is therefore sufficient to show $\mu_n(f(S_f(Q))) = 0$ for a cube $Q \subset \Omega$, since $f(S_f(\Omega)) = \bigcup_i f(S_f(Q_i))$. Let ϱ be the lateral length of Q. By the uniform continuity of f' on Q, given $\varepsilon > 0$, we then find $m \in \mathbb{N}$ such that $|f'(x) - f'(\bar{x})| \leqq \varepsilon$ for all $x, \bar{x} \in Q$ with $|x - \bar{x}| \leqq \delta = \sqrt{n}\,\varrho/m$, and therefore

$$|f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x})| \leqq \int_0^1 |f'(\bar{x} + t(x - \bar{x})) - f'(\bar{x})|\,|x - \bar{x}|\,dt$$
$$\leqq \varepsilon |x - \bar{x}|$$

for any such $x, \bar{x}$. So let us decompose Q into r cubes Q^k of diameter δ. Since $\delta/\sqrt{n}$ is the lateral length of Q^k, we have $r = m^n$ and

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + R(x, \bar{x}) \quad \text{with} \quad |R(x, \bar{x})| \leqq \varepsilon\delta \quad \text{for} \quad x, \bar{x} \in Q^k.$$

Now, suppose that $Q^k \cap S_f \neq \emptyset$, choose $\bar{x} \in Q^k \cap S_f$, let $A = f'(\bar{x})$ and $g(y) = f(\bar{x} + y) - f(\bar{x})$ for $y \in \tilde{Q}^k = Q^k - \bar{x}$. Then we have

$$g(y) = Ay + \tilde{R}(y) \quad \text{with} \quad |\tilde{R}(y)| = |R(\bar{x} + y, \bar{x})| \leqq \varepsilon\delta \quad \text{on} \quad \tilde{Q}^k.$$

Since $\det A = 0$, we know that $A(\tilde{Q}^k)$ is contained in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Hence, there exists $b^1 \in \mathbb{R}^n$ with $|b^1| = 1$ and $(x, b^1) = \sum_{i=1}^{n} x_i b_i^1 = 0$ for all $x \in A(\tilde{Q}^k)$. Extending b^1 to an orthonormal base $\{b^1, \ldots, b^n\}$ of $\mathbb{R}^n$, we have $g(y) = \sum_{i=1}^{n} (g(y), b^i)\, b^i$ with

$$|(g(y), b^1)| = |(\tilde{R}(y), b^1)| \leqq |\tilde{R}(y)|\,|b^1| \leqq \varepsilon\delta$$

and

$$|(g(y), b^i)| \leqq |A|\,|y| + |\tilde{R}(y)| \leqq |A|\,\delta + \varepsilon\delta \quad \text{for } i = 2, \ldots, n,$$

where $|A| = |(a_{ij})| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}$. Thus, $f(Q^k) = f(\bar{x}) + g(\tilde{Q}^k)$ is contained in an interval J_k around $f(\bar{x})$ satisfying

$$\mu_n(J_k) = [2(|A|\,\delta + \varepsilon\delta)]^{n-1} \cdot 2\varepsilon\delta = 2^n(|A| + \varepsilon)^{n-1}\,\varepsilon\delta^n.$$

Since f' is bounded on the large cube Q, we have $|f'(x)| \leqq c$ for some c, in particular $|A| \leqq c$. Therefore, $f(S_f(Q)) \subset \bigcup_{k=1}^{r} J_k$ with

$$\sum_{k=1}^{r} \mu_n(J_k) \leqq r \cdot 2^n(c+\varepsilon)^{n-1}\varepsilon\delta^n = 2^n(c+\varepsilon)^{n-1}(\sqrt{n}\,\varrho)^n\varepsilon,$$

i.e. $f(S_f(Q))$ has measure zero, since $\varepsilon > 0$ is arbitrary. □

Let us remark that Proposition 1.4 is a special case of *Sard's lemma*: If $\Omega \subset \mathbb{R}^n$ is open, $f \in C^1(\Omega)$ and $\Omega^* \subset \Omega$ measurable, then $f(\Omega^*)$ is measurable and $\mu_n(f(\Omega^*)) \leqq \int_{\Omega^*} |J_f(x)|\,dx$; see e.g. Schwartz [2] for a complete proof.

1.4 From C^∞-Maps to Linear Maps. At the present level we only need to consider $f \in \bar{C}^\infty(\Omega)$ and $y \notin f(\partial\Omega \cup S_f)$.

Suppose first that $f^{-1}(y) = \emptyset$. From (d2) with $\Omega_1 = \Omega$ and $\Omega_2 = \emptyset$ we obtain $d(f, \emptyset, y) = 0$, and therefore $d(f, \Omega, y) = d(f, \Omega_1, y)$ whenever Ω_1 is an open subset of Ω such that $y \notin f(\bar{\Omega}\backslash\Omega_1)$. Hence $f^{-1}(y) = \emptyset$ implies $d(f, \Omega, y) = d(f, \emptyset, y) = 0$. In case $f^{-1}(y) = \{x^1, \ldots, x^n\}$, we choose disjoint neighbourhoods U_i of x^i and obtain $d(f, \Omega, y) = \sum_{i=1}^{n} d(f, U_i, y)$ from (d2). To compute $d(f, U_i, y)$, let $A = f'(x^i)$ and notice that

$$f(x) = y + A(x - x^i) + o(|x - x^i|) \quad \text{as} \quad |x - x^i| \to 0.$$

Since $\det A \neq 0$ we know that A^{-1} exists, and therefore $|z| = |A^{-1}Az| \leqq |A^{-1}|\,|Az|$, i.e. $|Az| \geqq c|z|$ on $\mathbb{R}^n$ for some $c > 0$. By means of this estimate we see that $y(t) = ty$ and $h(t, x) = tf(x) + (1-t)A(x - x^i)$ satisfy

$$|h(t,x) - y(t)| = |A(x - x^i) + t \cdot o(|x - x^i|)| \geqq c|x - x^i| - o(|x - x^i|) > 0$$

for all $t \in [0, 1]$ provided that $|x - x^i| \leqq \delta$ with $\delta > 0$ sufficiently small. Hence $d(f, B_\delta(x^i), y) = d(A - Ax^i, B_\delta(x^i), 0)$ by (d3). Since $f(x) \neq y$ in $\bar{U}_i\backslash B_\delta(x^i)$, we also have $d(f, U_i, y) = d(f, B_\delta(x^i), y)$ by (d2), and therefore

$$d(f, U_i, y) = d(A - Ax^i, B_\delta(x^i), 0).$$

Since x^i is the only solution of $Ax - Ax^i = 0$, (d2) implies

$$d(A - Ax^i, B_\delta(x^i), 0) = d(A - Ax^i, B_r(0), 0)$$

for $B_r(0) \supset B_\delta(x^i)$, and $A(x - tx^i) \neq 0$ on $[0, 1] \times \partial B_r(0)$ yields

$$d(f, U_i, y) = d(f'(x^i), B_r(0), 0),$$

by (d3). Finally, $r > 0$ may now be arbitrary, by (d2). Thus, we have arrived at a very simple situation and you will see that

1.5 Linear Algebra May Help. The only thing that remains to be shown is that $d(A, B_r(0), 0)$ is uniquely determined if A is a linear map with $\det A \neq 0$. It turns out that $d(A, \Omega, 0) = \operatorname{sgn} \det A$, the sign of $\det A$. The proof of this result requires some basic facts from linear algebra which you will certainly have seen unless you slept through those lessons which prepared, for example, Jordan's canonical form of a matrix. If you did, it is sufficient to accept that our next proposition is true since we shall prove a more general result in a later chapter.

Proposition 1.5. *Let A be a real $n \times n$ matrix with $\det A \neq 0$, let $\lambda_1, \ldots, \lambda_m$ be the negative eigenvalues of A and $\alpha_1, \ldots, \alpha_m$ their multiplicities as zeros of $\det(A - \lambda\,\mathrm{id})$, provided that A has such eigenvalues at all. Then $\mathbb{R}^n$ is the direct sum of two subspaces N and M, $\mathbb{R}^n = N \oplus M$, such that*

(a) *N and M are invariant under A.*
(b) *$A_{|N}$ has only the eigenvalues $\lambda_1, \ldots, \lambda_m$ and $A_{|M}$ has no negative eigenvalues.*
(c) $\dim N = \sum_{k=1}^{m} \alpha_k$.

Let $\det(A - \lambda\,\mathrm{id}) = (-1)^n \prod_{k=1}^{m} (\lambda - \lambda_k)^{\alpha_k} \prod_{j=m+1}^{n} (\lambda - \mu_j)^{\beta_j}$. Then

$$\det A = (-1)^\alpha \prod_{k=1}^{m} |\lambda_k|^{\alpha_k} \prod_{j=m+1}^{n} \mu_j^{\beta_j} \quad \text{with} \quad \alpha = \sum_{k=1}^{m} \alpha_k = \dim N,$$

hence $\operatorname{sgn} \det A = (-1)^\alpha$.

Now, if A has no negative eigenvalues then $\det(tA + (1-t)\,\mathrm{id}) \neq 0$ in $[0, 1]$, and therefore $d(A, B_r(0), 0) = d(\mathrm{id}, B_r(0), 0) = 1 = \operatorname{sgn} \det A$ by (d 3) and (d 1). So, let us consider the case $N \neq \{0\}$ and let us write Ω for $B_r(0)$.

Step 1. Suppose that $\alpha = \dim N$ is even. Since $\mathbb{R}^n = N \oplus M$, every $x \in \mathbb{R}^n$ has a unique representation $x = P_1 x + P_2 x$ with $P_1 x \in N$ and $P_2 x \in M$. Thus we have defined linear projections $P_1 \colon \mathbb{R}^n \to N$ and $P_2 = \mathrm{id} - P_1 \colon \mathbb{R}^n \to M$. Then $A = AP_1 + AP_2$ is a direct decomposition of A since $A(N) \subset N$ and $A(M) \subset M$ by Proposition 1.5(a). Now, since AP_1 has only negative eigenvalues and AP_2 has no negative eigenvalues by Proposition 1.5(b), it is easy to see that A is homotopic to $-P_1 + P_2$. We claim that

$$(2) \qquad h(t, x) = tAx + (1-t)(-P_1 x + P_2 x) \neq 0 \quad \text{on} \quad [0, 1] \times \partial\Omega .$$

To see this, notice first that $h(0, x) = 0$ implies $P_1 x = P_2 x$, hence $P_1 x = P_2 x \in N \cap M = \{0\}$ and therefore $x = 0$. Next, $h(t, x) = 0$ with $t \neq 0$ means

$$AP_1 x = \lambda P_1 x \in N \quad \text{and} \quad AP_2 x = -\lambda P_2 x \in M \quad \text{with} \quad \lambda = t^{-1}(1-t) > 0$$

which is possible only for $P_1 x = P_2 x = 0$, by the remark on the eigenvalues of AP_1 and AP_2. Hence, (2) holds and (d 3) implies $d(A, \Omega, 0) = d(-P_1 + P_2, \Omega, 0)$. Now, since $\alpha = 2p$ for some $p \geqq 1$, we find an $\alpha \times \alpha$ matrix B such that $B^2 = -\mathrm{id}_{|N}$. Indeed, for $p = 1$ you may choose a rotation by $\pi/2$, i.e. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and for general

p you may arrange p such blocks along the main diagonal, i.e.

$$b_{2j-1,2j} = 1 = -b_{2j,2j-1} \quad \text{for } j = 1, \dots, p \quad \text{and } b_{jk} = 0$$

for all other j, k. Since B has only complex eigenvalues we find homotopies from $-P_1 + P_2$ to $BP_1 + P_2$ and from $BP_1 + P_2$ to $\mathrm{id} = P_1 + P_2$, namely $tBP_1 - (1-t)P_1 + P_2$ and $tBP_1 + (1-t)P_1 + P_2$, as you may easily check. Hence

$$d(A, \Omega, 0) = d(-P_1 + P_2, \Omega, 0) = d(\mathrm{id}, \Omega, 0) = 1 = (-1)^{2p} = \operatorname{sgn} \det A.$$

Step 2. Let us finally assume that $\alpha = \dim N = 2p + 1$ for some $p \geqq 0$. Then we may decompose $N = N_1 \oplus N_2$, with $\dim N_1 = 1$ and $\dim N_2 = 2p$, which yields projections $\tilde{Q}_1 : N \to N_1$ and $\tilde{Q}_2 = \mathrm{id}|_N - \tilde{Q}_1 : N \to N_2$. Then $P_1 = \tilde{Q}_1 P_1 + \tilde{Q}_2 P_1$ and as in the first step we find homotopies, indicated by $\to$, such that

$$A \to -P_1 + P_2 \to -\tilde{Q}_1 P_1 + B\tilde{Q}_2 P_1 + P_2 \to -\tilde{Q}_1 P_1 + \tilde{Q}_2 P_1 + P_2.$$

Hence $d(A, \Omega, 0) = d(-Q_1 + Q_2, \Omega, 0)$ with $Q_1 = \tilde{Q}_1 P_1$ and $Q_2 = \tilde{Q}_2 P_1 + P_2$. Notice that Q_1 and $Q_2 = \mathrm{id} - Q_1$ are the projections from the decomposition $\mathbb{R}^n = N_1 \oplus (N_2 \oplus M)$. Since $x = 0$ is the only zero of $-Q_1 + Q_2$ we may also replace $\Omega = B_r(0)$ by any open bounded set containing $x = 0$, without changing d, for example by $B_r(0) \cap N_1 + \tilde{B}_r(0)$ with $\tilde{B}_r(0) = B_r(0) \cap (N_2 \oplus M)$; recall that $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$.

Now, you will see immediately that we are essentially in a one-dimensional situation. Indeed, given $\Omega \subset N_1$ open and bounded and $g : \bar{\Omega} \to N_1$ continuous with $0 \notin g(\partial\Omega)$, let $\tilde{d}(g, \Omega, 0) = d(g \circ Q_1 + Q_2, \Omega + \tilde{B}_r(0), 0)$.

Then you will convince yourself that (d 1) – (d 3) imply

($\tilde{\mathrm{d}}$1) $\tilde{d}(\mathrm{id}|_{N_1}, \Omega, 0) = 1$ for $0 \in \Omega$.

($\tilde{\mathrm{d}}$2) $\tilde{d}(g, \Omega, 0) = \tilde{d}(g, \Omega_1, 0) + \tilde{d}(g, \Omega_2, 0)$ whenever Ω_1, Ω_2 are disjoint open subsets of $\Omega \subset N_1$ and $0 \notin g(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

($\tilde{\mathrm{d}}$3) $d(h(t, \cdot), \Omega, 0)$ is constant on $J = [0, 1]$ whenever $h : J \times \bar{\Omega} \to N_1$ is continuous and $0 \notin h(J \times \partial\Omega)$.

In this notation we have to compute

$$\tilde{d}(-\mathrm{id}|_{N_1}, \Omega, 0) = d(-Q_1 + Q_2, \Omega + \tilde{B}_r(0), 0),$$

where $\Omega \subset N$ is any open bounded set with $0 \in \Omega$. Since we guess $\tilde{d}(-\mathrm{id}|_{N_1}, \Omega, 0) = -1 = (-1)^{2p+1} = \operatorname{sgn} \det A$ and since ($\tilde{\mathrm{d}}$1) is the only concrete thing we have at hand, it is natural to look for a function g and sets $\Omega \supset \Omega_1 \cup \Omega_2$ such that $\tilde{d}(g, \Omega, 0) = 0$, $g|_{\Omega_1}$ is homotopic to $-\mathrm{id}|_{\Omega_1}$ and $g|_{\Omega_2}$ is homotopic $\mathrm{id}|_{\Omega_2}$, since then $\tilde{d}(-\mathrm{id}|_{N_1}, \Omega_1, 0) = -\tilde{d}(\mathrm{id}|_{N_1}, \Omega_2, 0) = -1$, by ($\tilde{\mathrm{d}}$2) and ($\tilde{\mathrm{d}}$3). This is roughly the idea of the

Last step. Since $\dim N_1 = 1$, we have $N_1 = \{\lambda e : \lambda \in \mathbb{R}\}$ for some $e \in \mathbb{R}^n$ with $|e| = 1$. Consider

$$\Omega = \{\lambda e : \lambda \in (-2, 2)\}, \quad \Omega_1 = \{\lambda e : \lambda \in (-2, 0)\}, \quad \Omega_2 = \{\lambda e : \lambda \in (0, 2)\}$$

and $f(\lambda e) = (|\lambda| - 1)\, e$. Since $f(0) = -e \neq 0$ and $h(t, \lambda e) = t(|\lambda| - 2)\, e + e \neq 0$ on $[0, 1] \times \partial\Omega$, we have

$$0 = \tilde{d}(e, \Omega, 0) = \tilde{d}(f, \Omega, 0) = \tilde{d}(f, \Omega_1, 0) + \tilde{d}(f, \Omega_2, 0)$$

by ($\tilde{\text{d}}$2), ($\tilde{\text{d}}$3) and ($\tilde{\text{d}}$2) again. Now, $f|_{\Omega_1}(\lambda e) = -(\lambda + 1)\, e$ has the only zero $-e \in \Omega_1 \subset \Omega$, whence

$$\tilde{d}(f, \Omega_1, 0) = \tilde{d}(-\operatorname{id}|_{N_1} - e, \Omega, 0) = \tilde{d}(-\operatorname{id}|_{N_1}, \Omega, 0),$$

since also $-\lambda e - te \neq 0$ on $[0, 1] \times \partial\Omega$. By the same argument we obtain $\tilde{d}(f, \Omega_2, 0) = \tilde{d}(\operatorname{id}|_{N_1}, \Omega, 0)$, and therefore $\tilde{d}(-\operatorname{id}|_{N_1}, \Omega, 0) = -1$, as we wanted to show. Thus we have proved

Theorem 1.1. *Let*

$$M = \{(f, \Omega, y) \colon \Omega \subset \mathbb{R}^n \textit{ open bounded, } f \in C(\bar{\Omega}) \textit{ and } y \in \mathbb{R}^n \backslash f(\partial\Omega)\}.$$

Then there exists at most one function $d \colon M \to \mathbb{Z}$ with the properties (d 1) – (d 3). *Furthermore, these properties imply that $d(A, \Omega, 0) = \operatorname{sgn} \det A$ for linear maps A with $\det A \neq 0$ and $0 \in \Omega$.*

Having seen that homotopies and linear algebra are useful, you will certainly enjoy the following

Exercises

1. Let A be a real $n \times n$ matrix and $e^A = \sum_{m \geq 0} \frac{A^m}{m!}$. Then $\det e^A > 0$. *Hint*: Consider e^{tA}.

2. Let A be a real $n \times n$ matrix with $\det A > 0$. Then there exists a continuous map H from $[0, 1]$ into the space of all $n \times n$ matrices such that $H(0) = \operatorname{id}$, $H(1) = A$ and $\det H(t) > 0$ in $[0, 1]$. *Hint*: The proof is hidden in § 1.5.

§ 2. Construction of the Degree

At the end of § 1 we reached the simplest situation. Now, progress by stages to the general case.

2.1 The Regular Case. It will be convenient to start with

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in \bar{C}^1(\Omega)$ and $y \in \mathbb{R}^n \backslash f(\partial\Omega \cup S_f)$. Then we define

$$d(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x) \qquad \left(\text{agreement: } \sum_{\emptyset} = 0\right).$$

In the sequel, the main difficulty will be to get rid of the assumption $y \notin f(S_f)$. We already know that this exceptional set has measure zero, and since such sets are immaterial when we integrate, let us replace $\sum \operatorname{sgn} J_f(x)$ by a suitable integral.

Proposition 2.1. *Let Ω, f and y be as in Definition 2.1 and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be the mollifiers from the proof to Proposition 1.2. Then there exists $\varepsilon_0 = \varepsilon_0(y, f)$ such that $d(f, \Omega, y) = \int_\Omega \varphi_\varepsilon(f(x) - y)\, J_f(x)\, dx$ for $0 < \varepsilon \leqq \varepsilon_0$.*

Proof. The case $f^{-1}(y) = \emptyset$ is trivial since $\varphi_\varepsilon(f(x) - y) \equiv 0$ for $\varepsilon < \alpha = \varrho(y, f(\bar{\Omega}))$. If $f^{-1}(y) = \{x^1, \ldots, x^m\}$, then we find disjoint balls $B_\varrho(x^i)$ such that $f|_{B_\rho(x^i)}$ is a homeomorphism onto a neighbourhood V_i of y and such that $\operatorname{sgn} J_f(x) = \operatorname{sgn} J_f(x^i)$ in $B_\varrho(x^i)$. Let $B_r(y) \subset \bigcap_{i=1}^m V_i$ and $U_i = B_\varrho(x^i) \cap f^{-1}(B_r(y))$. Then $|f(x) - y| \geqq \beta$ on $\bar{\Omega} \setminus \bigcup_{i=1}^m U_i$ for some $\beta > 0$, and therefore $\varepsilon < \beta$ implies

$$\int_\Omega \varphi_\varepsilon(f(x) - y)\, J_f(x)\, dx = \sum_{i=1}^m \operatorname{sgn} J_f(x^i) \int_{U_i} \varphi_\varepsilon(f(x) - y)\, |J_f(x)|\, dx .$$

Since $J_f(x) = J_{f-y}(x)$ and $f(U_i) - y = B_r(0)$, the well-known substitution formula for integrals yields

$$\int_{U_i} \varphi_\varepsilon(f(x) - y)\, |J_{f-y}(x)|\, dx = \int_{B_r(0)} \varphi_\varepsilon(x)\, dx = 1 \qquad \text{for } \varepsilon < \min\{\beta, r\}. \quad \square$$

2.2 From Regular to Singular Values. Consider $f \in \bar{C}^2(\Omega)$ and $y_0 \notin f(\partial\Omega)$. Let $\alpha = \varrho(y_0, f(\partial\Omega))$ and suppose that $y^1, y^2 \in B_\alpha(y_0)$ are two regular values of f. Finally, let $\delta = \alpha - \max\{|y^i - y_0|: i = 1, 2\}$. By Proposition 2.1 we find $\varepsilon < \delta$ such that

$$d(f, \Omega, y^i) = \int_\Omega \varphi_\varepsilon(f(x) - y^i)\, J_f(x)\, dx \qquad \text{for } i = 1, 2.$$

We shall show that these integrals are equal and then we may define $d(f, \Omega, y_0)$ as $d(f, \Omega, y^1)$ since we know that regular values y^1 exist in $B_\alpha(y_0)$. To prove that the difference of the integrals is zero, notice first that

$$\varphi_\varepsilon(x - y^2) - \varphi_\varepsilon(x - y^1) = \operatorname{div} w(x)$$
$$\text{for } w(x) = (y^1 - y^2) \int_0^1 \varphi_\varepsilon(x - y^1 + t(y^1 - y^2))\, dt,$$

since $\operatorname{div} w(x) = \sum_{i=1}^n \partial w_i(x)/\partial x_i$ be definition. Furthermore $\operatorname{supp} w \subset \bar{B}_r(y_0)$ for $r = \alpha - (\delta - \varepsilon) < \alpha$, since $\operatorname{supp} \varphi_\varepsilon = \bar{B}_\varepsilon(0)$. This implies in particular that $f(\partial\Omega) \cap \operatorname{supp} w = \emptyset$. We shall show in a minute that this property enables us to find a map $v \in C^1(\mathbb{R}^n)$ such that $\operatorname{supp} v \subset \Omega$ and

$$(1) \qquad [\varphi_\varepsilon(f(x) - y^2) - \varphi_\varepsilon(f(x) - y^1)]\, J_f(x) = \operatorname{div} v(x) \quad \text{in} \quad \Omega.$$

Then we are done, since integration over a cube $Q = [-a, a]^n$ such that $\Omega \subset Q$ yields

$$d(f, \Omega, y^2) - d(f, \Omega, y^1) = \int_\Omega \operatorname{div} v(x)\, dx = \int_Q \operatorname{div} v(x)\, dx$$
$$= \sum_{i=1}^n \int_{-a}^a \cdots \int_{-a}^a \left(\int_{-a}^a \frac{\partial v_i}{\partial x_i}\, dx_i \right) dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n = 0.$$

To find v we need an old formula which is well-known for people familiar with differential forms. Since others may not have seen it, let us prove

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in C^2(\Omega)$ and $d_{ij}(x)$ the cofactor of $\partial f_j(x)/\partial x_i$ in $J_f(x)$, i.e. $d_{ij}(x)$ is $(-1)^{i+j}$ times the determinant which you obtain from $J_f(x)$ cancelling the jth row and the ith column. Then*

$$\sum_{i=1}^{n} \frac{\partial d_{ij}(x)}{\partial x_i} = 0 \qquad \text{for } j = 1, \ldots, n.$$

Proof. Fix j, let $\partial_k = \partial/\partial x_k$ and let f_{x_k} denote the column $(\partial_k f_1, \ldots, \partial_k f_{j-1}, \partial_k f_{j+1}, \ldots, \partial_k f_n)$. Then

$$d_{ij}(x) = (-1)^{i+j} \det (f_{x_1}, \ldots, f_{x_{i-1}}, \hat{f}_{x_i}, f_{x_{i+1}}, \ldots, f_{x_n}),$$

where the hat indicates cancellation. Since a determinant is linear in each column, you may easily check that

$$\partial_i d_{ij}(x) = (-1)^{i+j} \sum_{k=1}^{n} \det (f_{x_1}, \ldots, \hat{f}_{x_i}, \ldots, f_{x_{k-1}}, \partial_i f_{x_k}, f_{x_{k+1}}, \ldots, f_{x_n}).$$

Let $c_{ki} = \det(\partial_i f_{x_k}, f_{x_1}, \ldots, \hat{f}_{x_i}, \ldots, \hat{f}_{x_k}, \ldots, f_{x_n})$. Then $c_{ki} = c_{ik}$ since $f \in C^2(\Omega)$, and since the sign of det changes whenever we permute two adjacent columns, we obtain

$$(-1)^{i+j} \partial_i d_{ij}(x) = \sum_{k<i} (-1)^{k-1} c_{ki} + \sum_{k>i} (-1)^{k-2} c_{ki} = \sum_{k=1}^{n} (-1)^{k-1} \sigma_{ki} c_{ki}$$

with $\sigma_{ki} = 1$ for $k < i$, $\sigma_{ii} = 0$ and $\sigma_{ki} = -\sigma_{ik}$ for all i, k. Therefore

$$\begin{aligned}(-1)^j \sum_{i=1}^{n} \partial_i d_{ij}(x) &= \sum_{i,k=1}^{n} (-1)^{k-1+i} \sigma_{ki} c_{ki} = \sum_{k,i=1}^{n} (-1)^{i-1+k} \sigma_{ik} c_{ik} \\ &= - \sum_{i,k=1}^{n} (-1)^{k-1+i} \sigma_{ki} c_{ki}\end{aligned}$$

i.e. the sum is zero. □

Now, let us define $v_i(x) = \sum_{j=1}^{n} w_j(f(x))\, d_{ij}(x)$ on $\bar{\Omega}$ and $v_i(x) = 0$ on $\mathbb{R}^n \setminus \bar{\Omega}$, for $i = 1, \ldots, n$. Then $\operatorname{supp} w \subset \bar{B}_r(y_0) \subset B_\alpha(y_0)$ implies $\operatorname{supp} v \subset \Omega$, and we have

$$\partial_i v_i(x) = \sum_{j,k=1}^{n} d_{ij}(x)\, \partial_k w_j(f(x))\, \partial_i f_k(x) + \sum_{j=1}^{n} w_j(f(x))\, \partial_i d_{ij}(x).$$

Since $\sum_{i=1}^{n} d_{ij}(x)\, \partial_i f_k(x) = \delta_{jk} J_f(x)$ with Kronecker's δ_{jk}, Proposition 2.2 yields

$$\operatorname{div} v(x) = \sum_{k,j=1}^{n} \partial_k w_j(f(x))\, \delta_{jk} J_f(x) = \operatorname{div} w(f(x))\, J_f(x),$$

i.e. the formula (1). Thus we have justified

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in \bar{C}^2(\Omega)$ and $y \notin f(\partial\Omega)$. Then we define $d(f, \Omega, y) = d(f, \Omega, y^1)$, where y^1 is any regular value of f such that $|y^1 - y| < \varrho(y, f(\partial\Omega))$ and $d(f, \Omega, y^1)$ is given by Definition 1.1.

2.3 From $\bar{C}^2(\Omega)$ to $C(\bar{\Omega})$. In this final step we shall show that the degree of Definition 2.2 is the same for all $\bar{C}^2(\Omega)$-functions sufficiently near to a given continuous map. To this end we use a special case of the implicit function theorem which is appropriate for the present purpose. A more general result will be proved in a later chapter.

Proposition 2.3. *Let $h: \mathbb{R} \times \Omega \to \mathbb{R}^n$ be continuously differentiable, $h(t_0, x_0) = 0$ and $J_{h(t_0,\cdot)}(x_0) \neq 0$ for some $(t_0, x_0) \in \mathbb{R} \times \Omega$. Then there exist an interval $(t_0 - r, t_0 + r)$, a ball $B_\delta(x_0) \subset \Omega$ and a continuous function $x: (t_0 - r, t_0 + r) \to B_\delta(x_0)$ such that $x(t_0) = x_0$ and $x(t)$ is the only solution in $B_\delta(x_0)$ of $h(t, x) = 0$.*

Now, let us prove

Proposition 2.4. *Let $f \in \bar{C}^2(\Omega)$ and $y \notin f(\partial\Omega)$. Then, for $g \in \bar{C}^2(\Omega)$ there exists a $\delta = \delta(f, y, g) > 0$ such that $d(f + tg, \Omega, y) = d(f, \Omega, y)$ for $|t| < \delta$.*

Proof. 1. In case $f^{-1}(y) = \emptyset$ it is obvious that $f(x) + tg(x) \neq y$ in $\bar{\Omega}$ for $|t|$ sufficiently small, and therefore both degrees are zero.

2. Let $f^{-1}(y) = \{x^1, \ldots, x^m\}$ and $J_f(x^i) \neq 0$ for $i = 1, \ldots, m$, $f_t = f + tg$ and $h(t, x) = f_t(x) - y$. We have $h(0, x^i) = 0$ and $J_{h(0,\cdot)}(x^i) = J_f(x^i) \neq 0$. By Proposition 2.3 we therefore find an interval $(-r, r)$, disjoint balls $\bar{B}_\varrho(x^i)$ and continuous functions $z^i: (-r, r) \to B_\varrho(x^i)$ such that $f_t^{-1}(y) \cap V = \{z^1(t), \ldots, z^m(t)\}$ for $V = \bigcup_{i=1}^{m} B_\varrho(x^i)$. We choose ϱ also so small that $\operatorname{sgn} J_f(x) = \operatorname{sgn} J_f(x^i)$ on $\bar{B}_\varrho(x^i)$. Since $|f(x) - y| > \beta$ in $\bar{\Omega} \backslash V$ for some $\beta > 0$, we even have

$$f_t^{-1}(y) = \{z^1(t), \ldots, z^m(t)\} \qquad \text{for } |t| < \delta_0 = \min\{r, \beta |g|_0^{-1}\}.$$

Finally, since $J_{f_t}(x)$ is continuous in (t, x), we find $\delta \leqq \delta_0$ such that $|J_{f_t}(x) - J_f(x)| < \min\{|J_f(z)|: z \in \bar{V}\}$ for $|t| < \delta$ and $x \in \bar{V}$. Hence, $\operatorname{sgn} J_{f_t}(z^i(t)) = \operatorname{sgn} J_f(z^i(t)) = \operatorname{sgn} J_f(x^i)$, that is, $d(f_t, \Omega, y) = d(f, \Omega, y)$ for $|t| < \delta$, by Definition 2.1.

3. For the last case, suppose that y is not regular. Then we choose a regular $y_0 \in B_{\alpha/3}(y)$, where $\alpha = \varrho(y, f(\partial\Omega))$, and we find a $\delta_0 > 0$ such that $d(f_t, \Omega, y_0) = d(f, \Omega, y_0) = d(f, \Omega, y)$ for $|t| < \delta_0$, by the second step. Let $\delta = \min\{\delta_0, \frac{1}{3}|g|_0^{-1}\alpha\}$. Then $|y_0 - f_t(x)| > \alpha/3$ for $x \in \partial\Omega$ and $|t| < \delta$, and therefore $|y_0 - y| < \varrho(y_0, f_t(\partial\Omega))$. Thus, $d(f_t, \Omega, y_0) = d(f_t, \Omega, y)$ by Definition 2.2. □

By means of this result it is now easy to see that the degree is constant on all C^2-maps sufficiently close to a continuous map. Indeed, let $f \in C(\bar{\Omega})$, $y \notin f(\partial\Omega)$ and $\alpha = \varrho(y, f(\partial\Omega))$. Consider two functions $g, \tilde{g} \in \bar{C}^2(\Omega)$ such that $|g - f|_0 < \alpha$ and $|\tilde{g} - f|_0 < \alpha$, let $h(t, x) = g(x) + t(\tilde{g}(x) - g(x))$ and $\varphi(t) = d(h(t, \cdot), \Omega, y)$ for $t \in [0, 1]$. Since $h(t, \cdot) = h(t_0, \cdot) + (t - t_0)(\tilde{g} - g)$, Proposition 2.4 tells us that $\varphi(t)$

is constant in a neighbourhood of t_0. Thus, φ is continuous on $[0, 1]$ and since this interval is a connected set, $\varphi([0, 1])$ is connected too, i.e. φ is constant in $[0, 1]$; in particular, $d(g, \Omega, y) = d(\tilde{g}, \Omega, y)$. Hence, we have our final

Definition 2.3. Let $f \in C(\bar{\Omega})$ and $y \in \mathbb{R}^n \setminus f(\partial\Omega)$. Then we define $d(f, \Omega, y) := d(g, \Omega, y)$, where $g \in \bar{C}^2(\Omega)$ is any map such that $|g - f|_0 < \varrho(y, f(\partial\Omega))$ and $d(g, \Omega, y)$ is given by Definition 2.2.

Now, you will have no difficulty in proving that d satisfies (d 1)–(d 3), by reduction to the regular case. After so much theory you will find some light relief in the following

Exercises

1. (a) Let $\Omega \subset \mathbb{R}$ be an open interval with $0 \in \Omega$ and let $f(x) = \alpha x^k$ with $\alpha \neq 0$. Then $d(f, \Omega, 0) = 0$ if k is even and $d(f, \Omega, 0) = \operatorname{sgn} \alpha$ if k is odd.
(b) Let $g(x) = f(x) + \sum_{i=0}^{k-1} \alpha_i x^i$ for $x \in \mathbb{R}$, with f from (a). Then $d(g, (-r, r), 0) = d(f, (-r, r), 0)$ for sufficiently large r.
(c) Let $[a, b] \subset \mathbb{R}$, $f: [a, b] \to \mathbb{R}$ continuous and such that $f(a) f(b) \neq 0$. Then $d(f, (a, b), 0) = \frac{1}{2}(\operatorname{sgn} f(b) - \operatorname{sgn} f(a))$. *Hint*: Consider $g(x) = \alpha x + \beta$ such that $g(a) = f(a)$ and $g(b) = f(b)$ and show that g is homotopic to f.

2. Let $n = 1$ and show that d is surjective, i.e. for $m \in \mathbb{Z}$ there exists an admissible $(f, \Omega, 0)$ such that $d(f, \Omega, 0) = m$.

3. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x, y) = x^3 - 3xy^2$ and $f_2(x, y) = -y^3 + 3x^2 y$, and let $a = (1, 0)$. Then $d(f, B_2(0), a) = 3$.

4. Let $\Omega = B_1(0) = \{z \in \mathbb{C} \cong \mathbb{R}^2 : |z| < 1\}$, $y = 0$ and

$$h(t, z) = \begin{cases} |z| & \text{for } t = 0, \quad z \in \bar{\Omega} \\ |z| \exp(i\varphi/t) & \text{for } 0 < t \leqq 1,\ z = |z| e^{i\varphi} \text{ and } \quad 0 \leqq \varphi \leqq 2\pi t \\ |z| & \text{for } 0 < t < 1,\ z = |z| e^{i\varphi} \text{ and } 2\pi t < \varphi \leqq 2\pi . \end{cases}$$

You will easily verify that $h(t, \cdot)$ and $h(\cdot, z)$ are continuous on $\bar{\Omega}$ and $[0, 1]$, respectively. Furthermore, $h(t, z) \neq 0$ on $[0, 1] \times \partial\Omega$ and $d(h(t, \cdot), \Omega, 0) = 1$. Finally, $h(0, \cdot)$ is homotopic to $f(z) \equiv (1, 0)$; consider, for example, $g(s, z) = s(|z|, 0) + (1 - s)(1, 0)$. Therefore $d(h(0, \cdot), \Omega, 0) = 0$, a contradiction with (d 3)?

§ 3. Further Properties of the Degree

This is an appropriate point to show that the degree is useful. Let us start with

3.1 Consequences of (d 1)–(d 3). The basic properties (d 1)–(d 3) immediately yield some simple consequences which we are going to list as (d 4)–(d 7) in the following

Theorem 3.1. *Let $M = \{(f, \Omega, y): \Omega \subset \mathbb{R}^n$ open bounded, $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)\}$ and $d: M \to \mathbb{Z}$ the topological degree defined by Definition 2.3. Then d has the following properties.*

(d1) *$d(\mathrm{id}, \Omega, y) = 1$ for $y \in \Omega$.*

(d2) *$d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} \backslash (\Omega_1 \cup \Omega_2))$.*

(d3) *$d(h(t, \cdot), \Omega, y(t))$ is independent of t whenever $h: [0,1] \times \bar{\Omega} \to \mathbb{R}^n$ and $y: [0,1] \to \mathbb{R}^n$ are continuous and $y(t) \notin h(t, \partial\Omega)$ for every $t \in [0,1]$.*

(d4) *$d(f, \Omega, y) \neq 0$ implies $f^{-1}(y) \neq \emptyset$.*

(d5) *$d(\cdot, \Omega, y)$ and $d(f, \Omega, \cdot)$ are constant on $\{g \in C(\bar{\Omega}): |g - f|_0 < r\}$ and $B_r(y) \subset \mathbb{R}^n$, respectively, where $r = \varrho(y, f(\partial\Omega))$. Moreover, $d(f, \Omega, \cdot)$ is constant on every connected component of $\mathbb{R}^n \backslash f(\partial\Omega)$.*

(d6) *$d(g, \Omega, y) = d(f, \Omega, y)$ whenever $g|_{\partial\Omega} = f|_{\partial\Omega}$*

(d7) *$d(f, \Omega, y) = d(f, \Omega_1, y)$ for every open subset Ω_1 of Ω such that $y \notin f(\bar{\Omega} \backslash \Omega_1)$.*

Proof. At the beginning of § 1.4 we saw that (d2) implies (d7) and $d(f, \Omega, y) = 0$ if $f^{-1}(y) = \emptyset$, and so (d4) follows. Next, (d6) follows from (d3) with $y(t) \equiv y$ and $h(t, \cdot) = tf + (1-t)g$. The first two parts of (d5) are obvious by Definition 2.3 or by (d3), as you prefer. For the last part, recall first that a (connected) component is a connected set which is maximal (with respect to inclusion) in the connected sets. Since $\mathbb{R}^n \backslash f(\partial\Omega)$ is open, its components are open, and for open sets in $\mathbb{R}^n$ connectedness is the same as arcwise connectedness. Therefore, if C is a component of $\mathbb{R}^n \backslash f(\partial\Omega)$ and y^1, y^2 are points in C, we find a continuous curve $y: [0,1] \to C$ with $y(0) = y^1$ and $y(1) = y^2$; hence the last part follows from (d3) again. $\square$

3.2 Brouwer's Fixed Point Theorem. You have no doubt met situations where one wants to solve equations of type $f(x) = x$, and you know that such points x are called fixed points of the map f. Before we state a fairly general result on existence of fixed points of a continuous map $f: D \subset \mathbb{R}^n \to D$, let us recall that D is said to be *convex* if $\lambda x + (1-\lambda) y \in D$ whenever $x, y \in D$ and $\lambda \in [0,1]$, that the intersection of convex sets is also convex and that the *convex hull* of D, conv D for short, is defined as the intersection of all convex sets which contain D. From these definitions it is clear that D is convex iff $D = \operatorname{conv} D$, and it is easy to see that

$$\operatorname{conv} D = \left\{ \sum_{i=1}^{n} \lambda_i x^i : x^i \in D;\ \lambda_i \in [0,1] \text{ and } \sum_{i=1}^{n} \lambda_i = 1;\ n \in \mathbb{N} \right\}.$$

Theorem 3.2 (Brouwer). *Let $D \subset \mathbb{R}^n$ be a nonempty compact convex set and $f: D \to D$ continuous. Then f has a fixed point. The same is true if D is only homeomorphic to a compact convex set.*

Proof. Suppose first that $D = \bar{B}_r(0)$. We may assume that $f(x) \neq x$ on ∂D since otherwise we are done. Let $h(t, x) = x - tf(x)$. This defines a continuous $h: [0,1] \times D \to \mathbb{R}^n$ such that $0 \notin h([0,1] \times \partial D)$, since by assumption $|h(t,x)| \geqq |x| - t|f(x)| \geqq (1-t) r > 0$ in $[0,1) \times \partial D$ and $f(x) \neq x$ for $|x| = r$. Therefore $d(\mathrm{id} - f, \mathring{D}, 0) = d(\mathrm{id}, B_r(0), 0) = 1$, and this proves the existence of an $x \in B_r(0)$ such that $x - f(x) = 0$, by (d4).

Next, let D be a general compact and convex set. By Proposition 1.1 we have a continuous extension $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ of f, and if you look at the defining formula in the proof of this result you see that $\tilde{f}(\mathbb{R}^n) \subset \overline{\operatorname{conv} f(D)} \subset D$ since

$\left[\sum_{i=1}^{m} 2^{-i}\varphi_i(x)\right]^{-1} \sum_{i=1}^{m} 2^{-i}\varphi_i(x)\, f(a^i)$ is defined for $m = m(x)$ sufficiently large, and belongs to conv $f(D)$. Now, we choose a ball $\bar{B}_r(0) \supset D$, and we find a fixed point x of $\tilde{f}$ in $\bar{B}_r(0)$, by the first step. But $\tilde{f}(x) \in D$ and therefore $x = \tilde{f}(x) = f(x)$.

Finally, assume that $D = h(D_0)$ with D_0 compact convex and h a homeomorphism. Then $h^{-1}fh\colon D_0 \to D_0$ has a fixed point x by the second step and therefore $f(h(x)) = h(x) \in D$. □

Let us illustrate this important theorem by some examples.

Example 3.1 (Perron-Frobenius). Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} \geqq 0$ for all i, j. Then there exist $\lambda \geqq 0$ and $x \neq 0$ such that $x_i \geqq 0$ for every i and $Ax = \lambda x$. In other words, A has a nonnegative eigenvector corresponding to a nonnegative eigenvalue.

To prove this result, let

$$D = \left\{x \in \mathbb{R}^n\colon x_i \geqq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} x_i = 1\right\}.$$

If $Ax = 0$ for some $x \in D$, then we are done, with $\lambda = 0$. If $Ax \neq 0$ in D, then $\sum_{i=1}^{n} (Ax)_i \geqq \alpha$ in D for some $\alpha > 0$. Therefore, $f\colon x \to Ax / \sum_{i=1}^{n} (Ax)_i$ is continuous in D, and $f(D) \subset D$ since $a_{ij} \geqq 0$ for all i, j. By Theorem 3.2 we have a fixed point of f, i.e. an $x_0 \in D$ such that $Ax_0 = \lambda x_0$ with $\lambda = \sum_{i=1}^{n} (Ax_0)_i$. You will find more results of this type e.g. in Varga [1] and Schäfer [3].

Example 3.2. Consider the system of ordinary differential equations $u' = f(t, u)$, where $u' = \dfrac{du}{dt}$ and $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is ω-periodic in t, i.e. $f(t + \omega, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Then it is natural to look for ω-periodic solutions. Suppose, for simplicity, that f is continuous and that there is a ball $B_r(0)$ such that the initial value problems

$$(1) \qquad u' = f(t, u), \qquad u(0) = x \in \bar{B}_r(0)$$

have a unique solution $u(t; x)$ on $[0, \infty)$. If you do not remember conditions on f which guarantee this property of (1), you will meet them in a later chapter as easy exercises to Banach's fixed point theorem.

Now, let $P_t x = u(t; x)$ and suppose also that f satisfies the boundary condition $(f(t, x), x) = \sum_{i=1}^{n} f_i(t, x)\, x_i < 0$ for $t \in [0, \omega]$ and $|x| = r$.

Then, we have $P_t\colon \bar{B}_r(0) \to \bar{B}_r(0)$ for every $t \in \mathbb{R}^+$, since

$$\frac{d}{dt}|u(t)|^2 = 2(u'(t), u(t)) = 2(f(t, u(t)), u(t)) < 0$$

if the solution u of (1) takes a value in $\partial B_r(0)$ at time t. Furthermore, P_t is continuous, as follows easily from our assumption that (1) has only one solution. Thus

P_ω has a fixed point $x_\omega \in \bar{B}_r(0)$, i.e. $u' = f(t, u)$ has a solution such that $u(0; x_\omega) = x_\omega = u(\omega; x_\omega)$. Now, you may easily check that $v: [0, \infty) \to \mathbb{R}^n$, defined by $v(t) = u(t - k\omega; x_\omega)$ on $[k\omega, (k+1)\omega]$, is an ω-periodic solution of (1). The map P_ω is usually called the *Poincaré operator* of $u' = f(t, u)$, and it is now evident that $u(\cdot; x)$ is an ω-periodic solution iff x is a fixed point of P_ω. The problem of existence of periodic solutions to differential equations will be considered in later chapters too.

Example 3.3. It is impossible to retract the whole unit ball continuously onto its boundary such that the boundary remains pointwise fixed, i.e. there is no continuous $f: \bar{B}_1(0) \to \partial B_1(0)$ such that $f(x) = x$ for all $x \in \partial B_1(0)$.

Otherwise $g = -f$ would have a fixed point x_0, by Theorem 3.2, but this implies $|x_0| = 1$ and therefore $x_0 = -f(x_0) = -x_0$, which is nonsense. This result is in fact equivalent to Brouwer's theorem for the ball. To see this, suppose that $f: \bar{B}_1(0) \to \bar{B}_1(0)$ is continuous and has no fixed point. Let $g(x)$ be the point where the line segment from $f(x)$ to x hits $\partial B_1(0)$, i.e. $g(x) = f(x) + t(x)(x - f(x))$, where $t(x)$ is the positive root of

$$t^2 |x - f(x)|^2 + 2t(f(x), x - f(x)) + |f(x)|^2 = 1 .$$

Since $t(x)$ is continuous, g would be such a retraction which does not exist by assumption.

3.3 Surjective Maps. In this section we shall show that a certain growth condition on $f \in C(\mathbb{R}^n)$ implies $f(\mathbb{R}^n) = \mathbb{R}^n$. Let us consider first $f_0(x) = Ax$ with a positive definite matrix A. Since $\det A \neq 0$, f_0 is surjective. We also have $(f_0(x), x) \geqq c|x|^2$ for some $c > 0$ and every $x \in \mathbb{R}^n$, and therefore $(f_0(x), x)/|x| \to \infty$ as $|x| \to \infty$. This condition is sufficient for surjectivity in the nonlinear case too, since we can prove

Theorem 3.3. *Let $f \in C(\mathbb{R}^n)$ be such that $(f(x), x)/|x| \to \infty$ as $|x| \to \infty$. Then $f(\mathbb{R}^n) = \mathbb{R}^n$.*

Proof. Given $y \in \mathbb{R}^n$, let $h(t, x) = tx + (1 - t) f(x) - y$. At $|x| = r$ we have

$$(h(t, x), x) \geqq r[tr + (1 - t)(f(x), x)/|x| - |y|] > 0$$

for $t \in [0, 1]$ and $r > |y|$ sufficiently large. Therefore, $d(f, B_r(0), y) = 1$ for such an r, i.e. $f(x) = y$ has a solution. □

Another way to prove $f(\mathbb{R}^n) = \mathbb{R}^n$ is to look for conditions on f implying that $f(\mathbb{R}^n)$ is both open and closed and to use the connectedness of $\mathbb{R}^n$. This will be done later.

3.4 The Hedgehog Theorem. Up to now we have applied the homotopy invariance of the degree as it stands. However, it is also useful to use the converse namely: if two maps f and g have different degree then a certain h that connects f and g cannot be a homotopy. Along these lines we shall prove

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open bounded with $0 \in \Omega$ and let $f: \partial\Omega \to \mathbb{R}^n \setminus \{0\}$ be continuous. Suppose also that the space dimension n is odd. Then there exist $x \in \partial\Omega$ and $\lambda \neq 0$ such that $f(x) = \lambda x$.*

Proof. Without loss of generality we may assume $f \in C(\bar{\Omega})$, by Proposition 1.1. Since n is odd, we have $d(-\mathrm{id}, \Omega, 0) = -1$. If $d(f, \Omega, 0) \neq -1$, then $h(t, x) = (1-t) f(x) - tx$ must have a zero $(t_0, x_0) \in (0,1) \times \partial\Omega$. Therefore, $f(x_0) = t_0(1-t_0)^{-1} x_0$. If, however, $d(f, \Omega, 0) = -1$ then we apply the same argument to $h(t, x) = (1-t) f(x) + tx$. □

Since the dimension is odd in this theorem, it does not apply in $\mathbb{C}^n$. In fact, the rotation by $\frac{\pi}{2}$ of the unit sphere in $\mathbb{C} = \mathbb{R}^2$, i.e. $f(x_1, x_2) = (-x_2, x_1)$, is a simple counterexample. In case $\Omega = B_1(0)$ the theorem tells us that there is at least one normal such that f changes at most its orientation. In other words: there is no continuous nonvanishing tangent vector field on $S = \partial B_1(0)$, i.e. an $f: S \to \mathbb{R}^n$ such that $f(x) \neq 0$ and $(f(x), x) = 0$ on S. In particular, if $n = 3$ this means, that 'a hedgehog cannot be combed without leaving tufts or whorls'. However, $f(x) = (x_2, -x_1, \ldots, x_{2m}, -x_{2m-1})$ is a nonvanishing tangent vector field on $S \subset \mathbb{R}^{2m}$.

Having reached this level you should have no difficulty with the following

Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be open bounded, $f \in C(\bar{\Omega})$, $g \in C(\bar{\Omega})$ and $|g(x)| < |f(x)|$ on $\partial\Omega$. Then $d(f+g, \Omega, 0) = d(f, \Omega, 0)$. For analytic functions this result is known as Rouché's theorem. *Hint*: Use (d3).

2. The system $2x + y + \sin(x+y) = 0$, $x - 2y + \cos(x+y) = 0$ has a solution in $B_r(0) \subset \mathbb{R}^2$, where $r > 1/\sqrt{5}$.

3. Let $\Omega = B_1(0) \subset \mathbb{R}^n$, $f \in C(\bar{\Omega})$ and $0 \notin f(\bar{\Omega})$. Then there exist $x, y \in \partial\Omega$ and $\lambda > 0$, $\mu < 0$ such that $f(x) = \lambda x$ and $f(y) = \mu y$, i.e. f has a positive and negative eigenvalue, each with an eigenvector in $\partial\Omega$.

4. Let $\Omega = B_1(0) \subset \mathbb{R}^{2m+1}$ and $f: \partial\Omega \to \partial\Omega$ continuous. Then there exists an $x \in \partial\Omega$ such that either $x = f(x)$ or $x = -f(x)$.

5. Let A be a real $n \times n$ matrix with $\det A \neq 0$ and $f \in C(\mathbb{R}^n)$ such that $|x - Af(x)| \leqq \alpha |x| + \beta$ on $\mathbb{R}^n$ for some $\alpha \in [0, 1)$ and $\beta \geqq 0$. Then $f(\mathbb{R}^n) = \mathbb{R}^n$.

6. Consider, as in Example 3.2, the ODE $u' = f(t, u)$ in $\mathbb{R}^n$ with ω-periodic f such that the IVPs $u' = f(t, u)$, $u(0) = x$ have a unique solution $u(t; x)$ on $[0, \infty)$. Let us call $x \in \mathbb{R}^n$ ω-irreversible if $u(t; x) \neq x$ in $(0, \omega]$. Suppose that $\Omega \subset \mathbb{R}^n$ is open bounded, $0 \notin f(0, \partial\Omega)$ and every $x \in \partial\Omega$ is ω-irreversible. Then $d(\mathrm{id} - P_\omega, \Omega, 0) = d(-f(0, \cdot), \Omega, 0)$. Example 3.2 is a special case of this result, which is from Krasnoselskii [3]. *Hint*: Consider the homotopy, defined by

$$h(t, x) = \begin{cases} (x - u(\omega t; x)) \cdot \left(\dfrac{1-t}{t\omega} + t\right) & \text{for } t \neq 0 \\ -f(0, x) & \text{for } t = 0. \end{cases}$$

7. Let A be a symmetric $n \times n$ matrix and let $s_1 > s_2 > \ldots > s_n$ be given real numbers. Some applications require the determination of a diagonal matrix $V = \mathrm{diag}(v_1, \ldots, v_n)$ such that $A + V$ has the eigenvalues $s_1, \ldots, s_n$ (inverse eigenvalue problem).

Let $g_j = \sum_{k \neq j} |a_{jk}|$ and $s_j - s_{j+1} > 2 \max\{g_j, g_{j+1}\}$ for $j = 1, \ldots, n-1$. Then such a V exists, satisfying in addition $|v_j - s_j| \leqq g_j$ for $j = 1, \ldots, n$.

Similarly, given a positive definite A and $s_1 > \ldots > s_n > 0$, find a positive diagonal matrix V such that VA has the eigenvalues $s_1, \ldots, s_n$. This problem has a solution if $s_j - s_{j+1} > 2 \max\{g_j, g_{j+1}\} s_1$ for $j = 1, \ldots, n-1$. *Hint*: Without loss of generality, $a_{ii} = 0$ in the first problem and $a_{ii} = 1$ in the second one; let $D = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ and consider $DV(D^{-1/2} A D^{-1/2}) D^{1/2}$ in the second case to see this. Consider

$$C = \{v \in \mathbb{R}^n : s_1 + \varepsilon \geqq v_1 \geqq v_2 \geqq \ldots \geqq v_n \geqq s_n - \varepsilon\} \quad \text{for some } \varepsilon > 0$$

and $H(t, v) = (\lambda_1(t), \ldots, \lambda_n(t)) \in \mathbb{R}^n$, where $\lambda_1(t) \geqq \ldots \geqq \lambda_n(t)$ are the eigenvalues of $tA + V$ and $V(I + t(A - I))$ in the first and second problem, respectively. Notice that $s = (s_1, \ldots, s_n) \in \mathring{C}$ and $H(0, \cdot) = \mathrm{id}$. The verification of $s \notin H(t, \partial C)$ for $t \in (0, 1]$ requires some knowledge about the Gerschgorin discs $\{\lambda : |\lambda - v_j| \leqq g_j\}$. These results are from Hadeler [2] where you will find the proofs. Applications are indicated in, for example, Hadeler [1].

§4. Borsuk's Theorem

Whenever we want to show by means of degree theory that $f(x) = y$ has a solution in Ω, we have to verify $d(f, \Omega, y) \neq 0$. The following result of Borsuk [2] helps a lot.

4.1 Borsuk's Theorem. Recall that Ω is said to be *symmetric* with respect to the origin if $\Omega = -\Omega$, and a map f on Ω is said to be *odd* if $f(-x) = -f(x)$ on Ω.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open bounded symmetric with $0 \in \Omega$. Let $f \in C(\bar{\Omega})$ be odd and $0 \notin f(\partial\Omega)$. Then $d(f, \Omega, 0)$ is odd.*

Proof. 1. We may assume that $f \in \bar{C}^1(\Omega)$ and $J_f(0) \neq 0$. To see this, approximate $f \in C(\bar{\Omega})$ by $g_1 \in \bar{C}^1(\Omega)$, consider the odd part $g_2(x) = \frac{1}{2}(g_1(x) - g_1(-x))$ and choose a δ which is not an eigenvalue of $g_2'(0)$. Then $\tilde{f} = g_2 - \delta \cdot \mathrm{id}$ is in $\bar{C}^1(\Omega)$, odd with $J_{\tilde{f}}(0) \neq 0$, and close to f if δ and $|g_1 - f|_0$ are chosen sufficiently small. Hence $d(f, \Omega, 0) = d(\tilde{f}, \Omega, 0)$.

2. Let $f \in \bar{C}^1(\Omega)$ and $J_f(0) \neq 0$. To prove the theorem, it suffices to show that there is an odd $g \in \bar{C}^1(\Omega)$ sufficiently close to f such that $0 \notin g(S_g)$, since then

$$d(f, \Omega, 0) = d(g, \Omega, 0) = \operatorname{sgn} J_g(0) + \sum_{0 \neq x \in g^{-1}(0)} \operatorname{sgn} J_g(x),$$

where the sum is even since $g(x) = 0$ iff $g(-x) = 0$ and $J_g(\cdot)$ is even.

3. Such a map g will be defined by induction as follows. Consider $\Omega_k = \{x \in \Omega : x_i \neq 0 \text{ for some } i \leqq k\}$ and an odd $\varphi \in C^1(\mathbb{R})$ such that $\varphi'(0) = 0$ and $\varphi(t) = 0$ iff $t = 0$.

Consider $\tilde{f}(x) = f(x)/\varphi(x_1)$ on the open bounded $\Omega_1 = \{x \in \Omega : x_1 \neq 0\}$. By Proposition 1.4, we find $y^1 \notin \tilde{f}(S_{\tilde{f}}(\Omega_1))$ with $|y^1|$ as small as necessary in the sequel. Hence, 0 is a regular value for $g_1(x) = f(x) - \varphi(x_1) y^1$ on Ω_1, since $g_1'(x) = \varphi(x_1) \tilde{f}'(x)$ for $x \in \Omega_1$ such that $g_1(x) = 0$. Now, suppose that we have already an odd $g_k \in \bar{C}^1(\Omega)$ close to f on $\bar{\Omega}$ such that $0 \notin g_k(S_{g_k}(\Omega_k))$, for some $k < n$. Then we define $g_{k+1}(x) = g_k(x) - \varphi(x_{k+1}) y^{k+1}$ with $|y^{k+1}|$ small and such that 0 is a regular value for g_{k+1} on $\{x \in \Omega : x_{k+1} \neq 0\}$.

Evidently, $g_{k+1} \in \bar{C}^1(\Omega)$ is odd and close to f on $\bar{\Omega}$. If $x \in \Omega_{k+1}$ and $x_{k+1} = 0$ then $x \in \Omega_k$, $g_{k+1}(x) = g_k(x)$ and $g'_{k+1}(x) = g'_k(x)$, hence $J_{g_{k+1}}(x) \neq 0$, and therefore $0 \notin g_{k+1}(S_{g_{k+1}}(\Omega_{k+1}))$. Thus, $g = g_n$ is odd, close to f on $\bar{\Omega}$ and such that $0 \notin g(S_g(\Omega \setminus \{0\}))$, since $\Omega_n = \Omega \setminus \{0\}$. By the induction step you see that we also have $g'(0) = g'_1(0) = f'(0)$; hence $0 \notin g(S_g(\Omega))$. □

This proof is from Gromes [1]. The following generalization is an immediate consequence of Theorem 4.1 and the homotopy invariance.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open bounded symmetric and $0 \in \Omega$. Let $f \in C(\bar{\Omega})$ be such that $0 \notin f(\partial\Omega)$ and $f(-x) \neq \lambda f(x)$ on $\partial\Omega$ for all $\lambda \geqq 1$. Then $d(f, \Omega, 0)$ is odd.*

Proof. $h(t, x) = f(x) - tf(-x)$ for $t \in [0, 1]$ defines a homotopy in $\mathbb{R}^n \setminus \{0\}$ between f and the odd g, defined by $g(x) = f(x) - f(-x)$. □

4.2 Some Applications of Borsuk's Theorem. The first result is known as the *Borsuk-Ulam theorem* and reads as follows:

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^n$ be as in Theorem* 4.1, *$f\colon \partial\Omega \to \mathbb{R}^m$ continuous and $m < n$. Then $f(x) = f(-x)$ for some $x \in \partial\Omega$.*

Proof. Suppose, on the contrary, that $g(x) = f(x) - f(-x) \neq 0$ on $\partial\Omega$ and let g be any continuous extension to $\bar{\Omega}$ of these boundary values. Then $d(g, \Omega, y) = d(g, \Omega, 0) \neq 0$ for all y in some ball $B_r(0)$, by Theorem 4.1 and (d5). Thus, (d4) implies that the $\mathbb{R}^n$-ball $B_r(0)$ is contained in $g(\bar{\Omega}) \subset \mathbb{R}^m$, which is nonsense. □

In the literature you will find the metereological interpretation that at two opposite ends of the earth we have the same weather, i.e. temperature and pressure ($n = 3$ and $m = 2$). Our second result tells us something about coverings of the boundary $\partial\Omega$. Sometimes it is called the *Lusternik-Schnirelmann-Borsuk theorem*, and it will play a role in later chapters.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open bounded and symmetric with respect to $0 \in \Omega$, and let $\{A_1, \ldots, A_p\}$ be a covering of $\partial\Omega$ by closed sets $A_i \subset \partial\Omega$ such that $A_i \cap (-A_i) = \emptyset$ for $i = 1, \ldots, p$. Then, $p \geqq n + 1$.*

Proof. Suppose that $p \leqq n$; let $f_i(x) = 1$ on A_i and $f_i(x) = -1$ on $-A_i$ for $i = 1, \ldots, p-1$ and $f_i(x) = 1$ on $\bar{\Omega}$ for $i = p, \ldots, n$. Extend the f_i with $i \leqq p - 1$ continuously to $\bar{\Omega}$ and let us show that f satisfies $f(-x) \neq \lambda f(x)$ on $\partial\Omega$ for every $\lambda \geqq 0$. Then $d(f, \Omega, 0) \neq 0$ by Corollary 4.1, i.e. $f(x) = 0$ for some $x \in \Omega$; a contradiction to $f_n(x) \equiv 1$ in $\bar{\Omega}$.

Now, $x \in A_p$ implies $-x \notin A_p$ and therefore $-x \in A_i$ for some $i \leqq p - 1$, i.e. $x \in -A_i$. Hence $\partial\Omega \subset \bigcup_{i=1}^{p-1} \{A_i \cup (-A_i)\}$. Let $x \in \partial\Omega$. Then $x \in A_i$ implies $f_i(x) = 1$ and $f_i(-x) = -1$, and $x \in -A_j$ implies $f_j(x) = -1$ and $f_j(-x) = 1$. Hence, $f(-x)$ doesn't point into the same direction as $f(x)$ in both cases. □

Thus you have seen, in particular, that you need at least $n + 1$ closed subsets A_i containing no antipodal points if you want to cover $\partial B_r(0) \subset \mathbb{R}^n$ by such sets. In this special case $n + 1$ of them are also enough; consider, for example, three arcs of length $\frac{2}{3}\pi$ in case $n = 2$.

Finally let us apply Theorem 4.1 to the problem of finding conditions sufficient for a continuous map f to be open, i.e. to map open subsets of its domain onto open sets, a property which does not follow from continuity alone as you will convince yourself by simple examples. The result is the *domain-invariance theorem* for maps f which are locally one-to-one, i.e. such that to every x in the domain of f there exists a neighbourhood $U(x)$ such that $f|_{U(x)}$ is one-to-one.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $f: \Omega \to \mathbb{R}^n$ continuous and locally one-to-one. Then f is an open map.*

Proof. It is sufficient to show that to $x_0 \in \Omega$ there exists a ball $B_r(x_0)$ such that $f(B_r(x_0))$ contains a ball with centre $f(x_0)$. Passing to $\Omega - x_0$ and $\tilde{f}(x) = f(x + x_0) - f(x_0)$ for $x \in \Omega - x_0$, if necessary, we see that we may assume $x_0 = 0$ and $f(0) = 0$. Let us choose $r > 0$ such that $f|_{\bar{B}_r(0)}$ is one-to-one and consider

$$h(t, x) = f\left(\frac{1}{1+t}x\right) - f\left(-\frac{t}{1+t}x\right) \quad \text{for } t \in [0, 1], \quad x \in \bar{B}_r(0).$$

Evidently, h is continuous in (t, x), $h(0, \cdot) = f$ and $h(1, x) = f(\frac{1}{2}x) - f(-\frac{1}{2}x)$ is odd. If $h(t, x) = 0$ for some $(t, x) \in [0, 1] \times \partial B_r(0)$, then $x/(1 + t) = -xt/(1 + t)$ since f is one-to-one, i.e. $x = 0$, a contradiction. Therefore,

$$d(f, B_r(0), y) = d(h(1, \cdot), B_r(0), 0) \neq 0$$

for every y in some ball $B_s(0)$ and this implies $B_s(0) \subset f(B_r(0))$. □

Theorem 4.3 may be used, for example, to prove surjectivity results for continuous maps $f: \mathbb{R}^n \to \mathbb{R}^n$. Suppose, for example, that f is locally one-to-one and $|f(x)| \to \infty$ as $|x| \to \infty$. Then we have $f(\mathbb{R}^n) = \mathbb{R}^n$. Indeed, $f(\mathbb{R}^n)$ is open by Theorem 4.3, but also closed since $f(x_n) \to y$ implies that (x_n) is bounded, hence $x_n \to x_0$ without loss of generality, and therefore $y = f(x_0)$. Thus, $f(\mathbb{R}^n) = \mathbb{R}^n$ since $\mathbb{R}^n$ is connected.

Now, you would no doubt like to do something by yourself. Here are some

Exercises

1. Let $P = \mathbb{C}^n \to \mathbb{C}$ be a homogenous polynomial of degree m, i.e. $P(z) = \sum_{|\alpha| = m} a_\alpha \cdot z^\alpha$ with $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, α_j nonnegative integers and $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Such a polynomial is said to be elliptic if $P(x) \neq 0$ in $\mathbb{R}^n \setminus \{0\}$. Show that m is even if P is elliptic and $n > 2$. *Hint*: Suppose m is odd; apply Theorem 4.1 to

$$f(x_1, x_2) = (\operatorname{Re} P(x_1, x_2, 0, \ldots, 0), \operatorname{Im} P(x_1, x_2, 0, \ldots, 0))$$

and consider

$$\tilde{f}(x_1, x_2) = (\operatorname{Re} P(x_1, x_2, \xi, 0, \ldots, 0), \operatorname{Im} P(x_1, x_2, 0, \ldots, 0))$$

for a sufficiently small $\xi \neq 0$. Such polynomials play an important role in the study of differential operators $\sum_{|\alpha| \leqq m} b_\alpha \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$ via Fourier transform. Notice that m may be odd if $n = 2$; consider, for example, $P(z_1, z_2) = z_1 + i z_2$, which corresponds to the Cauchy-Riemann operator $\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}$.

2. If $f\colon \{x \in \mathbb{R}^n \colon |x| = r\} \to \mathbb{R}^m$ with $m < n$ is continuous and odd then f has a zero.

3. 'The sandwich problem': Given n measurable bounded sets $A_1, \ldots, A_n$ in $\mathbb{R}^n$, there exists a hyperplane which cuts their volumes into equal halves ($n = 3$: A_1 = bread, A_2 = ham, A_3 = cheese, the hyperplane = a long knife). *Hint*: For $x \in \partial B_1(0) \subset \mathbb{R}^{n+1}$, let $H_x = \{y \in \mathbb{R}^{n+1} \colon (y, x) = x_{n+1}\}$ and $H_x^+ = \{y \in \mathbb{R}^{n+1} \colon (y, x) > x_{n+1}\}$. Then $f\colon \partial B_1(0) \to \mathbb{R}^n$ defined by $f_i(x) = \mu_n(A_i \cap H_x^+)$ is continuous.

4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, $J_f(x) \neq 0$ in $\mathbb{R}^n$ and $|f(x)| \to \infty$ as $|x| \to \infty$. Then $f(\mathbb{R}^n) = \mathbb{R}^n$. In a later chapter, you will prove that f is in fact a homeomorphism.

§ 5. The Product Formula

In this section we present a useful formula that relates the degree of a composed map gf to those of g and f. By means of this formula it is easy to prove Jordan's curve theorem, as you will see.

5.1 Preliminaries. Let $\Omega \subset \mathbb{R}^n$ be open bounded and $f \in C(\bar{\Omega})$. By (d5) we know that $d(f, \Omega, y)$ is the same integer for every y in a connected component K of $\mathbb{R}^n \backslash f(\partial\Omega)$. It will therefore be convenient to denote this integer by $d(f, \Omega, K)$. Since $f(\partial\Omega)$ is compact, we have one unbounded component K_∞ of $\mathbb{R}^n \backslash f(\partial\Omega)$ if $n > 1$, and two such components if $n = 1$, in which case K_∞ will denote the union of these two. In the sequel, K_∞ will play no role since it contains points $y \notin f(\bar{\Omega})$ and therefore $d(f, \Omega, K_\infty) = 0$.

5.2 The Product Formula. We shall write gf for the composition of g and f, i.e. $(gf)(x) = g(f(x))$. Then we have

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C(\bar{\Omega})$, $g \in C(\mathbb{R}^n)$ and K_i the bounded connected components of $\mathbb{R}^n \backslash f(\partial\Omega)$. Suppose that $y \notin (gf)(\partial\Omega)$. Then*

$$d(gf, \Omega, y) = \sum_i d(f, \Omega, K_i)\, d(g, K_i, y), \tag{1}$$

where only finitely many terms are different from zero.

Proof. 1. Let $f(\bar{\Omega}) \subset B_r(0)$. Since $M = \bar{B}_r(0) \cap g^{-1}(y)$ is compact and $M \subset \mathbb{R}^n \backslash f(\partial\Omega) = \bigcup_i K_i$, there are finitely many i, say $i = 1, \ldots, p$, such that $\bigcup_{i=1}^{p} K_i$ and $K_{p+1} = K_\infty \cap B_{r+1}(0)$ cover M. Then $d(f, \Omega, K_{p+1}) = 0$, and $d(g, K_j, y) = 0$ for $j \geqq p + 2$ since $K_j \subset B_r(0)$ and $g^{-1}(y) \cap K_j = \emptyset$ for these j. Therefore, the summation in (1) is finite.

2. Formula (1) is easy to check in the regular case. Therefore let us start with $f \in \bar{C}^1(\Omega)$, $g \in C^1(\mathbb{R}^n)$ and $y \notin gf(S_{gf})$. We have $(gf)'(x) = g'(f(x))\,f'(x)$ and therefore

$$\begin{aligned} d(gf, \Omega, y) &= \sum_{x \in (gf)^{-1}(y)} \operatorname{sgn} J_{gf}(x) = \sum_{\dots} \operatorname{sgn} J_g(f(x)) \operatorname{sgn} J_f(x) \\ &= \sum_{\substack{x \in f^{-1}(z) \\ z \in g^{-1}(y)}} \operatorname{sgn} J_g(z) \operatorname{sgn} J_f(x) = \sum_{\substack{z \in g^{-1}(y) \\ z \in f(\Omega)}} \operatorname{sgn} J_g(z) \left[\sum_{x \in f^{-1}(z)} \operatorname{sgn} J_f(x) \right] \\ &= \sum_{\substack{z \in f(\Omega) \\ g(z) = y}} \operatorname{sgn} J_g(z)\, d(f, \Omega, z). \end{aligned}$$

In the last sum we may replace '$z \in f(\Omega)$' by '$z \in \bar{B}_r(0) \backslash f(\partial\Omega)$' since $d(f, \Omega, z) = 0$ for $z \notin f(\Omega)$, and since the K_i are disjoint, we obtain

$$\begin{aligned} d(gf, \Omega, y) &= \sum_{i=1}^{p} \sum_{\substack{z \in K_i \\ z \in g^{-1}(y)}} \operatorname{sgn} J_g(z)\, d(f, \Omega, z) = \sum_{i=1}^{p} d(f, \Omega, K_i) \left[\sum_{\substack{z \in K_i \\ z \in g^{-1}(y)}} \operatorname{sgn} J_g(z) \right] \\ &= \sum_i d(f, \Omega, K_i)\, d(g, K_i, y). \end{aligned}$$

By definition of the degree, it is clear that (1) is also true if $y \in gf(S_{gf})$.

3. Now, let us consider the general case $f \in C(\bar{\Omega})$ and $g \in C(\mathbb{R}^n)$. Since the components may change when we pass to C^1-approximations, we shall write down all details. It will be convenient to rearrange the right-hand side of (1) as follows. Let

$$S_m = \{z \in B_{r+1}(0) \backslash f(\partial\Omega) : d(f, \Omega, z) = m\} \quad \text{and} \quad N_m = \{i \in \mathbb{N} : d(f, \Omega, K_i) = m\}.$$

Since $S_m = \bigcup_{i \in N_m} K_i$, we have by (d 3)

$$\sum_i d(f, \Omega, K_i)\, d(g, K_i, y) = \sum_m m \left[\sum_{i \in N_m} d(g, K_i, y) \right] = \sum_m m \cdot d(g, S_m, y).$$

Thus, we have to show

$$d(gf, \Omega, y) = \sum_m m \cdot d(g, S_m, y). \tag{2}$$

Since $\partial S_m \subset f(\partial\Omega)$, we find $g_0 \in C^1(\mathbb{R}^n)$ such that

$$d(g_0 f, \Omega, y) = d(gf, \Omega, y) \quad \text{and} \quad d(g_0, S_m, y) = d(g, S_m, y) \quad \text{for all } m, \tag{3}$$

and we may assume that $M_0 = \bar{B}_{r+1}(0) \cap g_0^{-1}(y)$ is not empty; otherwise (2) is trivially $0 = 0$ by (3). Since M_0 is compact and $y \notin g_0 f(\partial\Omega)$, we have

$$\varrho(M_0, f(\partial\Omega)) = \inf\{|x - z| : x \in M_0, z \in f(\partial\Omega)\} > 0.$$

Now, we choose $f_0 \in \bar{C}^1(\Omega)$ such that

$$|f - f_0|_0 = \max_{\bar{\Omega}} |f(x) - f_0(x)| < \varrho(M_0, f(\partial\Omega)) \quad \text{and} \quad f_0(\bar{\Omega}) \subset B_{r+1}(0)$$

and define

$$\tilde{S}_m = \{z \in B_{r+1}(0) \backslash f_0(\partial\Omega) \colon d(f_0, \Omega, z) = m\}.$$

Then we have the essential equality $S_m \cap M_0 = \tilde{S}_m \cap M_0$, since $z \in M_0$ implies $\varrho(z, f(\partial\Omega)) \geqq \varrho(M_0, f(\partial\Omega)) > |f - f_0|_0$ and therefore $d(f_0, \Omega, z) = d(f, \Omega, z)$ by (d5).

Evidently $S_m \cap M_0 = \tilde{S}_m \cap M_0$ implies that both sets are contained in $S_m \cap \tilde{S}_m$ and therefore

$$d(g_0, S_m, y) = d(g_0, S_m \cap \tilde{S}_m, y) = d(g_0, \tilde{S}_m, y), \tag{4}$$

by (d7). Thus, the second step, (3) and (4) yield

$$d(g_0 f_0, \Omega, y) = \sum_m m \cdot d(g_0, \tilde{S}_m, y) = \sum_m m \cdot d(g, S_m, y),$$

and by the first part of (3) it remains to be shown that $d(g_0 f_0, \Omega, y) = d(g_0 f, \Omega, y)$. But this follows from (d3) with $h(t, \cdot) = g_0(f + t(f_0 - f))$, since $y \in h([0,1] \times \partial\Omega)$ would imply $f(x) + t(f_0(x) - f(x)) \in M_0$ for some $(t, x) \in [0,1] \times \partial\Omega$, but

$$|z - f(x) - t(f_0(x) - f(x))| \geqq \varrho(M_0, f(\partial\Omega)) - |f - f_0|_0 > 0 \quad \text{for all } z \in M_0. \quad \square$$

5.3 Jordan's Separation Theorem. You will remember the famous 'obvious but hard to prove' curve theorem of C. Jordan, which says that a simple closed curve C in the plane divides the plane into two regions G_1 and G_2 such that $C = \partial G_1 = \partial G_2$ and $G_2 = \mathbb{R}^2 \backslash \bar{G}_1$. Since such a curve is homeomorphic to the unit circle $\partial B_1(0)$, and since $B_1(0)$ and $\mathbb{R}^2 \backslash \bar{B}_1(0)$ are the components of $\mathbb{R}^2 \backslash \partial B_1(0)$, the curve theorem may also be formulated as follows: if $C \subset \mathbb{R}^2$ is homeomorphic to $\partial B_1(0)$ then $\mathbb{R}^2 \backslash C$ has precisely two components. This version can be extended to $\mathbb{R}^n$, i.e. we have

Theorem 5.2. *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$ be compact sets which are homeomorphic to each other. Then $\mathbb{R}^n \backslash \Omega_1$ and $\mathbb{R}^n \backslash \Omega_2$ have the same number of connected components.*

Proof. Let $h \colon \Omega_1 \to \Omega_2$ be a homeomorphism onto Ω_2; $\tilde{h}$ a continuous extension of h to $\mathbb{R}^n$; $\widetilde{h^{-1}}$ a continuous extension to $\mathbb{R}^n$ of $h^{-1} \colon \Omega_2 \to \Omega_1$; K_j the bounded components of $\mathbb{R}^n \backslash \Omega_1$ and L_i those of $\mathbb{R}^n \backslash \Omega_2$. Notice that $\partial K_j \subset \Omega_1$ and $\partial L_i \subset \Omega_2$. Now, let us fix j and let G_q denote the components of $\mathbb{R}^n \backslash h(\partial K_j)$. Since

$$\bigcup_i L_i = \mathbb{R}^n \backslash \Omega_2 \subset \mathbb{R}^n \backslash h(\partial K_j) = \bigcup_q G_q,$$

we see that to every i there exists a q such that $L_i \subset G_q$; remember that components are maximal connected sets. In particular, $L_\infty \subset G_\infty$. Consider any $y \in K_j$.

Then (d 6) and Theorem 5.1 imply that

$$1 = d(\widetilde{h^{-1}}\tilde{h}, K_j, y) = \sum_q d(\tilde{h}, K_j, G_q)\, d(\widetilde{h^{-1}}, G_q, y).$$

If $N_q = \{i : L_i \subset G_q\}$, then $d(\widetilde{h^{-1}}, G_q, y) = \sum_{i \in N_q} d(\widetilde{h^{-1}}, L_i, y)$ by (d 2) and $d(\tilde{h}, K_j, G_q) = d(\tilde{h}, K_j, L_i)$ for every $i \in N_q$, by the definition of $d(\cdot, \cdot, K)$ for a component K. Therefore

$$1 = \sum_q \sum_{i \in N_q} d(\tilde{h}, K_j, L_i)\, d(\widetilde{h^{-1}}, L_i, y) = \sum_i d(\tilde{h}, K_j, L_i)\, d(\widetilde{h^{-1}}, L_i, K_j), \tag{5}$$

since $y \in K_j \subset \mathbb{R}^n \backslash h^{-1}(\Omega_2) \subset \mathbb{R}^n \backslash h^{-1}(\partial L_i)$. We may repeat the same argument with fixed L_i instead of K_j to obtain

$$1 = \sum_j d(\tilde{h}, K_j, L_i)\, d(\widetilde{h^{-1}}, L_i, K_j) \quad \text{for every } i. \tag{6}$$

If there are only m components L_i, then (5) and summation over i in (6) yields

$$m = \sum_{i=1}^m 1 = \sum_j \sum_{i=1}^m d(\tilde{h}, K_j, L_i)\, d(\widetilde{h^{-1}}, L_i, K_j) = \sum_j 1,$$

i.e. there are only m components K_j too, and conversely. Therefore $\mathbb{R}^n \backslash \Omega_1$ and $\mathbb{R}^n \backslash \Omega_2$ either have the same finite number of components or they both have countably many. □

You will find some simple consequences in the following

Exercises

1. Let $f \in C(\mathbb{R}^n)$ be such that f maps $\partial B_r(0)$ onto itself, for some $r > 0$. Then $d(f^m, B_r(0), 0) = [d(f, B_r(0), 0)]^m$.

2. If $\Omega \subset \mathbb{R}^n$ is open bounded and $f \in C(\bar{\Omega})$ is one-to-one, then $d(f, \Omega, y) \in \{1, -1\}$ for every $y \in f(\Omega)$. *Hint*: f is a homeomorphism onto $f(\bar{\Omega})$; let $y_0 = f(x_0)$, K_j the component of $\mathbb{R}^n \backslash f(\partial\Omega)$ that contains y_0 and $\widetilde{f^{-1}}$ an extension of f^{-1}; notice that $d(\widetilde{f^{-1}}, K_i, x_0) \neq 0$ implies $f(\bar{\Omega}) \cap K_i = \emptyset$ for $i \neq j$.

§ 6. Concluding Remarks

This last section on finite-dimensional degree theory is a mixture of various extensions of earlier results and of clarifying final remarks.

6.1 Degree on Unbounded Sets. So far we have always assumed that the open sets $\Omega \subset \mathbb{R}^n$ in the second argument of d are bounded, so that $f^{-1}(y)$ is a compact subset of Ω whenever $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)$. Now, suppose that $\Omega \subset \mathbb{R}^n$ is open but possibly unbounded. Then $f^{-1}(y)$ will still be compact if f does not grow too fast. More precisely, let us assume that $f \in C(\bar{\Omega})$, $\sup_{\Omega} |x - f(x)| < \infty$ and

$y \notin f(\partial\Omega)$. Then $f^{-1}(y)$ is compact and $d(f, \Omega \cap \Omega_0, y)$ is the same integer for all open bounded $\Omega_0 \supset f^{-1}(y)$, by (d 7). Therefore, we have the following extension of Definition 2.3.

Definition 6.1. For $\Omega \subset \mathbb{R}^n$ open, let $\tilde{C}(\bar{\Omega})$ be the set of all $f \in C(\bar{\Omega})$ such that $\sup_{\Omega} |x - f(x)| < \infty$. Let $\tilde{M} = \{(f, \Omega, y) : \Omega \subset \mathbb{R}^n$ open, $f \in \tilde{C}(\bar{\Omega})$ and $y \notin f(\partial\Omega)\}$. Then we define $\tilde{d}: \tilde{M} \to \mathbb{Z}$ by $\tilde{d}(f, \Omega, y) = d(f, \Omega \cap \Omega_0, y)$, where Ω_0 is any open bounded set that contains $f^{-1}(y)$.

Obviously, $\tilde{d}$ has all properties of d and coincides with d on triplets $(f, \Omega, y) \in \tilde{M}$ with bounded Ω. For example, the homotopy invariance (d 3) says that $\tilde{d}(h(t, \cdot), \Omega, y(t))$ is constant on $[0, 1]$ if $h: [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ and $y: [0, 1] \to \mathbb{R}^n$ are continuous, $\sup\{|x - h(t, x)| : (t, x) \in [0, 1] \times \bar{\Omega}\} < \infty$ and $y(t) \notin h(t, \partial\Omega)$ on $[0, 1]$. This extension of d is needed if one wants to extend degree theory to maps between spaces where all open sets $\neq \emptyset$ are unbounded – e.g. proper locally convex spaces, as you will see in a later chapter.

6.2 Degree in Finite-Dimensional Topological Vector Spaces. We always used the natural base $\{e^1, \ldots, e^n\}$ of $\mathbb{R}^n$, where $e^i_j = \delta_{ij}$. It is immediately seen that we obtain the same degree function if we consider a different base, say $\{\tilde{e}^1, \ldots, \tilde{e}^n\}$, since there is a matrix A with $\det A \neq 0$ such that $\tilde{x} = Ax$, $\tilde{\Omega} = A\Omega$ and $g(\tilde{x}) = Af(A^{-1}\tilde{x})$ for $\tilde{x} \in \tilde{\Omega}$ are the representations of x, Ω and f with respect to the new base, and

$$J_g(\tilde{x}) = \det A \cdot J_f(A^{-1}\tilde{x}) \det A^{-1} = J_f(A^{-1}\tilde{x})$$

in the differentiable case.

Now, let X be an n-dimensional real topological vector space, i.e. a real vector space X of $\dim X = n$ with a topology τ such that addition and multiplication by scalars are continuous. In the references given in § 10.2 you will find that X is homeomorphic to $\mathbb{R}^n$; indeed, choose a base $\{x^1, \ldots, x^n\}$ for X and show that $h: \sum_{i=1}^{n} \alpha_i(x)\, x^i \to \sum_{i=1}^{n} \alpha_i(x)\, e^i$ is a homeomorphism. Now, let $\Omega \subset X$ be open bounded, $F: \bar{\Omega} \to X$ continuous and $y \notin F(\partial\Omega)$. Then $d(f, h(\Omega), h(y))$ is defined for $f = hFh^{-1}$, and if we choose another base $\{\tilde{x}^1, \ldots, \tilde{x}^n\}$ and the corresponding $\tilde{h}$, then $h = A\tilde{h}$ with $\det A \neq 0$, and therefore we get the same integer as before. Thus, it is natural to introduce

Definition 6.2. Let X be a real n-dimensional topological vector space and $M = \{(F, \Omega, y) : \Omega \subset X$ open bounded, $F: \bar{\Omega} \to X$ continuous and $y \in X \backslash F(\partial\Omega)\}$. Then we define $d: M \to \mathbb{Z}$ by $d(F, \Omega, y) = d(hFh^{-1}, h(\Omega), h(y))$, where $h: X \to \mathbb{R}^n$ is the linear homeomorphism defined by $h(x^i) = e^i$, with $\{x^1, \ldots, x^n\}$ a base for X and $\{e^1, \ldots, e^n\}$ the natural base of $\mathbb{R}^n$.

Finally, suppose that we have two real n-dimensional topological vector spaces X and Y, $\Omega \subset X$ open bounded, $F: \bar{\Omega} \to Y$ continuous and $y \in Y \backslash F(\partial\Omega)$. We consider bases $\{x^1, \ldots, x^n\}$ for X and $\{y^1, \ldots, y^n\}$ for Y and the corresponding homeomorphisms $X \xrightarrow{h} \mathbb{R}^n \xleftarrow{\tilde{h}} Y$. Then $d(f, h(\Omega), \tilde{h}(y))$ is defined for $f = \tilde{h}Fh^{-1}$.

Therefore we may define $d(F, \Omega, y)$ as $d(\tilde{h} F h^{-1}, h(\Omega), \tilde{h}(y))$. However, if we change the bases, then $h = A\hat{h}$, $\tilde{h} = Bh^*$ and $B^{-1}fA$ is the new f. Therefore

$$d(B^{-1}fA, \hat{h}(\Omega), h^*(y)) = \operatorname{sgn}(\det A \cdot \det B)\, d(f, h(\Omega), \tilde{h}(y)),$$

i.e. our last definition depends on the choice of the bases. In this situation the widely used terminology is as follows. Say that two bases for X have the same orientation if A, defined by $Ax^i = \tilde{x}^i$ for $i = 1, \ldots, n$, has $\det A > 0$. Evidently, this gives you an equivalence relation with exactly two equivalence classes. Call X 'oriented' if you have chosen which class is admissible for you, so that you ignore the other one. Then the degree of continuous maps between oriented spaces X and Y of the same dimension is defined, since you only have $\det A > 0$ and $\det B > 0$ above.

6.3 A Relation Between the Degrees for Spaces of Different Dimension. Suppose that $\Omega \subset \mathbb{R}^n$ is open bounded, that $f: \bar{\Omega} \to \mathbb{R}^m$ with $m < n$ is continuous and that $y \in \mathbb{R}^m \backslash f(\partial\Omega)$. Let $g = \mathrm{id} - f$. Then $g(x) = y$ for some $x \in \Omega$ implies $x = f(x) + y \in \mathbb{R}^m$, i.e. all solutions of $g(x) = y$ are already in $\Omega \cap \mathbb{R}^m$ and therefore it is to be expected that $d(\mathrm{id} - f, \Omega, y)$ can be computed by means of the m-dimensional degree of $(\mathrm{id} - f)|_{\overline{\Omega \cap \mathbb{R}^m}}$. This is in fact easy to prove, i.e. we have

Theorem 6.1. *Let X_n be a real topological vector space of* $\dim X_n = n$, *X_m a subspace with* $\dim X_m = m < n$, *$\Omega \subset X_n$ open bounded, $f: \bar{\Omega} \to X_m$ continuous and $y \in X_m \backslash g(\partial\Omega)$, where $g = \mathrm{id} - f$. Then $d(g, \Omega, y) = d(g|_{\overline{\Omega \cap X_m}}, \Omega \cap X_m, y)$.*

Proof. By § 6.2 we may assume that $X_n = \mathbb{R}^n$ and $X_m = \mathbb{R}^m = \{x \in \mathbb{R}^n : x_{m+1} = \ldots = x_n = 0\}$, and since the reduction to the regular case presents no difficulties, let us assume that $f \in \bar{C}^1(\Omega)$ and $y \notin g(S_g)$. Suppose that $g(x) = y$ for some $x \in \Omega \cap \mathbb{R}^m$, let $g_m = g|_{\overline{\Omega \cap \mathbb{R}^m}}$, I_k the $k \times k$ identity matrix and (0) the $(n - m) \times m$ zero matrix. Then we have $J_{g_m}(x) = \det(I_m - (\partial_j f_i(x)))$ and

$$J_g(x) = \det \left[\begin{array}{c|c} I_m - (\partial_j f_i(x)) & -(\partial_j f_i(x)) \\ \hline (0) & I_{n-m} \end{array} \right].$$

Developing with respect to the last $n - m$ rows, we obtain $J_g(x) = J_{g_m}(x)$ and therefore $d(g_m, \Omega \cap \mathbb{R}^m, y) = d(g, \Omega, y)$. □

We shall need this observation as early as the next chapter.

6.4 Hopf's Theorem and Generalizations of Borsuk's Theorem. You have seen that homotopic maps have the same degree. H. Hopf has shown that the converse is also true for spheres, that is

Theorem 6.2. *Let $\Omega = B_r(0) \subset \mathbb{R}^n$ with $n \geqq 2$. Suppose that $f \in C(\bar{\Omega})$ and $g \in C(\bar{\Omega})$ are such that $d(f, \Omega, 0) = d(g, \Omega, 0)$. Then there is a continuous $h: [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ such that $0 \notin h([0, 1] \times \partial\Omega)$ and $h(0, \cdot) = f$, $h(1, \cdot) = g$.*

This result can be extended to Jordan regions, i.e. regions Ω such that $\mathbb{R}^n \setminus \bar{\Omega}$ is connected; see e.g. § 5 of Krasnoselskii and Zabreiko [1]. An analytic proof of Theorem 6.2 may be found e.g. in § 3.6 of Guillemin and Pollack [1] and Zeidler [1]; for another proof see § 7.7 of Dugundji and Granas [1].

Let us also remark that Borsuk's Theorem 4.1 and Theorem 4.2 have been generalized with respect to the assumptions concerning antipodal points. For example, let $S = \partial B_1(0) \subset \mathbb{R}^n$ and $f: S \to S$ continuous such that $f(x) \neq f(-x)$. Then Theorem 4.1 implies that $d(\tilde{f}, B_1(0), 0)$ is odd for every continuous extension $\tilde{f}$ of f to $\bar{B}_1(0)$, as you see by means of $\tilde{f}(x) - t\tilde{f}(-x)$ on $[0, 1] \times \bar{B}_1(0)$. If you denote by $\varrho(x, y) = \arccos\left(\sum_{i=1}^{n} x_i y_i\right) \in [0, \pi]$ the spherical distance of $x, y \in S$, then you may rewrite the condition '$f(x) \neq f(-x)$ on S' as '$f(x) \neq f(y)$ whenever $\varrho(x, y) = \pi$'.

H. Hopf conjectured that $d(\tilde{f}, B_1(0), 0) \neq 0$ if, given $\alpha \in (0, \pi]$, $f: S \to S$ satisfies $f(x) \neq f(y)$ whenever $\varrho(x, y) = \alpha$. This conjecture has been proved recently by Wille [3]. Similarly, it has been shown in Wille [1] that Theorem 4.2 remains true for $\partial B_r(0)$ if you replace the assumption on the A_i by the condition that, given $\alpha \in (0, 2r]$, the A_i do not contain pairs x, y such that $|x - y| = \alpha$. Related generalizations of these results are also contained in §§ 8, 9 of Krasnoselskii and Zabreiko [1].

6.5 The Index of an Isolated Solution. Suppose that $f \in C(\bar{B}_r(x_0))$, $y = f(x_0)$ and $y \neq f(x)$ in $\bar{B}_r(x_0) \setminus \{x_0\}$. Then we know that $d(f, B_\varrho(x_0), y)$ is the same integer for all $\varrho \in (0, r]$. This number is called the *index* of x_0 and is denoted by $j(f, x_0, y)$.

Obviously, $j(f, x_0, y) = \operatorname{sgn} J_f(x_0)$ if $f \in C^1(B_r(x_0))$ and $J_f(x_0) \neq 0$. Let us note, for example, the following special case of the product formula, which you can verify without difficulty: If $f \in C(\bar{\Omega})$, $g \in C(\mathbb{R}^n)$, $y \notin gf(\partial\Omega)$ and $g^{-1}(y) = \{z^1, \ldots, z^p\}$ then

$$d(gf, \Omega, y) = \sum_{k=1}^{p} d(f, \Omega, z^k)\, j(g, z^k, y).$$

In the next section you will see that the index of a zero may be regarded as the natural extension of the multiplicity of a zero.

6.6 Degree and Winding Number. At the beginning of this chapter we used the winding number of plane curves as a motivation for (d 1)–(d 3) and we claimed that it is a special case of the degree. The precise relation between these two concepts is the following one.

A continuous closed oriented curve $\gamma: [0, 1] \to \mathbb{C}$ may be regarded as a continuous image of the oriented unit circle $S = \partial B_1(0) \subset \mathbb{C}$, since $h: s \to e^{2\pi i s}$ is a homeomorphism from $(0, 1)$ onto $S \setminus \{1\}$, and therefore f, defined by $f(z) = \gamma(h^{-1}(z))$ for $z \neq 1$ and $f(1) = \gamma(1)$, is continuous on S. If $a \notin \gamma = f(S)$ then $d(f, B_1(0), a)$ is the same integer for all continuous extensions of f to $\bar{B}_1(0)$, by (d 6). We claim that

$$d(f, B_1(0), a) = w(f(S), a). \tag{1}$$

By the definitions of d and w it is sufficient to prove (1) in case $f \in \bar{C}^1(B_1(0))$ and $a \notin f(S_f)$. Let $f^{-1}(a) = \{z_1, \ldots, z_p\}$. Then we have to show

$$\frac{1}{2\pi i} \int_{f(S)} \frac{dz}{z-a} = \sum_{k=1}^{p} \operatorname{sgn} J_f(z_k). \tag{2}$$

Let $\delta > 0$ be so small that the $U_k = B_\delta(z_k)$ are disjoint, $\operatorname{sgn} J_f(z) = \operatorname{sgn} J_f(z_k)$ on $\bar{U}_k \subset B_1(0)$ and $f|_{\bar{U}_k}$ is a homeomorphism. Let $S_k = \partial U_k$. Then $f(S_k)$ is a closed Jordan curve such that the point a lies in its interior region, $f(S_k)$ has the same orientation as S_k if $J_f(z_k) > 0$ and the opposite orientation if $J_f(z_k) < 0$. To see this, let S_k: $\varphi(t) - z_k = (\delta \cos t, \delta \sin t)$ for $t \in [0, 2\pi]$ and let $\psi(t) = f(\varphi(t))$. Then

$$(\psi(t) - a) \times (\psi(t) - a)' = [J_f(z_k) + o(|\delta|)](\varphi(t) - z_k) \times (\varphi(t) - z_k)' \quad \text{as} \quad \delta \to 0,$$

where

$$u \times v = \left(0, 0, \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \in \mathbb{R}^3 \qquad \text{for } u, v \in \mathbb{R}^2.$$

Now, let $G = \bar{B}_1(0) \setminus \bigcup_{k=1}^{p} U_k$. Then $|f(z) - a| \geqq \alpha$ in G for some $\alpha > 0$ and since f is uniformly continuous on G, we can divide G into rectangles R such that

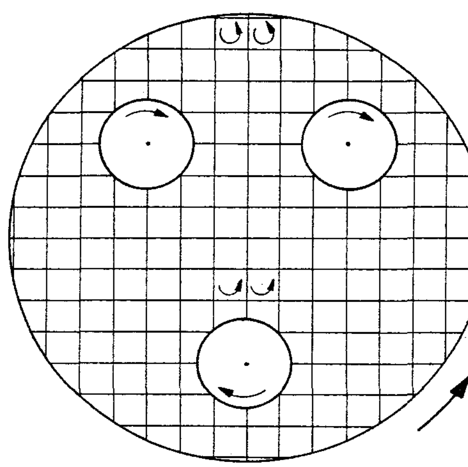

Fig. 6.1

$\sup |f(z) - f(\bar{z})|$ is less than α on each R; see Fig. 6.1, where we have also indicated the orientation. Since the image $f(\Gamma_R)$ of the boundary $\Gamma_R = \partial(R \cap G)$ does not wind around a, we have $w(f(\Gamma_R), a) = 0$ and summation over all R yields

$$\int_{f(S)} \frac{dz}{z-a} + \sum_{k=1}^{p} \int_{f(\bar{S}_k)} \frac{dz}{z-a} = 0, \quad \text{that is} \quad \int_{f(S)} \frac{dz}{z-a} = \sum_{k=1}^{p} \int_{f(S_k)} \frac{dz}{z-a};$$

but $f(S_k)$ winds exactly once around a and since the orientation of $f(S_k)$ is determined by $\operatorname{sgn} J_f(z_k)$, we have $\int_{f(S_k)} \frac{dz}{z-a} = 2\pi i \operatorname{sgn} J_f(z_k)$, and therefore (2). □

Now, the relation between index and multiplicity of a zero of an analytic function becomes evident. Suppose that f is analytic in $B_r(z_0) \subset \mathbb{C}$, $f(z_0) = 0$ and $f(z) \neq 0$ in $B_r(z_0) \backslash \{z_0\}$, and let p be the multiplicity of z_0. Then we have for $\varrho < r$ and $\varphi(z) = z_0 + \varrho z$, by the product formula,

$$\begin{aligned} j(f, z_0, 0) &= d(f, B_\varrho(z_0), 0) = d(f\varphi, B_1(0), 0) = w(f\varphi(S), 0) \\ &= w(f(\partial B_\varrho(z_0)), 0) = p. \end{aligned}$$

6.7 Index of Gradient Maps. Suppose that $\varphi: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Recall that $f = \operatorname{grad} \varphi$, i.e. $f_i = \partial_i \varphi$ for $i = 1, \ldots, n$, is said to be a gradient map and φ is said to be a potential of f, and you may have seen that such maps play a prominent role in various disciplines, e.g. physics.

Now, suppose that $\operatorname{grad} \varphi(x) \neq 0$ for all sufficiently large $x \in \mathbb{R}^n$. Then $\operatorname{ind} \varphi = d(\operatorname{grad} \varphi, B_r(0), 0)$ is the same integer for all sufficiently large $r > 0$ and is called the *index* of φ.

In the simplest case, $\varphi(x) = (x, b) = \sum_{i=1}^{n} x_i b_i$ with $b \neq 0$, we have $\operatorname{grad} \varphi = b$ and therefore $\operatorname{ind} \varphi = 0$. In the quadratic case, $\varphi(x) = \frac{1}{2}(Ax, x)$ with A symmetric and $\det A \neq 0$, we have $\operatorname{grad} \varphi(x) = Ax$ and therefore $\operatorname{ind} \varphi = \operatorname{sgn} \det A$. As a less obvious result, let us prove

Theorem 6.3. *Let* $\varphi: \mathbb{R}^n \to \mathbb{R}^1$ *be continuously differentiable*, $\operatorname{grad} \varphi(x) \neq 0$ *for* $|x| \geqq \varrho$ *and* $\varphi(x) \to \infty$ *as* $|x| \to \infty$. *Then* $\operatorname{ind} \varphi = 1$.

Proof. We may assume $\varphi \in C^\infty(\mathbb{R}^n)$, if necessary replacing φ by $\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(\xi)\, \varrho_\varepsilon(\xi - x)\, d\xi$ with mollifiers ϱ_ε as in the proof to Proposition 1.2 and $\varepsilon > 0$ small. Hence $f = \operatorname{grad} \varphi \in C^1(\mathbb{R}^n)$ and the initial value problems

$$u' = -f(u), \qquad u(0) = x \in \mathbb{R}^n \tag{3}$$

have unique local solutions $u(t) = u(t; x)$. Now, $\psi(t) = \varphi(u(t))$ satisfies $\psi'(t) = -|f(u(t))|^2 \leqq 0$, hence $\varphi(u(t)) \leqq \varphi(x)$ on the interval where $u(\cdot; x)$ exists. Therefore u remains bounded since $\varphi(y) \to \infty$ as $|y| \to \infty$. Consequently, $u' = -f(u)$ remains bounded, and therefore u can be extended to a unique solution $u(\cdot; x)$ of (3) on $[0, \infty)$.

Without loss of generality we also have $\varphi(x) \geqq 0$ since addition of a constant does not change f. Now let $M_1 = \max_{\bar{B}_\varrho(0)} \varphi(x)$, choose $r > \varrho$ so large that $\varphi(x) \geqq M_1 + 1$ for $|x| \geqq r$ and let $M_2 = \max_{\partial B_r(0)} \varphi(x)$. You have already seen that

$$\varphi(u(t; x)) \leqq \varphi(x) - \int_0^t |f(u(s; x))|^2 \, ds \qquad \text{for } t \geqq 0. \tag{4}$$

Thus, the solutions starting at $x \in \partial B_r(0)$ satisfy $\varphi(u(t; x)) \leqq \varphi(x) \leqq M_2$ in $[0, \infty)$. In fact, we get much more. Since $r > \varrho$ and $f(y) \neq 0$ in $|y| \geqq \varrho$, let $\alpha = \min\{|f(y)|: |y| \geqq \varrho \text{ and } \varphi(y) \leqq M_2\}$. Then (4) and $|x| = r$ imply

$$0 \leqq \varphi(u(t; x)) \leqq \varphi(x) - \alpha^2 t \leqq M_2 - \alpha^2 t \qquad \text{as long as } |u(t; x)| \geqq \varrho.$$

Thus, $|u(t_0; x)| \leqq \varrho$ for some $t_0 = t_0(x) < \omega = \alpha^{-2} M_2$, hence $\varphi(u(t_0; x)) \leqq M_1$ and therefore $\varphi(u(t; x)) \leqq M_1$ for all $t \geqq t_0$. This means that the Poincaré P_ω satisfies $P_\omega(\partial B_r(0)) \subset B_r(0)$, since

$$\varphi(P_\omega x) \leqq M_1 \leqq M_1 + 1 \leqq \varphi(x) \quad \text{for } |x| = r.$$

Now, we are done since $\operatorname{ind} \varphi = d(f, B_r(0), 0) = d(\mathrm{id} - P_\omega, B_r(0), 0)$ by Exercise 3.6 and $d(\mathrm{id} - P_\omega, B_r(0), 0) = 1$ by (d3) with $h(t, x) = x - t P_\omega x$. □

Results of this type have been applied to obtain existence of periodic solutions, bounded solutions, etc. of ordinary differential equations; see e.g. Amann [8], Krasnoselskii [3], Mawhin [1]. Related ideas will play an essential role in § 27.

6.8 Final Remarks. This chapter is an improved version of Chap. 2 in Deimling [8]. In § 1 we profited by Amann and Weiss [1]. You have become familiar with one of the basic concepts in the study of nonlinear equations and you have seen that the topological degree may be useful to solve nontrivial existence problems, especially in situations where one doesn't expect that the problem has a unique solution. Uniqueness will be studied later on by other means. You will have noticed that in nearly every case we exploited the fact that bounded subsets of $\mathbb{R}^n$ are relatively compact. This is not the case in most of the interesting infinite dimensional spaces and therefore large portions of the following chapters centre around the problem of finding powerful substitutes motivated by 'concrete' problems. Before you leave finite dimensions you should perhaps convince yourself that you are ready to enter infinite dimensions. The final exercises may help you to clarify this point.

Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be open bounded and $f \in C(\bar{\Omega})$. Suppose there exists an $x_0 \in \Omega$ such that f satisfies the following boundary condition: 'If $f(x) - x_0 = \lambda(x - x_0)$ for some $x \in \partial\Omega$ then $\lambda \leqq 1$.'

   Then f has a fixed point. This is the most general fixed point theorem for continuous f on open sets. Two special cases are:

   (i) $0 \in \Omega$ and $|x - f(x)|^2 \geqq |f(x)|^2 - |x|^2$ on $\partial\Omega$ or equivalently (for the Euclidean norm) $0 \in \Omega$ and $(f(x), x) \leqq |x|^2$ on $\partial\Omega$;
   (ii) $0 \in \Omega$, $\lambda\bar{\Omega} \subset \Omega$ for $\lambda \in (0, 1)$ and $f(\partial\Omega) \subset \bar{\Omega}$.

2. Let $[a, b] \subset \mathbb{R}^n$ such that $a_i < b_i$ for $i = 1, \ldots, n$; $f\colon [a, b] \to \mathbb{R}^n$ continuous,

$$f_i(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \geqq 0 \text{ and } f_i(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n) \leqq 0 \text{ for } i = 1, \ldots, n.$$

   Then f has a zero. *Hint*: Find a suitable x_0 in Exercise 1.

3. Let $\Omega \subset \mathbb{R}^n$ open bounded, $f \in C(\bar{\Omega})$, $f(\bar{\Omega}) \subset \bar{\Omega}$ and $f(x) = x$ on $\partial\Omega$. Then $f(\bar{\Omega}) = \bar{\Omega}$.

4. Let $\Omega \subset \mathbb{R}^n$ open bounded and $0 \in \Omega$, $f \in C(\bar{\Omega})$ and $(f(x), x) \geqq 0$ on $\partial\Omega$. Then f has a zero.

5. Let $\Omega \subset \mathbb{R}^n$ open bounded, $0 \in \Omega$ and $\bar{\Omega}$ star-shaped with respect to 0, i.e. $y \in \bar{\Omega}$ implies $ty \in \bar{\Omega}$ for $t \in [0, 1]$. Suppose also that $\partial\Omega$ is simple, i.e. $y \in \partial\Omega$ implies $ty \in \Omega$ for $t \in [0, 1)$. Then $f \in C(\bar{\Omega})$ and $f(\partial\Omega) \subset \bar{\Omega}$ imply that f has a fixed point. It is still an open problem whether this result remains true if $\partial\Omega$ is not simple.

6. Let $\Omega_m \subset \mathbb{R}^m$ and $\Omega_n \subset \mathbb{R}^n$ be open bounded, $f\colon \bar{\Omega}_m \to \mathbb{R}^m$ and $g\colon \bar{\Omega}_n \to \mathbb{R}^n$ continuous, $y \in \mathbb{R}^m \setminus f(\partial\Omega_m)$ and $z \in \mathbb{R}^n \setminus g(\partial\Omega_n)$. Then

$$d((f, g), \Omega_m \times \Omega_n, (y, z)) = d(f, \Omega_m, y)\, d(g, \Omega_n, z).$$

7. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable, grad $\varphi(x) \neq 0$ for all $|x| \geqq \varrho$. Then (i) ind φ is odd if φ is even. (ii) ind $\varphi = (-1)^n$ if $\varphi(x) \to -\infty$ as $|x| \to \infty$. *Hint*: Product formula for A with $Ax = -x$. (iii) ind $\varphi = 1$ if φ is homogenous of degree $\alpha > 0$, *i.e.* $\varphi(tx) = t^\alpha \varphi(x)$ for $t > 0$, and $\varphi(x) > 0$ for $x \neq 0$.

8. Consider $u' = f(t, u)$, where $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$ and f is ω-periodic in t. Suppose that (grad $\varphi(x)$, $f(t, x)) \geqq 0$ for all $t \in [0, \omega]$ and $|x| \geqq \varrho$, where $\varphi: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $\varphi(x) \to -\infty$ as $|x| \to \infty$. Then $u' = f(t, u)$ has an ω-periodic solution.

9. Let $\Omega \subset \mathbb{C}$ be open bounded, $f \in C(\bar{\Omega})$ and f analytic in Ω, $a \notin f(\partial\Omega)$. Then $d(f, \Omega, a) \geqq 0$. If Ω is connected and $f(z) \not\equiv a$ in Ω, then $d(f, \Omega, a)$ is the number of solutions of $f(z) = a$. *Hint*: Remember the Cauchy-Riemann differential equations for the real and the imaginary part of f.

10. The following problem arises in a model for generation of sound near an infinite compliant wall; see Möhring and Rahman [1]. For $z \in \bar{\Omega} = \{z \in \mathbb{C}: \operatorname{Im} z \geqq 0\}$, let

$$f(z) = i\omega^2(z)\, g(z, k)\, \varrho c + [\omega^2(z) - c^2 |k|^2]^{1/2}, \quad \omega(z) = z - k_1 u_0, \quad k = (k_1, k_2) \in \mathbb{R}^2,$$

where g is analytic in Ω and such that

$$\overline{\lim_{|z| \to \infty}} |g(z, k)\, z^{-2}| < \infty, \quad \operatorname{Im} g(z, k) \neq 0 \quad \text{for } z \neq 0 \quad \text{and } x \cdot \operatorname{Im} g(x, k) > 0 \text{ on } \mathbb{R}\backslash\{0\},$$

c is the constant speed of sound and u_0, ϱ are constants. The square root has to be chosen such that its imaginary part is negative in Ω. Does f have a zero in Ω? You might be able to show that there is a zero if $u_0^2 k_1^2 \varrho g(0, k) > [(1 - M_0)^2 k_1^2 + k_2^2]^{1/2}$ with $M_0 = u_0/c$. *Hint*: A tedious calculation shows that the choice of the square root implies $x \operatorname{Re}[\omega^2(x) - c^2 |k|^2]^{1/2} \leqq 0$ on $\mathbb{R}$. Consider this fact as given. Notice that $\sup\{|f(z) - z|: z \in \bar{\Omega}\} < \infty$. Choose ϱ as the homotopy parameter. If ϱ is large, notice that f can have zeros for small $\omega(z)$ only, and consider $\omega^2 = |k|\, [\varrho\, g(u_0 k_1, k)]^{-1}$.

11. Let $\Omega \subset \mathbb{R}^n$ be open bounded. For $n = 1$ we have

$$\{d(f, \Omega, y): f \in C(\bar{\Omega}), y \notin f(\partial\Omega)\} = \mathbb{Z}$$

iff Ω has infinitely many components. However, $d(\cdot, \Omega, \cdot)$ may be surjective even if Ω is connected but $n \geqq 2$.