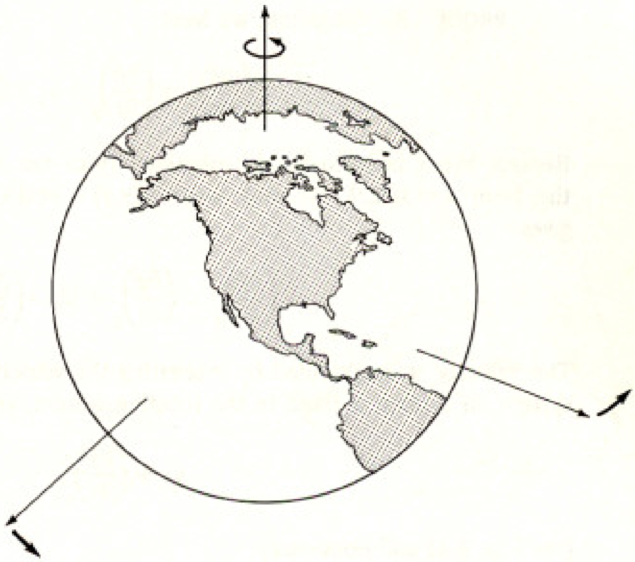

FIGURE 7.1
A cartesian system of axes fixed to the earth and rotating with it.

orthogonal cartesian system whose origin is at the center of the earth, but which rotates with the earth. This rotation is brought into the calculations with an angular velocity vector $\mathbf{\Omega}$, which is parallel to the earth's axis and has magnitude given by the angular speed of the earth. This vector is taken to point from the center of the earth toward the North Pole, and so the counterclockwise rotation seen looking down on the North Pole is considered positive (Fig. 7.1).

7.1.1 Differential Relations

We shall denote the fixed inertial coordinates by $\hat{\mathbf{x}}$ and those rotating with the earth by $\mathbf{x}$. Our calculations are based upon the observation that the velocity $\mathbf{v}_a$ of the air as seen in the fixed or absolute system will be the sum of the velocity $\mathbf{v}_r$, due to the rotation of the earth, and the velocity $\mathbf{v}$, relative to the earth. Thus

$$\mathbf{v}_a = \mathbf{v} + \mathbf{v}_r \tag{1}$$

In Chap. 2 we defined the total or material derivative of a scalar quantity as the derivative obtained following the fluid motion, and thus it should not depend on motions of the coordinate system.

Theorem For a scalar ϕ, the material derivative $(d\phi/dt)_a$ in the absolute system is identical to the material derivative $d\phi/dt$ in the rotating system.

PROOF By definition, we have

$$\left(\frac{d\phi}{dt}\right)_a = \left(\frac{\partial\phi}{\partial t}\right)_{\hat{\mathbf{x}}} + \mathbf{v}_a \cdot \hat{\nabla}\phi \tag{2}$$

Because the $\mathbf{x}$ coordinates are obtained from the $\hat{\mathbf{x}}$ coordinates by a relation of the form $\mathbf{x} = \mathbf{x}(\hat{\mathbf{x}},t)$, we may write $\phi(\mathbf{x},t) = \phi(\mathbf{x}(\hat{\mathbf{x}},t),t)$ and thus the chain rule gives

$$\left(\frac{\partial\phi}{\partial t}\right)_{\hat{\mathbf{x}}} = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{x}} + \nabla\phi \cdot \left(\frac{\partial\mathbf{x}}{\partial t}\right)_{\hat{\mathbf{x}}} \tag{3}$$

The velocity $\mathbf{v}_r$ is obtained by measuring the velocity, with respect to the inertial system, of a point $\mathbf{x}$ fixed in the rotating system, and so

$$\mathbf{v}_r = \left(\frac{\partial\hat{\mathbf{x}}}{\partial t}\right)_{\mathbf{x}} \tag{4}$$

(see Fig. 7.2) and conversely

$$\left(\frac{\partial\mathbf{x}}{\partial t}\right)_{\hat{\mathbf{x}}} = -\mathbf{v}_r \tag{5}$$

When Eqs. (3) and (5) are substituted into Eq. (2), we find

$$\left(\frac{d\phi}{dt}\right)_a = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{x}} - \mathbf{v}_r \cdot \nabla\phi + \mathbf{v}_a \cdot \hat{\nabla}\phi \tag{6}$$

But the gradient vector is identical in both coordinate systems so that $\nabla\phi = \hat{\nabla}\phi$, and hence Eqs. (6) and (1) imply that

$$\left(\frac{d\phi}{dt}\right)_a = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{x}} + \mathbf{v} \cdot \nabla\phi = \frac{d\phi}{dt} \tag{7}$$

completing the proof. ////

Now let us turn to the material derivative of vectors. A unit vector $\mathbf{i}$ in the rotating system will obviously appear to be changing when viewed from the inertial system. As can be seen from Fig. 7.2, a point fixed in the rotating coordinate system with position vector $\mathbf{r}$ from the center of the earth has velocity $\mathbf{\Omega} \times \mathbf{r}$. Note here that the direction of the cross product is obviously correct; the tip of the vector $\mathbf{r}$ travels a distance $2\pi r \sin\theta$ in time $2\pi/\Omega$ so that its speed is $\Omega r \sin\theta$, which is the same as the magnitude of $\mathbf{\Omega} \times \mathbf{r}$. Hence the velocity of the tip of an arbitrary unit vector $\mathbf{i}$ with tail at $\mathbf{r}$ is $\mathbf{\Omega} \times (\mathbf{r} + \mathbf{i})$ and the velocity of the tail is $\mathbf{\Omega} \times \mathbf{r}$. But

$$\mathbf{i} = \mathbf{r}_{\text{tip}} - \mathbf{r}_{\text{tail}} \tag{8}$$

so that