Chapter 14
Oscillations

Conceptual Problems

1 •

Determine the Concept The acceleration of an oscillator of amplitude A and frequency f is zero when it is passing through its equilibrium position and is a maximum when it is at its turning points.

When $v = v_{max}$: $a = \boxed{0}$

When $x = x_{max}$: $a = \omega^2 A = \boxed{4\pi^2 f^2 A}$

2 •

Determine the Concept The condition for simple harmonic motion is that there be a linear restoring force; i.e., that $F = -kx$. Thus, the acceleration and displacement (when they are not zero) are always oppositely directed. v and a can be in the same direction, as can v and x.

3 •

(*a*) False. In simple harmonic motion, the period is independent of the amplitude.

(*b*) True. In simple harmonic motion, the frequency is the reciprocal of the period which, in turn, is independent of the amplitude.

(*c*) True. The condition that the acceleration of a particle is proportional to the displacement and oppositely directed is equivalent to requiring that there be a linear restoring force; i.e., $F = -kx \Leftrightarrow ma = -kx$ or $a = -(k/m)x$.

***4 •**

Determine the Concept The energy of a simple harmonic oscillator varies as the square of the amplitude of its motion. Hence, tripling the amplitude increases the energy by a factor of 9.

5 ••

Picture the Problem The total energy of an object undergoing simple harmonic motion is given by $E_{tot} = \frac{1}{2}kA^2$, where k is the stiffness constant and A is the amplitude of the motion. The potential energy of the oscillator when it is a distance x from its equilibrium position is $U(x) = \frac{1}{2}kx^2$.

Express the ratio of the potential energy of the object when it is 2 cm from the equilibrium position to its total energy:

$$\frac{U(x)}{E_{\text{tot}}}=\frac{\frac{1}{2}kx^2}{\frac{1}{2}kA^2}=\frac{x^2}{A^2}$$

Evaluate this ratio for $x = 2$ cm and $A = 4$ cm:

$$\frac{U(2\,\text{cm})}{E_{\text{tot}}}=\frac{(2\,\text{cm})^2}{(4\,\text{cm})^2}=\frac{1}{4}$$

$$\frac{U(2\,\text{cm})}{E_{\text{tot}}}=\frac{(2\,\text{cm})^2}{(4\,\text{cm})^2}=\frac{1}{4}$$

and $\boxed{(a)\text{ is correct.}}$

6 •

(*a*) True. The factors determining the period of the object, i.e., its mass and the spring constant, are independent of the oscillator's orientation.

(*b*) True. The factors determining the maximum speed of the object, i.e., its amplitude and angular frequency, are independent of the oscillator's orientation.

7 •

False. In order for a simple pendulum to execute simple harmonic motion, the restoring force must be linear. This condition is satisfied, at least approximately, for small initial angular displacements.

8 •

True. In order for a simple pendulum to execute periodic motion, the restoring force must be linear. This condition is satisfied for any initial angular displacement.

***9** ••

Determine the Concept Assume that the first cart is given an initial velocity v by the blow. After the initial blow, there are no external forces acting on the carts, so their center of mass moves at a constant velocity $v/2$. The two carts will oscillate about their center of mass in simple harmonic motion where the amplitude of their velocity is $v/2$. Therefore, when one cart has velocity $v/2$ with respect to the center of mass, the other will have velocity $-v/2$. The velocity with respect to the laboratory frame of reference will be $+v$ and 0, respectively. Half a period later, the situation is reversed; one cart will move as the other stops, and vice-versa.

***10** ••

Determine the Concept The period of a simple pendulum depends on the reciprocal of the length of the pendulum. Increasing the length of the pendulum will decrease its period and the clock would run slow.

11 •

True. The mechanical energy of a damped, undriven oscillator varies with time according to $E = E_0 e^{-t/\tau}$ where E_0 is the oscillator's energy at $t = 0$ and τ is the time constant.

12 •

(*a*) True. The amplitude of the motion of a driven oscillator depends on the driving (ω) and natural (ω_0) frequencies according to $A = F_0 \big/ \sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}$. When $\omega = \omega_0$, the amplitude of the motion is a maximum and is given by $A = F_0 \big/ \sqrt{b^2\omega^2}$.

(*b*) True. The width of the resonance curve ($\Delta\omega$) depends on the Q value according to $\Delta\omega / \omega_0 = 1/Q$. Thus when Q is large, $\Delta\omega$ is small and the resonance is sharp.

13 •

Determine the Concept Examples of driven oscillators include the pendulum of a clock, a bowed violin string, and the membrane of any loudspeaker.

14 •

Determine the Concept The shattering of a crystal wineglass is a consequence of the glass being driven at or near its resonant frequency. $\boxed{(a) \text{ is correct.}}$

***15** •

Determine the Concept We can use the expression for the frequency of a spring-and-mass oscillator to determine the effect of the mass of the spring.

If m represents the mass of the object attached to the spring in a spring-and-mass oscillator, the frequency is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

If the mass of the spring is taken into account, the effective mass is greater than the mass of the object alone.

$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{m_{\text{eff}}}}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{f'}{f} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{m_{\text{eff}}}}}{\frac{1}{2\pi}\sqrt{\frac{k}{m}}} = \sqrt{\frac{m}{m_{\text{eff}}}}$$

Solve for f':

$$f' = f\sqrt{\frac{m}{m_{\text{eff}}}}$$

$\boxed{\text{Because } f' \text{ varies inversely with the square root of } m, \text{ taking into account the effective mass of the spring predicts that the frequency will be reduced.}}$

16 ••

Determine the Concept The period of the lamp varies inversely with the square root of the effective value of the local gravitational field.

1. greater than T_0 when — B. the train rounds a curve of radius R with speed v.

2. less than T_0 when — D. the train goes over the crest of a hill of radius of curvature R with constant speed.

3. equal to T_0 when — A. the train moves horizontally with constant velocity.

C. the train climbs a hill of inclination θ at constant speed.

17 ••

Picture the Problem We can use $f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$ to express the frequencies of the two mass-spring systems in terms of their masses. Dividing one of the equations by the other will allow us to express M_A in terms of M_B.

Express the frequency of mass-spring system A as a function of its mass:

$$f_A = \frac{1}{2\pi}\sqrt{\frac{k}{M_A}}$$

Express the frequency of mass-spring system B as a function of its mass:

$$f_B = \frac{1}{2\pi}\sqrt{\frac{k}{M_B}}$$

Divide the second of these equations by the first to obtain:

$$\frac{f_B}{f_A} = \sqrt{\frac{M_A}{M_B}}$$

Solve for M_A:

$$M_A = \left(\frac{f_B}{f_A}\right)^2 M_B = \left(\frac{f_B}{2f_B}\right)^2 M_B = \tfrac{1}{4}M_B$$

and $\boxed{(d)\text{ is correct.}}$

18 ••

Picture the Problem We can relate the energies of the two mass-spring systems through either $E=\frac{1}{2}kA^2$ or $E=\frac{1}{2}M\omega^2A^2$ and investigate the relationship between their amplitudes by equating the expressions, substituting for M_A, and expressing A_A in terms of A_B.

Express the energy of mass-spring system A:

$$E_A=\tfrac{1}{2}k_AA_A^2=\tfrac{1}{2}M_A\omega_A^2A_A^2$$

Express the energy of mass-spring system B:

$$E_B=\tfrac{1}{2}k_BA_B^2=\tfrac{1}{2}M_B\omega_B^2A_B^2$$

Divide the first of these equations by the second to obtain:

$$\frac{E_A}{E_B}=1=\frac{\frac{1}{2}M_A\omega_A^2A_A^2}{\frac{1}{2}M_B\omega_B^2A_B^2}$$

Substitute for M_A and simplify:

$$1=\frac{2M_B\omega_A^2A_A^2}{M_B\omega_B^2A_B^2}=\frac{2\omega_A^2A_A^2}{\omega_B^2A_B^2}$$

Solve for A_A:

$$A_A=\frac{\omega_B}{\sqrt{2}\omega_A}A_B$$

Without knowing how ω_A and ω_B, or k_A and k_B, are related, we cannot simplify this expression further. $\boxed{(d)\text{ is correct.}}$

19 ••

Picture the Problem We can express the energy of each system using $E=\frac{1}{2}kA^2$ and, because the energies are equal, equate them and solve for A_A.

Express the energy of mass-spring system A in terms of the amplitude of its motion:

$$E_A=\tfrac{1}{2}k_AA_A^2$$

Express the energy of mass-spring system B in terms of the amplitude of its motion:

$$E_B=\tfrac{1}{2}k_BA_B^2$$

Because the energies of the two systems are equal we can equate them to obtain:

$$\tfrac{1}{2}k_A A_A^2 = \tfrac{1}{2}k_B A_B^2$$

Solve for A_A:

$$A_A = \sqrt{\frac{k_B}{k_A}}A_B$$

Substitute for k_A and simplify to obtain:

$$A_A = \sqrt{\frac{k_B}{2k_B}}A_B = \frac{A_B}{\sqrt{2}}$$

and $\boxed{(b)\text{ is correct.}}$

20 ••

Picture the Problem The period of a simple pendulum is independent of the mass of its bob and is given by $T = 2\pi\sqrt{L/g}$.

Express the period of pendulum A:

$$T_A = 2\pi\sqrt{\frac{L_A}{g}}$$

Express the period of pendulum B:

$$T_B = 2\pi\sqrt{\frac{L_B}{g}}$$

Divide the first of these equations by the second and solve for L_A/L_B:

$$\frac{L_A}{L_B} = \left(\frac{T_A}{T_B}\right)^2$$

Substitute for T_A and solve for L_B to obtain:

$$L_A = \left(\frac{2T_B}{T_B}\right)^2 L_B = 4L_B$$

and $\boxed{(c)\text{ is correct.}}$

Estimation and Approximation

21 ••

Picture the Problem The Q factor for this system is related to the decay constant τ through $Q = \omega_0\tau = 2\pi\tau/T$ and the amplitude of the child's damped motion varies with time according to $A = A_0 e^{-t/2\tau}$. We can set the ratio of two displacements separated by eight periods equal to $1/e$ to determine τ in terms of T.

Express Q as a function of τ:

$$q = \omega_0\tau = \frac{2\pi\tau}{T} \qquad (1)$$

The amplitude of the oscillations varies with time according to:

$$A = A_0 e^{-t/2\tau}$$

The amplitude after eight periods is:

$$A_8 = A_0 e^{-(t+8T)/2\tau}$$

Express and simplify the ratio A_8/A:

$$\frac{A_8}{A} = \frac{A_0 e^{-(t+8T)/2\tau}}{A_0 e^{-t/2\tau}} = e^{-4T/\tau}$$

Set this ratio equal to $1/e$ and solve for τ:

$$e^{-4T/\tau} = e^{-1} \Rightarrow \tau = 4T$$

Substitute in equation (1) and evaluate Q:

$$Q = \frac{2\pi(4T)}{T} = \boxed{8\pi}$$

***22 ••**

Picture the Problem Assume that an average length for an arm is about 0.8 m, and that it can be treated as a uniform stick, pivoted at one end. We can use the expression for the period of a physical pendulum to derive an expression for the period of the swinging arm. When carrying a heavy briefcase, the mass is concentrated mostly at the end of the pivot (i.e., in the briefcase), so we can treat the arm-plus-briefcase as a simple pendulum.

(*a*) Express the period of a uniform rod pivoted at one end:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

where I is the moment of inertia of the stick about an axis through one end, M is the mass of the stick, and D (= $L/2$) is the distance from the end of the stick to its center of mass.

Express the moment of inertia of the stick with respect to an axis through its end:

$$I = \tfrac{1}{3}ML^2$$

Substitute the values for I and D to find T:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\left(\frac{1}{2}L\right)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.8\,\text{m})}{3(9.81\,\text{m/s}^2)}} = \boxed{1.47\,\text{s}}$$

(*b*) Express the period of a simple pendulum:

$$T' = 2\pi\sqrt{\frac{L'}{g}}$$

where L' is slightly longer than the arm length due to the size of the briefcase.

Assuming $L' = 1$ m, evaluate the period of the simple pendulum:

$$T' = 2\pi\sqrt{\frac{1\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{2.01\,\text{s}}$$

$$\boxed{\text{From observation of people as they walk, these estimates seem reasonable.}}$$

Simple Harmonic Motion

23 •

Picture the Problem The position of the particle is given by $x = A\cos(\omega t + \delta)$ where A is the amplitude of the motion, ω is the angular frequency, and δ is a phase constant.

(*a*) Use the definition of ω to determine f:

$$f = \frac{\omega}{2\pi} = \frac{6\pi\,\text{s}^{-1}}{2\pi} = \boxed{3.00\,\text{Hz}}$$

(*b*) Evaluate the reciprocal of the frequency:

$$T = \frac{1}{f} = \frac{1}{3.00\,\text{Hz}} = \boxed{0.333\,\text{s}}$$

(*c*) Compare $x = (7\text{ cm})\cos 6\pi t$ to $x = A\cos(\omega t + \delta)$:

$$A = \boxed{7.00\,\text{cm}}$$

(*d*) $x = 0$ when $\cos\omega t = 0$:

$$\omega t = \cos^{-1} 0 = \frac{\pi}{2}$$

Solve for t:

$$t = \frac{\pi}{2\omega} = \frac{\pi}{2(6\pi)} = \boxed{0.0833\,\text{s}}$$

Differentiate x to find $v(t)$:

$$v = \frac{d}{dt}[(7\,\text{cm})\cos 6\pi t]$$
$$= -(42\pi\,\text{cm/s})\sin 6\pi t$$

Evaluate $v(0.0833\text{ s})$:

$$v(0.0833\,\text{s}) = -(42\pi\,\text{cm/s})\sin 6\pi(0.0833\,\text{s}) < 0$$

$$\boxed{\text{Because } v < 0\text{, the particle is moving in the negative direction at } t = 0.0833\,\text{s}.}$$

24 •

Picture the Problem The initial position of the oscillating particle is related to the amplitude and phase constant of the motion by $x_0 = A\cos\delta$ where $0 \le \delta < 2\pi$.

(*a*) For $x_0 = 0$:

$\cos\delta = 0$

and

$\delta = \cos^{-1} 0 = \boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$

(*b*) For $x_0 = -A$:

$-A = A\cos\delta$

and

$\delta = \cos^{-1}(-1) = \boxed{\pi}$

(*c*) For $x_0 = A$:

$A = A\cos\delta$

and

$\delta = \cos^{-1}(1) = \boxed{0}$

(*d*) When $x = A/2$:

$\frac{A}{2} = A\cos\delta$

and

$\delta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$

***25 •**

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. Its velocity as a function of time is given by $v = -A\omega\sin(\omega t + \delta)$ and its acceleration by $a = -A\omega^2\cos(\omega t + \delta)$. The initial position and velocity give us two equations from which to determine the amplitude A and phase constant δ.

(*a*) Express the position, velocity, and acceleration of the particle as a function of t:

$$x = A\cos(\omega t + \delta) \quad (1)$$
$$v = -A\omega\sin(\omega t + \delta) \quad (2)$$
$$a = -A\omega^2\cos(\omega t + \delta) \quad (3)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\text{s}^{-1} = 4.19\,\text{s}^{-1}$$

Relate the initial position and velocity to the amplitude and phase constant:

$x_0 = A\cos\delta$

and

$v_0 = -\omega A\sin\delta$

Divide these equations to eliminate A:

$$\frac{v_0}{x_0} = \frac{-\omega A\sin\delta}{A\cos\delta} = -\omega\tan\delta$$

Solve for δ and substitute numerical values to obtain:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right) = \tan^{-1}\left(-\frac{0}{x_0\omega}\right) = 0$$

Substitute in equation (1) to obtain:

$$x = (25\,\text{cm})\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{(25\,\text{cm})\cos\left[\left(4.19\,\text{s}^{-1}\right)t\right]}$$

(*b*) Substitute in equation (2) to obtain:

$$v = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)\sin\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-(105\,\text{cm/s})\sin\left[\left(4.19\,\text{s}^{-1}\right)t\right]}$$

(*c*) Substitute in equation (3) to obtain:

$$a = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)^2\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-\left(439\,\text{cm/s}^2\right)\cos\left[\left(4.19\,\text{s}^{-1}\right)t\right]}$$

26 •

Picture the Problem The maximum speed and maximum acceleration of the particle in are given by $v_{\text{max}} = A\omega$ and $a_{\text{max}} = A\omega^2$. The particle's position is given by $x = A\cos(\omega t + \delta)$ where A = 7 cm, $\omega = 6\pi\,\text{s}^{-1}$, and δ = 0, and its velocity is given by $v = -A\omega\sin(\omega t + \delta)$.

(*a*) Express v_{max} in terms of A and ω:

$$v_{\text{max}} = A\omega = (7\,\text{cm})\left(6\pi\,\text{s}^{-1}\right) = 42\pi\,\text{cm/s} = \boxed{1.32\,\text{m/s}}$$

(*b*) Express a_{max} in terms of A and ω:

$$a_{\text{max}} = A\omega^2 = (7\,\text{cm})\left(6\pi\,\text{s}^{-1}\right)^2 = 252\pi^2\,\text{cm/s}^2 = \boxed{24.9\,\text{m/s}^2}$$

(*c*) When $x = 0$:

$$\cos\omega t = 0$$

and

$$\omega t = \cos^{-1} 0 = \frac{\pi}{2}, \frac{3\pi}{2}$$

Evaluate v at $\omega t = \dfrac{\pi}{2}$:

$$v = -A\omega\sin\left(\frac{\pi}{2}\right) = -A\omega$$

i.e., the particle is moving to the left.

Evaluate v at $\omega t = \frac{3\pi}{2}$:

$$v = -A\omega\sin\left(\frac{3\pi}{2}\right) = A\omega$$

i.e., the particle is moving to the right.

Solve for t:

$$t = \frac{3\pi}{2\omega} = \frac{3\pi}{2\left(6\pi\,\text{s}^{-1}\right)} = \boxed{0.250\,\text{s}}$$

27 ••

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. Its velocity as a function of time is given by $v = -A\omega\sin(\omega t + \delta)$ and its acceleration by $a = -A\omega^2\cos(\omega t + \delta)$. The initial position and velocity give us two equations from which to determine the amplitude A and phase constant δ.

(*a*) Express the position, velocity, and acceleration of the particle as functions of t:

$$x = A\cos(\omega t + \delta) \quad (1)$$
$$v = -A\omega\sin(\omega t + \delta) \quad (2)$$
$$a = -A\omega^2\cos(\omega t + \delta) \quad (3)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\text{s}^{-1} = 4.19\,\text{s}^{-1}$$

Relate the initial position and velocity to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

and

$$v_0 = -\omega A\sin\delta$$

Divide these equations to eliminate A:

$$\frac{v_0}{x_0} = \frac{-\omega A\sin\delta}{A\cos\delta} = -\omega\tan\delta$$

Solve for δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right)$$

Substitute numerical values and evaluate δ:

$$\delta = \tan^{-1}\left(-\frac{50\,\text{cm/s}}{(25\,\text{cm})\left(4.192\,\text{s}^{-1}\right)}\right) = -0.445\,\text{rad}$$

Use either the x_0 or v_0 equation (x_0 is used here) to find the amplitude:

$$A = \frac{x_0}{\cos\delta} = \frac{25\,\text{cm}}{\cos(-0.445\,\text{rad})} = 27.7\,\text{cm}$$

Substitute in equation (1) to obtain:

$$x = \boxed{(27.7\,\text{cm})\cos\left[\left(4.19\,\text{s}^{-1}\right)t - 0.445\right]}$$

(*b*) Substitute in equation (2) to obtain:

$$v = -(27.7\,\text{cm})\left(\frac{4\pi}{3}\,\text{s}^{-1}\right) \times \sin\left[\left(\frac{4\pi}{3}\,\text{s}^{-1}\right)t - 0.445\right] = \boxed{-(116\,\text{cm/s})\sin\left[\left(4.19\,\text{s}^{-1}\right)t - 0.445\right]}$$

(*c*) Substitute in equation (3) to obtain:

$$a = -(27.7\,\text{cm})\left(\frac{4\pi}{3}\,\text{s}^{-1}\right)^2 \cos\left[\left(\frac{4\pi}{3}\,\text{s}^{-1}\right)t - 0.445\right] = \boxed{-\left(486\,\text{cm/s}^2\right)\cos\left[\left(4.19\,\text{s}^{-1}\right)t - 0.445\right]}$$

28 ••

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. We're given the amplitude A of the motion and can use the initial position of the particle to determine the phase constant δ. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of t:

$$x = (12\,\text{cm})\cos(\omega t + \delta) \qquad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8\,\text{s}} = \frac{\pi}{4}\,\text{s}^{-1}$$

Relate the initial position of the particle to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

Solve for δ:

$$\delta = \cos^{-1}\frac{x_0}{A} = \cos^{-1}\frac{0}{A} = \frac{\pi}{2}$$

Substitute in equation (1) to obtain:

$$x = (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t + \frac{\pi}{2}\right]$$

Express the distance the particle travels in terms of t_f and t_i:

$$\Delta x = \left|(12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f + \frac{\pi}{2}\right] - (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i + \frac{\pi}{2}\right]\right|$$

$$= \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i + \frac{\pi}{2}\right]\right\}\right|$$

(*a*) Evaluate Δx for $t_f = 2$ s, $t_i = 1$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(2\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right|$$

$$= \left|(12\,\text{cm})\{-1 - 0\}\right|$$

$$= \boxed{12.0\,\text{cm}}$$

(*b*) Evaluate Δx for $t_f = 4$ s, $t_i = 2$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(4\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(2\,\text{s}) + \frac{\pi}{2}\right]\right\}\right|$$

$$= \left|(12\,\text{cm})\{0 - 1\}\right|$$

$$= \boxed{12.0\,\text{cm}}$$

(*c*) Evaluate Δx for $t_f = 1$ s, $t_i = 0$:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(1\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right|$$

$$= \left|(12\,\text{cm})\{-0.7071 - 0\}\right|$$

$$= \boxed{8.49\,\text{cm}}$$

(*d*) Evaluate Δx for $t_f = 2$ s, $t_i = 1$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(2\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)(1\,\text{s}) + \frac{\pi}{2}\right]\right\}\right|$$
$$= \left|(12\,\text{cm})\{-1 + 0.7071\}\right|$$
$$= \boxed{3.51\,\text{cm}}$$

29 ••

Picture the Problem The position of the particle as a function of time is given by $x = (10\,\text{cm})\cos(\omega t + \delta)$. We can determine the angular frequency ω from the period of the motion and the phase constant δ from the initial position and velocity. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of *t*:

$$x = (10\,\text{cm})\cos(\omega t + \delta) \qquad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8\,\text{s}} = \frac{\pi}{4}\,\text{s}^{-1}$$

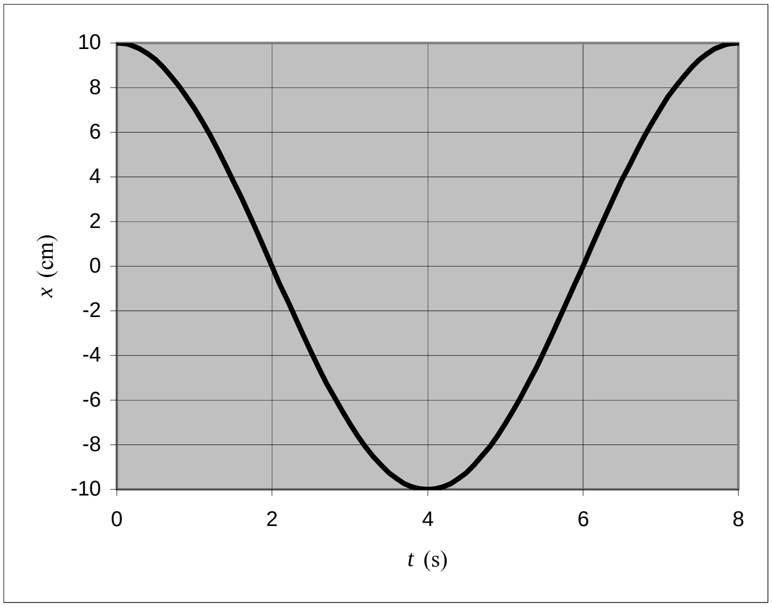
Find the phase constant of the motion:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right) = \tan^{-1}\left(-\frac{0}{x_0\omega}\right) = 0$$

Substitute in equation (1) to obtain:

$$x = (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t\right]$$

(*a*) A graph of $x = (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t\right]$ follows:



(*b*) Express the distance the particle travels in terms of t_f and t_i:

$$\Delta x = \left|(10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f\right] - (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i\right]\right|$$
$$= \left|(10\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i\right]\right\}\right| \quad (2)$$

Substitute numerical values in equation (2) and evaluate Δx in each of the given time intervals to obtain:

t_f	t_i	Δx
(s)	(s)	(cm)
1	0	$\boxed{2.93}$
2	1	$\boxed{7.07}$
3	2	$\boxed{7.07}$
4	3	$\boxed{2.93}$

***30 ••**

Picture the Problem We can use the expression for the maximum acceleration of an oscillator to relate the 10*g* military specification to the compliance frequency.

Express the maximum acceleration of an oscillator:

$$a_{max} = A\omega^2$$

Express the relationship between the angular frequency and the frequency of the vibrations:

$$\omega = 2\pi f$$

Substitute to obtain:

$$a_{max} = 4\pi^2 A f^2$$

Solve for f:

$$f = \frac{1}{2\pi}\sqrt{\frac{a_{max}}{A}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{98.1\,\text{m/s}^2}{1.5\times10^{-2}\,\text{m}}} = \boxed{12.9\,\text{Hz}}$$

31 ••

Picture the Problem The maximum speed and acceleration of the particle are given by $v_{max} = A\omega$ and $a_{max} = A\omega^2$. The velocity and acceleration of the particle are given by $v = -A\omega \sin\omega t$ and $a = -A\omega^2 \cos\omega t$.

(*a*) Find v_{max} from A and ω:

$$v_{max} = A\omega = (2.5\,\text{m})(\pi\,\text{s}^{-1}) = \boxed{7.85\,\text{m/s}}$$

Find a_{max} from A and ω:

$$a_{max} = A\omega^2 = (2.5\,\text{m})(\pi\,\text{s}^{-1})^2 = \boxed{24.7\,\text{m/s}^2}$$

(*b*) Use the equation for the position of the particle to relate its position at $x = 1.5$ m to the time t' to reach this position:

$$1.5\,\text{m} = (2.5\,\text{m})\cos\pi t'$$

Solve for $\pi t'$:

$$\pi t' = \cos^{-1} 0.6 = 0.9273\,\text{rad}$$

Evaluate v when $\pi t = \pi t'$:

$$v = -(2.5\,\text{m})(\pi\,\text{s}^{-1})\sin(0.9273\,\text{rad}) = \boxed{-6.28\,\text{m/s}}$$

where the minus sign indicates that the particle is moving in the negative direction.

Evaluate a when $\pi t = \pi t'$:

$$a = -(2.5\,\text{m})(\pi\,\text{s}^{-1})^2\cos(0.9273\,\text{rad}) = \boxed{-14.8\,\text{m/s}^2}$$

where the minus sign indicates that the particle's acceleration is in the negative direction.

***32 ••**

Picture the Problem We can use the formula for the cosine of the sum of two angles to write $x = A_0 \cos(\omega t + \delta)$ in the desired form. We can then evaluate x and dx/dt at $t = 0$ to relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.

(*a*) Apply the trigonometric identity $\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$ to obtain:

$$\begin{aligned} x = A_0\cos(\omega t+\delta) &= A_0[\cos\omega t\cos\delta \\ &\quad -\sin\omega t\sin\delta] \\ &= -A_0\sin\delta\sin\omega t \\ &\quad + A_0\cos\delta\cos\omega t \\ &= \boxed{A_s\sin\omega t + A_c\cos\omega t} \end{aligned}$$

provided

$A_s = -A_0\sin\delta$ and $A_c = A_0\cos\delta$

(*b*) At $t = 0$:

$$x(0) = \boxed{A_0\cos\delta = A_c}$$

Evaluate dx/dt:

$$\begin{aligned} v = \frac{dx}{dt} &= \frac{d}{dt}[A_s\sin\omega t + A_c\cos\omega t] \\ &= A_s\omega\cos\omega t - A_c\omega\sin\omega t \end{aligned}$$

Evaluate $v(0)$ to obtain:

$$v(0) = \omega A_s = \boxed{-\omega A_0\sin\delta}$$

Simple Harmonic Motion and Circular Motion

33 •

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The frequency of the motion is the reciprocal of its period and the *x*-component of the particle's position is given by $x = A\cos(\omega t + \delta)$.

(*b*) Use the definition of speed to find the period of the motion:

$$T = \frac{2\pi r}{v} = \frac{2\pi(0.4\,\text{m})}{0.8\,\text{m/s}} = \boxed{3.14\,\text{s}}$$

(*a*) Because the frequency and the period are reciprocals of each other:

$$f = \frac{1}{T} = \frac{1}{3.14\,\text{s}} = \boxed{0.318\,\text{Hz}}$$

(*c*) Express the *x* component of the

$$x = A\cos(\omega t + \delta)$$

position of the particle:

Assuming that the particle is on the positive x axis at time $t = 0$:

$$A = A\cos\delta \Rightarrow \delta = \cos^{-1}1 = 0$$

Substitute for A, ω, and δ to obtain:

$$x = A\cos(2\pi f t) = \boxed{(40\,\text{cm})\cos\left[\left(2\,\text{s}^{-1}\right)t\right]}$$

***34** •

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The angular frequency of the motion is 2π times the reciprocal of its period and the x-component of the particle's position is given by $x = A\cos(\omega t + \delta)$.

(a) Use the definition of speed to express and evaluate the speed of the particle:

$$v = \frac{2\pi r}{T} = \frac{2\pi(15\,\text{cm})}{3\,\text{s}} = \boxed{31.4\,\text{cm/s}}$$

(b) Express the angular velocity of the particle:

$$\omega = \frac{2\pi}{T} = \boxed{\frac{2\pi}{3}\,\text{rad/s}}$$

(c) Express the x component of the position of the particle:

$$x = A\cos(\omega t + \delta)$$

Assuming that the particle is on the positive x axis at time $t = 0$:

$$A = A\cos\delta \Rightarrow \delta = \cos^{-1}1 = 0$$

Substitute to obtain:

$$x = \boxed{(15\,\text{cm})\cos\left(\frac{2\pi}{3}\,\text{s}^{-1}\right)t}$$

Energy in Simple Harmonic Motion

35 •

Picture the Problem The total energy of the object is given by $E_{\text{tot}} = \frac{1}{2}kA^2$, where A is the amplitude of the object's motion.

Express the total energy of the system:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E_{tot}:

$$E_{\text{tot}} = \tfrac{1}{2}(4.5\,\text{kN/m})(0.1\,\text{m})^2 = \boxed{22.5\,\text{J}}$$

36 •

Picture the Problem The total energy of an oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{tot} = \frac{1}{2}mv_{max}^2$. Its maximum speed, in turn, can be expressed in terms of its angular frequency and the amplitude of its motion.

Express the total energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{max}^2$$

Express v_{max}:

$$v_{max} = A\omega = 2\pi Af$$

Substitute to obtain:

$$E = \tfrac{1}{2}m(2\pi Af)^2 = 2mA^2\pi^2 f^2$$

Substitute numerical values and evaluate E:

$$E = 2(3\,\text{kg})(0.1\,\text{m})^2\pi^2(2.4\,\text{s}^{-1})^2 = \boxed{3.41\,\text{J}}$$

37 •

Picture the Problem The total mechanical energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{tot} = \frac{1}{2}mv_{max}^2$. Its total energy is also given by $E_{tot} = \frac{1}{2}kA^2$. We can equate these expressions to obtain an expression for A.

(*a*) Express the total mechanical energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{max}^2$$

Substitute numerical values and evaluate E:

$$E = \tfrac{1}{2}(1.5\,\text{kg})(0.7\,\text{m/s})^2 = \boxed{0.368\,\text{J}}$$

(*b*) Express the total energy of the object in terms of the amplitude of its motion:

$$E_{tot} = \tfrac{1}{2}kA^2$$

Solve for A:

$$A = \sqrt{\frac{2E_{tot}}{k}}$$

Substitute numerical values and evaluate A:

$$A = \sqrt{\frac{2(0.368\,\text{J})}{500\,\text{N/m}}} = \boxed{3.84\,\text{cm}}$$

38 •

Picture the Problem The total energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{tot} = \frac{1}{2}mv_{max}^2$. Its total energy is also given by $E_{tot} = \frac{1}{2}kA^2$. We can solve the latter equation to find A and solve the former equation for v_{max}.

(*a*) Express the total energy of the object as a function of the amplitude of its motion:

$$E_{tot} = \frac{1}{2}kA^2$$

Solve for A:

$$A = \sqrt{\frac{2E_{tot}}{k}}$$

Substitute numerical values and evaluate A:

$$A = \sqrt{\frac{2(0.9\,\text{J})}{2000\,\text{N/m}}} = \boxed{3.00\,\text{cm}}$$

(*b*) Express the total energy of the object in terms of its maximum speed:

$$E_{tot} = \frac{1}{2}mv_{max}^2$$

Solve for v_{max}:

$$v_{max} = \sqrt{\frac{2E_{tot}}{m}}$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = \sqrt{\frac{2(0.9\,\text{J})}{3\,\text{kg}}} = \boxed{0.775\,\text{m/s}}$$

39 •

Picture the Problem The total energy of the object is given by $E_{tot} = \frac{1}{2}kA^2$. We can solve this equation for the force constant k and substitute the numerical data to determine its value.

Express the total energy of the oscillator as a function of the amplitude of its motion:

$$E_{tot} = \frac{1}{2}kA^2$$

Solve for k:

$$k = \frac{2E_{tot}}{A^2}$$

Substitute numerical values and evaluate k:

$$k = \frac{2(1.4\,\text{J})}{(0.045\,\text{m})^2} = \boxed{1.38\,\text{kN/m}}$$

***40 ••**

Picture the Problem The total energy of the object is given, in terms of its maximum kinetic energy by $E_{tot} = \frac{1}{2}mv_{max}^2$. We can express v_{max} in terms of A and ω and, in turn, express ω in terms of a_{max} to obtain an expression for E_{tot} in terms of a_{max}.

Express the total energy of the object in terms of its maximum kinetic energy:

$$E_{tot} = \frac{1}{2}mv_{max}^2$$

Relate the maximum speed of the object to its angular frequency:

$$v_{max} = A\omega$$

Substitute to obtain:

$$E_{tot} = \frac{1}{2}m(A\omega)^2 = \frac{1}{2}mA^2\omega^2$$

Relate the maximum acceleration of the object to its angular frequency:

$$a_{max} = A\omega^2$$

or

$$\omega^2 = \frac{a_{max}}{A}$$

Substitute and simplify to obtain:

$$E_{tot} = \frac{1}{2}mA^2\frac{a_{max}}{A} = \frac{1}{2}mAa_{max}$$

Substitute numerical values and evaluate E_{tot}:

$$E_{tot} = \frac{1}{2}(3\,\text{kg})(0.08\,\text{m})(3.50\,\text{m/s}^2) = \boxed{0.420\,\text{J}}$$

Springs

41 •

Picture the Problem The frequency of the object's motion is given by $f = \frac{1}{2\pi}\sqrt{k/m}$. Its period is the reciprocal of its frequency. The maximum velocity and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{4.5\,\text{kN/m}}{2.4\,\text{kg}}} = \boxed{6.89\,\text{Hz}}$$

(*b*) The period of the motion to is the reciprocal of its frequency:

$$T=\frac{1}{f}=\frac{1}{6.89\,\mathrm{s}^{-1}}=\boxed{0.145\,\mathrm{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 10 cm:

$$A=\boxed{0.100\,\mathrm{m}}$$

(*d*) Express the object's maximum speed:

$$v_{\max}=A\omega=2\pi fA$$

Substitute numerical values and evaluate $v_{\max}$:

$$v_{\max}=2\pi\left(6.89\,\mathrm{s}^{-1}\right)(0.1\,\mathrm{m})=\boxed{4.33\,\mathrm{m/s}}$$

(*e*) Express the object's maximum acceleration:

$$a_{\max}=A\omega^2=\omega v_{\max}=2\pi f v_{\max}$$

Substitute numerical values and evaluate $a_{\max}$:

$$a_{\max}=2\pi\left(6.89\,\mathrm{s}^{-1}\right)(4.33\,\mathrm{m/s})$$
$$=\boxed{187\,\mathrm{m/s^2}}$$

(*f*) The object first reaches its equilibrium when:

$$t=\tfrac{1}{4}T=\tfrac{1}{4}(0.145\,\mathrm{s})=\boxed{36.3\,\mathrm{ms}}$$

Because the resultant force acting on the object as it passes through its equilibrium point is zero, the acceleration of the object is:

$$a=\boxed{0}$$

42 •

Picture the Problem The frequency of the object's motion is given by $f=\frac{1}{2\pi}\sqrt{k/m}$.

Its period is the reciprocal of its frequency. The maximum velocity and acceleration of an object executing simple harmonic motion are $v_{\max}=A\omega$ and $a_{\max}=A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f=\frac{1}{2\pi}\sqrt{\frac{700\,\mathrm{N/m}}{5\,\mathrm{kg}}}=\boxed{1.88\,\mathrm{Hz}}$$

(*b*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{1.88\,\text{s}^{-1}} = \boxed{0.531\,\text{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 8 cm:

$$A = \boxed{0.0800\,\text{m}}$$

(*d*) Express the object's maximum speed:

$$v_{\max} = A\omega = 2\pi f A$$

Substitute numerical values and evaluate $v_{\max}$:

$$v_{\max} = 2\pi\left(1.88\,\text{s}^{-1}\right)(0.08\,\text{m}) = \boxed{0.945\,\text{m/s}}$$

(*e*) Express the object's maximum acceleration:

$$a_{\max} = A\omega^2 = \omega v_{\max} = 2\pi f v_{\max}$$

Substitute numerical values and evaluate $a_{\max}$:

$$a_{\max} = 2\pi\left(1.88\,\text{s}^{-1}\right)(0.945\,\text{m/s}) = \boxed{11.2\,\text{m/s}^2}$$

(*f*) The object first reaches its equilibrium when:

$$t = \tfrac{1}{4}T = \tfrac{1}{4}(0.531\,\text{s}) = \boxed{0.133\,\text{s}}$$

Because the resultant force acting on the object as it passes through its equilibrium point is zero, the acceleration of the object is:

$$a = \boxed{0}$$

43 •

Picture the Problem The angular frequency, in terms of the force constant of the spring and the mass of the oscillating object, is given by $\omega^2 = k/m$. The period of the motion is the reciprocal of its frequency. The maximum velocity and acceleration of an object executing simple harmonic motion are $v_{\max} = A\omega$ and $a_{\max} = A\omega^2$, respectively.

(*a*) Relate the angular frequency of the motion to the force constant of the spring:

$$\omega^2 = \frac{k}{m}$$

or

$$k = m\omega^2 = 4\pi^2 f^2 m$$

Substitute numerical values to obtain:

$$k = 4\pi^2\left(2.4\,\text{s}^{-1}\right)^2(3\,\text{kg}) = \boxed{682\,\text{N/m}}$$

(*b*) Relate the period of the motion to its frequency:

$$T = \frac{1}{f} = \frac{1}{2.4\,\text{s}^{-1}} = \boxed{0.417\,\text{s}}$$

(*c*) Express the maximum speed of the object:

$$v_{max} = A\omega = 2\pi fA$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = 2\pi\left(2.4\,\text{s}^{-1}\right)\left(0.1\,\text{m}\right) = \boxed{1.51\,\text{m/s}}$$

(*d*) Express the maximum acceleration of the object:

$$a_{max} = A\omega^2 = 4\pi^2 f^2 A$$

Substitute numerical values and evaluate a_{max}:

$$a_{max} = 4\pi^2\left(2.4\,\text{s}^{-1}\right)^2\left(0.1\,\text{m}\right) = \boxed{22.7\,\text{m/s}^2}$$

***44 •**

Picture the Problem We can find the frequency of vibration of the car-and-passenger system using $f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$, where M is the total mass of the system. The spring constant can be determined from the compressing force and the amount of compression.

Express the frequency of the car-and-passenger system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

Express the spring constant:

$$k = \frac{F}{\Delta x} = \frac{mg}{\Delta x}$$

where m is the person's mass.

Substitute to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{M\Delta x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{\left(85\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{\left(2485\,\text{kg}\right)\left(2.35\times10^{-2}\,\text{m}\right)}} = \boxed{0.601\,\text{Hz}}$$

45 •

Picture the Problem We can relate the force constant k to the maximum acceleration by eliminating ω^2 between $\omega^2 = k/m$ and $a_{max} = A\omega^2$. We can also express the frequency f

of the motion by substituting ma_{max}/A for k in $f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

(*a*) Relate the angular frequency of the motion to the force constant and the mass of the oscillator:

$$\omega^2=\frac{k}{m}\text{ or }k=\omega^2 m$$

Relate the object's maximum acceleration to its angular frequency and amplitude and solve for the square of the angular frequency:

$$a_{max}=A\omega^2$$

or

$$\omega^2=\frac{a_{max}}{A} \qquad (1)$$

Substitute to obtain:

$$k=\frac{ma_{max}}{A}$$

Substitute numerical values and evaluate *k*:

$$k=\frac{(4.5\,\text{kg})(26\,\text{m/s}^2)}{3.8\times10^{-2}\,\text{m}}=\boxed{3.08\,\text{kN/m}}$$

(*b*) Replace ω in equation (1) by $2\pi f$ and solve for f to obtain:

$$f=\frac{1}{2\pi}\sqrt{\frac{a_{max}}{A}}$$

Substitute numerical values and evaluate *f*:

$$f=\frac{1}{2\pi}\sqrt{\frac{26\,\text{m/s}^2}{3.8\times10^{-2}\,\text{m}}}=\boxed{4.16\,\text{Hz}}$$

(*c*) The period of the motion is the reciprocal of its frequency:

$$T=\frac{1}{f}=\frac{1}{4.16\,\text{s}^{-1}}=\boxed{0.240\,\text{s}}$$

46 •

Picture the Problem We can find the frequency of the motion from its maximum speed and the relationship between frequency and angular frequency. The mass of the object can be found by eliminating ω between $\omega^2=k/m$ and $v_{max}=A\omega$.

(*b*) Express the object's maximum speed as a function of the frequency of its motion:

$$v_{max}=A\omega=2\pi fA \qquad (1)$$

Solve for *f*:

$$f=\frac{v_{max}}{2\pi A}$$

Substitute numerical values and evaluate *f*:

$$f=\frac{2.2\,\text{m/s}}{2\pi(5.8\times10^{-2}\,\text{m})}=\boxed{6.04\,\text{Hz}}$$

(*a*) Relate the square of the angular frequency of the motion to the force constant and the mass of the object:

$$\omega^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega^2} \qquad (2)$$

Eliminate ω between equations (1) and (2) to obtain:

$$m = \frac{kA^2}{v_{\text{max}}^2}$$

Substitute numerical values and evaluate *m*:

$$m = \frac{(1.8\times10^3\ \text{N/m})(5.8\times10^{-2}\ \text{m})^2}{(2.2\ \text{m/s})^2} = \boxed{1.25\ \text{kg}}$$

(*c*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{6.04\ \text{s}^{-1}} = \boxed{0.166\ \text{s}}$$

47 ••

Picture the Problem The maximum speed of the block is given by $v_{\text{max}} = A\omega$ and the angular frequency of the motion is $\omega = \sqrt{k/m} = 5.48\ \text{rad/s}$. We'll assume that the position of the block is given by $x = A\cos\omega t$ and solve for ωt for x = 4 cm and x = 0. We can use these values for ωt to find the time for the block to travel from x = 4 cm to its equilibrium position.

(*a*) Express the maximum speed of the block as a function of the system's angular frequency:

$$v_{\text{max}} = A\omega$$

Substitute numerical values and evaluate v_{max}:

$$v_{\text{max}} = (0.08\ \text{m})(5.48\ \text{rad/s}) = \boxed{0.438\ \text{m/s}}$$

(*b*) Assuming that $x = A\cos\omega t$, evaluate ωt for x = 4 cm = A/2:

$$\frac{A}{2} = A\cos\omega t \Rightarrow \omega t = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

Evaluate v for $\omega t = \pi/3$:

$$v = v_{\text{max}}\sin\omega t = (0.438\ \text{m/s})\sin\frac{\pi}{3}$$
$$= (0.438\ \text{m/s})\frac{\sqrt{3}}{2} = \boxed{0.379\ \text{m/s}}$$

Express a as a function of v_{max} and ω:

$$a = A\omega^2\cos\omega t = v_{\text{max}}\,\omega\cos\omega t$$

Substitute numerical values and evaluate a:

$$a = (0.438\,\text{m/s})(5.48\,\text{rad/s})\cos\frac{\pi}{3} = \boxed{1.20\,\text{m/s}^2}$$

(c) Evaluate ωt for $x = 0$:

$$0 = A\cos\omega t \Rightarrow \omega t = \cos^{-1}0 = \frac{\pi}{2}$$

Let Δt = time to go from $\omega t = \pi/3$ to $\omega t = \pi/2$. Then:

$$\omega\Delta t = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

Solve for and evaluate Δt:

$$\Delta t = \frac{\pi}{6\omega} = \frac{\pi}{6(5.48\,\text{rad/s})} = \boxed{95.5\,\text{ms}}$$

***48 ••**

Picture the Problem Choose a coordinate system in which upward is the positive y direction. We can find the mass of the object using $m = k/\omega^2$. We can apply a condition for translational equilibrium to the object when it is at its equilibrium position to determine the amount the spring has stretched from its natural length. Finally, we can use the initial conditions to determine A and δ and express $x(t)$ and then differentiate this expression to obtain $v(t)$ and $a(t)$.

(a) Express the angular frequency of the system in terms of the mass of the object fastened to the vertical spring and solve for the mass of the object:

$$\omega^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega^2}$$

Express ω^2 in terms of f:

$$\omega^2 = 4\pi^2 f^2$$

Substitute to obtain:

$$m = \frac{k}{4\pi^2 f^2}$$

Substitute numerical values and evaluate m:

$$m = \frac{1800\,\text{N/m}}{4\pi^2(5.5\,\text{s}^{-1})^2} = \boxed{1.51\,\text{kg}}$$

(b) Letting Δx represent the amount the spring is stretched from its natural length when the object is in equilibrium, apply $\sum F_y = 0$ to the object when it is in equilibrium:

$$k\Delta x - mg = 0$$

Solve for Δx:

$$\Delta x = \frac{mg}{k}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{(1.51\,\text{kg})(9.81\,\text{m/s}^2)}{1800\,\text{N/m}} = \boxed{8.23\,\text{mm}}$$

(*c*) Express the position of the object as a function of time:

$$x = A\cos(\omega t + \delta)$$

Use the initial conditions ($x_0 = -2.5$ cm and $v_0 = 0$) to find δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) = \tan^{-1} 0 = \pi$$

Evaluate ω:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1800\,\text{N/m}}{1.51\,\text{kg}}} = 34.5\,\text{rad/s}$$

Substitute to obtain:

$$x = (2.5\,\text{cm})\cos[(34.5\,\text{rad/s})t + \pi]$$
$$= \boxed{-(2.5\,\text{cm})\cos[(34.5\,\text{rad/s})t]}$$

Differentiate $x(t)$ to obtain v:

$$v = \boxed{(86.4\,\text{cm/s})\sin[(34.5\,\text{rad/s})t]}$$

Differentiate $v(t)$ to obtain a:

$$a = \boxed{(29.8\,\text{m/s}^2)\cos[(34.5\,\text{rad/s})t]}$$

49 ••

Picture the Problem Let the system include the object and the spring. Then, the net external force acting on the system is zero. Choose $E_i = 0$ and apply the conservation of mechanical energy to the system.

Express the period of the motion in terms of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply conservation of energy to the system:

$$E_i = E_f \text{ or } 0 = U_g + U_{\text{spring}}$$

Substitute for U_g and U_{spring}:

$$0 = -mg\Delta x + \tfrac{1}{2}k(\Delta x)^2$$

Solve for $\omega^2 = k/m$:

$$\omega^2 = \frac{k}{m} = \frac{2g}{\Delta x}$$

Substitute numerical values and evaluate ω^2:

$$\omega^2 = \frac{2(9.81\,\text{m/s}^2)}{3.42\times10^{-2}\,\text{m}} = 574\,\text{rad/s}^2$$

Substitute in equation (1) to obtain:

$$T=\frac{2\pi}{\sqrt{574\,\text{rad/s}^2}}=\boxed{0.262\,\text{s}}$$

50 ••

Picture the Problem Let the system include the object and the spring. Then the net external force acting on the system is zero. Because the net force acting on the object when it is at its equilibrium position is zero, we can apply a condition for translational equilibrium to determine the distance from the starting point to the equilibrium position. Letting $E_\text{i} = 0$, we can apply conservation of energy to the system to determine how far down the object moves before coming momentarily to rest. We can find the period of the motion and the maximum speed of the object from $T = 2\pi\sqrt{m/k}$ and $v_\text{max} = A\sqrt{k/m}$.

(*a*) Apply $\sum F_y = 0$ to the object when it is at the equilibrium position:

$$ky_0 - mg = 0$$

Solve for y_0:

$$y_0=\frac{mg}{k}$$

Substitute numerical values and evaluate y_0:

$$y_0=\frac{(1\,\text{kg})(9.81\,\text{m/s}^2)}{250\,\text{N/m}}=\boxed{3.92\,\text{cm}}$$

(*b*) Apply conservation of energy to the system:

$$E_\text{i}=E_\text{f}$$

or

$$0=U_\text{g}+U_\text{spring}$$

Substitute for U_g and U_spring:

$$0=-mgy_\text{f}+\tfrac{1}{2}ky_\text{f}^2$$

Solve for y_f:

$$y_\text{f}=\frac{2mg}{k}$$

Substitute numerical values and evaluate y_f:

$$y_\text{f}=\frac{2(1\,\text{kg})(9.81\,\text{m/s}^2)}{250\,\text{N/m}}=\boxed{7.85\,\text{cm}}$$

(*c*) Express the period T of the motion in terms of the mass of the object and the spring constant:

$$T=2\pi\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{1\,\text{kg}}{250\,\text{N/m}}} = \boxed{0.397\,\text{s}}$$

(d) The object will be moving with its maximum speed when it reaches its equilibrium position:

$$v_{\text{max}} = A\omega = A\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate v_{max}:

$$v_{\text{max}} = (3.92\,\text{cm})\sqrt{\frac{250\,\text{N/m}}{1\,\text{kg}}} = \boxed{62.0\,\text{cm/s}}$$

(e) The time required for the object to reach equilibrium is one-fourth of its period:

$$t = \tfrac{1}{4}T = \tfrac{1}{4}(0.397\,\text{s}) = \boxed{99.3\,\text{ms}}$$

51 ••

Picture the Problem The stunt woman's kinetic energy, after 2 s of flight, is $K_{2\,\text{s}} = \frac{1}{2}mv_{2\,\text{s}}^2$. We can evaluate this quantity as soon as we know how fast she is moving after two seconds. Because her motion is oscillatory, her velocity as a function of time is $v(t) = -A\omega\sin(\omega t + \delta)$. We can find the amplitude of her motion from her distance of fall and the angular frequency of her motion by applying conservation of energy to her fall to the ground.

Express the kinetic energy of the stunt woman when she has fallen for 2 s:

$$K_{2\,\text{s}} = \tfrac{1}{2}mv_{2\,\text{s}}^2 \qquad (1)$$

Express her velocity as a function of time:

$$v(t) = -A\omega\sin(\omega t + \delta)$$

where $\delta = 0$ (she starts from rest with positive displacement) and

$$A = \tfrac{1}{2}(192\,\text{m}) = 96\,\text{m}$$

$$\therefore v(t) = -(96\,\text{m})\omega\sin(\omega t) \qquad (2)$$

Letting $E_i = 0$, use conservation of energy to find the force constant of the elastic band:

$$0 = U_g + U_{\text{elastic}}$$

or

$$0 = -mgh + \tfrac{1}{2}kh^2 = 0$$

Solve for k:

$$k = \frac{2mg}{h}$$

Substitute numerical values and evaluate k:

$$k = \frac{2(60\,\text{kg})(9.81\,\text{m/s}^2)}{192\,\text{m}} = 6.13\,\text{N/m}$$

Express the angular frequency of her motion:

$$\omega = \sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{6.13\,\text{N/m}}{60\,\text{kg}}} = 0.320\,\text{rad/s}$$

Substitute in equation (2) to obtain:

$$\begin{aligned} v(t) &= -(96\,\text{m})(0.320\,\text{rad/s}) \\ &\quad \times \sin[(0.320\,\text{rad/s})t] \\ &= (30.7\,\text{m/s})\sin[(0.320\,\text{rad/s})t] \end{aligned}$$

Evaluate $v(2\text{ s})$:

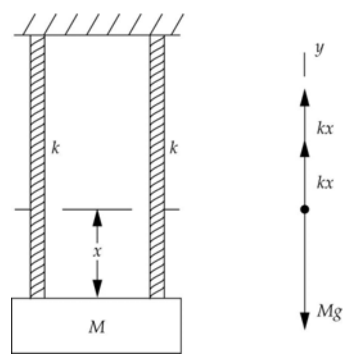
$$\begin{aligned} v(2\,\text{s}) &= (30.7\,\text{m/s})\sin[(0.320\,\text{rad/s})(2\,\text{s})] \\ &= 18.3\,\text{m/s} \end{aligned}$$

Substitute in equation (1) and evaluate $K(2\text{ s})$:

$$K(2\,\text{s}) = \tfrac{1}{2}(60\,\text{kg})(18.3\,\text{m/s})^2 = \boxed{10.1\,\text{kJ}}$$

***52 ••**

Picture the Problem The diagram shows the stretched bungie cords supporting the suitcase under equilibrium conditions. We can use $f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$ to express the frequency of the suitcase in terms of the effective "spring" constant k_{eff} and apply a condition for translational equilibrium to the suitcase to find k_{eff}.



Express the frequency of the suitcase oscillator:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$$

Apply $\sum F_y = 0$ to the suitcase to obtain:

$$kx + kx - Mg = 0$$

or

$$2kx - Mg = 0$$

or

$$k_{\text{eff}}x - Mg = 0$$

where $k_{\text{eff}} = 2k$

Solve for k_{eff} to obtain:

$$k_{eff} = \frac{Mg}{x}$$

Substitute to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{0.05\,\text{m}}} = \boxed{2.23\,\text{Hz}}$$

53 ••

Picture the Problem The frequency of the motion of the stone and block depends on the force constant of the spring and the mass of the stone plus block. The force constant can be determined from the equilibrium of the system when the spring is stretched additionally by the addition of the stone to the mass. When the block is at the point of maximum upward displacement, it is momentarily at rest and the net force acting on it is its weight.

(*a*) Express the frequency of the motion in terms of k and m:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{tot}}}$$

where m_{tot} is the total mass suspended from the spring.

Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0$$

Solve for k:

$$k = \frac{mg}{\Delta y}$$

Substitute numerical values and evaluate k:

$$k = \frac{(0.03\,\text{kg})(9.81\,\text{m/s}^2)}{0.05\,\text{m}} = 5.89\,\text{N/m}$$

Substitute and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{5.89\,\text{N/m}}{0.15\,\text{kg}}} = \boxed{0.997\,\text{Hz}}$$

(*b*) The time to travel from its lowest point to its highest point is one-half its period:

$$t = \tfrac{1}{2}T = \frac{1}{2f} = \frac{1}{2(0.997\,\text{s}^{-1})} = \boxed{0.502\,\text{s}}$$

(*c*) When the stone is at a point of maximum upward displacement:

$$F_{\text{net}} = mg = (0.03\,\text{kg})(9.81\,\text{m/s}^2) = \boxed{0.294\,\text{N}}$$

54 ••

Picture the Problem We can use the maximum acceleration of the oscillator $a_{\max} = A\omega^2$ to express $a_{\max}$ in terms of A, k, and m. k can be determined from the equilibrium of the system when the spring is stretched additionally by the addition of the stone to the mass. If the stone is to remain in contact with the block, the block's maximum downward acceleration must not exceed g.

Express the maximum acceleration in terms of the angular frequency and amplitude of the motion:

$$a_{\max} = A\omega^2$$

Relate ω^2 to the force constant and the mass of the stone:

$$\omega^2 = \frac{k}{m}$$

Substitute to obtain:

$$a_{\max} = A\frac{k}{m}$$

Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0$$

Solve for k:

$$k = \frac{mg}{\Delta y}$$

Substitute numerical values and evaluate k:

$$k = \frac{(0.03\,\text{kg})(9.81\,\text{m/s}^2)}{0.05\,\text{m}} = 5.89\,\text{N/m}$$

Substitute numerical values to express $a_{\max}$ in terms of A:

$$a_{\max} = A\frac{5.89\,\text{N/m}}{0.15\,\text{kg}} = (39.3\,\text{s}^{-2})A$$

Set $a_{\max} = g$ and solve for $A_{\max}$:

$$A_{\max} = \frac{g}{39.3\,\text{s}^{-2}}$$

Substitute for g and evaluate $A_{\max}$:

$$A_{\max} = \frac{9.81\,\text{m/s}^2}{39.3\,\text{s}^{-2}} = \boxed{25.0\,\text{cm}}$$

55 ••

Picture the Problem The maximum height above the floor to which the object rises is the sum of its initial distance from the floor and the amplitude of its motion. We can find the amplitude of its motion by relating it to the object's maximum speed. Because the object initially travels downward, it will be three-fourths of the way through its cycle when it first reaches its maximum height. We can find the minimum initial speed the object would need to be given in order for the spring to become uncompressed by applying conservation of energy.

(*a*) Relate *h*, the maximum height above the floor to which the object rises, to the amplitude of its motion:

$$h = A + 5.0\text{ cm} \qquad (1)$$

Relate the maximum speed of the object to the angular frequency and amplitude of its motion and solve for the amplitude:

$$v_{\max} = A\omega$$

or

$$A = v_{\max}\sqrt{\frac{m}{k}} \qquad (2)$$

Using its definition, express and evaluate the force constant of the spring:

$$k = \frac{mg}{\Delta y} = \frac{(2\text{ kg})(9.81\text{ m/s}^2)}{0.03\text{ m}} = 654\text{ N/m}$$

Substitute numerical values in equation (2) and evaluate *A*:

$$A = 0.3\text{ m/s}\sqrt{\frac{2\text{ kg}}{654\text{ N/m}}} = 1.66\text{ cm}$$

Substitute in equation (1) to obtain:

$$h = 1.66\text{ cm} + 5.00\text{ cm} = \boxed{6.66\text{ cm}}$$

(*b*) Express the time required for the object to reach its maximum height the first time:

$$t = \tfrac{3}{4}T$$

Express the period of the motion:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate *T*:

$$T = 2\pi\sqrt{\frac{2\text{ kg}}{654\text{ N/m}}} = 0.347\text{ s}$$

Substitute to obtain:

$$t = \tfrac{3}{4}(0.347\text{ s}) = \boxed{0.261\text{ s}}$$

(*c*) Because $h < 8.0$ cm:

$$\boxed{\text{the spring is never uncompressed.}}$$

Using conservation of energy and letting U_g be zero 5 cm above the floor, relate the height to which the object rises, Δy, to its initial kinetic energy:

$$\Delta K + \Delta U_g + \Delta U_s = 0$$

or, because $K_f = U_i = 0$,

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(L - y_i)^2 = 0$$

Because $\Delta y = L - y_i$:

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(\Delta y)^2 = 0$$

and

$$\tfrac{1}{2}mv_i^2 - mg\Delta y = 0$$

Solve for and evaluate v_i for $\Delta y = 3$ cm:

$$v_i = \sqrt{2g\Delta y} = \sqrt{2(9.81\,\text{m/s}^2)(3\,\text{cm})} = 0.767\,\text{m/s}$$

i.e., the minimum initial velocity that must be given to the object for the spring to be uncompressed at some time is $\boxed{0.767\,\text{m/s}}$

***56 ••**

Picture the Problem We can relate the elongation of the cable to the load on it using the definition of Young's modulus and use the expression for the frequency of a spring and mass oscillator to find the oscillation frequency of the engine block at the end of the wire.

(*a*) Using the definition of Young's modulus, relate the elongation of the cable to the applied stress:

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta\ell/\ell}$$

Solve for $\Delta\ell$:

$$\Delta\ell = \frac{F\ell}{AY} = \frac{Mg\ell}{AY}$$

Substitute numerical values and evaluate $\Delta\ell$:

$$\Delta\ell = \frac{(950\,\text{kg})(9.81\,\text{m/s}^2)(2.5\,\text{m})}{(1.5\,\text{cm}^2)(150\,\text{GN/m}^2)} = \boxed{1.04\,\text{mm}}$$

(*b*) Express the oscillation frequency of the wire-engine block system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$$

Express the effective "spring" constant of the cable:

$$k_{\text{eff}} = \frac{F}{\Delta\ell} = \frac{Mg}{\Delta\ell}$$

Substitute to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta\ell}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{1.04\,\text{mm}}} = \boxed{15.5\,\text{Hz}}$$

Energy of an Object on a Vertical Spring

57 ••

Picture the Problem Let the origin of our coordinate system be at y_0, where y_0 is the equilibrium position of the object and let $U_g = 0$ at this location. Because $F_{\text{net}} = 0$ at equilibrium, the extension of the spring is then $y_0 = mg/k$, and the potential energy stored in the spring is $U_s = \frac{1}{2}ky_0^2$. A further extension of the spring by an amount y increases U_s to $\frac{1}{2}k(y+y_0)^2 = \frac{1}{2}ky^2 + kyy_0 + \frac{1}{2}ky_0^2 = \frac{1}{2}ky^2 + mgy + \frac{1}{2}ky_0^2$. Consequently, if we set $U = U_g + U_s = 0$, a further extension of the spring by y increases U_s by $\frac{1}{2}ky^2 + mgy$ while decreasing U_g by mgy. Therefore, if $U = 0$ at the equilibrium position, the change in U is given by $\frac{1}{2}k(y')^2$, where $y' = y - y_0$.

(*a*) Express the total energy of the system:

$$E = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E:

$$E = \tfrac{1}{2}(600\,\text{N/m})(0.03\,\text{m})^2 = \boxed{0.270\,\text{J}}$$

(*b*) Express and evaluate U_g when the object is at its maximum downward displacement:

$$\begin{aligned} U_g &= -mgA \\ &= -(2.5\,\text{kg})(9.81\,\text{m/s}^2)(0.03\,\text{m}) \\ &= \boxed{-0.736\,\text{J}} \end{aligned}$$

(*c*) When the object is at its maximum downward displacement:

$$\begin{aligned} U_s &= \tfrac{1}{2}kA^2 + mgA \\ &= \tfrac{1}{2}(600\,\text{N/m})(0.03\,\text{m})^2 \\ &\quad + (2.5\,\text{kg})(9.81\,\text{m/s}^2)(0.03\,\text{m}) \\ &= \boxed{1.01\,\text{J}} \end{aligned}$$

(*d*) The object has its maximum kinetic energy when it is passing through its equilibrium position:

$$K_{max} = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(600\,\text{N/m})(0.03\,\text{m})^2 = \boxed{0.270\,\text{J}}$$

58 ••

Picture the Problem Let the origin of our coordinate system be at y_0, where y_0 is the equilibrium position of the object and let $U_g = 0$ at this location. Because $F_{net} = 0$ at equilibrium, the extension of the spring is then $y_0 = mg/k$, and the potential energy stored in the spring is $U_s = \tfrac{1}{2}ky_0^2$. A further extension of the spring by an amount y increases U_s to $\tfrac{1}{2}k(y+y_0)^2 = \tfrac{1}{2}ky^2 + kyy_0 + \tfrac{1}{2}ky_0^2 = \tfrac{1}{2}ky^2 + mgy + \tfrac{1}{2}ky_0^2$. Consequently, if we set $U = U_g + U_s = 0$, a further extension of the spring by y increases U_s by $\tfrac{1}{2}ky^2 + mgy$ while decreasing U_g by mgy. Therefore, if $U = 0$ at the equilibrium position, the change in U is given by $\tfrac{1}{2}k(y')^2$, where $y' = y - y_0$.

(*a*) Express the total energy of the system:

$$E = \tfrac{1}{2}kA^2$$

Letting Δy represent the amount the spring is stretched from its natural length by the 1.5-kg object, apply $\sum F_y = ma_y$ to the object when it is in its equilibrium position:

$$k\Delta y - mg = 0$$

Solve for k:

$$k = \frac{mg}{\Delta y}$$

Substitute for k to obtain:

$$E = \frac{mgA^2}{2\Delta y}$$

Substitute numerical values and evaluate E:

$$E = \frac{(1.5\,\text{kg})(9.81\,\text{m/s}^2)(0.022\,\text{m})^2}{2(0.028\,\text{m})} = \boxed{0.127\,\text{J}}$$

(*b*) Express U_g when the object is at its maximum downward displacement:

$$U_g = -mgA$$

Substitute numerical values and evaluate U_g:

$$U_g = -(1.5\,\text{kg})(9.81\,\text{m/s}^2)(0.022\,\text{m}) = \boxed{-0.324\,\text{J}}$$

(*c*) When the object is at its maximum downward displacement:

$$U_s = \tfrac{1}{2}kA^2 + mgA$$

Substitute numerical values and evaluate U_s:

$$U_s = \tfrac{1}{2}(526\,\text{N/m})(0.022\,\text{m})^2 + (1.5\,\text{kg})(9.81\,\text{m/s}^2)(0.022\,\text{m}) = \boxed{0.451\,\text{J}}$$

(*d*) The object has its maximum kinetic energy when it is passing through its equilibrium position:

$$K_{max} = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(526\,\text{N/m})(0.022\,\text{m})^2 = \boxed{0.127\,\text{J}}$$

***59 ••**

Picture the Problem We can find the amplitude of the motion by relating it to the maximum speed of the object. Let the origin of our coordinate system be at y_0, where y_0 is the equilibrium position of the object and let $U_g = 0$ at this location. Because $F_{net} = 0$ at equilibrium, the extension of the spring is then $y_0 = mg/k$, and the potential energy stored in the spring is $U_s = \tfrac{1}{2}ky_0^2$. A further extension of the spring by an amount y increases U_s to $\tfrac{1}{2}k(y + y_0)^2 = \tfrac{1}{2}ky^2 + kyy_0 + \tfrac{1}{2}ky_0^2 = \tfrac{1}{2}ky^2 + mgy + \tfrac{1}{2}ky_0^2$. Consequently, if we set $U = U_g + U_s = 0$, a further extension of the spring by y increases U_s by $\tfrac{1}{2}ky^2 + mgy$ while decreasing U_g by mgy. Therefore, if $U = 0$ at the equilibrium position, the change in U is given by $\tfrac{1}{2}k(y')^2$, where $y' = y - y_0$.

(*a*) Relate the maximum speed of the object to the amplitude of its motion:

$$v_{max} = A\omega$$

Solve for A:

$$A = \frac{v_{max}}{\omega} = v_{max}\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate A:

$$A = (0.3\,\text{m/s})\sqrt{\frac{1.2\,\text{kg}}{300\,\text{N/m}}} = \boxed{1.90\,\text{cm}}$$

(*b*) Express the energy of the object at maximum displacement:

$$E = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E:

$$E = \tfrac{1}{2}(300\,\text{N/m})(0.019\,\text{m})^2 = \boxed{0.0542\,\text{J}}$$

(*c*) At maximum displacement from equilibrium:

$$U_g = -mgA$$

Substitute numerical values and evaluate U_g:

$$U_g = -(1.2\,\text{kg})(9.81\,\text{m/s}^2)(0.019\,\text{m}) = \boxed{-0.224\,\text{J}}$$

(*d*) Express the potential energy in the spring when the object is at its maximum downward displacement:

$$U_s = \tfrac{1}{2}kA^2 + mgA$$

Substitute numerical values and evaluate U_s:

$$U_s = \tfrac{1}{2}(300\,\text{N/m})(0.019\,\text{m})^2 + (1.2\,\text{kg})(9.81\,\text{m/s}^2)(0.019\,\text{m}) = \boxed{0.278\,\text{J}}$$

Simple Pendulums

60 •

Picture the Problem We can determine the required length of the pendulum from the expression for the period of a simple pendulum.

Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Solve for L:

$$L = \frac{T^2 g}{4\pi^2}$$

Substitute numerical values and evaluate L:

$$L = \frac{(5\,\text{s})^2(9.81\,\text{m/s}^2)}{4\pi^2} = \boxed{6.21\,\text{m}}$$

61 •

Picture the Problem We can find the period of the pendulum from $T = 2\pi\sqrt{L/g_{\text{moon}}}$ where $g_{\text{moon}} = \frac{1}{6}g$ and L = 6.21 m.

Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g_{\text{moon}}}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{6.21\,\text{m}}{\frac{1}{6}(9.81\,\text{m/s}^2)}} = \boxed{12.2\,\text{s}}$$

62 •

Picture the Problem We can find the value of g at the location of the pendulum by solving the equation $T = 2\pi\sqrt{L/g}$ for g and evaluating it for the given length and period.

Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Solve for g:

$$g = \frac{4\pi^2 L}{T^2}$$

Substitute numerical values and evaluate g:

$$g = \frac{4\pi^2(0.7\,\text{m})}{(1.68\,\text{s})^2} = \boxed{9.79\,\text{m/s}^2}$$

***63** •

Picture the Problem We can use $T = 2\pi\sqrt{L/g}$ to find the period of this pendulum.

Express the period of a simple pendulum:

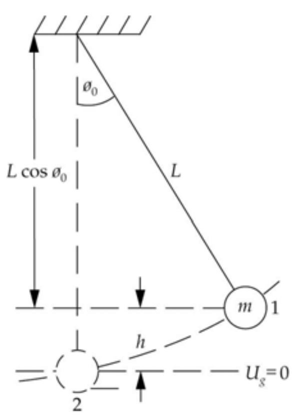
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{34\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{11.7\,\text{s}}$$

64 ••

Picture the Problem The figure shows the simple pendulum at maximum angular displacement ϕ_0. The total energy of the simple pendulum is equal to its initial gravitational potential energy. We can apply the definition of gravitational potential energy and use the small-angle approximation to show that $E \approx \frac{1}{2}mgL\phi_0^2$.



Express the total energy of the simple pendulum at maximum displacement:

$$E = U_{\text{max displacement}} = mgh$$
$$= mgL[1 - \cos\phi_0]$$

For $\phi << 1$:

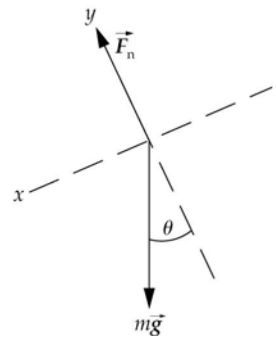
$$\cos\phi \approx 1 - \tfrac{1}{2}\phi^2$$

Substitute and simplify to obtain:

$$E = mgL\left[1 - \left(1 - \tfrac{1}{2}\phi_0^2\right)\right] = \boxed{\tfrac{1}{2}mgL\phi_0^2}$$

65 ••

Picture the Problem Because the cart is accelerating down the incline, the period of the simple pendulum will be given by $T = 2\pi\sqrt{L/g_{\text{eff}}}$ where g_{eff} is less than g by the acceleration of the cart. We can apply Newton's 2nd law to the cart to find its acceleration down the incline and then subtract this acceleration from g to find g_{eff}.



Express the period of a simple pendulum in terms of its length and the effective value of the acceleration of gravity:

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}}$$

Relate g_{eff} to the acceleration of the cart:

$$g_{\text{eff}} = g - a$$

Apply $\sum F_x = ma_x$ to the cart and solve for its acceleration:

$$mg\sin\theta = ma$$

and

$$a = g\sin\theta$$

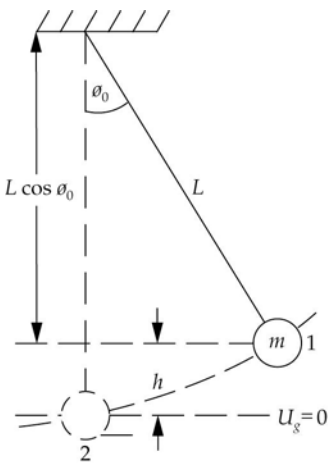
Substitute to obtain:

$$T = 2\pi\sqrt{\frac{L}{g-a}} = 2\pi\sqrt{\frac{L}{g - g\sin\theta}}$$

$$= \boxed{2\pi\sqrt{\frac{L}{g(1-\sin\theta)}}}$$

66 ••

Picture the Problem The figure shows the simple pendulum at maximum angular displacement ϕ_0. We can express the angular position of the pendulum's bob in terms of its initial angular position and time and differentiate this expression to find the maximum speed of the bob. We can use conservation of energy to find an exact value for v_{max} and the approximation $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ to show that this value reduces to the former value for small ϕ.



(*a*) Relate the speed of the pendulum's bob to its angular speed:

$$v = L\frac{d\phi}{dt} \qquad (1)$$

Express the angular position of the pendulum as a function of time:

$$\phi = \phi_0 \cos\omega t$$

Differentiate this expression to express the angular speed of the pendulum:

$$\frac{d\phi}{dt} = -\phi_0\omega\sin\omega t$$

Substitute in equation (1) to obtain:

$$v = -L\phi_0\omega\sin\omega t = -v_{max}\sin\omega t$$

Simplify v_{max} to obtain:

$$v_{max} = L\phi_0\sqrt{\frac{g}{L}} = \boxed{\phi_0\sqrt{gL}}$$

(*b*) Use conservation of energy to relate the potential energy of the pendulum at point 1 to its kinetic energy at point 2:

$$\Delta K + \Delta U = 0$$

or, because $K_1 = U_2 = 0$,

$$K_2 - U_1 = 0$$

Substitute for K_2 and U_1:

$$\tfrac{1}{2}mv_2^2 - mgh = 0$$

Express h in terms of L and ϕ_0:

$$h = L(1-\cos\phi_0)$$

Substitute for h and solve for $v_2 = v_{max}$ to obtain:

$$v_{max} = \boxed{\sqrt{2gL(1-\cos\phi_0)}} \qquad (2)$$

(*c*) For $\phi_0 << 1$:

$$1-\cos\phi_0 \approx \tfrac{1}{2}\phi_0^2$$

Substitute in equation (2) to obtain:

$$v_{\max} = \sqrt{2gL\left(\tfrac{1}{2}\phi_0^2\right)} = \boxed{\phi_0\sqrt{gL}}$$

in agreement with our result in part (*a*).

(*d*) Express the difference in the results from (*a*) and (*b*):

$$\Delta v = v_{\max,a} - v_{\max,b} \qquad (3)$$

Using $\phi_0 = 0.20$ rad and $L = 1$ m, evaluate the result in (*b*):

$$v_{\max,b} = \sqrt{2\left(9.81\,\text{m/s}^2\right)(1\,\text{m})(1-\cos 0.2)}$$
$$= 0.6254\,\text{m/s}$$

Using $\phi_0 = 0.20$ rad and $L = 1$ m, evaluate the result in part (*a*):

$$v_{\max.a} = (0.20\,\text{rad})\sqrt{\left(9.81\,\text{m/s}^2\right)(1\,\text{m})}$$
$$= 0.6264\,\text{m/s}$$

Substitute in equation (3) to obtain:

$$\Delta v = 0.6264\,\text{m/s} - 0.6254\,\text{m/s}$$
$$= 0.001\,\text{m/s} = \boxed{1.00\,\text{mm/s}}$$

Physical Pendulums

67 •

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the thin disk with respect to an axis through its pivot point. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the disk with respect to its center of mass and the distance from its center of mass to its pivot point.

Express the period of physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the thin disk about an axis through the pivot point:

$$I = I_{\text{cm}} + MR^2 = \tfrac{1}{2}MR^2 + MR^2$$
$$= \tfrac{3}{2}MR^2$$

Substitute to obtain:

$$T = 2\pi\sqrt{\frac{\tfrac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{3(0.2\,\text{m})}{2\left(9.81\,\text{m/s}^2\right)}} = \boxed{1.10\,\text{s}}$$

68 •

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the circular hoop with respect to an axis through its pivot point. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the hoop with respect to its center of mass and the distance from its center of mass to its pivot point.

Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the circular hoop about an axis through the pivot point:

$$I = I_{\text{cm}} + MR^2 = MR^2 + MR^2 = 2MR^2$$

Substitute to obtain:

$$T = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.5\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{2.01\,\text{s}}$$

69 •

Picture the Problem The period of a physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is its moment of inertia with respect to an axis through its pivot point. We can solve this equation for I and evaluate it using the given numerical data.

Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Solve for I:

$$I = \frac{MgDT^2}{4\pi^2}$$

Substitute numerical values and evaluate I:

$$I = \frac{(3\,\text{kg})(9.81\,\text{m/s}^2)(0.1\,\text{m})(2.6\,\text{s})^2}{4\pi^2} = \boxed{0.504\,\text{kg}\cdot\text{m}^2}$$

***70** ••

Picture the Problem We can use the expression for the period of a simple pendulum to find the period of the clock.

(*a*) Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Substitute numerical values and evaluate *T*:

$$T = 2\pi\sqrt{\frac{4\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{4.01\,\text{s}}$$

(*b*) By effectively raising the center of mass of the pendulum, placing coins in the tray shortens the period.

71 ••

Picture the Problem Let *x* be the distance of the pivot from the center of the rod, *m* the mass at each end of the rod, and *L* the length of the rod. We can express the period of the physical pendulum as a function of the distance *x* and then differentiate this expression with respect to *x* to show that, when $x = L/2$, the period is a minimum.

(*a*) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \qquad (1)$$

Express the moment of inertia of the dumbbell with respect to an axis through its center of mass:

$$I_{\text{cm}} = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 = \tfrac{1}{2}mL^2$$

Using the parallel-axis theorem, express the moment of inertia of the dumbbell with respect to an axis through the pivot point:

$$I = I_{\text{cm}} + 2mx^2 = \tfrac{1}{2}mL^2 + 2mx^2$$

Substitute in equation (1) to obtain:

$$T = 2\pi\sqrt{\frac{\tfrac{1}{2}mL^2 + 2mx^2}{2mgx}}$$

$$= \frac{2\pi}{\sqrt{g}}\sqrt{\frac{\tfrac{1}{4}L^2 + x^2}{x}} \qquad (2)$$

$$= C\sqrt{\frac{\tfrac{1}{4}L^2 + x^2}{x}}$$

where $C = \dfrac{2\pi}{\sqrt{g}}$

Set $dT/dx = 0$ to find the condition for minimum T:

$$\frac{dT}{dx} = C \cdot \frac{d}{dx}\sqrt{\frac{\frac{1}{4}L^2 + x^2}{x}} = 0 \text{ for extrema}$$

Evaluate the derivative to obtain:

$$\frac{2x^2 - \left(\frac{1}{4}L^2 + x^2\right)}{x^2\sqrt{\frac{\frac{1}{4}L^2 + x^2}{x}}} = 0$$

Because the denominator of this expression cannot be zero, it must be true that:

$$2x^2 - \left(\frac{1}{4}L^2 + x^2\right) = 0$$

Solve for x to obtain:

$$x = \boxed{\tfrac{1}{2}L}$$

i.e., the period is a minimum when the pivot point is at one of the masses.

(*b*) Substitute $x = L/4$ in equation (2) and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{4}L^2 + \left(\frac{1}{4}L\right)^2}{\frac{1}{2}gL}} = \pi\sqrt{\frac{5L}{g}}$$

Substitute numerical values and evaluate T:

$$T = \pi\sqrt{\frac{5(2\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{3.17\,\text{s}}$$

Remarks: In (*a*), we've shown that $x = L/2$ corresponds to an *extreme* value; i.e., to either a maximum or a minimum. To complete the demonstration that this value of x corresponds to a minimum, we can either (1) show that d^2T/dx^2 evaluated at $x = L/2$ is positive, or (2) graph T as a function of x and note that the graph is a minimum at $x = L/2$.

72 ••

Picture the Problem Let x be the distance of the pivot from the center of the rod. We'll express the period of the physical pendulum as a function of the distance x and then differentiate this expression with respect to x to find the location of the pivot point that minimizes the period of the physical pendulum.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \qquad (1)$$

Express the moment of inertia of the dumbbell with respect to an axis through its center of mass:

$$I_{\text{cm}} = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 + \tfrac{1}{12}(2m)L^2 = \tfrac{2}{3}mL^2$$

Using the parallel-axis theorem, express the moment of inertia of the dumbbell with respect to an axis through the pivot point:

$$I = I_{\text{cm}} + 4mx^2 = \tfrac{2}{3}mL^2 + 4mx^2$$

Substitute in equation (1) to obtain:

$$T = 2\pi\sqrt{\frac{\tfrac{2}{3}mL^2 + 4mx^2}{4mgx}} = \frac{\pi}{\sqrt{g}}\sqrt{\frac{\tfrac{2}{3}L^2 + 4x^2}{x}}$$

or

$$T = C\sqrt{\frac{\tfrac{2}{3}L^2 + 4x^2}{x}} \text{ where } C = \frac{\pi}{\sqrt{g}}$$

Set $dT/dx = 0$ to find the condition for minimum T:

$$\frac{dT}{dx} = C \times \frac{d}{dx}\sqrt{\frac{\tfrac{2}{3}L^2 + 4x^2}{x}} = 0 \text{ for extrema}$$

Evaluate the derivative to obtain:

$$\frac{8x^2 - \left(\tfrac{2}{3}L^2 + 4x^2\right)}{2x^2\sqrt{\frac{\tfrac{2}{3}L^2 + 4x^2}{x}}} = 0$$

Because the denominator of this expression cannot be zero, it follows that:

$$8x^2 - \left(\tfrac{2}{3}L^2 + 4x^2\right) = 0$$

Solve for x to obtain:

$$x = \frac{L}{\sqrt{6}}$$

The distance to the pivot point from the nearer mass is:

$$d = \frac{L}{2} - \frac{L}{\sqrt{6}} = \boxed{0.0918L}$$

Remarks: We've shown that $x = L/\sqrt{6}$ corresponds to an *extreme* value; i.e., to either a maximum or a minimum. To complete the demonstration that this value of x corresponds to a minimum, we can either (1) show that d^2T/dx^2 evaluated at

$x = L/\sqrt{6}$ is positive, or (2) graph T as a function of x and note that the graph is a minimum at $x = L/\sqrt{6}$.

***73 ••**

Picture the Problem Let x be the distance of the pivot from the center of the meter stick, m the mass of the meter stick, and L its length. We'll express the period of the meter stick as a function of the distance x and then differentiate this expression with respect to x to determine where the hole should be drilled to minimize the period.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \qquad (1)$$

Express the moment of inertia of the meter stick with respect to its center of mass:

$$I_{\mathrm{cm}} = \tfrac{1}{12}mL^2$$

Using the parallel-axis theorem, express the moment of inertia of the meter stick with respect to the pivot point:

$$I = I_{\mathrm{cm}} + mx^2 = \tfrac{1}{12}mL^2 + mx^2$$

Substitute in equation (1) to obtain:

$$T = 2\pi\sqrt{\frac{\tfrac{1}{12}mL^2 + mx^2}{mgx}} = \frac{2\pi}{\sqrt{g}}\sqrt{\frac{\tfrac{1}{12}L^2 + x^2}{x}} = C\sqrt{\frac{\tfrac{1}{12}L^2 + x^2}{x}}$$

where $C = \dfrac{2\pi}{\sqrt{g}}$

Set $dT/dx = 0$ to find the condition for minimum T:

$$\frac{dT}{dx} = C \times \frac{d}{dx}\sqrt{\frac{\tfrac{1}{12}L^2 + x^2}{x}} = 0 \text{ for extrema}$$

Evaluate the derivative to obtain:

$$\frac{2x^2 - \left(\tfrac{1}{12}L^2 + x^2\right)}{x^2\sqrt{\dfrac{\tfrac{1}{12}L^2 + x^2}{x}}} = 0$$

Because the denominator of this expression cannot be zero, it follows

$$2x^2 - \left(\tfrac{1}{12}L^2 + x^2\right) = 0$$

that:

Solve for and evaluate x to obtain:

$$x = \frac{L}{\sqrt{12}} = \frac{100\,\text{cm}}{\sqrt{12}} = 28.9\,\text{cm}$$

The hole should be drilled at a distance:

$$d = 50\,\text{cm} - 28.9\,\text{cm} = \boxed{21.1\,\text{cm}}$$

from the center of the meter stick.

74 ••

Picture the Problem Let m represent the mass and r the radius of the uniform disk. We'll use the expression for the period of a physical pendulum and the parallel-axis theorem to obtain a quadratic equation that we can solve for d. We will then treat our expression for the period of the pendulum as an extreme-value problem, setting its derivative equal to zero in order to determine the value for d that will minimize the period.

(a) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem, relate the moment of inertia with respect to an axis through the hole to the moment of inertia with respect to the disk's center of mass:

$$I = I_{\text{cm}} + md^2 = \tfrac{1}{2}mR^2 + md^2$$

Substitute to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}mR^2 + md^2}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + d^2}{gd}} \qquad (1)$$

Square both sides of this equation, simplify, and substitute numerical values to obtain:

$$d^2 - \frac{gT^2}{4\pi^2}d + \frac{R^2}{2} = 0$$

or

$$d^2 - (1.553\,\text{m})d + 0.320\,\text{m}^2 = 0$$

Solve the quadratic equation to obtain:

$$d = \boxed{0.245\,\text{m}}$$

The second root, $d = 1.31$ m, is too large to be physically meaningful.

(*b*) Set the derivative of equation (1) equal to zero to find relative maxima and minima:

$$\frac{dT}{dd} = \frac{2\pi}{\sqrt{g}} \cdot \frac{d}{dd}\sqrt{\frac{\frac{1}{2}R^2 + d^2}{d}} = 0 \text{ for extrema}$$

Evaluate the derivative to obtain:

$$\frac{2d^2 - \left(\frac{1}{2}R^2 + d^2\right)}{2d^2\sqrt{\frac{\frac{1}{2}R^2 + d^2}{d}}} = 0$$

Because the denominator of this fraction cannot be zero:

$$2d^2 - \left(\frac{1}{2}R^2 + d^2\right) = 0$$

Solve this equation to obtain:

$$d = \boxed{\frac{R}{\sqrt{2}}}$$

Evaluate equation (1) with $d = R/\sqrt{2}$ to obtain an expression for the shortest possible period of this physical pendulum:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + \frac{1}{2}R^2}{g\frac{R}{\sqrt{2}}}} = 2\pi\sqrt{\frac{\sqrt{2}R}{g}}$$

Substitute numerical values and evaluate *T*:

$$T = 2\pi\sqrt{\frac{\sqrt{2}(0.8\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{2.13\,\text{s}}$$

Remarks: We've shown that $d = R/\sqrt{2}$ corresponds to an *extreme* value; i.e., to either a maximum or a minimum. To complete the demonstration that this value of *d* corresponds to a minimum, we can either (1) show that d^2T/dd^2 evaluated at $d = R/\sqrt{2}$ is positive, or (2) graph *T* as a function of *d* and note that the graph is a minimum at $d = R/\sqrt{2}$.

75 •••

Picture the Problem We can use the equation for the period of a physical pendulum and the parallel-axis theorem to show that $h_1 + h_2 = gT^2/4\pi^2$.

Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem, relate the moment of inertia with respect to an axis through P_1 to the

$$I = I_{\text{cm}} + mh_1^2$$

moment of inertia with respect to the disk's center of mass:

Substitute to obtain:

$$T = 2\pi\sqrt{\frac{I_{\rm cm} + mh_1^2}{mgh_1}}$$

Square both sides of this equation and rearrange to obtain:

$$\frac{mgT^2}{4\pi^2} = \frac{I_{\rm cm}}{h_1} + mh_1 \qquad (1)$$

Because the period of oscillation is the same for point P_2:

$$\frac{I_{\rm cm}}{h_1} + mh_1 = \frac{I_{\rm cm}}{h_2} + mh_2$$

Solve this equation for $I_{\rm cm}$:

$$I_{\rm cm} = mh_1h_2$$

Substitute in equation (1) to obtain:

$$\frac{mgT^2}{4\pi^2} = \frac{mh_1h_2}{h_1} + mh_1$$

or

$$\boxed{h_2 + h_1 = \frac{gT^2}{4\pi^2}}$$

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Picture the Problem We can find the period of the physical pendulum in terms of the period of a simple pendulum by starting with $T = 2\pi\sqrt{I/mgL}$ and applying the parallel-axis theorem. Performing a binomial expansion for $r << L$ on the radicand of our expression for T will lead to $T \approx T_0(1 + r^2/5L^2)$.

(*a*) Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

Using the parallel-axis theorem, relate the moment of inertia of the pendulum about an axis through its center of mass to its moment of inertia with respect to an axis through its point of support:

$$I = I_{\rm cm} + mL^2 = \tfrac{2}{5}mr^2 + mL^2$$

Substitute and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{2}{5}mr^2 + mL^2}{mgL}} = 2\pi\sqrt{\frac{\frac{2}{5}r^2 + L^2}{gL}}$$
$$= 2\pi\sqrt{\frac{L}{g}\left(1+\frac{2r^2}{5L^2}\right)} = 2\pi\sqrt{\frac{L}{g}}\sqrt{1+\frac{2r^2}{5L^2}}$$
$$= \boxed{T_0\sqrt{1+\frac{2r^2}{5L^2}}}$$

(*b*) Using the binomial expansion, expand $\left(1+\frac{2r^2}{5L^2}\right)^{1/2}$:

$$\left(1+\frac{2r^2}{5L^2}\right)^{1/2} = 1+\frac{1}{2}\left(\frac{2r^2}{5L^2}\right)+\frac{1}{8}\left(\frac{2r^2}{5L^2}\right)^2 + \text{higher-order terms}$$
$$\approx 1+\frac{r^2}{5L^2}$$

provided $r \ll L$

Substitute in our result from (*a*) to obtain:

$$T \approx \boxed{T_0\left(1+\frac{r^2}{5L^2}\right)}$$

(*c*) Express the fractional error when the approximation $T = T_0$ is used for this pendulum:

$$\frac{\Delta T}{T} \approx \frac{T-T_0}{T_0} = \frac{T}{T_0} - 1$$
$$= 1+\frac{r^2}{5L^2} - 1 = \frac{r^2}{5L^2}$$

Substitute numerical values and evaluate $\Delta T/T$:

$$\frac{\Delta T}{T} \approx \frac{(2\,\text{cm})^2}{5(100\,\text{cm})^2} = \boxed{0.008\%}$$

For an error of 1%:

$$\frac{r^2}{5L^2} = 0.01$$

Solve for and evaluate r with $L = 100$ cm:

$$r = L\sqrt{0.05} = (100\,\text{cm})\sqrt{0.05}$$
$$= \boxed{22.4\,\text{cm}}$$

77 •••

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$. We can express its period as a function of the distance d by using the definition of the center of mass of the pendulum to find D in terms of d and the parallel-axis theorem to express I in terms of d. Solving the resulting quadratic equation yields d.

In (*b*), because the clock is losing 5 minutes per day, one would reposition the disk so that the clock runs faster; i.e., so the pendulum has a shorter period. We can determine the appropriate correction to make in the position of the disk by relating the fractional time loss to the fractional change in its position.

(*a*) Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{m_{\text{tot}}gx_{\text{cm}}}}$$

Solve for $\dfrac{I}{x_{\text{cm}}}$:

$$\frac{I}{x_{\text{cm}}} = \frac{T^2 g m_{\text{tot}}}{4\pi^2} \qquad (1)$$

Express the moment of inertia of the physical pendulum, relative to an axis through the pivot point, as a function of *d*:

$$I = I_{\text{cm}} + Md^2 = \tfrac{1}{3}mL^2 + \tfrac{1}{2}Mr^2 + Md^2$$

Substitute numerical values and evaluate *I*:

$$\begin{aligned} I &= \tfrac{1}{3}(0.8\,\text{kg})(2\,\text{m})^2 + \tfrac{1}{2}(1.2\,\text{kg})(0.15\,\text{m})^2 \\ &\quad + (1.2\,\text{kg})d^2 \\ &= 1.0802\,\text{kg}\cdot\text{m}^2 + (1.2\,\text{kg})d^2 \end{aligned}$$

Locate the center of mass of the physical pendulum relative to the pivot point:

$$(2\,\text{kg})x_{\text{cm}} = (0.8\,\text{kg})(1\,\text{m}) + (1.2\,\text{kg})d$$

and

$$x_{\text{cm}} = 0.4\,\text{m} + 0.6d$$

Substitute in equation (1) to obtain:

$$\frac{1.0802\,\text{kg}\cdot\text{m}^2 + (1.2\,\text{kg})d^2}{0.4\,\text{m} + 0.6d} = \frac{T^2(9.81\,\text{m/s}^2)(2\,\text{kg})}{4\pi^2} = (0.49698\,\text{kg}\cdot\text{m/s}^2)T^2 \qquad (2)$$

Setting $T = 2.5$ s and solving for *d* yields:

$$d = \boxed{1.63572\,\text{m}}$$

where we have kept more than three significant figures for use in part (*b*).

(*b*) There are 1440 minutes per day. If the clock loses 5 minutes per day, then the period of the clock is related to the perfect period of the clock by:

$$1435T = 1440T_{\text{perfect}}$$

where $T_{\text{perfect}} = 3.5$ s.

Solve for and evaluate T:

$$T = \frac{1440}{1435}T_{\text{perfect}} = \frac{1440}{1435}(3.5\,\text{s}) = 3.51220\,\text{s}$$

Substitute $T = 3.51220$ s in equation (2) and solve for d to obtain:

$$d = 3.40140\,\text{m}$$

Substitute $T = 3.50$ s in equation (2) and solve for d' to obtain:

$$d' = 3.37825\,\text{m}$$

Express the distance the disk needs to be moved upward to correct the period:

$$\Delta d = d - d' = 3.40140\,\text{m} - 3.37825\,\text{m} = \boxed{2.32\,\text{cm}}$$

***78 ••**

Picture the Problem The period of a simple pendulum depends on its amplitude ϕ_0 according to $T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0 + \frac{1}{2^2}\left(\frac{3}{4}\right)^2\sin^4\frac{1}{2}\phi_0 + \ldots\right]$. We can approximate T to the second-order term and express $\Delta T/T = (T_{\text{slow}} - T_{\text{accurate}})/T$. Equating this expression to $\Delta T/T$ calculated from the fractional daily loss of time will allow us to solve for and evaluate the amplitude of the pendulum that corresponds to keeping perfect time.

Express the fractional daily loss of time:

$$\frac{\Delta T}{T} = \frac{48\,\text{s}}{\text{day}} \times \frac{1\,\text{day}}{24\,\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}} = \frac{48}{86400}$$

Approximate the period of the clock to the second-order term:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]$$

Express the difference in the periods of the slow and accurate clocks:

$$\begin{aligned}\Delta T &= T_{\text{slow}} - T_{\text{accurate}} \\ &= 2\pi\sqrt{\frac{L}{g}}\left\{\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}(8.4°)\right] - \left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]\right\} \\ &= 2\pi\sqrt{\frac{L}{g}}\left[\frac{1}{2^2}\sin^2\frac{1}{2}(8.4°) - \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]\end{aligned}$$

Divide both sides of this equation by T to obtain:

$$\frac{\Delta T}{T} = \frac{1}{4}\sin^2 4.2° - \frac{1}{4}\sin^2 \frac{1}{2}\phi_0$$

Substitute for $\frac{\Delta T}{T}$ and simplify to obtain:

$$\frac{1}{4}\sin^2 4.2° - \frac{1}{4}\sin^2 \frac{1}{2}\phi_0 = \frac{48}{86400}$$

and

$$\sin\frac{1}{2}\phi_0 = 0.05605$$

Solve for ϕ_0:

$$\phi_0 = \boxed{6.43°}$$

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Picture the Problem The period of a simple pendulum depends on its amplitude ϕ_0 according to $T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0 + \frac{1}{2^2}\left(\frac{3}{4}\right)^2\sin^4\frac{1}{2}\phi_0 + \ldots\right]$. We'll approximate T to the second-order term and express $\Delta T/T = (T_{slow} - T_{correct})/T$. Equating this expression to $\Delta T/T$ calculated from the fractional daily loss of time will allow us to solve for and evaluate the amplitude of the pendulum that corresponds to keeping correct time.

Express the fractional daily loss of time:

$$\frac{\Delta T}{T} = \frac{5\,\text{min}}{\text{day}} \times \frac{1\,\text{day}}{24\,\text{h}} \times \frac{1\,\text{h}}{60\,\text{min}} = \frac{5}{1440}$$

Approximate the period of the clock to the second-order term:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]$$

Assuming that the amplitude of the slow-running clock's pendulum is small enough to ignore, express the difference in the periods of the slow and corrected clocks:

$$\begin{aligned}\Delta T &= T_{slow} - T_{correct}\\ &= 2\pi\sqrt{\frac{L}{g}}\left\{1 - \left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]\right\}\\ &= 2\pi\sqrt{\frac{L}{g}}\left[-\frac{1}{2^2}\sin^2\frac{1}{2}\phi_0\right]\end{aligned}$$

Divide both sides of this expression by T to obtain:

$$\frac{\Delta T}{T} = -\frac{1}{4}\sin^2\frac{1}{2}\phi_0$$

Substitute for $\frac{\Delta T}{T}$ and simplify to obtain:

$$-\frac{1}{4}\sin^2\frac{1}{2}\phi_0 = \frac{-5}{1440}$$

and

$$\sin\frac{1}{2}\phi_0 = 0.1178$$

Solve for ϕ_0: $\phi_0 = \boxed{13.5°}$

Damped Oscillations

80 •

Picture the Problem We can use the definition of the damping constant and its dimensions to show that it has units of kg/s.

Using its definition, relate the decay constant τ to the damping constant b:

$$\tau = \frac{m}{b} \Rightarrow b = \frac{m}{\tau}$$

Substitute the units of m and τ to obtain:

$$\text{Dimensionally, } b = \frac{[M]}{[T]} = \boxed{\frac{\text{kg}}{\text{s}}}$$

81 •

Picture the Problem For small damping, $Q = \frac{2\pi}{(|\Delta E|/E)_{\text{cycle}}}$ where $\Delta E/E$ is the fractional energy loss per cycle.

Relate the Q factor to the fractional energy loss per cycle:

$$Q = \frac{2\pi}{(|\Delta E|/E)_{\text{cycle}}}$$

Solve for and evaluate the fractional energy loss per cycle:

$$(|\Delta E|/E)_{\text{cycle}} = \frac{2\pi}{Q} = \frac{2\pi}{200} = \boxed{3.14\%}$$

82 •

Picture the Problem We can find the period of the oscillator from $T = 2\pi\sqrt{m/k}$ and its total initial energy from $E_0 = \frac{1}{2}kA^2$. The Q factor can be found from its definition $Q = 2\pi/(|\Delta E|/E)_{\text{cycle}}$ and the damping constant from $Q = \omega_0 m/b$.

(*a*) The period of the oscillator is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2\,\text{kg}}{400\,\text{N/m}}} = \boxed{0.444\,\text{s}}$$

(*b*) Relate the initial energy of the

$$E_0 = \tfrac{1}{2}kA^2$$

oscillator to its amplitude:

Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}(400\,\text{N/m})(0.03\,\text{m})^2 = \boxed{0.180\,\text{J}}$$

(*c*) Relate the fractional rate at which the energy decreases to the Q value and evaluate Q:

$$Q = \frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}} = \frac{2\pi}{0.01} = \boxed{628}$$

Express the Q value in terms of b:

$$Q = \frac{\omega_0 m}{b}$$

Solve for the damping constant b:

$$b = \frac{\omega_0 m}{Q} = \frac{2\pi m}{TQ}$$

Substitute numerical values and evaluate b:

$$b = \frac{2\pi(2\,\text{kg})}{(0.444\,\text{s})(628)} = \boxed{0.0451\,\text{kg/s}}$$

83 ••

Picture the Problem The amplitude of the oscillation at time t is $A(t) = A_0 e^{-t/2\tau}$ where $\tau = m/b$ is the decay constant. We'll express the amplitudes one period apart and then show that their ratio is constant.

Relate the amplitude of a given oscillation peak to the time at which the peak occurs:

$$A(t) = A_0 e^{-t/2\tau}$$

Express the amplitude of the oscillation peak at $t' = t + T$:

$$A(t+T) = A_0 e^{-(t+T)/2\tau}$$

Express the ratio of these consecutive peaks:

$$\frac{A(t)}{A(t+T)} = \frac{A_0 e^{-t/2\tau}}{A_0 e^{-(t+T)/2\tau}} = e^{-T/2\tau} = \boxed{\text{constant}}$$

84 ••

Picture the Problem We can relate the fractional change in the energy of the oscillator each cycle to the fractional change in its amplitude. Both the Q value and the decay constant τ can be found from their definitions.

(*a*) Relate the energy of the oscillator to its amplitude:

$$E = \tfrac{1}{2}kA^2$$

Take the differential of this relationship to obtain:

$$dE = kAdA$$

Divide both sides of this equation by *E*:

$$\frac{dE}{E} = \frac{kAdA}{\frac{1}{2}kA^2} = 2\frac{dA}{A}$$

Approximate *dE* and *dA* by Δ*E* and Δ*A* and evaluate Δ*E*/*E*:

$$\frac{\Delta E}{E} = 2(5\%) = \boxed{10\%}$$

(*b*) For small damping:

$$\frac{|\Delta E|}{E} = \frac{T}{\tau}$$

and

$$\tau = \frac{T}{|\Delta E|/E} = \frac{3\,\text{s}}{0.01} = \boxed{30\,\text{s}}$$

(*c*) Using its definition, express and evaluate *Q*:

$$A = \omega_0\tau = \frac{2\pi}{T}\tau = \frac{2\pi}{3\,\text{s}}(30\,\text{s}) = \boxed{62.8}$$

85 ••

Picture the Problem We can use the physical interpretation of *Q* for small damping $Q = \dfrac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}}$ to find the fractional decrease in the energy of the oscillator each cycle.

(*a*) Express the fractional decrease in energy each cycle as a function of the *Q* factor and evaluate $|\Delta E|/E$:

$$\frac{|\Delta E|}{E} = \frac{2\pi}{Q} = \frac{2\pi}{20} = \boxed{0.314}$$

(*b*) Using the definition of the *Q* factor, use Equation 14-35 to express ω' as a function of *Q*:

$$\omega' = \omega_0\left[1 - \frac{1}{4}\left(\frac{b^2}{m^2\omega_0^2}\right)\right]^{1/2}$$

$$= \omega_0\left[1 - \frac{1}{4Q^2}\right]^{1/2}$$

Use the approximation $(1 + x)^{½} \approx 1 + ½x$ for small *x* to obtain:

$$\omega' = \omega_0\left[1 - \frac{1}{8Q^2}\right]$$

Express and evaluate $\omega' - \omega_0$:

$$\omega' - \omega_0 = \omega_0\left[1 - \frac{1}{8Q^2}\right] - \omega_0 = -\frac{1}{8Q^2}$$
$$= -\frac{1}{8(20)^2}$$
$$= \boxed{-3.13\times10^{-2}\,\text{percent}}$$

86 ••

Picture the Problem The amplitude of the spring-and-mass oscillator varies with time according to $A = A_0 e^{-t/2\tau}$ and its energy according to $E = E_0 e^{-t/\tau}$.

(*a*) Express the amplitude of the oscillations as a function of time:

$$A = (6\,\text{cm})e^{-t/4\text{s}}$$

Evaluate the amplitude when $t = 2$ s:

$$A(2\,\text{s}) = (6\,\text{cm})e^{-2\,\text{s}/4\,\text{s}} = (6\,\text{cm})e^{-1/2}$$
$$= \boxed{3.64\,\text{cm}}$$

Evaluate the amplitude when $t = 4$ s:

$$A(4\,\text{s}) = (6\,\text{cm})e^{-4\,\text{s}/4\,\text{s}} = (6\,\text{cm})e^{-1}$$
$$= \boxed{2.21\,\text{cm}}$$

(*b*) Express the energy of the system at $t = 0$:

$$E(0) = E_0 e^{-0/2\,\text{s}} = E_0 = 60\,\text{J}$$

Express the energy in the system at t =2 s:

$$E(2\,\text{s}) = E_0 e^{-2\,\text{s}/2\,\text{s}} = E_0 e^{-1}$$

The energy dissipated in the first 2 s is:

$$\Delta E_{0-2\text{s}} = E(0) - E(2\,\text{s})$$
$$= E_0\left(1 - e^{-1}\right)$$
$$= (60\,\text{J})\left(1 - e^{-1}\right)$$
$$= \boxed{37.9\,\text{J}}$$

The energy dissipated in the second 2-s interval is:

$$\Delta E_{2-4\,\text{s}} = E_{2\,\text{s}}\left(1 - e^{-2\text{s}/2\,\text{s}}\right)$$
$$= (37.9\,\text{J})\left(1 - e^{-1}\right) = \boxed{24.0\,\text{J}}$$

***87** ••

Picture the Problem We can find the fractional loss of energy per cycle from the physical interpretation of Q for small damping. We will also find a general expression for the earth's vibrational energy as a function of the number of cycles it has completed. We can then solve this equation for the earth's vibrational energy after any number of days.

(*a*) Express the fractional change in energy as a function of *Q*:

$$\frac{\Delta E}{E}=\frac{2\pi}{Q}=\frac{2\pi}{400}=\boxed{1.57\%}$$

(*b*) Express the energy of the damped oscillator after one cycle:

$$E_1=E_0\left(1-\frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2=E_1\left(1-\frac{\Delta E}{E}\right)=E_0\left(1-\frac{\Delta E}{E}\right)^2$$

Generalizing to *n* cycles:

$$E_n=E_0\left(1-\frac{\Delta E}{E}\right)^n=E_0(1-0.0157)^n$$
$$=\boxed{E_0(0.9843)^n}$$

(*c*) Express 2 d in terms of the number of cycles; i.e., the number of vibrations the earth will have experienced:

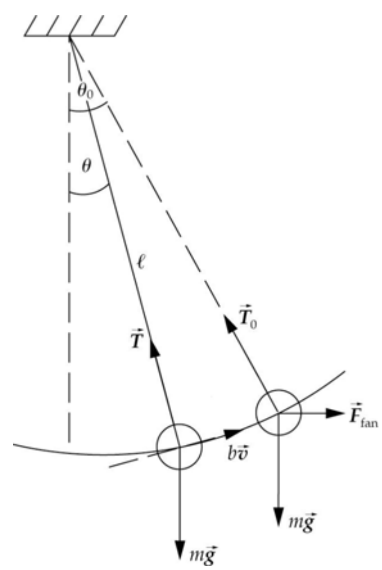
$$2\,\text{d}=2\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{60\,\text{m}}{\text{h}}$$
$$=2880\,\text{min}\times\frac{1T}{54\,\text{min}}$$
$$=53.3T$$

Evaluate $E(2\text{ d})$:

$$E(2\,\text{d})=E_0(0.9843)^{53.3}=\boxed{0.430E_0}$$

88 ••

Picture the Problem The diagram shows 1) the pendulum bob displaced through an angle θ_0 and held in equilibrium by the force exerted on it by the air from the fan and 2) the bob accelerating, under the influence of gravity, tension force, and drag force, toward its equilibrium position. We can apply Newton's 2nd law to the bob to obtain the differential equation of motion of the damped pendulum and then use its solution to find the decay time constant and the time required for the amplitude of oscillation to decay to 1°.



(*a*) Apply $\sum\tau=I\alpha$ to the pendulum to obtain:

$$-mg\ell\sin\theta+\ell F_{\text{d}}=I\frac{d^2\theta}{dt^2}$$

Express the moment of inertia of the pendulum with respect to an axis through its point of support:

$$I = m\ell^2$$

Substitute for I and F_d to obtain:

$$m\ell^2 \frac{d^2\theta}{dt^2} + \ell bv + mg\ell \sin\theta = 0$$

Because $\theta << 1$ and $v = \ell\omega = \ell d\theta/dt$:

$$m\ell^2 \frac{d^2\theta}{dt^2} + \ell^2 b \frac{d\theta}{dt} + mg\ell\theta = 0$$

or

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{\ell}\theta = 0$$

The solution to this second-order homogeneous differential equation with constant coefficients is:

$$\theta = \theta_0 e^{-t/2\tau}\cos(\omega' t + \delta) \qquad (1)$$

where θ_0 is the maximum amplitude, $\tau = m/b$ is the time constant, and the frequency $\omega' = \omega_0\sqrt{1-(b/2m\omega_0)^2}$.

Apply $\sum \vec{F} = m\vec{a}$ to the bob when it is at its maximum angular displacement to obtain:

$$\sum F_x = F_{\text{fan}} - T\sin\theta_0 = 0$$

and

$$\sum F_y = T\cos\theta_0 - mg = 0$$

Divide the x equation by the y equation to obtain:

$$\frac{F_{\text{fan}}}{mg} = \frac{T\sin\theta_0}{T\cos\theta_0} = \tan\theta_0$$

or

$$F_{\text{fan}} = mg\tan\theta_0$$

When the bob is in equilibrium, the drag force on it equals F_{fan}:

$$bv = mg\tan\theta_0$$

Solve for m/b in the definition of τ to obtain:

$$\tau = \frac{m}{b} = \frac{v}{g\tan\theta_0}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{7\,\text{m/s}}{(9.81\,\text{m/s}^2)\tan 5°} = \boxed{8.16\,\text{s}}$$

(*b*) From equation (1) we have:

$$\theta = \theta_0 e^{-t/2\tau}$$

When the amplitude has decreased to 1°:

$$5° e^{-t/2\tau} = 1° \text{ or } e^{-t/2\tau} = 0.2$$

Take the natural logarithm of both sides of the equation to obtain:

$$-\frac{t}{2\tau} = \ln(0.2)$$

Solve for t:

$$t = -2\tau \ln(0.2)$$

Substitute for τ and evaluate t:

$$t = -2(8.16\,\text{s})\ln(0.2) = \boxed{26.3\,\text{s}}$$

Driven Oscillations and Resonance

89 •

Picture the Problem The resonant frequency of a vibrating system depends on the mass of the system and on a "stiffness" constant according to $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or, in the case of a simple pendulum oscillating with small-amplitude vibrations, $f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

(*a*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{400\,\text{N/m}}{10\,\text{kg}}} = \boxed{1.01\,\text{Hz}}$$

(*b*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{800\,\text{N/m}}{5\,\text{kg}}} = \boxed{2.01\,\text{Hz}}$$

(*c*) For this simple pendulum we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{2\,\text{m}}} = \boxed{0.352\,\text{Hz}}$$

90 •

Picture the Problem We can use the physical interpretation of Q for small damping to find the Q factor for this damped oscillator. The width of the resonance curve depends on the Q factor according to $\Delta\omega = \omega_0/Q$.

(*a*) Using the physical interpretation of Q for small damping, relate Q to the fractional loss of energy of the damped oscillator per cycle:

$$Q = \frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}}$$

Evaluate this expression for $\left(|\Delta E|/E\right)_{\text{cycle}} = 2\%$:

$$Q = \frac{2\pi}{0.02} = \boxed{314}$$

(*b*) Relate the width of the resonance curve to the Q value of the oscillatory system:

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{2\pi f_0}{Q}$$

Substitute numerical values and evaluate $\Delta\omega$:

$$\Delta\omega = \frac{2\pi\left(300\,\text{s}^{-1}\right)}{3.14} = \boxed{6.00\,\text{rad/s}}$$

91 ••

Picture the Problem The amplitude of the damped oscillations is related to the damping constant, mass of the system, the amplitude of the driving force, and the natural and driving frequencies through $A = \dfrac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}}$. Resonance occurs when $\omega = \omega_0$. At resonance, the amplitude of the oscillations is $A = F_0 \big/ \sqrt{b^2\omega^2}$ and the width of the resonance curve is related to the damping constant and the mass of the system according to $\Delta\omega = b/m$.

(*a*) Express the amplitude of the oscillations as a function of the driving frequency:

$$A = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}}$$

Determine ω_0:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{400\,\text{N/m}}{2\,\text{kg}}} = 14.14\,\text{rad/s}$$

Evaluate the radicand in the expression for A to obtain:

$$\begin{aligned}&(2\,\text{kg})^2\left[(14.14\,\text{rad/s})^2 - (10\,\text{rad/s})^2\right]^2 \\ &\quad + (2\,\text{kg/s})^2(10\,\text{rad/s})^2 \\ &\qquad = 4.04\times10^4\,\text{kg}^2/\text{s}^4\end{aligned}$$

Substitute numerical values and evaluate A:

$$A = \frac{10\,\text{N}}{\sqrt{4.04\times10^4\,\text{kg}^2/\text{s}^4}} = \boxed{4.98\,\text{cm}}$$

(*b*) Resonance occurs when:

$$\omega = \omega_0 = \boxed{14.1\,\text{rad/s}}$$

(*c*) Express the amplitude of the motion at resonance:

$$A = \frac{F_0}{\sqrt{b^2\omega_0^2}}$$

Substitute numerical values and evaluate A:

$$A = \frac{10\,\text{N}}{\sqrt{(2\,\text{kg/s})^2(14.14\,\text{rad/s})^2}} = \boxed{35.4\,\text{cm}}$$

(*d*) The width of the resonance curve is:

$$\Delta\omega = \frac{b}{m} = \frac{2\,\text{kg/s}}{2\,\text{kg}} = \boxed{1.00\,\text{rad/s}}$$

92 ••

Picture the Problem We'll find a general expression for the damped oscillator's energy as a function of the number of cycles it has completed. We can then solve this equation for the number of cycles corresponding to the loss of half the oscillator's energy. The Q factor is related to the fractional energy loss per cycle through $\Delta E/E = 2\pi/Q$ and the width of the resonance curve is $\Delta\omega = \omega_0/Q$ where ω_0 is the oscillator's natural angular frequency.

(*a*) Express the energy of the damped oscillator after one cycle:

$$E_1 = E_0\left(1 - \frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2 = E_1\left(1 - \frac{\Delta E}{E}\right) = E_0\left(1 - \frac{\Delta E}{E}\right)^2$$

Generalizing to n cycles:

$$E_n = E_0\left(1 - \frac{\Delta E}{E}\right)^n$$

Substitute numerical values:

$$0.5E_0 = E_0(1 - 0.035)^n$$

or

$$0.5 = (0.965)^n$$

Solve for n to obtain:

$$n = \frac{\ln 0.5}{\ln 0.965} = 19.5$$

$$\approx \boxed{20 \text{ complete cycles.}}$$

(*b*) Apply the physical interpretation of Q for small damping to obtain:

$$Q = \frac{2\pi}{\Delta E/E} = \frac{2\pi}{0.035} = \boxed{180}$$

(*c*) The width of the resonance curve is given by:

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{2\pi f_0}{Q} = \frac{2\pi(100\,\text{Hz})}{180}$$

$$= \boxed{3.49\,\text{rad/s}}$$

Collisions

93 •••

Picture the Problem Let the system include the spring-and-mass oscillator and the second object of mass m. Because the net external force acting on this system is zero, momentum is conserved during the collision of the second object with the oscillator.

Because the collision is elastic, we can also apply conservation of energy. Let the subscript 1 refer to the object attached to the spring and the subscript 2 identify the second object.

(*a*) Using momentum conservation, relate the speeds of the objects before and after their collision:

$$mv_{1\mathrm{i}} + mv_{2\mathrm{i}} = mv_{2\mathrm{f}}$$

or

$$v_{1\mathrm{i}} + v_{2\mathrm{i}} = v_{2\mathrm{f}} \qquad (1)$$

Using conservation of energy, obtain a second relationship between the speeds of the objects before and after their collision:

$$\tfrac{1}{2}mv_{1\mathrm{i}}^2 + \tfrac{1}{2}mv_{2\mathrm{i}}^2 = \tfrac{1}{2}mv_{2\mathrm{f}}^2$$

or

$$v_{1\mathrm{i}}^2 + v_{2\mathrm{i}}^2 = v_{2\mathrm{f}}^2 \qquad (2)$$

Solve equation (2) for $v_{2\mathrm{i}}^2$:

$$v_{2\mathrm{i}}^2 = v_{2\mathrm{f}}^2 - v_{1\mathrm{i}}^2 = (v_{2\mathrm{f}} + v_{1\mathrm{i}})(v_{2\mathrm{f}} - v_{1\mathrm{i}})$$

Substitute for $v_{2\mathrm{f}}$ from equation (1):

$$v_{2\mathrm{i}}^2 = (v_{1\mathrm{i}} + v_{2\mathrm{i}} + v_{1\mathrm{i}})(v_{1\mathrm{i}} + v_{2\mathrm{i}} - v_{1\mathrm{i}})$$
$$= (2v_{1\mathrm{i}} + v_{2\mathrm{i}})(v_{2\mathrm{i}}) = 2v_{1\mathrm{i}}v_{2\mathrm{i}} + v_{2\mathrm{i}}^2$$

or

$$2v_{1\mathrm{i}}v_{2\mathrm{i}} = 0$$

Because $v_{1\mathrm{i}} \neq 0$, it follows that:

$$v = v_{2\mathrm{i}} = \boxed{0}$$

i.e., the second object must be initially at rest.

(*b*) Because $v_{2\mathrm{i}} = 0$, we have, from equation (1):

$$v_{2\mathrm{f}} = v_{1\mathrm{i}}$$

Because the object connected to the spring was moving through its equilibrium position at the time of collision:

$$v_{1\mathrm{i}} = v_{\max} = A\omega = (0.1\,\mathrm{m})(40\,\mathrm{s}^{-1})$$
$$= \boxed{4.00\,\mathrm{m/s}}$$

94 •••

Picture the Problem Let the system include the spring-and-mass oscillator and the second object of mass *m*. Because the net external force acting on this system is zero, momentum is conserved during the collision of the second object with the oscillator. Because the collision is elastic, we can also apply conservation of energy. Let the subscript 1 refer to the object attached to the spring and the subscript 2 identify the second object.

Using momentum conservation, relate the speeds of the objects

$$mv_{1\mathrm{i}} + mv_{2\mathrm{i}} = mv_{2\mathrm{f}}$$

or

before and after their collision:

$$v_{1i} + v_{2i} = v_{2f} \qquad (1)$$

Using conservation of energy, obtain a second relationship between the speeds of the objects before and after their collision:

$$\tfrac{1}{2}mv_{1i}^2 + \tfrac{1}{2}mv_{2i}^2 = \tfrac{1}{2}mv_{2f}^2$$

or

$$v_{1i}^2 + v_{2i}^2 = v_{2f}^2 \qquad (2)$$

Solve equation (2) for v_{2i}^2:

$$v_{2i}^2 = v_{2f}^2 - v_{1i}^2 = (v_{2f} + v_{1i})(v_{2f} - v_{1i})$$

Substitute for v_{2f} from equation (1):

$$v_{2i}^2 = (v_{1i} + v_{2i} + v_{1i})(v_{1i} + v_{2i} - v_{1i})$$
$$= (2v_{1i} + v_{2i})(v_{2i}) = 2v_{1i}v_{2i} + v_{2i}^2$$

or

$$2v_{1i}v_{2i} = 0$$

Because $v_{1i} \neq 0$, it follows that:

$$v = v_{2i} = 0$$

i.e., the second object must be initially at rest.

Because the object connected to the spring was moving through its equilibrium position at the time of collision:

$$v_{1i} = v_{max} = A\omega = (0.1\,\text{m})(40\,\text{s}^{-1})$$
$$= 4\,\text{m/s}$$

Express the total energy of the system just before the collision:

$$E = \tfrac{1}{2}mv_{1i}^2$$

Solve for *m*:

$$m = \frac{2E}{v_{1i}^2}$$

Substitute numerical values and evaluate *m*:

$$m = \frac{2(8\,\text{J})}{(4\,\text{m/s})^2} = \boxed{1.00\,\text{kg}}$$

Relate the spring constant to the angular frequency of the oscillator:

$$k = m\omega^2$$

Substitute numerical values and evaluate *k*:

$$k = (1\,\text{kg})(40\,\text{s}^{-1})^2 = \boxed{1.60\,\text{kN/m}}$$

95 •••

Picture the Problem Let the system include the spring-and-mass oscillator and the 1-kg object. Because the net external force acting on this system is zero, momentum is conserved during the collision of the second object with the oscillator. Let the subscript 1 refer to the 1-kg object and the subscript 2 to the 2-kg object. We can relate the amplitude of the motion to the maximum speed of the oscillator (which we can find from conservation of momentum) and the angular frequency of the oscillator, which we can determine from its definition. Once we have found the amplitudes and angular frequencies for both collisions, we express the position of each as a function of time, using the initial conditions to find the phase constants.

(*a*) Relate the amplitude of the motion to the angular frequency and maximum speed of the oscillator:

$$A=\frac{v_{\max}}{\omega} \qquad (1)$$

Because the 2-kg object is initially at rest, the maximum speed of the oscillator will be its speed immediately after the collision. Use conservation of momentum to relate this maximum speed to the speed of the 1-kg object before the collision:

$$m_1v_{1i}=(m_1+m_2)v_{\max}$$

Solve for $v_{\max}$:

$$v_{\max}=\frac{m_1}{m_1+m_2}v_{1i}$$

Substitute numerical values and evaluate $v_{\max}$:

$$v_{\max}=\frac{1\,\text{kg}}{1\,\text{kg}+2\,\text{kg}}(6\,\text{m/s})=2\,\text{m/s}$$

Express the angular frequency of the oscillator:

$$\omega=\sqrt{\frac{k}{m_1+m_2}}$$

Substitute numerical values and evaluate ω:

$$\omega=\sqrt{\frac{600\,\text{N/m}}{3\,\text{kg}}}=14.14\,\text{rad/s}$$

Substitute in equation (1) and evaluate A:

$$A=\frac{2\,\text{m/s}}{14.14\,\text{s}^{-1}}=\boxed{14.1\,\text{cm}}$$

Express and evaluate the period of the oscillator's period:

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{14.14\,\text{s}^{-1}}=\boxed{0.444\,\text{s}}$$

(*b*) For an elastic collision:

$$v_{\max}=v_{2f}=\frac{2m_1}{m_1+m_2}v_{1i}$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = \frac{2(1\,\text{kg})}{3\,\text{kg}}(6\,\text{m/s}) = 4\,\text{m/s}$$

Using its definition, evaluate the angular frequency of the oscillator:

$$\omega = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{600\,\text{N/m}}{2\,\text{kg}}} = 17.32\,\text{rad/s}$$

Substitute in equation (1) and evaluate A:

$$A = \frac{4\,\text{m/s}}{17.32\,\text{s}^{-1}} = \boxed{23.1\,\text{cm}}$$

Express and evaluate the period of the oscillator's period:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{17.32\,\text{s}^{-1}} = \boxed{0.363\,\text{s}}$$

(*c*) For the perfectly inelastic collision:

$$x(t) = (14.1\,\text{cm})\cos\left[\left(14.1\,\text{s}^{-1}\right)t + \delta\right] \quad (2)$$

Use the initial conditions to evaluate δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) = \tan^{-1}\left(-\frac{v_0}{\omega(0)}\right) = -\frac{\pi}{2}$$

Substitute in equation (2) to obtain:

$$x(t) = (14.1\,\text{cm})\cos\left[\left(14.1\,\text{s}^{-1}\right)t - \frac{\pi}{2}\right] = \boxed{(14.1\,\text{cm})\sin\left[\left(14.1\,\text{s}^{-1}\right)t\right]}$$

For the elastic collision:

$$x(t) = (23.1\,\text{cm})\cos\left[\left(17.3\,\text{s}^{-1}\right)t + \delta\right] \quad (3)$$

Use the initial conditions to evaluate δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) = \tan^{-1}\left(-\frac{v_0}{\omega(0)}\right) = -\frac{\pi}{2}$$

Substitute in equation (3) to obtain:

$$x(t) = (23.1\,\text{cm})\cos\left[\left(17.3\,\text{s}^{-1}\right)t - \frac{\pi}{2}\right] = \boxed{(23.1\,\text{cm})\sin\left[\left(17.3\,\text{s}^{-1}\right)t\right]}$$

General Problems

96 •

Picture the Problem The particle's displacement is of the form $x = A\cos(\omega t + \delta)$. Thus, we have $A = 0.4$ m, $\omega = 3$ rad/s, and $\delta = \pi/4$. We can find the frequency of the motion from its angular frequency and the period from the frequency. The particle's position at $t = 0$ and $t = 0.5$ s can be found directly from its displacement function.

(*a*) Express and evaluate the frequency of the particle's motion:

$$f=\frac{\omega}{2\pi}=\frac{3\,\text{rad/s}}{2\pi}=\boxed{0.477\,\text{Hz}}$$

Use the relationship between the frequency and the period of the particle's motion to find its period:

$$T=\frac{1}{f}=\frac{1}{0.477\,\text{s}^{-1}}=\boxed{2.09\,\text{s}}$$

(*b*) Using the expression for the particle's displacement, find its position at $t = 0$:

$$x(0)=(0.4\,\text{m})\cos\left[(3\,\text{rad/s})(0)+\frac{\pi}{4}\right]$$
$$=(0.4\,\text{m})\cos\left[\frac{\pi}{4}\right]=\boxed{0.283\,\text{m}}$$

(*c*) Using the expression for the particle's displacement, find its position at $t = 0.5$ s:

$$x(0)=(0.4\,\text{m})\cos\left[(3\,\text{rad/s})(0.5\,\text{s})+\frac{\pi}{4}\right]$$
$$=(0.4\,\text{m})\cos[2.29\,\text{rad}]$$
$$=\boxed{-0.264\,\text{m}}$$

97 •

Picture the Problem We can express the velocity of the particle by differentiating its displacement with respect to time.

(*a*) Differentiate the particle's displacement to obtain:

$$v=\frac{dx}{dt}$$
$$=\frac{d}{dt}\left\{(0.4\,\text{m})\sin\left[(3\,\text{rad/s})t+\frac{\pi}{4}\right]\right\}$$
$$=\boxed{-(1.2\,\text{m/s})\sin\left[(3\,\text{rad/s})t+\frac{\pi}{4}\right]}$$

(*b*) Evaluate the result in part (*a*) at $t = 0$:

$$v(0)=-(1.2\,\text{m/s})\sin\left[(3\,\text{rad/s})(0)+\frac{\pi}{4}\right]$$
$$=-(1.2\,\text{m/s})\sin\left[\frac{\pi}{4}\right]$$
$$=\boxed{-0.849\,\text{m/s}}$$

(*c*) By inspection of the result in part (*a*) (or from $v_{max}=A\omega$):

$$v_{max}=\boxed{1.20\,\text{m/s}}$$

(*d*) Substitute v_{max} for v to obtain:

$$1.2\,\text{m/s}=-(1.2\,\text{m/s})\sin\left[(3\,\text{rad/s})t'+\frac{\pi}{4}\right]$$

or

$$(3\,\text{rad/s})t' + \frac{\pi}{4} = \sin^{-1}(-1) = \frac{3\pi}{2}$$

Solve for t' to obtain:

$$t' = \boxed{1.31\,\text{s}}$$

98 •

Picture the Problem Let Δy represent the amount by which the spring stretches. We'll apply a condition for equilibrium to the object to relate the amount the spring has stretched to the angular frequency of its motion and then solve this equation for Δy.

Apply $\sum_i F_y = 0$ to the object when it is in its equilibrium position and solve for the elongation of the spring:

$$k\Delta y - mg = 0$$

or

$$\Delta y = \frac{m}{k}g = \frac{g}{\omega^2}$$

Relate the angular frequency of the object's motion to its period:

$$\omega = \frac{2\pi}{T}$$

Substitute to obtain:

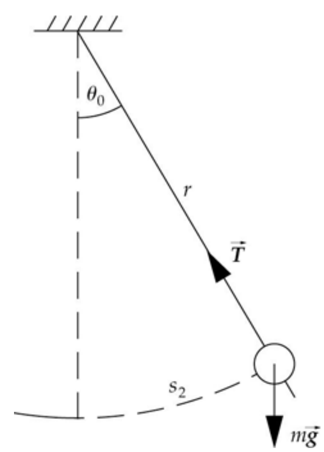
$$\Delta y = \left(\frac{T}{2\pi}\right)^2 g$$

Substitute numerical values and evaluate Δx:

$$\Delta y = \left(\frac{4.5\,\text{s}}{2\pi}\right)^2 (9.81\,\text{m/s}^2) = \boxed{5.03\,\text{m}}$$

***99** ••

Picture the Problem Compare the forces acting on the particle to the right in Figure 14-36 with the forces shown acting on the bob of the simple pendulum shown in the free-body diagram to the right. Because there is no friction, the only forces acting on the particle are mg and the normal force acting radially inward. In (*b*), we can think of the particles as the bobs of simple pendulums of equal length.

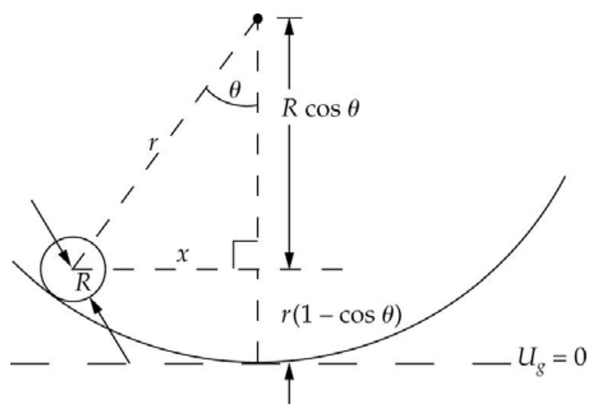


(*a*) The normal force is identical to the tension in a string of length r that keeps the particle moving in a circular path and a component of mg provides, for small displacements θ_0 or s_2, the linear restoring force required for oscillatory motion.

(*b*) The particles meet at the bottom. Because s_1 and s_2 are both much smaller than *r*, the particles behave like the bobs of simple pendulums of equal length; therefore they have the same periods.

100 ••

Picture the Problem The diagram shows the ball when it is a horizontal distance *x* from the bottom of the bowl. Note that we've chosen the zero of gravitational potential energy to be at the bottom of the bowl. The total energy of the ball is the sum of its potential energy and kinetic energies due to translation and rotation. Once we've obtained an expression for the total energy of the rolling ball, we can require, because the surface is frictionless, that the total energy of the sliding object be the same as that of the rolling ball. Because the motion of the ball is simple harmonic motion, we can assume a solution to its differential equation of motion and express the total energy of the ball in terms of this assumed solution. Doing so will lead us to an expression that we can solve for the oscillation frequency of the ball.



(*a*) Express the total energy *E* of the ball:

$$E = U + K = U + K_{\text{trans}} + K_{\text{rot}} \qquad (1)$$

Referring to the diagram shown above and assuming that $R << r$, express the potential energy of the ball when it is a horizontal distance *x* from the bottom of the bowl:

$$U(x) = mgr(1 - \cos\theta)$$

Express $\cos\theta$ as a power series:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots$$

For $\theta << 1$:

$$\cos\theta \approx 1 - \frac{\theta^2}{2!}$$

Substitute to obtain:

$$U(x) \approx mgr\left[1 - \left(1 - \frac{\theta^2}{2!}\right)\right] = \tfrac{1}{2} mgr\theta^2$$

For $R << r$:

$$\theta \approx \frac{x}{r}$$

Substitute to obtain:

$$U(x) = \frac{mgx^2}{2r}$$

Substitute in equation (1):

$$E = \frac{mgx^2}{2r} + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Because the ball is rolling without slipping, $v = R\omega$. Substitute for ω and I to obtain:

$$E = \frac{mgx^2}{2r} + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

Simplify to obtain:

$$E = \boxed{\frac{mgx^2}{2r} + \frac{7}{10}mv^2}$$

(*b*) Because energy is conserved if the side of the bowl is frictionless:

$$E = \frac{mgx^2}{2r} + \frac{7}{10}mv^2 = \text{constant}$$

Because the motion is simple harmonic motion, assume a solution of the form:

$$x = x_{0.}\cos(\omega t + \delta)$$

Differentiate this assumed solution with respect to time to obtain:

$$v = -\omega x_0 \sin(\omega t + \delta)$$

Substitute to obtain:

$$\begin{aligned} E &= \frac{mg}{2r}(x_{0.}\cos(\omega t + \delta))^2 \\ &\quad + \frac{7}{10}m(-\omega x_0 \sin(\omega t + \delta))^2 \\ &= \frac{mgx_0^2}{2r}\cos^2(\omega t + \delta) \\ &\quad + \frac{7m\omega^2 x_0^2}{10}\sin^2(\omega t + \delta) \end{aligned}$$

Express the condition the E = constant:

$$\frac{mgx_0^2}{2r} = \frac{7m\omega^2 x_0^2}{10} \text{ or } \frac{g}{r} = \frac{7\omega^2}{5}$$

Solve for ω to obtain:

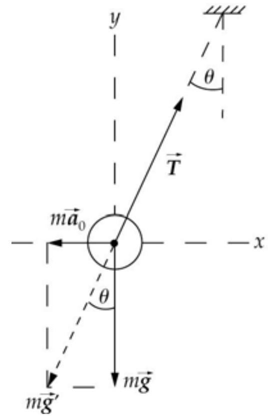
$$\omega = \boxed{\sqrt{\frac{5g}{7r}}}$$

101 ••

Picture the Problem Assume that the plane is accelerating to the right with an acceleration a_0. The free-body diagram shows the forces on the bob as seen in the accelerated frame of the airplane. Let g' represent the effective value of the acceleration due to gravity. The period of the yo-yo is given by

$$T = 2\pi\sqrt{L/g'}$$

where g' is the effective value of the acceleration due to gravity.



Express the period of your yo-yo pendulum as a function of the effective value for the acceleration due to gravity:

$$T = 2\pi\sqrt{\frac{L}{g'}}$$

Using the FBD, relate g' and g:

$$mg = mg'\cos\theta \Rightarrow g' = \frac{g}{\cos\theta}$$

Substitute to obtain:

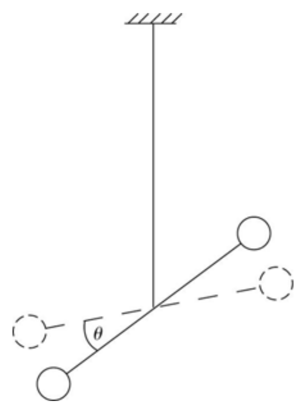
$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{(0.7\,\text{m})\cos 22°}{9.81\,\text{m/s}^2}} = \boxed{1.62\,\text{s}}$$

102 ••

Picture the Problem The diagram shows the wire described in the problem statement with an object of moment of inertia I suspended from its end. We can apply Newton's 2nd law to the suspended object to obtain its differential equation of motion. By comparing this equation to the equation of a simple harmonic oscillator, we can show that $\omega = \sqrt{\kappa/I}$.



Apply $\sum\tau = I\alpha$ to the object hung from the wire to obtain:

$$-\kappa\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

Divide both sides of this differential equation by I to obtain:

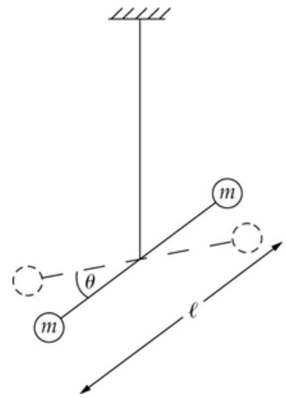
$$\frac{d^2\theta}{dt^2}+\frac{\kappa}{I}\theta=0$$

This equation can be written as:

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0 \text{ where } \omega=\boxed{\sqrt{\frac{\kappa}{I}}}$$

103 ••

Picture the Problem The diagram shows the torsion balance described in the problem statement. We can apply Newton's 2nd law to the suspended object to obtain its differential equation of motion. By comparing this equation and its solution to that of a simple harmonic oscillator, we can obtain an equation that we can solve for the torsion constant κ.



Apply $\sum\tau=I\alpha$ to the torsion pendulum:

$$-\kappa\theta=I\alpha=I\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2}+\frac{\kappa}{I}\theta=0 \qquad (1)$$

The differential equation of simple harmonic motion is:

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

where

$$x(t)=x_0\cos(\omega t+\delta) \text{ and } \omega=\frac{2\pi}{T}$$

The solution to equation (1) is:

$$\theta(t)=\theta_0\cos(\omega t+\delta)$$

where

$$\omega=\sqrt{\frac{\kappa}{I}}$$

Solve for κ to obtain:

$$\kappa=\omega^2I$$

Express the moment of inertia of the torsion pendulum:

$$I = 2m\left(\frac{\ell}{2}\right)^2 = \frac{m\ell^2}{2}$$

Substitute to obtain:

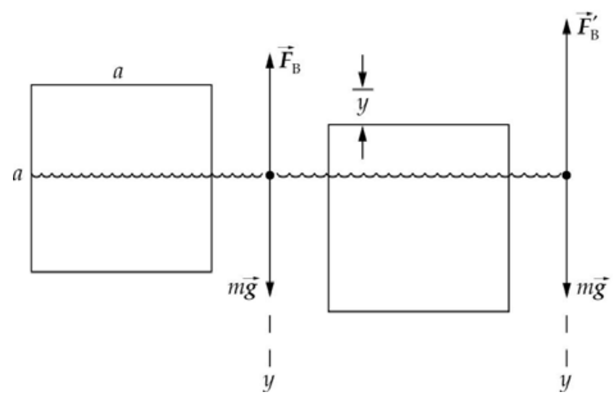
$$\kappa = \frac{\omega^2 m\ell^2}{2} = \frac{4\pi^2 m\ell^2}{2T^2} = \frac{2\pi^2 m\ell^2}{T^2}$$

Substitute numerical values and evaluate κ:

$$\kappa = \frac{2\pi^2(0.050\,\text{kg})(0.05\,\text{m})^2}{(80\,\text{s})^2} = \boxed{3.86\times10^{-7}\,\text{N}\cdot\text{m/rad}}$$

***104 ••**

Picture the Problem Choose a coordinate system in which the direction the cube is initially displaced (downward) is the positive *y* direction. The figure shows the forces acting on the cube when it is in equilibrium floating in the water and when it has been pushed down a small distance *y*. We can find the period of its oscillatory motion from its angular frequency. By applying Newton's 2nd law to the cube, we can obtain its equation of motion; from this equation we can determine the angular frequency of the cube's small-amplitude oscillations.



Express the period of oscillation in terms of the angular frequency of the oscillations:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum F_y = 0$ to the cube when it is floating in the water:

$$mg - F_B = 0$$

Apply $\sum T_y = ma_y$ to the cube when it is pushed down a small distance *y*:

$$mg - F_B' = ma_y$$

Eliminate mg between these equations to obtain:

$$F_B - F_B' = ma_y$$

or

$$\Delta F_B = F_B - F_B' = ma_y$$

For $y << 1$:

$$\Delta F_B \approx dF_B = -\rho Vg = -a^2\rho gy = m\frac{d^2y}{dt^2}$$

Rewrite the equation of motion as:

$$m\frac{d^2y}{dt^2} = -a^2\rho gy$$

or

$$\frac{d^2y}{dt^2} = -\frac{a^2\rho g}{m}y = -\omega^2 y$$

where $\omega^2 = \dfrac{a^2\rho g}{m}$

Solve for ω:

$$\omega = a\sqrt{\frac{\rho g}{m}}$$

Substitute in equation (1) to obtain:

$$T = \frac{2\pi}{a\sqrt{\dfrac{\rho g}{m}}} = \boxed{\frac{2\pi}{a}\sqrt{\frac{m}{\rho g}}}$$

105 ••

Picture the Problem Assume that the density of the earth ρ is constant and let m represent the mass of the clock. We can decide the question of where the clock is more accurate by applying the law of gravitation to the clock at a depth h below/above the surface of the earth and at the earth's surface and expressing the ratios of the acceleration due to gravity below/above the surface of the earth to its value at the surface of the earth.

Express the gravitational force acting on the clock when it is at a depth h in a mine:

$$mg' = \frac{GM'm}{(R_E - h)^2}$$

where M' is the mass between the location of the clock and the center of the earth.

Express the gravitational force acting on the clock at the surface of the earth:

$$mg = \frac{GM_Em}{R_E^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{g'}{g}=\frac{\dfrac{GM'}{(R_E-h)^2}}{\dfrac{GM_E}{R_E^2}}=\frac{M'}{M_E}\frac{R_E^2}{(R_E-h)^2}$$

Express M':

$$M'=\rho V'=\tfrac{4}{3}\pi\rho(R_E-h)^3$$

Express M_E:

$$M_E=\rho V=\tfrac{4}{3}\pi\rho R_E^3$$

Substitute to obtain:

$$\frac{g'}{g}=\frac{\tfrac{4}{3}\pi\rho(R_E-h)^3}{\tfrac{4}{3}\pi\rho R_E^3}\frac{R_E^2}{(R_E-h)^2}$$

Simplify and solve for g':

$$g'=g\left(\frac{R_E-h}{R_E}\right)=g\left(1-\frac{h}{R_E}\right)$$

or

$$g'=g\left(1-\frac{h}{R_E}\right) \qquad (1)$$

Express the gravitational force acting on the clock when it is at an elevation h:

$$mg''=\frac{GM_Em}{(R_E+h)^2}$$

Express the gravitational force acting on the clock at the surface of the earth:

$$mg=\frac{GM_Em}{R_E^2}$$

Divide the first of these equations by the second to obtain:

$$\frac{g''}{g}=\frac{\dfrac{GM_E}{(R_E+h)^2}}{\dfrac{GM_E}{R_E^2}}=\frac{R_E^2}{(R_E-h)^2}$$

$$=\frac{1}{\left(1+\dfrac{h}{R_E}\right)^2}$$

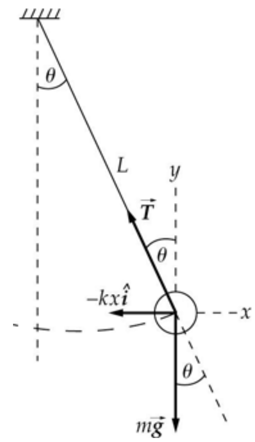
Solve for g'':

$$g''=g\left(1+\frac{h}{R_E}\right)^{-2}$$

Comparing equations (1) and (2), we see that g' is closer to g than is g''. Thus, the error is greater if the clock is elevated.

106 ••

Picture the Problem The figure shows this system when it has an angular displacement θ. The period of the system is related to its angular frequency according to $T = 2\pi/\omega$. We can find the equation of motion of the system by applying Newton's 2nd law. By writing this equation in terms of θ and using a small-angle approximation, we'll find an expression for ω that we can use to express T.



(*a*) Express the period of the system in terms of its angular frequency:

$$T = \frac{2\pi}{\omega} \quad (1)$$

Apply $\sum \vec{F} = m\vec{a}$ to the bob:

$$\sum F_x = -kx - T\sin\theta = Ma_x$$

and

$$\sum F_y = T\cos\theta - Mg = 0$$

Eliminate T between the two equations to obtain:

$$-kx - Mg\tan\theta = Ma_x$$

Noting that $x = L\theta$ and $a_x = L\alpha = L\dfrac{d^2\theta}{dt^2}$, eliminate the variable x in favor of θ:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\tan\theta$$

For $\theta \ll 1$, $\tan\theta \approx \theta$:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\theta = -(kL + Mg)\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\left(\frac{k}{M} + \frac{g}{L}\right)\theta = -\omega^2\theta$$

where

$$\omega = \sqrt{\frac{k}{M} + \frac{g}{L}}$$

Substitute in equation (1) to obtain:

$$T = \boxed{\frac{2\pi}{\sqrt{\frac{k}{M} + \frac{g}{L}}}}$$

(*b*) When $T = 2$ s and $M = 1$ kg we have:

$$2 = \frac{2\pi}{\sqrt{\frac{g}{L}}}$$

When $T = 1$ s we have:

$$1 = \frac{2\pi}{\sqrt{k + \frac{g}{L}}}$$

Solve these equations simultaneously to obtain:

$$k = \boxed{29.6\,\text{N/m}}$$

107 ••

Picture the Problem Applying Newton's 2nd law to the first object as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the system which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(*a*) Apply $\sum F_x = ma_x$ to the second object as it is about to slip:

$$f_{s,\max} = m_2 a_{\max}$$

Apply $\sum F_y = 0$ to the second object:

$$F_n - m_2 g = 0$$

Use $f_{s,\max} = \mu_s F_n$ to eliminate $f_{s,\max}$ and F_n between the two equations:

$$\mu_s m_2 g = m_2 a_{\max}$$

and

$$\mu_s = \frac{a_{\max}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency:

$$a_{\max} = A\omega^2 = A\frac{k}{m_1 + m_2}$$

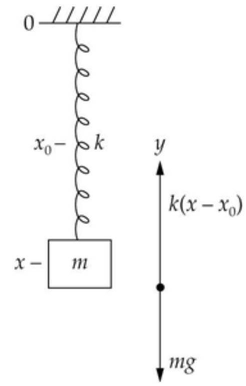
Substitute for $a_{\max}$ to obtain:

$$\mu_s = \boxed{\frac{Ak}{(m_1 + m_2)g}}$$

(*b*) $\boxed{A \text{ is unchanged. } E \text{ is unchanged because } E = \tfrac{1}{2}kA^2.\ \omega \text{ is reduced by increasing the total mass of the system and } T \text{ is increased.}}$

108 ••

Picture the Problem The diagram shows the box hanging from the stretched spring and the free-body diagram when the box is in equilibrium. We can apply $\sum F_y = 0$ to the box to derive an expression for *x*. In (*b*) and (*c*), we can proceed similarly to obtain expressions for the effective spring constant, the new equilibrium position of the box, and frequency of oscillations when the box is released.



(*a*) Apply $\sum F_y = 0$ to the box to obtain:

$$k(x - x_0) - mg = 0$$

Solve for *x*:

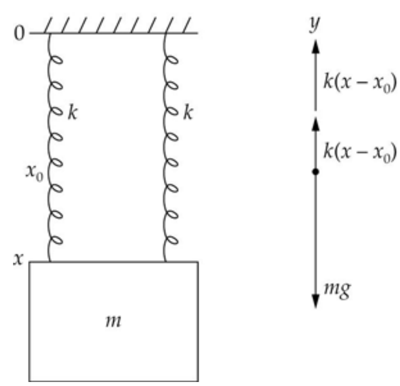
$$x = \frac{mg}{k} + x_0$$

Substitute numerical values and evaluate *x*:

$$x = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{500\,\text{N/m}} + 0.5\text{m}$$

$$= \boxed{2.46\,\text{m}}$$

(*b*) Draw the free-body diagram for the block with the two springs exerting equal upward forces on it:



Apply $\sum F_y = 0$ to the box to obtain:

$$k(x - x_0) + k(x - x_0) - mg = 0$$

or

$$k_{\text{eff}}(x - x_0) - mg = 0 \qquad (1)$$

where

$$k_{\text{eff}} = 2k$$

When the box is displaced from this equilibrium position and released, its motion is simple harmonic motion and its frequency is given by:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2k}{m}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{2(500\,\text{N/m})}{100\,\text{kg}}} = \boxed{3.16\,\text{rad/s}}$$

(c) Solve equation (1) for x:

$$x = \frac{mg}{2k} + x_0$$

Substitute numerical values and evaluate x:

$$x = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{2(500\,\text{N/m})} + 0.5\,\text{m}$$
$$= \boxed{1.48\,\text{m}}$$

109 ••

Picture the Problem We'll differentiate the expression for the period of simple pendulum $T = 2\pi\sqrt{\frac{L}{g}}$ with respect to g, separate the variables, and use a differential approximation to establish that $\frac{\Delta T}{T} \approx -\frac{1}{2}\frac{\Delta g}{g}$.

(a) Express the period of a simple pendulum in terms of its length and the local value of the acceleration due to gravity:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Differentiate this expression with respect to g to obtain:

$$\frac{dT}{dg} = \frac{d}{dg}\left[2\pi\sqrt{L}g^{-1/2}\right] = -\pi\sqrt{L}g^{-3/2}$$
$$= -\frac{T}{2g}$$

Separate the variables to obtain:

$$\frac{dT}{T} = -\frac{1}{2}\frac{dg}{g}$$

Approximate dT and dg by ΔT and Δg for $\Delta g << g$:

$$\frac{\Delta T}{T} \approx \boxed{-\frac{1}{2}\frac{\Delta g}{g}}$$

(b) Solve the result in part (a) for Δg:

$$\Delta g = -2g\frac{\Delta T}{T}$$

Express $\Delta T/T$:

$$\frac{\Delta T}{T} = -90\,\frac{\text{s}}{\text{d}} \times \frac{1\,\text{d}}{24\,\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}} = -1.04\times10^{-3}$$

Substitute and evaluate Δg:

$$\Delta g = -2\left(9.81\,\text{m/s}^2\right)\left(-1.04\times10^{-3}\right) = 0.0204\,\text{m/s}^2 = \boxed{2.04\,\text{cm/s}^2}$$

110 ••

Picture the Problem We can find the frequency of the vibrating system from its angular frequency; this depends on the spring constant and the total mass involved in the motion. The energy of the system can be found from the amplitude of its motion.

(*a*) Relate the frequency of the vibrating system to its angular frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{240\,\text{N/m}}{2(0.6\,\text{kg})}} = \boxed{2.25\,\text{Hz}}$$

Express the total energy of the system:

$$E = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate *E*:

$$E = \tfrac{1}{2}(240\,\text{N/m})(0.6\,\text{m})^2 = \boxed{43.2\,\text{J}}$$

(*b*) (1) The glue dissolves when the spring is at maximum compression:

Relate the frequency to the system's new angular frequency:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate f_1:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{240\,\text{N/m}}{0.6\,\text{kg}}} = \boxed{3.18\,\text{Hz}}$$

Express the system's new amplitude as a function of the oscillator's maximum speed and its new angular frequency:

$$A_1 = \frac{v_{\text{max}}}{\omega_1} = v_{\text{max}}\sqrt{\frac{m}{k}}$$

Find the maximum speed of the oscillator:

$$v_{\text{max}} = A\omega = 2\pi fA = 2\pi\left(2.25\,\text{s}^{-1}\right)(0.6\,\text{m}) = 8.48\,\text{m/s}$$

Substitute and evaluate A_1:

$$A_1 = (8.48\,\text{m/s})\sqrt{\frac{0.6\,\text{kg}}{240\,\text{N/m}}} = \boxed{42.4\,\text{cm}}$$

Express and evaluate the energy of the system:

$$E_1 = \tfrac{1}{2}kA_1^2 = \tfrac{1}{2}(240\,\text{N/m})(0.424\,\text{m})^2 = \boxed{21.6\,\text{J}}$$

(*b*) (2) The glue dissolves when the spring is at maximum extension and f_2 is the same as f_1:

$$f_2 = \boxed{3.18\,\text{Hz}}$$

Because the second object is at rest, the amplitude and energy of the system are unchanged:

$$A_2 = A = \boxed{0.600\,\text{m}}$$

and

$$E_2 = E = \boxed{43.2\,\text{J}}$$

111 ••

Picture the Problem Choose a coordinate system in which the positive *x* direction is to the right and assume that the object is displaced to the right. In case (*a*), note that the two springs undergo the same displacement whereas in (*b*) they experience the same force.

(*a*) Express the net force acting on the object:

$$F_{\text{net}} = -k_1x - k_2x = -(k_1 + k_2)x = -k_{\text{eff}}x$$

where $k_{\text{eff}} = \boxed{k_1 + k_2}$

(*b*) Express the force acting on each spring and solve for x_2:

$$F = -k_1x_1 = -k_2x_2$$

or

$$x_2 = \frac{k_1}{k_2}x_1$$

Express the total extension of the springs:

$$x_1 + x_2 = -\frac{F}{k_{\text{eff}}}$$

Solve for k_{eff}:

$$k_{\text{eff}} = -\frac{F}{x_1 + x_2} = -\frac{-k_1x_1}{x_1 + x_2} = \frac{k_1x_1}{x_1 + \frac{k_1}{k_2}x_1} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Take the reciprocal of both sides of the equation to obtain:

$$\frac{1}{k_{\text{eff}}}=\boxed{\frac{1}{k_1}+\frac{1}{k_2}}$$

***112 ••**

Picture the Problem If the displacement of the block is $y = A\sin\omega t$, its acceleration is $a = -\omega^2 A\sin\omega t$.

(*a*) At maximum upward extension, the block is momentarily at rest. Its downward acceleration is g. The downward acceleration of the piston is $\omega^2 A$. Therefore, if $\omega^2 A > g$, the block will separate from the piston.

(*b*) Express the acceleration of the small block:

$$a=-A\omega^2\sin\omega t$$

For $\omega^2 A = 3g$ and $A = 15$ cm:

$$a=-3g\sin\omega t=-g$$

Solve for t:

$$t=\frac{1}{\omega}\sin^{-1}\left(\frac{1}{3}\right)=\sqrt{\frac{A}{3g}}\sin^{-1}\left(\frac{1}{3}\right)$$

Substitute numerical values and evaluate t:

$$t=\sqrt{\frac{0.15\,\text{m}}{3\left(9.81\,\text{m/s}^2\right)}}\sin^{-1}\frac{1}{3}=\boxed{0.0243\,\text{s}}$$

113 ••

Picture the Problem The plunger and ball are moving with their maximum speed as they pass through their equilibrium position ($x = 0$). Once it has passed its equilibrium position, the acceleration of the plunger becomes negative; therefore it begins to slow down and the ball, continuing with speed v_s, separates from the plunger. We can find this separation speed by equating it to the maximum speed of the plunger. Application of conservation of energy to the motion of the plunger will allow us to express the distance at which the plunger comes momentarily to rest.

(*a*) The ball will leave the plunger when the plunger is moving with its maximum speed; i.e., at its equilibrium position:

$$x=\boxed{0}$$

(*b*) Express the speed of the ball upon separation in terms of the maximum speed of the plunger:

$$v_s=v_{\max}=A\omega=x_0\omega$$

The angular frequency is given by:

$$\omega = \sqrt{\frac{k}{m_b + m_p}}$$

Substitute to obtain:

$$v_s = \boxed{x_0\sqrt{\frac{k}{m_b + m_p}}}$$

(*c*) Apply conservation of energy to the plunger:

$$K_f - K_i + U_{f,s} - U_{i,s} = 0$$

or, because $K_f = U_i = 0$,

$$-\tfrac{1}{2}m_p v_s^2 + \tfrac{1}{2}kx_f^2 = 0$$

Solve for x_f:

$$x_f = \sqrt{\frac{m_p}{k}}v_s$$

Substitute for v_s and simplify to obtain:

$$x_f = \boxed{x_0\sqrt{\frac{m_p}{m_b + m_p}}}$$

114 ••

Picture the Problem Applying Newton's 2nd law to the box as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the platform which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(*a*) Apply $\sum F_x = ma_x$ to the box as it is about to slip:

$$f_{s,max} = ma_{max}$$

Apply $\sum F_y = 0$ to the box:

$$F_n - mg = 0$$

Use $f_{s,max} = \mu_s F_n$ to eliminate $f_{s,max}$ and F_n between the two equations:

$$\mu_s mg = ma_{max}$$

and

$$\mu_s = \frac{a_{max}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency:

$$a_{max} = A\omega^2$$

Substitute for a_{max} :

$$\mu_s = \frac{A\omega^2}{g} = \frac{4\pi^2 A}{T^2 g}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{4\pi^2(0.4\,\text{m})}{(0.8\,\text{s})^2(9.81\,\text{m/s}^2)} = \boxed{2.52}$$

(*b*) Solve the equation derived above for A_{max}:

$$A_{max} = \frac{\mu_s g}{\omega^2} = \frac{\mu_s g T^2}{4\pi^2}$$

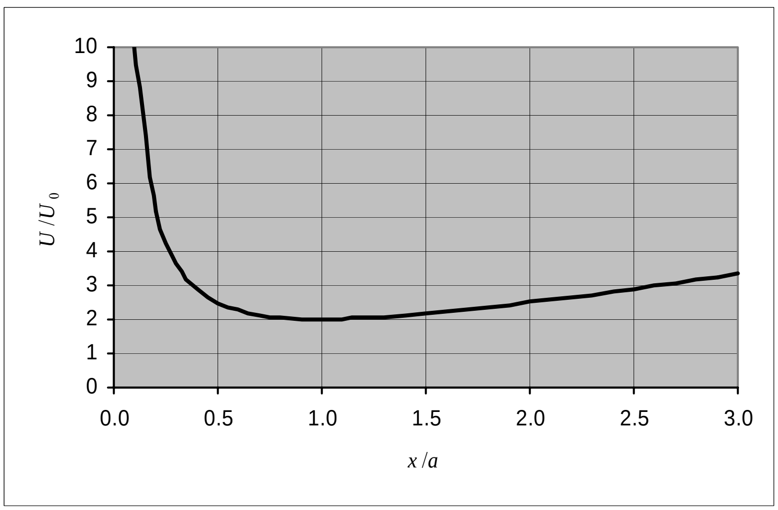
Substitute numerical values and evaluate A_{max}:

$$A_{max} = \frac{(0.4)(9.81\,\text{m/s}^2)(0.8\,\text{s})^2}{4\pi^2} = \boxed{6.36\,\text{cm}}$$

115 •••

Picture the Problem In (*b*), we can use the condition $F_{net} = dU/dx = 0$ for stable equilibrium to find the value of $x = x_0$ at stable equilibrium. In (*c*) and (*d*), we can simply follow the outline provided in the problem statement. In (*e*), we can obtain the frequency from $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ using the value for *k* from the potential function.

(*a*) A graph of $U(x)$ follows:



(*b*) Express the condition for equilibrium:

$$F = \frac{dU}{dx} = 0$$

Differentiate *U* with respect to *x*:

$$\frac{dU}{dx} = \frac{d}{dx}\left[U_0\left(\frac{x}{a} + \frac{a}{x}\right)\right] = U_0\left[\frac{1}{a} - \frac{a}{x^2}\right] = \frac{U_0}{a}\left(1 - \frac{a^2}{x^2}\right)$$

Set this derivative equal to zero and solve for x:

$$\frac{U_0}{a}\left(1-\frac{a^2}{x_0^2}\right)=0$$

and

$$x_0 = \boxed{a} \text{ or } \alpha = \boxed{1}$$

(c) Express $U(x_0 + \varepsilon)$:

$$U(x_0+\varepsilon)=U_0\left[\frac{x_0+\varepsilon}{a}+\frac{a}{x_0+\varepsilon}\right]$$

$$=U_0\left[\frac{x_0}{a}+\frac{\varepsilon}{a}+\frac{1}{\frac{x_0}{a}+\frac{\varepsilon}{a}}\right]$$

or, because $x_0 = a$,

$$U(x_0+\varepsilon)=U_0\left[1+\frac{\varepsilon}{a}+\frac{1}{1+\frac{\varepsilon}{a}}\right]$$

$$=\boxed{U_0\left[1+\beta+(1+\beta)^{-1}\right]}$$

where $\beta=\frac{\varepsilon}{a}$

(d) Expand $(1+\beta)^{-1}$ to obtain:

$$(1+\beta)^{-1}=1+(-1)\beta+\frac{(-1)(-2)}{2\times 1}\beta^2+\ldots$$

$$\approx 1-\beta+\beta^2$$

Substitute in $U(x_0 + \varepsilon)$:

$$U(x_0+\varepsilon)=U_0\left[1+\beta+1-\beta+\beta^2\right]$$

$$=U_0\left[2+\beta^2\right]$$

$$=2U_0+U_0\frac{\varepsilon^2}{a^2}$$

$$=\boxed{\text{constant}+U_0\frac{\varepsilon^2}{a^2}}$$

(e) Express the potential energy of a simple harmonic oscillator:

$$U=\text{constant}+\tfrac{1}{2}k\varepsilon^2$$

If the particle whose potential energy is given in part (d) is to undergo simple harmonic motion:

$$k=\frac{2U_0}{a^2}$$

Express the frequency of the simple harmonic motion, substitute for *k*, and simplify to obtain:

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{2U_0}{a^2m}}$$
$$=\boxed{\frac{1}{2\pi a}\sqrt{\frac{2U_0}{m}}}$$

116 •••

Picture the Problem Let *m* represent the mass of the cylindrical drum, *R* its radius, and *k* the stiffness constant of the spring. We can find the angular frequency of the oscillations by equating the maximum kinetic energy of the drum and the maximum energy stored in the spring. We can then express the frequency of the system in terms of its angular frequency. The application of Newton's 2nd law, under on-the-verge-of-sliding conditions, together with the introduction of the oscillator's total energy, will lead us to an expression for the minimum value of the coefficient of static friction.

(*a*) Express the frequency of oscillation of the system for small displacements from equilibrium:

$$f=\frac{\omega}{2\pi} \qquad (1)$$

Express the kinetic energy of the drum and simplify to obtain:

$$K=\tfrac{1}{2}I\omega^2+\tfrac{1}{2}mv^2$$
$$=\tfrac{1}{2}\left(\tfrac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2+\tfrac{1}{2}mv^2$$
$$=\tfrac{3}{4}mv^2$$

Apply conservation of energy to obtain:

$$K_{\max}=\tfrac{3}{4}mv_{\max}^2=\tfrac{1}{2}kA^2$$

Substitute $A\omega$ for $v_{\max}$:

$$\tfrac{3}{4}m(A\omega)^2=\tfrac{1}{2}kA^2$$

Solve for ω:

$$\omega=\sqrt{\frac{2k}{3m}}$$

Substitute in equation (1) to obtain:

$$f=\frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$$

Substitute numerical values and evaluate *f*:

$$f=\frac{1}{2\pi}\sqrt{\frac{2(4000\,\text{N/m})}{3(6\,\text{kg})}}=\boxed{3.36\,\text{Hz}}$$

(*b*) Apply $\sum F_x=0$ to the drum to

$$kA-f_{\text{s,max}}=0$$

or

establish the condition that governs slipping:

$$kA-\mu_s F_n=0$$

Using $F_n = mg$, solve for μ_s:

$$\mu_s=\frac{kA}{mg} \qquad (2)$$

Express the oscillator's total energy in terms of the amplitude of its motion:

$$E=\tfrac{1}{2}kA^2 \Rightarrow kA=\sqrt{2Ek}$$

Substitute in equation (2) to obtain:

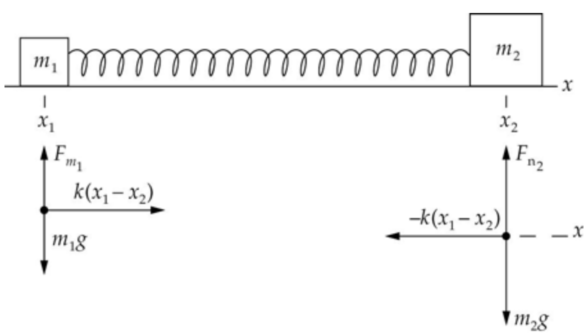
$$\mu_s=\frac{\sqrt{2Ek}}{mg}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s=\frac{\sqrt{2(5\,\text{J})(4000\,\text{N/m})}}{(6\,\text{kg})(9.81\,\text{m/s}^2)}=\boxed{3.40}$$

***117** •••

Picture the Problem The pictorial representation shows the two blocks connected by the spring and displaced from their equilibrium positions. We can apply Newton's 2nd law to each of these coupled oscillators and solve the resulting equations simultaneously to obtain the differential equation of motion of the coupled oscillators. We can then compare this differential equation and its solution to the differential equation of motion of the simple harmonic oscillator and its solution to show that the oscillation frequency is $\omega=(k/\mu)^{1/2}$ where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of the system.



Apply $\sum\vec{F}=m\vec{a}$ to the block whose mass is m_1 and solve for its acceleration:

$$k(x_1-x_2)=m_1a_1=m_1\frac{d^2x_1}{dt^2}$$

or

$$a_1=\frac{d^2x_1}{dt^2}=\frac{k}{m_1}(x_1-x_2)$$

Apply $\sum\vec{F}=m\vec{a}$ to the block whose mass is m_2 and solve for its

$$-k(x_1-x_2)=m_2a_2=m_1\frac{d^2x_2}{dt^2}$$

acceleration: or

$$a_2 = \frac{d^2x_2}{dt^2} = \frac{k}{m_2}(x_2 - x_1)$$

Subtract the first equation from the second to obtain:

$$\frac{d^2(x_2 - x_1)}{dt^2} = \frac{d^2x}{dt^2} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x$$

where $x = x_2 - x_1$

The reduced mass of the system is:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \text{ or } \mu = \frac{m_1m_2}{m_1 + m_2}$$

Substitute to obtain:

$$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x \qquad (1)$$

Compare this differential equation with the differential equation of the simple harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The solution to this equation is:

$$x = x_0\cos(\omega t + \delta)$$

where $\omega = \sqrt{\frac{k}{m}}$

Express the solution to equation (1):

$$x = x_0\cos(\omega t + \delta)$$

where $\omega = \boxed{\sqrt{\frac{k}{\mu}}}$

118 ••

Picture the Problem We can use $\omega = (k/\mu)^{1/2}$ and $\mu = m_1m_2/(m_1 + m_2)$ from Problem 117 to find the spring constant for the HCl molecule.

Use the result of Problem 118 to relate the oscillation frequency to the spring constant and reduced mass of the HCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

Solve for k to obtain:

$$k = \mu\omega^2$$

Express the reduced mass of the HCl molecule:

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

Substitute to obtain:

$$k = \frac{m_1m_2\omega^2}{m_1 + m_2}$$

Express the masses of the hydrogen and Cl atoms:

$m_1 = 1 \text{ amu} = 1.67\times10^{-27}$ kg
and
$m_2 = 35.45 \text{ amu} = 5.92\times10^{-26}$ kg

Substitute numerical values and evaluate k:

$$k = \frac{(1.67\times10^{-27}\text{ kg})(5.92\times10^{-26}\text{ kg})(8.969\times10^{13}\text{ s}^{-1})^2}{1.67\times10^{-27}\text{ kg} + 5.92\times10^{-26}\text{ kg}} = \boxed{13.1\text{ N/m}}$$

119 ••

Picture the Problem In Problem 117, we derived an expression for the oscillation frequency of a spring-and-two-block system as a function of the stiffness constant of the spring and the reduced mass of the two blocks. We can solve this problem, assuming that the "spring constant" does not change, by using the result of Problem 117 and the reduced mass of a deuterium atom and a Cl atom in the equation for the oscillation frequency.

Use the result of Problem 117 to relate the oscillation frequency to the spring constant and reduced mass of the HCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

Express the reduced mass of the HCl molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Express the masses of the deuterium and Cl atoms:

$m_1 = 2 \text{ amu} = 3.34\times10^{-27}$ kg
and
$m_2 = 35.45 \text{ amu} = 5.92\times10^{-26}$ kg

Evaluate the reduced mass of the molecule:

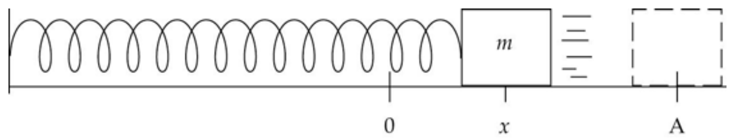
$$\mu = \frac{(3.34\times10^{-27}\text{ kg})(5.92\times10^{-26}\text{ kg})}{3.34\times10^{-27}\text{ kg} + 5.92\times10^{-26}\text{ kg}}$$
$$= 3.16\times10^{-27}\text{ kg}$$

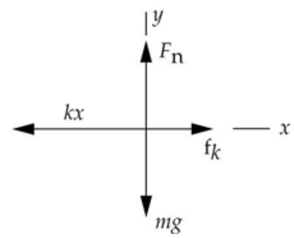
Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{13.1\text{ N/m}}{3.16\times10^{-27}\text{ kg}}}$$
$$= \boxed{6.44\times10^{13}\text{ rad/s}}$$

120 •••

Picture the Problem The pictorial representation shows the block moving from right to left with an instantaneous displacement x from its equilibrium position. The free-body diagram shows the forces acting on the block during the half-cycles that it moves from right to left. When the block is moving from left to right, the directions of the kinetic friction force and the restoring force exerted by the spring are reversed. We can apply Newton's 2nd law to these motions to obtain the differential equations given in the problem statements and then use their solutions to plot the graph called for in (*c*).





(*a*) Apply $\sum F_x = ma_x$ to the block while it is moving to the left to obtain:

$$f_k - kx = m\frac{d^2x}{dt^2}$$

Using $f_k = \mu_k F_n = \mu_k mg$, eliminate f_k in the differential equation of motion:

$$m\frac{d^2x}{dt^2} = -kx + \mu_k mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x - \frac{\mu_k mg}{k}\right)$$

Let $x_0 = \dfrac{\mu_k mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x - x_0)$$

or

$$\boxed{\frac{d^2x'}{dt^2} = -\frac{k}{m}x' = -\omega^2 x'}$$

provided $x' = x - x_0$ and

$$x_0 = \boxed{\frac{\mu_k mg}{k} = \frac{\mu_k g}{\omega^2}}$$

The solution to the differential equation is:

$$x' = x_0' \cos(\omega t + \delta)$$

and its derivative is

$$v' = -\omega x_0' \sin(\omega t + \delta)$$

The initial conditions are:

$$x'(0) = x - x_0 \text{ and } v'(0) = 0$$

Apply these conditions to obtain:

$$x'(0) = x_0' \cos\delta = x - x_0$$

and

$$v'(0) = -\omega x_0' \sin\delta = 0$$

Solve these equations simultaneously to obtain:

$$\delta = 0 \text{ and } x_0' = x - x_0$$

and

$$x' = (x - x_0)\cos\omega t$$

or

$$\boxed{x = (x - x_0)\cos\omega t + x_0} \qquad (1)$$

(*b*) Apply $\sum \vec{F} = m\vec{a}$ to the block while it is moving to the right to obtain:

$$-f_\text{k} - kx = m\frac{d^2x}{dt^2}$$

Using $f_\text{k} = \mu_\text{k}F_\text{n} = \mu_\text{k}mg$, eliminate f_k in the differential equation of motion:

$$m\frac{d^2x}{dt^2} = -kx - \mu_\text{k}mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x + \frac{\mu_\text{k}mg}{k}\right)$$

Let $x_0 = \dfrac{\mu_\text{k}mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x + x_0)$$

or

$$\boxed{\frac{d^2x''}{dt^2} = -\frac{k}{m}x'' = -\omega^2x''}$$

provided $x'' = x + x_0$ and

$$x_0 = \boxed{\frac{\mu_\text{k}mg}{k} = \frac{\mu_\text{k}g}{\omega^2}}\,.$$

The solution to the differential equation is:

$$x'' = x_0''\cos(\omega t + \delta)$$

and its derivative is

$$v'' = -\omega x_0''\sin(\omega t + \delta)$$

The initial conditions are:

$$x''(0) = x + x_0 \text{ and } v''(0) = 0$$

Apply these conditions to obtain:

$$x''(0) = x_0''\cos\delta = x + x_0$$

and

$$v''(0) = -\omega x_0''\sin\delta = 0$$

Solve these equations simultaneously to obtain:

$$\delta = 0 \text{ and } x_0'' = x + x_0$$

and

$$x'' = (x + x_0)\cos\omega t$$

or

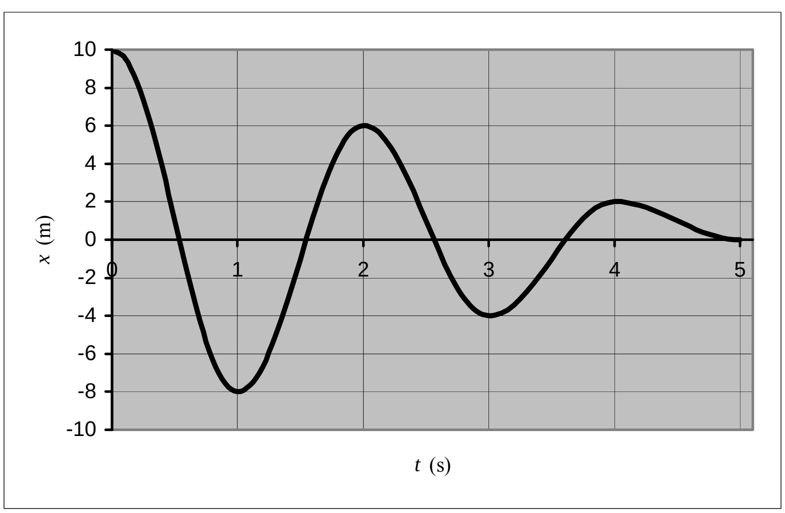
$$\boxed{x = (x + x_0)\cos\omega t - x_0} \qquad (2)$$

(*c*) A spreadsheet program to calculate the position of the oscillator as a function of time (equations (1) and (2)) is shown below. The constants used in the position functions (x_0 = 1 m and T = 2 s were used for simplicity) and the formulas used to calculate the positions are shown in the table. *After each half-period, one must compute a new amplitude for the oscillation, using the final value of the position from the last half-period.*

Cell	Content/Formula	Algebraic Form
B1	1	x_0
B2	10	A
C7	C6 + 0.1	$t+\Delta t$
D7	($B$2–$B$1)*COS(PI()*C7)+$B$1	$(A-x_0)\cos\pi t+x_0$
D17	(ABS($D$6+$B$1))*COS(PI()*C17)–$B$1	$\lvert x+x_0\rvert\cos\pi t-x_0$
D27	(ABS($D$6–$B$1))*COS(PI()*C27)+$B$1	$\lvert x-x_0\rvert\cos\pi t+x_0$
D37	(ABS($D$36+$B$1))*COS(PI()*C37)–$B$1	$\lvert x+x_0\rvert\cos\pi t-x_0$
D47	($D$46–$B$1)*COS(PI()*C47)+$B$1	$(x-x_0)\cos\pi t+x_0$

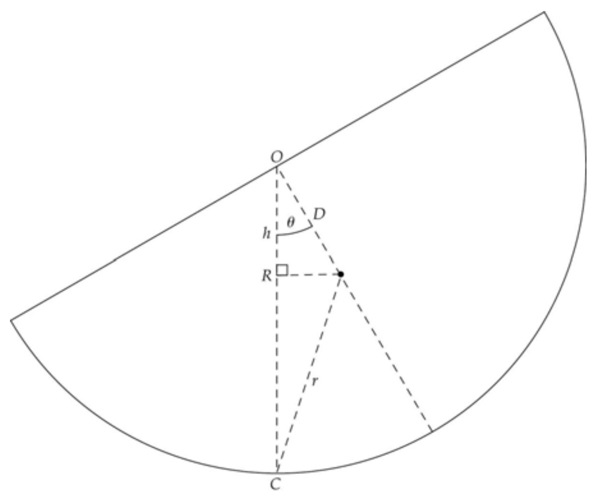
	A	B	C	D
1	x0=	1	m	
2	A=	10		
3				
4			*t*	*x*
5			(s)	(m)
6			0.0	10.00
7			0.1	9.56
8			0.2	8.28
9			0.3	6.29
10			0.4	3.78
53			4.7	0.41
54			4.8	0.19
55			4.9	0.05
56			5.0	0.00

The graph shown below was plotted using the data from columns C (t) and D (x). $\boxed{\text{Note that the motion of the block ceases after five half-cycles.}}$



121 •••

Picture the Problem The diagram shows the half-cylinder displaced from its equilibrium position through an angle θ. The frequency of its motion will be found by expressing the mechanical energy E in terms of θ and $d\theta/dt$. For small θ we will obtain an equation of the form $E = \frac{1}{2}\kappa\theta^2 + \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$. Differentiating both sides of this equation with respect to time will lead to $0 = \left(\kappa\theta + I\frac{d^2\theta}{dt^2}\right)\frac{d\theta}{dt}$, an equation that must be valid at all times. Because the situation of interest to us requires that $d\theta/dt$ is not always equal to zero, we have $0 = \kappa\theta + I\frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$, the differential equation of simple harmonic motion with $\omega^2 = \kappa/I$. The distance from O to the center of mass D, where, from Problem 8-39, $D = (4/3\pi)R$, and the distance from the contact point C to the center of mass is r. Finally, we'll take the potential energy to be zero where θ is zero and assume that there is no slipping.



Apply conservation of energy to obtain:

$$E = U + K$$
$$= Mg(h - D) + \frac{1}{2}I_C\left(\frac{d\theta}{dt}\right)^2 \qquad (1)$$

From Table 9-1, the moment of inertia of a solid cylinder about an axis perpendicular to its face and through its center is given by:

$$I_{0,\,\text{solid cylinder}} = \frac{1}{2}(2M)R^2 = MR^2$$

where M is the mass of the half-cylinder.

Express the moment of inertia of the half-cylinder about the same axis:

$$I_{0,\text{half cylinder}} = I_0 = \frac{1}{2}\left[MR^2\right] = \frac{1}{2}MR^2$$

Use the parallel-axis theorem to relate I_{cm} to I_0:

$$I_0 = I_{cm} + MD^2$$

Substitute for I_{cm} and solve for I_{cm}:

$$I_{cm} = I_0 - D^2M = \frac{1}{2}MR^2 - D^2M$$

Apply the parallel-axis theorem a second time to obtain an expression for I_C:

$$I_C = \frac{1}{2}MR^2 - D^2M + Mr^2 = M\left(\frac{1}{2}R^2 - D^2 + r^2\right) \qquad (2)$$

Apply the law of cosines to obtain:

$$r^2 = R^2 + D^2 - 2RD\cos\theta$$

Substitute for r^2 in equation (2) to obtain:

$$I_C = M\left(\frac{1}{2}R^2 - D^2 + R^2 + D^2 - 2RD\cos\theta\right) = MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)$$

Substitute for h and I_C in equation (1):

$$E = MgD(1-\cos\theta) + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)\left(\frac{d\theta}{dt}\right)^2$$

Use the small angle approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - \frac{D}{R}\left[2-\theta^2\right]\right)\left(\frac{d\theta}{dt}\right)^2$$

Because $\theta^2 << 2$, we can neglect the θ^2 in the square brackets to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d\theta}{dt}\right)^2$$

Differentiating both sides with respect to time yields:

$$0 = MgD\theta\frac{d\theta}{dt} + MR^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right),$$

$$R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d^2\theta}{dt^2}\right) + gD\theta = 0,$$

and

$$\frac{d^2\theta}{dt^2} + \frac{gD}{R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)}\theta = 0,$$

the differential equation of simple harmonic motion with $\omega^2 = \dfrac{gD}{R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)}$.

Substitute for D to obtain:

$$\omega^2 = \frac{\frac{4}{3\pi}}{\left(\frac{3}{2} - \frac{8}{3\pi}\right)}\frac{g}{R} = \left(\frac{8}{9\pi - 16}\right)\frac{g}{R}$$

Express the period of the motion in terms of ω and simplify to obtain:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{9\pi - 16}{8}\right)\frac{R}{g}}$$

$$= \boxed{7.78\sqrt{\frac{R}{g}}}$$

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Picture the Problem The net force acting on the particle as it moves in the tunnel is the x-component of the gravitational force acting on it. We can find the period of the particle from the angular frequency of its motion. We can apply Newton's 2nd law to the particle in order to express ω in terms of the radius of the earth and the acceleration due to gravity at the surface of the earth.

(a) From the figure we see that:

$$F_x = F_r \sin\theta = -\frac{GmM_E}{R_E^3}r\frac{x}{r}$$

$$= \boxed{-\frac{GmM_E}{R_E^3}x}$$

Because this force is a linear restoring force, the motion of the particle is simple harmonic motion.

(*b*) Express the period of the particle as a function of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum F_x = ma_x$ to the particle:

$$-\frac{GmM_E}{R_E^3}x = ma$$

Solve for *a*:

$$a = -\frac{GM_E}{R_E^3}x = -\omega^2 x$$

where

$$\omega = \sqrt{\frac{GM_E}{R_E^2}}$$

Use $GM_E = gR_E^2$ to simplify ω:

$$\omega = \sqrt{\frac{gR_E^2}{R_E^3}} = \sqrt{\frac{g}{R_E}}$$

Substitute in equation (1) to obtain:

$$T = \frac{2\pi}{\sqrt{\dfrac{g}{R_E}}} = \boxed{2\pi\sqrt{\frac{R_E}{g}}}$$

Substitute numerical values and evaluate *T*:

$$T = 2\pi\sqrt{\frac{6.37\times10^6\ \text{m}}{9.81\,\text{m/s}^2}} = 5.06\times10^3\ \text{s}$$

$$= \boxed{84.4\,\text{min}}$$

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Picture the Problem The amplitude of a damped oscillator decays with time according to $A = A_0e^{-(b/2m)t}$. We can find *b*/2*m* from $\omega' = \omega_0\sqrt{1-\left(\dfrac{b}{2m\omega_0}\right)^2}$ and then substitute in the amplitude equation to find the factor by which the amplitude is decreased during each oscillation. We'll use our result from (*a*), together with the dependence of the energy of the oscillator on the square of its amplitude, to find the factor by which its energy is reduced during each oscillation.

(*a*) Express the variation in amplitude with time:

$$A = A_0e^{-(b/2m)t} \qquad (1)$$

Relate the damped and undamped frequencies of the oscillator:

$$\omega' = \omega_0\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2} \qquad (14\text{-}46)$$

Solve for $b/2m$:

$$\frac{b}{2m} = \omega_0\sqrt{1-\frac{\omega'^2}{\omega_0^2}} = \omega_0\sqrt{1-(0.9)^2}$$
$$= 0.436\omega_0$$

Find the period of the damped oscillations:

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{0.9\omega_0}$$

Substitute in equation (1) with $t = T$ to obtain:

$$\frac{A}{A_0} = e^{-(0.436\omega_0)\left(\frac{2\pi}{0.9\omega_0}\right)} = e^{-3.04}$$
$$= \boxed{0.0478}$$

(b) Express the energy of the oscillator at time $t = 0$:

$$E_0 = \tfrac{1}{2}kA_0^2$$

Express the energy of the oscillator at time $t = T$:

$$E = \tfrac{1}{2}kA^2$$

Divide the second of these equations by the first, simplify, and substitute to evaluate E/E_0:

$$\frac{E}{E_0} = \frac{A^2}{A_0^2} = \left(\frac{A}{A_0}\right)^2 = (0.0477)^2$$
$$= \boxed{0.00228}$$

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Picture the Problem We can differentiate Equation 14-52 twice and substitute x and d^2x/dt^2 in Equation 14-51 to determine the condition that must be satisfied in order for Equation 14-52 to be a solution of Equation 14-51.

The differential equation of motion is Equation 14-51:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega_0^2x = F_0\cos\omega t$$

Its proposed solution is Equation 14-52:

$$x = A\cos(\omega t - \delta)$$

Obtain the first and second derivatives of x:

$$\frac{dx}{dt} = -A\omega\sin(\omega t - \delta)$$

and

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t - \delta)$$

Substitute in the differential equation to obtain:

$$-mA\omega^2\cos(\omega t-\delta) - bA\omega\sin(\omega t-\delta) + m\omega_0^2 A\cos(\omega t-\delta) = F_0\cos\omega t$$

Using trigonometric identities, expand $\cos(\omega t-\delta)$ and $\sin(\omega t-\delta)$ to obtain:

$$\begin{aligned}-mA\omega^2(\cos\omega t\cos\delta+\sin\omega t\sin\delta) - bA\omega(\sin\omega t\cos\delta-\cos\omega t\sin\delta)\\ +m\omega_0^2 A(\cos\omega t\cos\delta+\sin\omega t\sin\delta) = F_0\cos\omega t\end{aligned}$$

Factor $mA(\cos\omega t\cos\delta+\sin\omega t\sin\delta)$ from the first and third terms to obtain:

$$mA(\omega_0^2-\omega^2)(\cos\omega t\cos\delta+\sin\omega t\sin\delta) - bA\omega(\sin\omega t\cos\delta-\cos\omega t\sin\delta) = F_0\cos\omega t$$

Factor $\cos\omega t\cos\delta$ from the first term on the left-hand side of the equation and $\sin\omega t\cos\delta$ from the 2nd term:

$$\begin{aligned}mA(\omega_0^2-\omega^2)(\cos\omega t\cos\delta)\left(1+\frac{\sin\omega t\sin\delta}{\cos\omega t\cos\delta}\right) - bA\omega(\sin\omega t\cos\delta)\left(1-\frac{\cos\omega t\sin\delta}{\sin\omega t\cos\delta}\right)\\ = F_0\cos\omega t\end{aligned}$$

Simplify to obtain:

$$\begin{aligned}mA(\omega_0^2-\omega^2)(\cos\omega t\cos\delta)(1+\tan\omega t\tan\delta) - bA\omega(\sin\omega t\cos\delta)\left(1-\frac{\tan\delta}{\tan\omega t}\right)\\ = F_0\cos\omega t\end{aligned}$$

Divide both sides of the equation by $m(\omega_0^2-\omega^2)$:

$$\begin{aligned}A(\cos\omega t\cos\delta)(1+\tan\omega t\tan\delta) - \frac{bA\omega}{m(\omega_0^2-\omega^2)}(\sin\omega t\cos\delta)\left(1-\frac{\tan\delta}{\tan\omega t}\right)\\ = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t\end{aligned}$$

The phase constant for a driven oscillator is given by Equation 14-54:

$$\tan\delta = \frac{b\omega}{m(\omega_0^2-\omega^2)}$$

Substitute for $\tan\delta$:

$$A(\cos\omega t\cos\delta)\left(1+\tan\omega t\frac{b\omega}{m\left(\omega_0^2-\omega^2\right)}\right)-\frac{bA\omega}{m\left(\omega_0^2-\omega^2\right)}(\sin\omega t\cos\delta)$$
$$\times\left(1-\frac{\frac{b\omega}{m\left(\omega_0^2-\omega^2\right)}}{\tan\omega t}\right)=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

Simplify to obtain:

$$A(\cos\omega t\cos\delta)\left(1+\tan^2\delta\right)=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

Use the trigonometric identity $1+\tan^2\delta=\dfrac{1}{\cos^2\delta}$:

$$A(\cos\omega t\cos\delta)\frac{1}{\cos^2\delta}=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

Simplify to obtain:

$$A\cos\omega t=\frac{F_0\cos\delta}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

Thus $x=A\cos(\omega t-\delta)$ is a solution to Equation 14-51 provided:

$$A=\boxed{\frac{F_0\cos\delta}{m\left(\omega_0^2-\omega^2\right)}}$$

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Picture the Problem We can follow the step-by-step instructions provided in the problem statement to obtain the desired results.

(*a*) Express the average power delivered by a driving force to a driven oscillator:

$P=\vec{F}\cdot\vec{v}=Fv\cos\theta$

or, because θ is 0°,

$P=Fv$

Express F as a function of time:

$F=F_0\cos\omega t$

Express the position of the driven oscillator as a function of time:

$x=A\cos(\omega t-\delta)$

Differentiate this expression with respect to time to express the velocity of the oscillator as a function of time:

$$v = -A\omega\sin(\omega t - \delta)$$

Substitute to express the average power delivered to the driven oscillator:

$$P = (F_0\cos\omega t)[-A\omega\sin(\omega t - \delta)] = \boxed{-A\omega F_0\cos\omega t\sin(\omega t - \delta)}$$

(*b*) Expand $\sin(\omega t - \delta)$ to obtain:

$$\sin(\omega t - \delta) = \sin\omega t\cos\delta - \cos\omega t\sin\delta$$

Substitute in your result from (*a*) and simplify to obtain:

$$P = -A\omega F_0\cos\omega t(\sin\omega t\cos\delta - \cos\omega t\sin\delta) = \boxed{A\omega F_0\sin\delta\cos^2\omega t - A\omega F_0\cos\delta\cos\omega t\sin\omega t}$$

(*c*) Integrate $\sin\theta\cos\theta$ over one period to determine $\langle\sin\theta\cos\theta\rangle$:

$$\langle\sin\theta\cos\theta\rangle = \frac{1}{2\pi}\left[\int_0^{2\pi}\sin\theta\cos\theta d\theta\right] = \frac{1}{2\pi}\left[\frac{1}{2}\sin^2\theta\Big|_0^{2\pi}\right] = 0$$

Integrate $\cos^2\theta$ over one period to determine $\langle\cos^2\theta\rangle$:

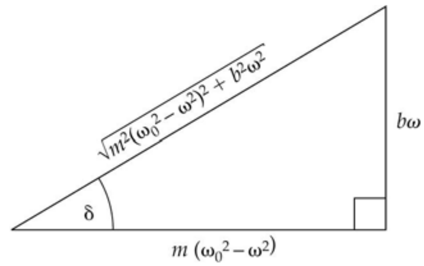
$$\langle\cos^2\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2\pi}\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta\right] = \frac{1}{2\pi}\left[\tfrac{1}{2}\int_0^{2\pi}d\theta + \tfrac{1}{2}\int_0^{2\pi}\cos 2\theta d\theta\right] = \frac{1}{2\pi}(\pi + 0) = \frac{1}{2}$$

Substitute and simplify to express P_{av}:

$$P_{\text{av}} = A\omega F_0\sin\delta\langle\cos^2\omega t\rangle - A\omega F_0\cos\delta\langle\cos\omega t\sin\omega t\rangle = \tfrac{1}{2}A\omega F_0\sin\delta - A\omega F_0\cos\delta(0) = \boxed{\tfrac{1}{2}A\omega F_0\sin\delta}$$

(*d*) Construct a triangle that is consistent with

$$\tan\delta = \frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)}:$$



Using the triangle, express sinδ:

$$\boxed{\sin\delta = \frac{b\omega}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}}}$$

Using equation 14-53, reduce this expression to the simpler form:

$$\boxed{\sin\delta = \frac{b\omega A}{F_0}}$$

(*e*) Solve $\sin\delta = \dfrac{b\omega A}{F_0}$ for ω:

$$\omega = \frac{F_0}{bA}\sin\delta$$

Substitute in the expression for P_{av} to eliminate ω:

$$P_{\text{av}} = \boxed{\frac{F_0^2}{2b}\sin^2\delta}$$

Substitute for $\sin\delta$ from (*d*) to obtain Equation 14-55:

$$P_{\text{av}} = \boxed{\frac{1}{2}\left[\frac{b\omega^2 F_0^2}{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}\right]}$$

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Picture the Problem We can follow the step-by-step instructions given in the problem statement to derive the given results.

(*a*) Express the condition on the denominator of Equation 14-55 when the power input is half its maximum value:

$$m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2 = 2b^2\omega_0^2$$

and, for a sharp resonance,

$$m^2\left(\omega_0^2 - \omega^2\right)^2 \approx b^2\omega_0^2$$

Factor the difference of two squares to obtain:

$$m^2\left[\left(\omega_0 - \omega\right)\left(\omega_0 + \omega\right)\right]^2 \approx b^2\omega_0^2$$

or

$$\boxed{m^2\left(\omega_0 - \omega\right)^2\left(\omega_0 + \omega\right)^2 \approx b^2\omega_0^2}$$

(*b*) Use the approximation $\omega + \omega_0 \approx 2\omega_0$ to obtain:

$$m^2\left(\omega_0 - \omega\right)^2\left(2\omega_0\right)^2 \approx b^2\omega_0^2$$

Solve for $\omega_0 - \omega$:

$$\omega_0 - \omega = \boxed{\pm\frac{b}{2m}} \qquad (1)$$

(c) Using its definition, express Q:

$$Q = \frac{\omega_0 m}{b}$$

Solve for b:

$$b = \boxed{\frac{\omega_0 m}{Q}}$$

(d) Substitute for b in equation (1) to obtain:

$$\omega_0 - \omega = \pm\frac{\omega_0}{2Q}$$

Solve for ω:

$$\omega = \omega_0 \pm \frac{\omega_0}{2Q}$$

Express the two values of ω:

$$\omega_+ = \boxed{\omega_0 + \frac{\omega_0}{2Q}}$$

and

$$\omega_- = \boxed{\omega_0 - \frac{\omega_0}{2Q}}$$

Remarks: Note that the width of the resonance at half-power is $\Delta\omega = \omega_+ - \omega_- = \omega_0/Q$, in agreement with Equation 14-49.

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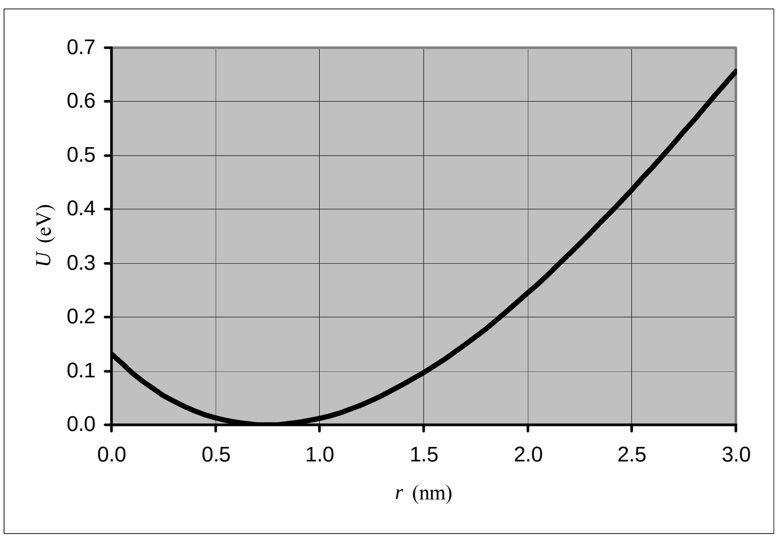
Picture the Problem We can find the equilibrium separation for the Morse potential by setting $dU/dr = 0$ and solving for r. The second derivative of U will give the "spring constant" for small displacements from equilibrium. In (c), we can use $\omega = \sqrt{k/\mu}$, where k is our result from (b) and μ is the reduced mass of a homonuclear diatomic molecule, to find the oscillation frequency of the molecule.

(a) A spreadsheet program to calculate the Morse potential as a function of r is shown below. The constants and cell formulas used to calculate the potential are shown in the table.

Cell	Content/Formula	Algebraic Form
B1	5	D
B2	0.2	β
C9	C8 + 0.1	$r + \Delta r$
D8	\$B\$1*(1–EXP(–\$B\$2*(C8–\$B\$3)))^2	$D\left[1 - e^{-\beta(r-r_0)}\right]^2$

	A	B	C	D
1	D=	5	eV	
2	Beta=	0.2	nm^{-1}	
3	r0=	0.75	nm	
4				
5				
6			r	U(r)
7			(nm)	(eV)
8			0.0	0.13095
9			0.1	0.09637
10			0.2	0.06760
11			0.3	0.04434
12			0.4	0.02629
235			22.7	4.87676
236			22.8	4.87919
237			22.9	4.88156
238			23.0	4.88390
239			23.1	4.88618

The graph shown below was plotted using the data from columns C (r) and D ($U(r)$).



(*b*) Differentiate the Morse potential with respect to r to obtain:

$$\frac{dU}{dr} = \frac{d}{dr}\left\{D\left[1-e^{-\beta(r-r_0)}\right]^2\right\}$$
$$= -2\beta D\left[1-e^{-\beta(r-r_0)}\right]$$

Set this derivative equal to zero for extrema:

$$-2\beta D\left[1-e^{-\beta(r-r_0)}\right] = 0$$

Solve for r to obtain:

$$r = \boxed{r_0}$$

Evaluate the second derivative of $U(r)$ to obtain:

$$\frac{d^2U}{dr^2} = \frac{d}{dr}\left\{-2\beta D\left[1 - e^{-\beta(r-r_0)}\right]\right\}$$
$$= 2\beta^2 De^{-\beta(r-r_0)}$$

Evaluate this derivative at $r = r_0$:

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 2\beta^2 D \qquad (1)$$

Recall that the potential function for a simple harmonic oscillator is:

$$U = \tfrac{1}{2}kx^2$$

Differentiate this expression twice to obtain:

$$\frac{d^2U}{dx^2} = k$$

By comparison with equation (1) we have:

$$k = \boxed{2\beta^2 D}$$

(*c*) Express the oscillation frequency of the diatomic molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass of the molecule.

Express the reduced mass of the homonuclear diatomic molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$$

Substitute and simplify to obtain:

$$\omega = \sqrt{\frac{2\beta^2 D}{\frac{m}{2}}} = \boxed{2\beta\sqrt{\frac{D}{m}}}$$

Remarks: An alternative approach in (*b*) is the expand the Morse potential in a Taylor series

$$U(r) = U(r_0) + (r - r_0)U'(r_0) + \frac{1}{2!}(r - r_0)^2 U''(r_0) + \text{higher order terms}$$

to obtain $U(r) \approx \beta^2 D(r - r_0)^2$. Comparing this expression to the energy of a spring-and-mass oscillator we see that, as was obtained above, $k = 2\beta^2 D$.