CHAPTER

23

Vector Analysis

23.1 Scalars and Vectors

23.1.1 Basic definitions

23.1.1.1 A **scalar** quantity is completely defined by a single real number (positive or negative) that measures its *magnitude.* Examples of scalars are length, mass, temperature, and electric potential. In print, scalars are represented by Roman or Greek letters like r, m, T, and ϕ.

A **vector** quantity is defined by giving its *magnitude* (a nonnegative scalar), and *its line of action* (a line in space) together with its **sense** (direction) along the line. Examples of vectors are velocity, acceleration, angular velocity, and electric field. In print, vector quantities are represented by Roman and Greek boldface letters like $\boldsymbol{\nu}$, $\mathbf{a}$, $\boldsymbol{\Omega}$, and $\mathbf{E}$. By convention, the magnitudes of vectors $\boldsymbol{\nu}$, $\mathbf{a}$, and $\boldsymbol{\Omega}$ are usually represented by the corresponding ordinary letters ν, a, and Ω, etc. The magnitude of a vector $\mathbf{r}$ is also denoted by $|\mathbf{r}|$, so that

1. $r = |\mathbf{r}|$.

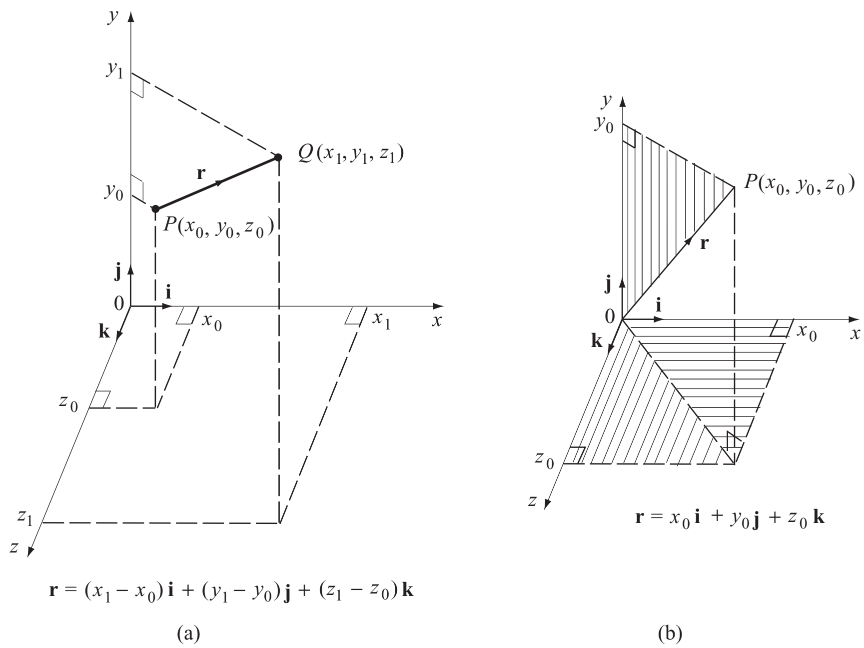

FIGURE 23.1

A vector of unit magnitude in the direction of $\boldsymbol{r}$, called a **unit vector**, is denoted by $\mathbf{e}_r$, so that

2. $\mathbf{r} = r\mathbf{e}_r$.

The **null vector (zero vector) 0** is a vector with zero magnitude and no direction.

A geometrical interpretation of a vector is obtained by using a straight-line segment parallel to the line of action of the vector, whose length is equal (or proportional) to the magnitude of the vector, with the sense of the vector being indicated by an arrow along the line segment. The end of the line segment from which the arrow is directed is called the **initial point** of the vector, while the other end (toward which the arrow is directed) is called the **terminal point** of the vector.

A **right-handed system** of rectangular cartesian coordinate axes $0\{x, y, z\}$ is one in which the positive direction along the z-axis is determined by the direction in which a right-handed screw advances when rotated from the x- to the y-axis. In such a system the signed lengths of the projections of a vector **r** with initial point $P(x_0, y_0, z_0)$ and terminal point $Q(x_1, y_1, z_1)$ onto the x-, y-, and z-axes are called the x, y, and z **components** of the vector. Thus the x, y, and z components of $\boldsymbol{r}$ directed from P to Q are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$, respectively (Figure 23.1(a)). A vector directed from the origin 0 to the point $P(x_0, y_0, z_0)$ has x_0, y_0, and z_0 as its respective x, y, and z components [Figure 23.1(b)]. Special unit vectors directed along the x-, y-, and z-axes are denoted by **i**, **j**, and **k**, respectively.

The cosines of the angles α, β, and γ between **r** and the respective x-, y-, and z-axes shown in Figure 23.2 are called the **direction cosines** of the vector **r**. If the

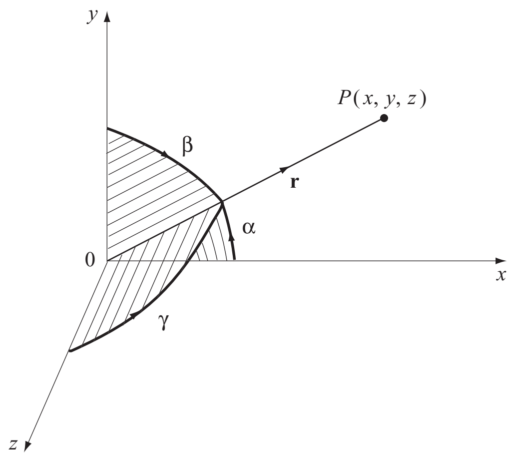

FIGURE 23.2

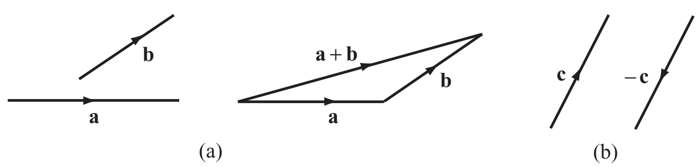

FIGURE 23.3

components of **r** are x, y, and z, then the respective direction cosines of **r**, denoted by l, m, and n, are

3. $l = \dfrac{x}{r}, \quad m = \dfrac{y}{r}, \quad n = \dfrac{z}{r} \quad \text{with } r = (x^2 + y^2 + z^2)^{1/2}.$

The direction cosines are related by

4. $l^2 + m^2 + n^2 = 1.$

Numbers u, v, and w proportional to l, m, and n, respectively, are called **direction ratios**.

23.1.2 Vector addition and subtraction

23.1.2.1 **Vector addition** of vectors **a** and **b**, denoted by $\mathbf{a} + \mathbf{b}$, is performed by first translating vector **b**, without rotation, so that its initial point coincides with the terminal point of **a**. The vector sum $\mathbf{a} + \mathbf{b}$ is then defined as the vector whose initial point is the initial point of **a**, and whose terminal point is the new terminal point of **b** (the **triangle rule** for vector addition) (Figure 23.3(a)).

The **negative** of vector **c**, denoted by $-\mathbf{c}$, is obtained from **c** by reversing its sense, as in Fig. 23.3(b), and so

1. $c = |\mathbf{c}| = |-\mathbf{c}|.$

The **difference** $\mathbf{a} - \mathbf{b}$ of vectors **a** and **b** is defined as the vector sum $\mathbf{a} + (-\mathbf{b})$. This corresponds geometrically to translating vector $-\mathbf{b}$, without rotation, until its

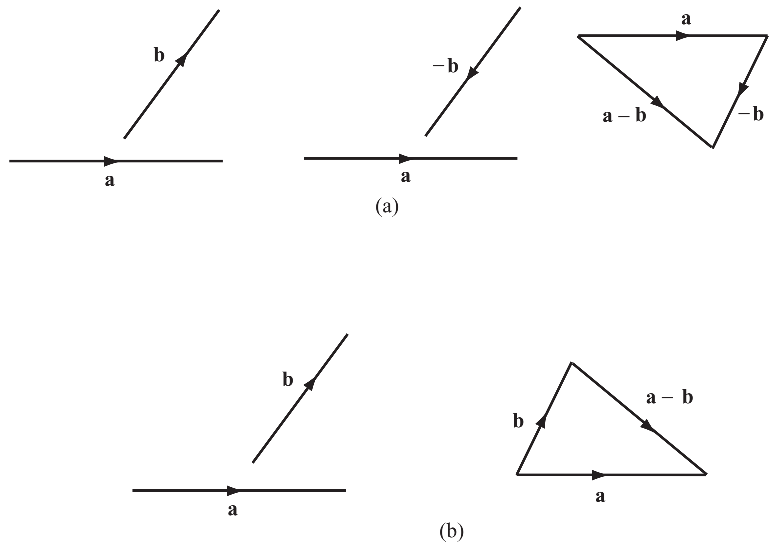

FIGURE 23.4

initial point coincides with the terminal point of **a**, when the vector **a** − **b** is the vector drawn from the initial point of **a** to the new terminal point of −**b** (Figure 23.4(a)). Equivalently, **a** − **b** is obtained by bringing into coincidence the initial points of **a** and **b** and defining **a** − **b** as the vector drawn from the terminal point of **b** to the terminal point of **a** [Figure 23.4(b)].

Vector addition obeys the following algebraic rules:

2. $\mathbf{a} + (-\mathbf{a}) = \mathbf{a} - \mathbf{a} = \mathbf{0}$
3. $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{a} + \mathbf{c} + \mathbf{b} = \mathbf{b} + \mathbf{c} + \mathbf{a}$ (commutative law)
4. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associative law)

The geometrical interpretations of laws 3 and 4 are illustrated in Figures 23.5(a) and 23.5(b).

23.1.3 Scaling vectors

23.1.3.1 A vector **a** may be **scaled** by the scalar λ to obtain the new vector $\mathbf{b} = \lambda\mathbf{a}$. The magnitude $b = |\mathbf{b}| = |\lambda\mathbf{a}| = |\lambda|a$. The sense of **b** is the same as that of **a** if $\lambda > 0$, but it is *reversed* if $\lambda < 0$. The scaling operation performed on vectors obeys the laws:

1. $\lambda\mathbf{a} = \mathbf{a}\lambda$ (commutative law)
2. $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ (distributive law)
3. $\lambda(\mu\mathbf{a}) = \mu(\lambda\mathbf{a}) = (\lambda\mu)\mathbf{a}$ (associative law)
4. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$ (distributive law)

where λ, μ are scalars and **a, b** are vectors.

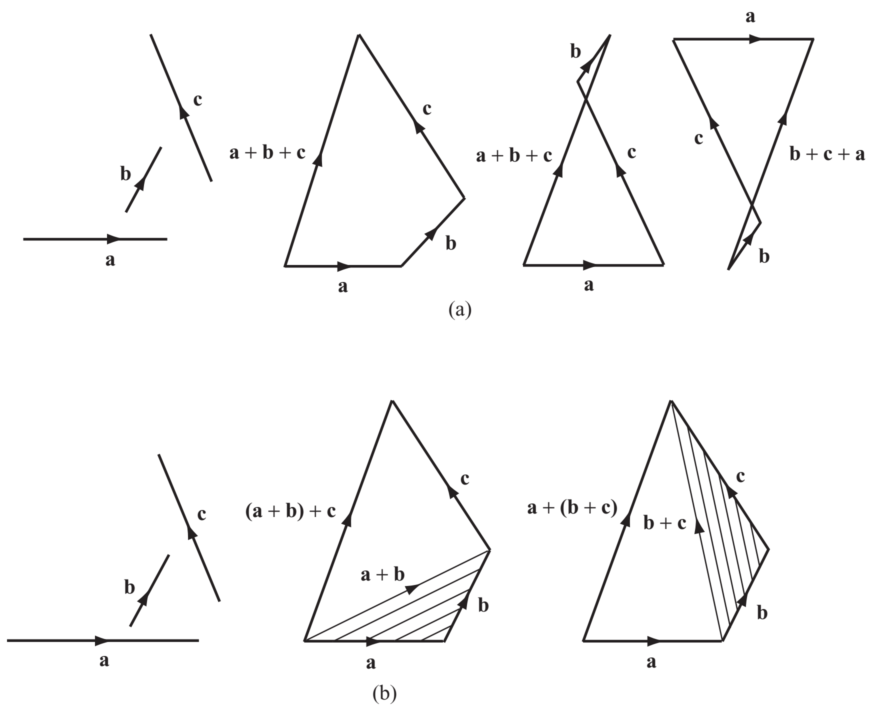

FIGURE 23.5

23.1.4 Vectors in component form

23.1.4.1 If **a**, **b**, and **c** are any three noncoplanar vectors, an arbitrary vector **r** may always be written in the form

1. $\mathbf{r} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}$,

where the scalars λ_1, λ_2, and λ_3 are the components of **r** in the **triad** of reference vectors **a**, **b**, **c**. In the important special case of rectangular Cartesian coordinates $0\{x, y, z\}$, with unit vectors **i**, **j**, and **k** along the x-, y-, and z-axes, respectively, the vector **r** drawn from point $P(x_0, y_0, z_0)$ to point $Q(x_1, y_1, z_1)$ can be written (Figure 23.1(a))

2. $\mathbf{r} = (x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}$.

Similarly, the vector drawn from the origin to the point $P(x_0, y_0, z_0)$ becomes (Figure 23.1(b))

3. $\mathbf{r} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$.

For 23.1.4.1.2 the magnitude of **r** is

4. $r = |\mathbf{r}| = [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{1/2}$,

whereas for 23.1.4.1.3 the magnitude of **r** is

$$r = |\mathbf{r}| = \left(x_0^2 + y_0^2 + z_0^2\right)^{1/2}.$$

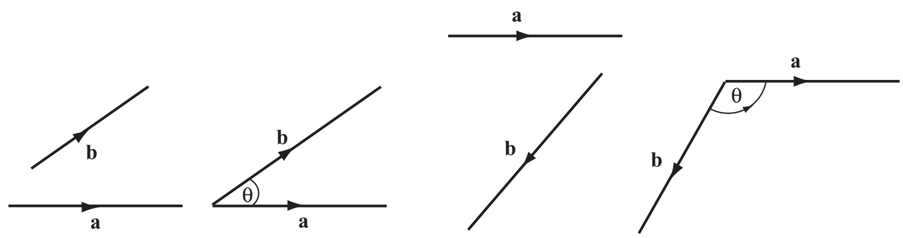

FIGURE 23.6

In terms of the direction cosines l, m, n (see 23.1.1.1.3) the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ becomes

$$\mathbf{r} = \mathbf{r}(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}),$$

where $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ is the unit vector in the direction of $\mathbf{r}$.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and λ and μ are scalars, then

5. $\lambda\mathbf{a} = \lambda a_1\mathbf{i} + \lambda a_2\mathbf{j} + \lambda a_3\mathbf{k}$,
6. $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$,
7. $\lambda\mathbf{a} + \mu\mathbf{b} = (\lambda a_1 + \mu b_1)\mathbf{i} + (\lambda a_2 + \mu b_2)\mathbf{j} + (\lambda a_3 + \mu b_3)\mathbf{k}$,

which are equivalent to the results in 23.1.3.

23.2 Scalar Products

23.2.1

The **scalar product (dot product** or **inner product)** of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ inclined at an angle θ to one another and written $\mathbf{a} \cdot \mathbf{b}$ is defined as the scalar (Figure 23.6)

1. $$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\ &= ab\cos\theta \\ &= a_1b_1 + a_2b_2 + a_3b_3.\end{aligned}$$

If required, the angle between $\mathbf{a}$ and $\mathbf{b}$ may be obtained from

2. $$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}\left(b_1^2 + b_2^2 + b_3^2\right)^{1/2}}.$$

Properties of the scalar product. If $\mathbf{a}$ and $\mathbf{b}$ are vectors and λ and μ are scalars, then:

3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative property)
4. $(\lambda\mathbf{a}) \cdot (\mu\mathbf{b}) = \lambda\mu\mathbf{a} \cdot \mathbf{b}$ (associative property)
5. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive property)

Special cases.

6. $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a}, \mathbf{b}$ are orthogonal ($\theta = \pi/2$)
7. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| = ab$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel ($\theta = 0$)
8. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.

23.3 Vector Products

23.3.1

The **vector product (cross product)** of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ inclined at an angle θ to one another and written $\mathbf{a} \times \mathbf{b}$ is defined as the vector

1. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\mathbf{n} = ab\ \sin\theta\,\mathbf{n}$,

where $\mathbf{n}$ is a unit vector normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$ directed in the sense in which a right-handed screw would advance if rotated from $\mathbf{a}$ to $\mathbf{b}$ (Figure 23.7).

An alternative and more convenient definition of $\mathbf{a} \times \mathbf{b}$ is

2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

If required, the angle θ between $\mathbf{a}$ and $\mathbf{b}$ follows from

3. $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{ab}$,

though the result 23.2.1.2 is usually easier to use.

Properties of the vector product. If $\mathbf{a}$ and $\mathbf{b}$ are vectors and λ and μ are scalars, then

4. $\mathbf{a} \times \mathbf{b} = -\mathbf{a} \times \mathbf{b}$ (noncommutative)
5. $(\lambda\mathbf{a}) \times (\mu\mathbf{b}) = \lambda\mu\mathbf{a} \times \mathbf{b}$ (associative property)
6. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive property)

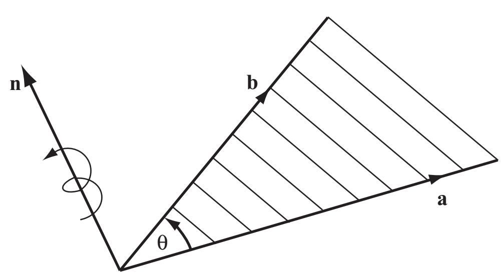

FIGURE 23.7

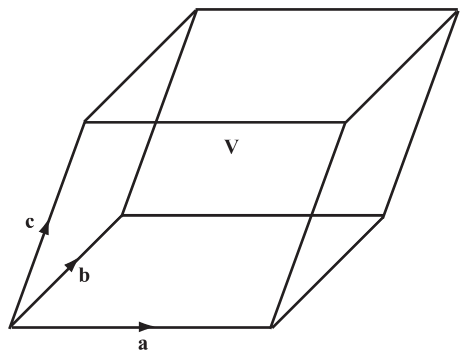

FIGURE 23.8

Special cases.

7. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if **a** and **b** are parallel ($\theta = 0$)
8. $\mathbf{a} \times \mathbf{b} = ab\mathbf{n}$ if **a** and **b** are orthogonal ($\theta = \pi/2$)
9. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$
10. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

23.4 Triple Products

23.4.1

The **scalar triple product** of the three vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, written $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, is the scalar

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

In terms of components

2. $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The alternative notation [**abc**] is also used for the scalar triple product in place of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

In geometrical terms the absolute value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ may be interpreted as the volume V of a parallelepiped in which **a**, **b**, and **c** form three adjacent edges meeting at a corner (Figure 23.8). This interpretation provides a useful test for the linear independence of any three vectors. The vectors **a**, **b**, and **c** are *linearly dependent* if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, because $V = 0$ implies that the vectors are coplanar, and so $\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}$ for some scalars λ and μ; whereas they are *linearly independent* if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$.

The **vector triple product** of the three vectors **a**, **b**, and **c**, denoted by $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, is given by

3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

The parentheses are essential in a vector triple product to avoid ambiguity, because $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

23.5 Products of Four Vectors

23.5.1

Two other products arise that involve the four vectors **a**, **b**, **c**, and **d**. The first is the *scalar product*

1. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$,

and the second is the *vector product*

2. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})\mathbf{c} - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\mathbf{d}$.

23.6 Derivatives of Vector Functions of a Scalar *t*

23.6.1

Let $x(t)$, $y(t)$, and $z(t)$ be continuous functions of t that are differentiable as many times as necessary, and let **i**, **j**, and **k** be the triad of fixed unit vectors introduced in 23.1.4. Then the vector $\mathbf{r}(t)$ given by

1. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

is a vector function of the scalar variable t that has the same continuity and differentiability properties as its components. The first- and second-order derivatives of $\mathbf{r}(t)$ with respect to t are

2. $\dfrac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}$

and

3. $\dfrac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \dfrac{d^2x}{dt^2}\mathbf{i} + \dfrac{d^2y}{dt^2}\mathbf{j} + \dfrac{d^2z}{dt^2}\mathbf{k}$.

Higher order derivatives are defined in similar fashion so that, in general,

4. $\dfrac{d^n\mathbf{r}}{dt^n} = \dfrac{d^nx}{dt^n}\mathbf{i} + \dfrac{d^ny}{dt^n}\mathbf{j} + \dfrac{d^nz}{dt^n}\mathbf{k}$.

If **r** is the **position vector** of a point in space of time t, then $\dot{\mathbf{r}}$ is its *velocity* and $\ddot{\mathbf{r}}$ is its *acceleration* (Figure 23.9).

Differentiation of combinations of vector functions of a scalar t. Let **u** and **v** be continuous functions of the scalar variable t that are differentiable as many times as necessary, and let $\phi(t)$ be a scalar function of t with the same continuity and differentiability properties as the components of the vector functions. Then the following differentiability results hold:

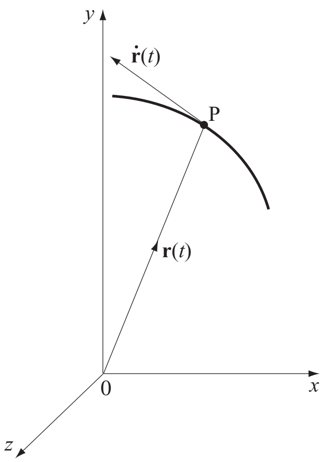

FIGURE 23.9

1. $\dfrac{d}{dt}(\mathbf{u}+\mathbf{v}) = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}$

2. $\dfrac{d}{dt}(\phi\mathbf{u}) = \dfrac{d\phi}{dt}\mathbf{u} + \phi\dfrac{d\mathbf{u}}{dt}$

3. $\dfrac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}$

4. $\dfrac{d}{dt}(\phi\mathbf{u}\cdot\mathbf{v}) = \dfrac{d\phi}{dt}\mathbf{u}\cdot\mathbf{v} + \phi\dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \phi\mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}$

5. $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\dfrac{d\mathbf{v}}{dt}$

6. $\dfrac{d}{dt}(\phi\mathbf{u}\times\mathbf{v}) = \dfrac{d\phi}{dt}\mathbf{u}\times\mathbf{v} + \phi\dfrac{d\mathbf{u}}{dt}\times\mathbf{v} + \phi\mathbf{u}\times\dfrac{d\mathbf{v}}{dt}$

23.7 Derivatives of Vector Functions of Several Scalar Variables

23.7.1

Let $u_i(x, y, z)$ and $v_i(x, y, z)$ for $i = 1, 2, 3$ be continuous functions of the scalar variables x, y, and z, and let them have as many partial derivatives as necessary. Define

1. $\mathbf{u}(x, y, z) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
2. $\mathbf{v}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,

where **i**, **j**, and **k** are the triad of fixed unit vectors introduced in 23.1.4. Then the following differentiability results hold:

3. $\dfrac{\partial \mathbf{u}}{\partial x} = \dfrac{\partial u_1}{\partial x}\mathbf{i} + \dfrac{\partial u_2}{\partial x}\mathbf{j} + \dfrac{\partial u_3}{\partial x}\mathbf{k}$

4. $\dfrac{\partial \mathbf{u}}{\partial y} = \dfrac{\partial u_1}{\partial y}\mathbf{i} + \dfrac{\partial u_2}{\partial y}\mathbf{j} + \dfrac{\partial u_3}{\partial y}\mathbf{k}$

5. $\dfrac{\partial \mathbf{u}}{\partial z} = \dfrac{\partial u_1}{\partial z}\mathbf{i} + \dfrac{\partial u_2}{\partial z}\mathbf{j} + \dfrac{\partial u_3}{\partial z}\mathbf{k}$,

with corresponding results of $\partial\mathbf{v}/\partial x$, $\partial\mathbf{v}/\partial y$, and $\partial\mathbf{v}/\partial z$.

Second-order and higher derivatives of **u** and **v** are defined in the obvious manner:

6. $\dfrac{\partial^2 \mathbf{u}}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{u}}{\partial x}\right), \quad \dfrac{\partial^2 \mathbf{u}}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{u}}{\partial y}\right), \quad \dfrac{\partial^2 \mathbf{u}}{\partial x \partial z} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{u}}{\partial z}\right), \ldots$

7. $\dfrac{\partial^3 \mathbf{u}}{\partial x^3} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 \mathbf{u}}{\partial x^2}\right), \quad \dfrac{\partial^3 \mathbf{u}}{\partial x^2 \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 \mathbf{u}}{\partial x \partial y}\right), \quad \dfrac{\partial^3 \mathbf{u}}{\partial x \partial z^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 \mathbf{u}}{\partial z^2}\right), \ldots.$

8. $\dfrac{\partial}{\partial x}(\mathbf{u}\cdot\mathbf{v}) = \dfrac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{v} + \mathbf{u}\cdot\dfrac{\partial \mathbf{v}}{\partial x}$

9. $\dfrac{\partial}{\partial x}(\mathbf{u}\times\mathbf{v}) = \dfrac{\partial \mathbf{u}}{\partial x}\times\mathbf{v} + \mathbf{u}\times\dfrac{\partial \mathbf{v}}{\partial x}$,

with corresponding results for derivatives with respect to y and z and for higher order derivatives.

10. $d\mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial x}\,dx + \dfrac{\partial \mathbf{u}}{\partial y}\,dy + \dfrac{\partial \mathbf{u}}{\partial z}\,dz$ (total differential)

and if $x = x(t)$, $y = y(t)$, $z = z(t)$,

11. $d\mathbf{u} = \left(\dfrac{\partial \mathbf{u}}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial \mathbf{u}}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial \mathbf{u}}{\partial z}\dfrac{dz}{dt}\right)dt$ (chain rule)

23.8 Integrals of Vector Functions of a Scalar Variable *t*

23.8.1

Let the vector function $f(t)$ of the scalar variable t be

1. $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$,

where f_1, f_2, and f_3 are scalar functions of t for which a function $\mathbf{F}(t)$ exists such that

2. $\mathbf{f}(t) = \dfrac{d\mathbf{F}}{dt}$.

Then

3. $\displaystyle\int \mathbf{f}(t)\,dt = \int \frac{d\mathbf{F}}{dt}\,dt = \mathbf{F}(t) + \mathbf{c},$

where **c** is an arbitrary vector constant. The function $\mathbf{F}(t)$ is called an **antiderivative** of $\mathbf{f}(t)$, and result 23.8.1.3 is called an **indefinite integral** of $\mathbf{f}(t)$. Expressed differently, 23.8.1.3 becomes

4. $\displaystyle\mathbf{F}(t) = \mathbf{i}\int f_1(t)\,dt + \mathbf{j}\int f_2(t)\,dt = \mathbf{k}\int f_3(t)\,dt + \mathbf{c}.$

The **definite integral** of $f(t)$ between the scalar limits $t = t_1$ and $t = t_2$ is

5. $\displaystyle\int_{t_1}^{t_2} \mathbf{f}(t)\,dt = \mathbf{F}(t_2) - \mathbf{F}(t_1).$

Properties of the definite integral. If λ is a scalar constant, t_3 is such that $t_1 < t_3 < t_2$, and $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector functions of the scalar variable t, then

1. $\displaystyle\int_{t_1}^{t_2} \lambda\mathbf{u}(t)\,dt = \lambda\int_{t_1}^{t_2} \mathbf{u}(t)\,dt$ (homogeneity)

2. $\displaystyle\int_{t_1}^{t_2} [\mathbf{u}(t) + \mathbf{v}(t)]\,dt = \int_{t_1}^{t_2} \mathbf{u}(t)\,dt + \int_{t_1}^{t_2} \mathbf{v}(t)\,dt$ (linearity)

3. $\displaystyle\int_{t_1}^{t_2} \mathbf{u}(t)\,dt = -\int_{t_2}^{t_1} \mathbf{u}(t)\,dt$ (interchange of limits)

4. $\displaystyle\int_{t_1}^{t_2} \mathbf{u}(t)\,dt = \int_{t_1}^{t_3} \mathbf{u}(t)\,dt + \int_{t_3}^{t_2} \mathbf{u}(t)\,dt$

(integration over contiguous intervals)

23.9 Line Integrals

23.9.1

Let **F** be a continuous and differentiable vector function of position $P(x, y, z)$ in space, and let C be a **path (arc)** joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then the **line integral** of **F** taken along the path C from P_1 to P_2 is defined as (Figure 23.10)

1. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz),$

where

2. $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},$

and

3. $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

is a differential vector displacement along the path C. It follows that

4. $\displaystyle\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = -\int_{P_2}^{P_1} \mathbf{F}\cdot dr,$

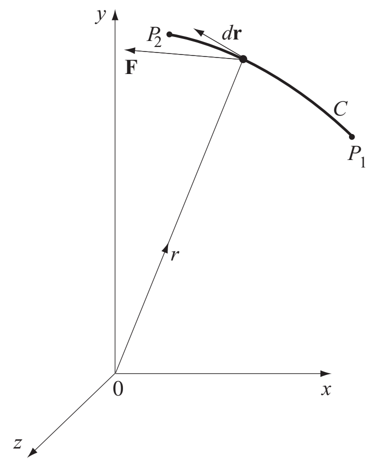

FIGURE 23.10

while for three points P_1, P_2, and P_3 on C,

5. $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3}^{P_2} \mathbf{F} \cdot d\mathbf{r}.$

A special case of a line integral occurs when **F** is given by

6. $\mathbf{F} = \text{grad}\,\phi = \nabla\phi,$

where in rectangular Cartesian coordinates

7. $\text{grad}\,\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z},$

for

8. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \phi(P_2) - \phi(P_1),$

and the line integral is *independent* of the path C, depending only on the initial point P_1 and terminal point P_2 of C. A **vector field** of the form

9. $\mathbf{F} = \text{grad}\,\phi$

is called a **conservative field**, and ϕ is then called a **scalar potential**. For the definition of grad ϕ in terms of other coordinate systems see 24.2.1 and 24.3.1.

In a conservative field, if C is a *closed curve*, it then follows that

10. $\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$

where the symbol $\oint$ indicates that the **curve** (contour) C is closed.

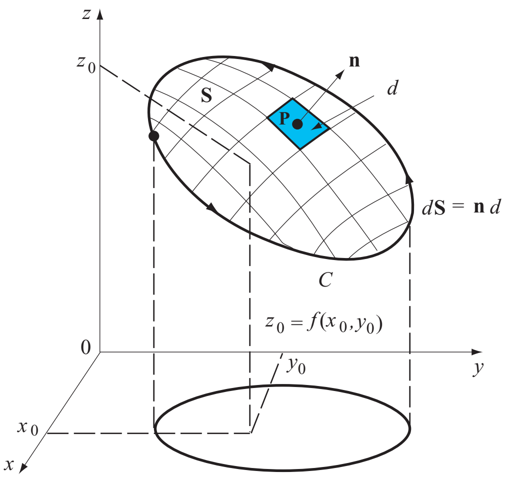

FIGURE 23.11

23.10 Vector Integral Theorems

23.10.1

Let a surface S defined by $z = f(x, y)$ that is bounded by a closed space curve C have an element of surface area $d\sigma$, and let $\mathbf{n}$ be a unit vector normal to S at a representative point P (Figure 23.11).

Then the **vector element of surface area** $d\mathbf{S}$ of surface S is defined as

1. $d\mathbf{S} = d\sigma\,\mathbf{n}$.

The **surface integral** of a vector function $\mathbf{F}(x, y, z)$ over the surface S is defined as

2. $\displaystyle\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$

The **Gauss divergence theorem** states that if S is a closed surface containing a volume V with volume element dV, and if the vector element of surface area $d\mathbf{S} = \mathbf{n}\,d\sigma$, where $\mathbf{n}$ is the unit normal directed out of V and $d\sigma$ is an element of surface area of S, then

3. $\displaystyle\int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$

The Gauss divergence theorem relates the volume integral of div $\mathbf{F}$ to the surface integral of the normal component of $\mathbf{F}$ over the closed surface S.

In terms of the rectangular Cartesian coordinates $0\{x, y, z\}$, the **divergence** of the vector $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, written div $\mathbf{F}$, is defined as

4. $\displaystyle\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$

For the definitions of div $\mathbf{F}$ in terms of other coordinate systems see 24.2.1 and 24.3.1.

Stokes's theorem states that if C is a closed curve spanned by an open surface S, and $\mathbf{F}$ is a vector function defined on S, then

5. $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma.$$

In this theorem the direction of unit normal $\mathbf{n}$ in the vector element of surface area $d\mathbf{S} = d\sigma\,\mathbf{n}$ is chosen such that it points in the direction in which a right-handed screw would advance when rotated in the sense in which the closed curve C is traversed. A surface for which the normal is defined in this manner is called an **oriented surface**.

The surface S shown in Figure 23.11 is oriented in this manner when C is traversed in the direction shown by the arrows. In terms of rectangular Cartesian coordinates $0\{x, y, z\}$, the **curl** of the vector $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, written either $\nabla\times\mathbf{F}$, or curl $\mathbf{F}$, is defined as

6. $$\nabla\times\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{F}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

For the definition of $\nabla\times\mathbf{F}$ in terms of other coordinate systems see 24.2.1 and 24.3.1.

Green's first and second theorems (identities). Let U and V be scalar functions of position defined in a volume V contained within a simple closed surface S, with an outward-drawn vector element of surface area $d\mathbf{S}$. Suppose further that the **Laplacians** $\nabla^2 U$ and $\nabla^2 V$ are defined throughout V, except on a finite number of surfaces inside V, across which the second-order partial derivatives of U and V are bounded but discontinuous. **Green's first theorem** states that

7. $$\int (U\nabla V)\cdot d\mathbf{S} = \int_V [U\nabla^2 V + (\nabla U)\cdot(\nabla V)]\,dV,$$

where in rectangular Cartesian coordinates

8. $$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}.$$

The Laplacian operator ∇^2 is also often denoted by Δ, or by Δ_n if it is necessary to specify the number n of space dimensions involved, so that in Cartesian coordinates $\Delta_2 U = \partial^2 U/\partial x^2 + \partial^2 U/\partial y^2$.

For the definition of the Laplacian in terms of other coordinate systems see 24.2.1 and 24.3.1.

Green's second theorem states that

9. $$\int_V (U\nabla^2 V - V\nabla^2 U)\,dV = \int_S (U\nabla V - V\nabla U)\cdot d\mathbf{S}.$$

In two dimensions $0\{x, y\}$, **Green's theorem in the plane** takes the form

10. $$\oint_C (P\,dx + Q\,dy) = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy,$$

where the scalar functions $P(x, y)$ and $Q(x, y)$ are defined and differentiable in some plane area A bounded by a simple closed curve C except, possibly, across an arc γ in A joining two distinct points of C, and the integration is performed counterclockwise around C.

23.11 A Vector Rate of Change Theorem

Let u be a continuous and differentiable scalar function of position and time defined throughout a moving volume $V(t)$ bounded by a surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of $\mathbf{u}$ is given by

1. $$\frac{d}{dt}\int_{V(t)} u\, dV = \int_{V(t)} \left\{\frac{\partial u}{\partial t} + \operatorname{div}(u\mathbf{v})\right\} dV.$$

23.12 Useful Vector Identities and Results

23.12.1

In each identity that follows the result is expressed first in terms of grad, div, and curl, and then in operator notation. $\mathbf{F}$ and $\mathbf{G}$ are suitably differentiable vector functions and V and W are suitably differentiable scalar functions.

1. $\operatorname{div}(\operatorname{curl}\mathbf{F}) \equiv \nabla\cdot(\nabla\times F) \equiv \mathbf{0}$
2. $\operatorname{curl}(\operatorname{grad} V) \equiv \nabla\times(\nabla V) \equiv \mathbf{0}$
3. $\operatorname{grad}(VW) \equiv V\operatorname{grad} W + W\operatorname{grad} V \equiv V\nabla W + W\nabla V$
4. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) \equiv \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F} \equiv \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$
5. $\operatorname{div}(\operatorname{grad} V) \equiv \nabla\cdot(\nabla V) \equiv \nabla^2 V$
6. $\operatorname{div}(V\mathbf{F}) \equiv V\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\operatorname{grad} V \equiv \nabla\cdot(V\mathbf{F}) \equiv V\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla V$
7. $\operatorname{curl}(V\mathbf{F}) \equiv V\operatorname{curl}\mathbf{F} - \mathbf{F}\times\operatorname{grad} V \equiv V\nabla\times\mathbf{F} - \mathbf{F}\times\nabla V$
8. $$\operatorname{grad}(\mathbf{F}\cdot\mathbf{G}) \equiv \mathbf{F}\times\operatorname{curl}\mathbf{G} + \mathbf{G}\times\operatorname{curl}\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$$
   $$\equiv \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$$
9. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) \equiv \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G} \equiv \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$
10. $$\operatorname{curl}(\mathbf{F}\times\mathbf{G}) \equiv \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$
    $$\equiv \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$
11. $\mathbf{F}\cdot\operatorname{grad} V = \mathbf{F}\cdot(\nabla V) = (\mathbf{F}\cdot\nabla)V$ is proportional to the *directional derivative* of V in the direction $\mathbf{F}$ and it becomes the directional derivative of V in the direction $\mathbf{F}$ when $\mathbf{F}$ is a unit vector.
12. $\mathbf{F}\cdot\operatorname{grad}\mathbf{G} = (\mathbf{F}\cdot\nabla)\mathbf{G}$ is proportional to the directional derivative of $\mathbf{G}$ in the direction of $\mathbf{F}$ and it becomes the directional derivative of $\mathbf{G}$ in the direction of $\mathbf{F}$ when $\mathbf{F}$ is a unit vector.