1

P1) i) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1+(\sqrt{x^2+1})^3}}$ $\quad u = x^2+1 \Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u+u^{3/2}}} = \frac{1}{2}\int \frac{du}{\sqrt{u}\sqrt{1+\sqrt{u}}} \qquad v = \sqrt{u} \Rightarrow dv = \frac{du}{2\sqrt{u}}$$

$$= \int \frac{dv}{\sqrt{1+v}} = 2\sqrt{1+v} + C = 2\sqrt{1+\sqrt{x^2+1}} + C$$

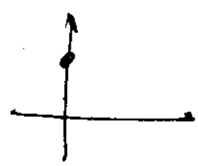
ii) $\displaystyle I = \int_0^4 \frac{dx}{\sqrt{2x^{3/2}-x^2}}$ $\quad x = u^2 \rightarrow dx = 2u\,du \Rightarrow u = \sqrt{x}$

$$= \int_0^2 \frac{2u\,du}{\sqrt{2u^3-u^4}} = \int_0^2 \frac{2\cancel{u}\,du}{\cancel{u}\sqrt{2u-u^2}} = 2\int_0^2 \frac{du}{\sqrt{2u-u^2}}$$

$-(u-1)^2 = -(u^2-2u+1)$

$$= 2\int_0^2 \frac{du}{\sqrt{1-(u-1)^2}} \qquad v = u-1,\quad dv = du$$

$$= 2\int_{-1}^1 \frac{dv}{\sqrt{1-v^2}} = 2\arcsin(v)\Big|_{-1}^{1} + C$$



$$= 2\left(\arcsin(1) - \arcsin(-1)\right) + C$$

$$= 2\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) + C = 2\pi + C$$

P3

$$J = \underbrace{\int_{-2a}^{2a} \underbrace{x}_{\text{impar}} \underbrace{\sqrt{4a^2 - x^2}}_{\text{par: área de circunferencia}} dx}_{\text{impar}} - \int_0^{2a} x \sqrt{a^2 - (x-a)^2}\, dx$$

$$\int_0^{2a} x \sqrt{a^2 - (x-a)^2}\, dx$$

CV:

$$\boxed{\begin{array}{ll} a^2+x^2, & x = a\tan u \\ a^2-x^2, & x = a \operatorname{sen} u \\ x^2-a^2, & x = a\sec u \end{array}}$$

$$x - a = a \operatorname{sen} u \xrightarrow{\text{arcsen}} \left(\frac{x-a}{a}\right) = u$$

$$\Rightarrow dx = a\cos u\, du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a(1+\operatorname{sen} u)\sqrt{a^2 - a^2\operatorname{sen}^2 u}\; a\cos u\, du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^3(1+\operatorname{sen} u)\cos^2 u\, du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^3 \cos^2 u\, du + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^3 \underbrace{\underbrace{\cos^2 u}_{\text{par}}\ \underbrace{\operatorname{sen} u}_{\text{impar}}}_{\text{impar}}\, du$$

$$= a^3\left(\frac{u}{2} + \frac{1}{4}\operatorname{sen}(2u)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0$$

$$= a^3\left(\frac{\pi}{4} + \frac{1}{4}\operatorname{sen}(\pi) - \frac{-\pi}{4} - \operatorname{sen}\frac{-\pi}{2}\right) = \frac{\pi}{2} a^3$$

②

P3]

ii) $I_{m,n} = \int_0^1 x^n (1+x)^m \, dx$

$u = x^n \rightarrow du = n\, x^{n-1}\, dx$

$dv = (1+x)^m\, dx \rightarrow v = \frac{(1+x)^{m+1}}{m+1}$

$\Rightarrow I_{m,n} = x^n \cdot \frac{(1+x)^{m+1}}{m+1} \Big|_0^1 - \int_0^1 \frac{(1+x)^{m+1}}{m+1} \cdot n \cdot x^{n-1}\, dx$

$= 1^n \cdot \frac{2^{m+1}}{m+1} - 0 - \frac{n}{m+1} \int_0^1 x^{n-1} (1+x)^{m+1}\, dx$

$I_{m,n} = \frac{2^{m+1}}{m+1} - \frac{n}{m+1} I_{m+1,n-1} \quad / \cdot m+1$

$(m+1) I_{m,n} + n\, I_{m+1,n-1} = 2^{m+1}$

$$\boxed{\text{Provare:} \quad \int \sin^n(x)\, dx \quad n \geq 2}$$

P4]

SRA $G(x) = \int_a^x f(t) - \int_x^b f(t)$ continua per TFC.

$G(a) = \cancel{\int_a^a f(t)}^{\,0} - \int_a^b f(t) = -\int_a^b f(t)$

$G(b) = \int_a^b f(t) - \cancel{\int_b^b f(t)}^{\,0} = \int_a^b f(t)$

$$G(a)\cdot G(b) \leq 0$$

Por TVM; $\exists\ \bar{x} \in [a,b]$ tal que:

$$G(\bar{x}) = 0$$

$$G(\bar{x}) = \int_a^{\bar{x}} f(t)\,dt - \int_{\bar{x}}^{b} f(t)\,dt = 0$$

$$\Rightarrow \exists\ \bar{x} \in [a,b] \text{ tal q}$$

$$\int_a^{\bar{x}} f(t)\,dt = \int_{\bar{x}}^{b} f(t)\,dt$$

ii) $1 + \int_0^{\pi/2} G(x)\,dx = \frac{\pi}{2}\int_0^{\pi/2} \frac{\operatorname{sen} t}{t}\,dt$

$$\int_0^{\pi/2} G(x)\,dx \qquad u = G(x) \Rightarrow du = \left(\frac{\operatorname{sen} x}{x}\cdot 1 - 0\right)dx$$

$$dv = dx \Rightarrow v = x$$

$$= x\,G(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} x\,\frac{\operatorname{sen} x}{x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\operatorname{sen} t}{t}\,dt + \cos x\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\operatorname{sen} t}{t}\,dt - 1$$

P4) ii)

$$\int_0^1 \frac{dx}{1+2x\cos(\alpha)+x^2} = \int_0^1 \frac{dx}{(x+\cos\alpha)^2-\cos^2\alpha+1}$$

$$\text{c.v.: } \operatorname{sen}\alpha\, u = x+\cos\alpha \Rightarrow \operatorname{sen}\alpha\, du = dx$$

$$= \int_{\frac{\cos\alpha}{\operatorname{sen}\alpha}}^{\frac{1+\cos\alpha}{\operatorname{sen}\alpha}} \frac{du\cdot \operatorname{sen}\alpha}{\operatorname{sen}^2\alpha\, u^2+\operatorname{sen}^2\alpha} = \frac{1}{\operatorname{sen}\alpha}\int_{\frac{\cos\alpha}{\operatorname{sen}\alpha}}^{\frac{1+\cos\alpha}{\operatorname{sen}\alpha}} \frac{du}{u^2+1}$$

$$= \frac{1}{\operatorname{sen}\alpha}\cdot \arctan u \Big|_{\frac{\cos\alpha}{\operatorname{sen}\alpha}}^{\frac{1+\cos\alpha}{\operatorname{sen}\alpha}} = \frac{1}{\operatorname{sen}\alpha}\cdot\left(\arctan\frac{1+\cos\alpha}{\operatorname{sen}\alpha} - \arctan\frac{\cos\alpha}{\operatorname{sen}\alpha}\right)$$

$$= \frac{1}{\operatorname{sen}\alpha}\left(\arctan\left(\frac{\cancel{2}\cos^{\cancel{2}}(\frac{\alpha}{2})}{\cancel{2}\operatorname{sen}(\frac{\alpha}{2})\cancel{\cos(\frac{\alpha}{2})}}\right) - \arctan\left(\frac{\operatorname{sen}(\frac{\pi}{2}-\alpha)}{\cos(\frac{\pi}{2}-\alpha)}\right)\right)$$

$$= \frac{1}{\operatorname{sen}\alpha}\left(\arctan\left(\frac{\cos(\frac{\alpha}{2})}{\operatorname{sen}(\frac{\alpha}{2})}\right) - \arctan\left(\tan\left(\frac{\pi}{2}-\alpha\right)\right)\right)$$

$$= \frac{1}{\operatorname{sen}\alpha}\left(\arctan\left(\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right) - \left(\frac{\pi}{2}-\alpha\right)\right)$$

$$= \frac{1}{\operatorname{sen}\alpha}\left(\cancel{\frac{\pi}{2}} - \frac{\alpha}{2} - \cancel{\frac{\pi}{2}} + \alpha\right) = \frac{\alpha}{2}\,\frac{1}{\operatorname{sen}\alpha}$$

P3

CALCULAR: $I = \displaystyle\int \frac{a_1 \operatorname{sen}^2 x + 2b_1 \operatorname{sen} x \cos x + c_1 \cos^2 x}{a \operatorname{sen} x + b \cos x}\, dx$

- DEMOSTRAR QUE $I = A \operatorname{sen} x + B \cos x + C \displaystyle\int \frac{dx}{a \operatorname{sen} x + b \cos x}$

- CALCULAR $\displaystyle\int \frac{dx}{a \operatorname{sen} x + b \cos x}$

•) TOMAMOS:

$$a_1 \operatorname{sen}^2 x + 2b_1 \operatorname{sen} x \cos x + c_1 \cos^2 x = A \cos x \,(a \operatorname{sen} x + b \cos x) - B \operatorname{sen} x \,(a \operatorname{sen} x + b \cos x) + C\,(\operatorname{sen}^2 x + \cos^2 x)$$

PARA OBTENER A, B y C BASTA RESOLVER:

$$\begin{array}{l} -aB + C = a_1 \\ aA - bB = 2b_1 \\ bA + C = c_1 \end{array} \longrightarrow \begin{cases} A = \dfrac{2ab_1 + b(c_1 - a_1)}{a^2 + b^2} \\ B = \dfrac{a}{b} A - \dfrac{2b_1}{b} \\ C = c_1 - bA \end{cases}$$

AHORA:

$$I = \int A \frac{\cos x \,\cancel{(a \operatorname{sen} x + b \cos x)}}{\cancel{(a \operatorname{sen} x + b \cos x)}} - \int B \frac{\operatorname{sen} x \,\cancel{(a \operatorname{sen} x + b \cos x)}}{\cancel{(a \operatorname{sen} x + b \cos x)}} + \int \frac{C}{a \operatorname{sen} x + b \cos x}$$

$$I = A \operatorname{sen} x + B \cos x + C \int \frac{dx}{a \operatorname{sen} x + b \cos x}$$

..)
$$\int \frac{dx}{a\sin x + b\cos x} = \int \frac{\frac{2t}{1+t^2}}{a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2}}$$

$t = \tan(\frac{x}{2})$ $dx = \frac{2dt}{1+t^2}$

$\operatorname{sen} x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

$$= \int \frac{2dt}{\cancel{1+t^2}} \cdot \frac{\cancel{1+t^2}}{2at + b(1-t^2)} = 2\int \frac{dt}{2at + b - bt^2}$$

→ temos que: $\frac{1}{2at + b - bt^2} = \frac{1}{\left(t - \underbrace{\frac{a - \sqrt{a^2+b^2}}{b}}_{\theta}\right) \cdot \left(t - \underbrace{\frac{a + \sqrt{a^2+b^2}}{b}}_{\varphi}\right)}$

$$= \frac{A}{t-\theta} + \frac{B}{t-\varphi} = \frac{A(t-\varphi) + B(t-\theta)}{(t-\theta)(t-\varphi)}$$

$\Rightarrow A + B = 0$

$\Rightarrow -A\varphi - B\theta = -A\varphi + A\theta = 1$

$$A = \frac{1}{\theta - \varphi} = \frac{1}{\frac{\cancel{a} - \sqrt{a^2+b^2}}{b} - \frac{\cancel{a} + \sqrt{a^2+b^2}}{b}} = -\frac{b}{2\sqrt{a^2+b^2}}$$

$\Rightarrow$ $\frac{1}{2at + b - bt^2} = \frac{A}{t-\theta} - \frac{A}{t-\varphi}$, com $A = \frac{1}{\theta - \varphi}$

$\therefore$ $\int \frac{dx}{a\sin x + b\cos x} = 2\int \frac{A}{t-\theta}\,dt - 2\int \frac{A}{t-\varphi}\,dt$

$$= 2A\ln(t-\theta) - 2A\ln(t-\varphi)$$

$$\boxed{= \frac{2\ln\left(\tan(\frac{x}{2}) - \theta\right) - 2\ln\left(\tan(\frac{x}{2}) - \varphi\right)}{\theta - \varphi}}$$