PROBLEM 7.3.5 The system (31) involves seven equations in seven variables. Can you reduce it to a five-equation system in the variables $\mathbf{v}_H$, ω, ϕ_p, and θ?

PROBLEM 7.3.6 Derive a system of equations, governing hydrostatic flow, that utilizes Exner's function $\pi = c_p(p/p_{00})^{R/c_p}$ as the vertical coordinate. Here we shall generally take $p_{00} = 1000$ mbar.

7.3.2 Isentropic Coordinates

Another commonly employed meteorological coordinate system utilizes the potential temperature as the measure of vertical position. The surfaces of constant potential temperature are also surfaces on which the entropy is constant and thus they are called isentropic. Hence one of the advantages of isentropic coordinates is that parcels in isentropic motion remain on the coordinate surface. Another is that representation in isentropic coordinates provides maximum resolution in the areas of greatest interest such as baroclinic zones and frontal areas, as illustrated in Fig. 7.10.

Many of the relations we derived for isobaric coordinates are valid with only slight modification. In this case the essential requirement for the validity of the transformation is that $\partial\theta/\partial z > 0$; in other words, that the stratification is everywhere statically stable.

Thus we have $z = h_\theta(x,y,\theta,t)$ along with the differential relations for a scalar function ϕ

$$\left(\frac{\partial\phi}{\partial z}\right)_{x,y,t} = \frac{\partial\phi}{\partial\theta}\left(\frac{\partial\theta}{\partial z}\right)_{x,y,t} = \frac{\partial\phi}{\partial\theta}\left(\frac{\partial h_\theta}{\partial\theta}\right)^{-1} \tag{33}$$

$$\nabla_z\phi = \nabla_\theta\phi + \frac{\partial\phi}{\partial\theta}\nabla_z\theta \tag{34}$$

and

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t_\theta} + \mathbf{v}\cdot\nabla_\theta\phi + \frac{d\theta}{dt}\frac{\partial\phi}{\partial\theta} \tag{35}$$

Furthermore, we have again

$$\frac{\partial h_\theta}{\partial z} = 1 \qquad \nabla_z h_\theta = 0 \tag{36}$$

and thus

$$\nabla_\theta h_\theta = -\frac{\partial h_\theta}{\partial\theta}\nabla_z\theta \tag{37}$$

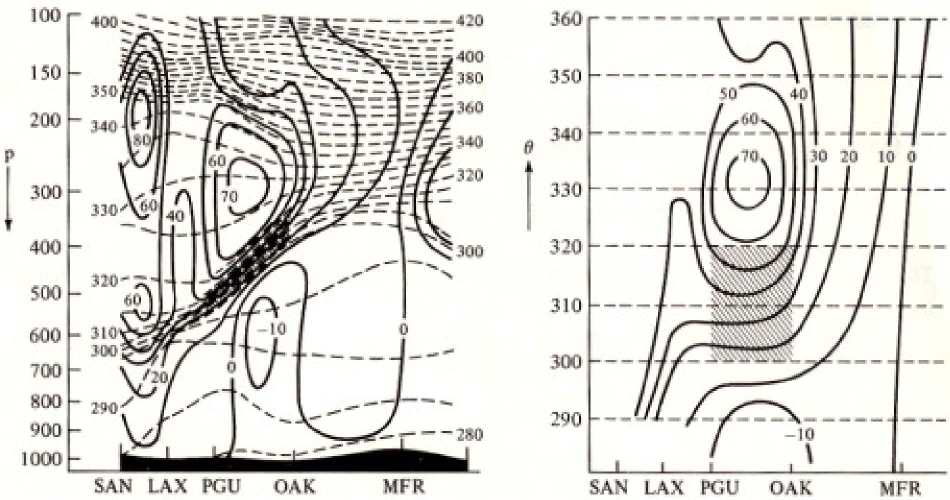

FIGURE 7.10
Comparison of resolution in isobaric and isentropic coordinates. The cross section on the left is the traditional form of presenting isotachs and isentropes with pressure as the vertical coordinate. An upper-level front or baroclinic zone is identified by the shaded area on the cross section. The figure on the right shows the same area as it would be represented in isentropic coordinates, making it clear that the dynamics of this structure could be analyzed or predicted more accurately in isentropic coordinates. *(Illustration provided by Dr. Rainer Bleck, National Center for Atmospheric Research. The figure on the left is from M. A. Shapiro and J. T. Hastings, 1973: "Objective Cross-Section Analysis by Hermite Polynomial Interpolation on Isentropic Surfaces," J. Appl. Meteorol.,* **12**:*753–762.)*

Use of this equation in Eq. (34) provides the result

$$\nabla_z \phi = \nabla_\theta \phi - \left(\frac{\partial h_\theta}{\partial \theta}\right)^{-1} \frac{\partial \phi}{\partial \theta} \nabla_\theta h_\theta = \nabla_\theta \phi - \frac{\partial \phi}{\partial z} \nabla_\theta h_\theta \tag{38}$$

In order to evaluate the forcing term in the equation of motion, we need to compute $\nabla_z p$, and with Eq. (38) we have

$$\nabla_z p = \nabla_\theta p - \frac{\partial p}{\partial z} \nabla_\theta h_\theta \tag{39}$$

and with Poisson's equation

$$\theta = T\left(\frac{p_{00}}{p}\right)^{R/c_p} \tag{40}$$

we find that

$$c_p \nabla_\theta T = \frac{T}{p} R \nabla_\theta p = \frac{1}{\rho} \nabla_\theta p \tag{41}$$

so that with the aid of the hydrostatic equation, Eq. (39) may be written

$$\frac{1}{\rho} \nabla_z p = c_p \nabla_\theta T + g \nabla_\theta h_\theta = \nabla_\theta (c_p T + g h_\theta) \tag{42}$$

This potential, known as the *Montgomery stream function*, is given a symbol of its own, $\Psi = c_p T + g h_\theta$.

When hydrostatic conditions prevail, we will want to express the fact in isentropic coordinates and not have to rely on the (x,y,z) representation. With Eq. (33) we find that in hydrostatic conditions

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} \frac{\partial h_\theta}{\partial \theta} = -g\rho \frac{\partial h_\theta}{\partial \theta} \tag{43}$$

and the logarithmic derivative of Eq. (40) with respect to θ gives

$$\frac{1}{\theta} = \frac{1}{T} \frac{\partial T}{\partial \theta} - \frac{R}{c_p p} \frac{\partial p}{\partial \theta} \tag{44}$$

With Eq. (43) this may be rearranged as

$$c_p \frac{\partial T}{\partial \theta} + g \frac{\partial h_\theta}{\partial \theta} = c_p \frac{T}{\theta} = c_p \left(\frac{p}{p_{00}} \right)^{R/c_p} \tag{45}$$

Thus we have derived a hydrostatic equation for isentropic coordinates in the form

$$\frac{\partial \Psi}{\partial \theta} = c_p \left(\frac{p}{p_{00}} \right)^{R/c_p} \tag{46}$$

The continuity equation in these coordinates follows from the basic principle of mass conservation, just as it did in isobaric coordinates. Thus for a material volume we have

$$M = \int_{V_\theta} \rho \frac{\partial h_\theta}{\partial \theta} \, dx \, dy \, d\theta \tag{47}$$

and so for a small volume

$$M = \rho \frac{\partial h_\theta}{\partial \theta} V_\theta \tag{48}$$

Hence $dM/dt = 0$ implies that [see Eq. (26)]

$$\frac{d}{dt}\left(\rho \frac{\partial h_\theta}{\partial \theta} \right) + \rho \frac{\partial h_\theta}{\partial \theta} \frac{1}{V_\theta} \frac{dV_\theta}{dt} = \frac{d}{dt}\left(\rho \frac{\partial h_\theta}{\partial \theta} \right) + \rho \frac{\partial h_\theta}{\partial \theta} (\nabla \cdot \mathbf{v})_\theta = 0 \tag{49}$$

Upon noting that the correct form of the divergence must be $(\nabla \cdot \mathbf{v})_\theta = \nabla_\theta \cdot \mathbf{v} + \partial\dot{\theta}/\partial\theta$, where $\dot{\theta} = d\theta/dt$, we have

$$\frac{\partial}{\partial t_\theta}\left(\rho\frac{\partial h_\theta}{\partial \theta}\right) + \nabla_\theta \cdot \left(\rho\frac{\partial h_\theta}{\partial \theta}\mathbf{v}\right) + \frac{\partial}{\partial \theta}\left(\rho\frac{\partial h_\theta}{\partial \theta}\frac{d\theta}{dt}\right) = 0 \tag{50}$$

For purely hydrostatic motion, Eq. (43) implies that this may be expressed as

$$\frac{\partial}{\partial t_\theta}\left(\frac{\partial p}{\partial \theta}\right) + \nabla_\theta \cdot \left(\frac{\partial p}{\partial \theta}\mathbf{v}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial \theta}\frac{d\theta}{dt}\right) = 0 \tag{51}$$

The first law of thermodynamics in isentropic coordinates is simply

$$\frac{d\theta}{dt} = \frac{\theta}{c_p T}(q + f_H) \tag{52}$$

In many cases in the study of large-scale flow the fact that motions tend to be isentropic allows Eq. (52) to be replaced by $\dot{\theta} = 0$.

The full set of equations for hydrostatic flow in isentropic coordinates is

$$\begin{aligned}
&\frac{\partial \mathbf{v}_H}{\partial t_\theta} + \mathbf{v}_H \cdot \nabla_\theta \mathbf{v}_H + \frac{d\theta}{dt}\frac{\partial \mathbf{v}_H}{\partial \theta} = -\nabla_\theta \Psi - f\mathbf{k} \times \mathbf{v}_H + (\mathbf{f}_{r,H})_\theta \\
&\frac{\partial \Psi}{\partial \theta} = c_p\left(\frac{p}{p_{00}}\right)^{R/c_p} \\
&\frac{\partial}{\partial t_\theta}\left(\frac{\partial p}{\partial \theta}\right) + \nabla_\theta \cdot \left(\frac{\partial p}{\partial \theta}\mathbf{v}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial \theta}\frac{d\theta}{dt}\right) = 0 \\
&\frac{d\theta}{dt} = \frac{\theta}{c_p T}(q + f_h)
\end{aligned} \tag{53}$$

Note that T and h_θ have been combined in Ψ and do not have to appear explicitly in order to complete the system. We can of course add the equations

$$\theta = T\left(\frac{p}{p_{00}}\right)^{-R/c_p} \qquad \Psi = c_p T + g h_\theta \tag{54}$$

to determine T and h_θ explicitly.

The equations in isentropic coordinates share the feature with those in isobaric coordinates that the forcing term is linear—in this case, linear in the Montgomery stream function Ψ. The price for this linearity, however, is the nonlinearity that appears in the hydrostatic equation with the R/c_p power of pressure. A major advantage of these equations is that the vertical velocity term appearing in the cartesian coordinate equations is replaced with a direct function of the heating, $d\theta/dt$.

The advantages of both isobaric and isentropic coordinates will appear as we proceed in the next two chapters. It is worth emphasizing again that these advantages are created primarily by the simplification introduced by the hydrostatic assumption.