Clase Auxiliar #1 Calc. Av. y Apps Prim'08 HR; GI, VR

Problemas Considere el campo vectorial

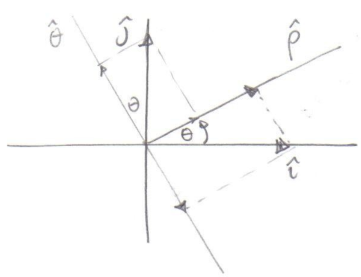
$$\vec{F}(x,y,z) = \frac{x^2}{x^2+y^2}\,\hat{\imath} + \frac{xy}{x^2+y^2}\,\hat{\jmath} + e^z\hat{k}$$

Expresarlo en coordenadas polares

Sol: Consideremos el cambio de coordenadas $\begin{cases} x = \rho\cos(\theta) \\ y = \rho\,\mathrm{sen}(\theta) \\ z = z \end{cases}$

Entonces, $\vec{F}(x,y,z) = \dfrac{\rho^2\cos^2(\theta)}{\rho^2}\,\hat{\imath} + \dfrac{\rho^2\cos(\theta)\,\mathrm{sen}(\theta)}{\rho^2}\,\hat{\jmath} + e^z\hat{k}$

Falta pasar los vectores unitarios $\hat{\imath},\hat{\jmath},\hat{k}$ a $\hat{\rho},\hat{\theta},\hat{k}$:



$$\begin{aligned} \hat{\imath} &= \hat{\rho}\cos(\theta) - \hat{\theta}\,\mathrm{sen}(\theta) \\ \hat{\jmath} &= \hat{\rho}\,\mathrm{sen}(\theta) + \hat{\theta}\cos(\theta) \\ \hat{k} &= \hat{k} \end{aligned}$$

$$\Rightarrow \vec{F}(\rho,\theta,z) = \cos^2(\theta)\left(\hat{\rho}\cos(\theta) - \hat{\theta}\,\mathrm{sen}(\theta)\right) + \cos(\theta)\,\mathrm{sen}(\theta)\left(\hat{\rho}\,\mathrm{sen}(\theta) + \hat{\theta}\cos(\theta)\right) + e^z\hat{k}$$

$$= \hat{\rho}\left(\cos^2(\theta)\cos(\theta) + \cos(\theta)\,\mathrm{sen}^2(\theta)\right) + \hat{\theta}\left(-\cos^2(\theta)\,\mathrm{sen}(\theta) + \cos^2\mathrm{sen}(\theta)\right) + e^z\hat{k}$$

$$= \hat{\rho}\cos(\theta) + e^z\hat{k}.$$

Problema 2 Probar las siguientes identidades:

> $\operatorname{div}(\operatorname{rot}(\vec{F}))=0$.

sol

$$\operatorname{rot}(\vec{F}) = \hat{\imath}(\partial_y F_z - \partial_z F_y) - \hat{\jmath}(\partial_x F_z - \partial_z F_x) + \hat{k}(\partial_x F_y - \partial_y F_x)$$

$$\Rightarrow \operatorname{div}(\operatorname{rot}(\vec{F})) = \partial_x(\partial_y F_z - \partial_z F_y) - \partial_y(\partial_x F_z - \partial_z F_x) + \partial_z(\partial_x F_y - \partial_y F_x)$$

$$= 0$$

> $\operatorname{div}(f\vec{F}) = f\operatorname{div}(\vec{F}) + \vec{F}\cdot\nabla f$

sol

$$\operatorname{div}(f\vec{F}) = \partial_x(fF_x) + \partial_y(fF_y) + \partial_z(fF_z)$$

$$= ((\partial_x f)F_x + f\,\partial_x F_x) + ((\partial_y f)F_y + f(\partial_y F_y)) + ((\partial_z f)F_z + f\,\partial_z F_z)$$

$$= ((\partial_x f)F_x + (\partial_y f)F_y + (\partial_z f)F_z) + f(\partial_x F_x + \partial_y F_y + \partial_z F_z)$$

$$= \nabla f\cdot\vec{F} + f\operatorname{div}(\vec{F})$$

Problema 3

(a) Sea $\varphi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ función de clase C^1. Dem. que

$$\text{rot}\left(\int_a^b \varphi(\vec{r},t)\,dt\right) = \int_a^b \text{rot}(\varphi)(\vec{r},t)\,dt$$

Sol. Supongamos $\varphi = \varphi_x \hat{\imath} + \varphi_y \hat{\jmath} + \varphi_z \hat{k}$. Entonces (definamos $I(f) = \int_a^b f\,dt$)

$$\int_a^b \varphi(\vec{r},t)\,dt = \int_a^b \varphi_x(\vec{r},t)\,dt\ \hat{\imath} + \int_a^b \varphi_y(\vec{r},t)\,dt\ \hat{\jmath} + \int_a^b \varphi_z(\vec{r},t)\,dt$$

En lo anterior, $I(\varphi) = \int_a^b \varphi(\vec{r},t)\,dt = I(\varphi_x)\hat{\imath} + I(\varphi_y)\hat{\jmath} + I(\varphi_z)\hat{k}$

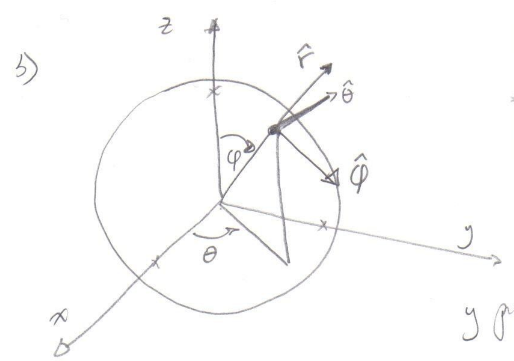
calculamos el rotor:

$$\text{rot}(I(\varphi)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ I(\varphi_x) & I(\varphi_y) & I(\varphi_z) \end{vmatrix} = \begin{pmatrix} \partial_y I(\varphi_z) - \partial_z I(\varphi_y) \\ \partial_z I(\varphi_x) - \partial_x I(\varphi_z) \\ \partial_x I(\varphi_y) - \partial_y I(\varphi_x) \end{pmatrix}$$

$$\overset{\text{regla de Leibniz}}{=} \begin{pmatrix} I(\partial_y \varphi_z) - I(\partial_z \varphi_y) \\ I(\partial_z \varphi_x) - I(\partial_x \varphi_z) \\ I(\partial_x \varphi_y) - I(\partial_y \varphi_x) \end{pmatrix} = I\left(\begin{pmatrix} \partial_y \varphi_z - \partial_z \varphi_y \\ \partial_z \varphi_x - \partial_x \varphi_z \\ \partial_x \varphi_y - \partial_y \varphi_x \end{pmatrix}\right)$$

$$= I(\text{rot}(\varphi)) = \int_a^b \text{rot}(\varphi)(\vec{r},t)\,dt$$

b)



$$\begin{cases} h_r = 1, \\ h_\varphi = r \\ h_\theta = r\,\text{sen}(\varphi) \end{cases}$$

El campo $\vec{F}(\vec{r}) = g(r)\hat{\theta}$. Verificar que $\text{div}(\vec{F}) = 0$

y probar que

$$\text{rot}\left(\vec{F}(t\vec{r}) \times t\vec{r}\right) = 2t\,\vec{F}(t\vec{r}) + t^2 \frac{d}{dt}\vec{F}(t\vec{r}).$$

Sol Primero: $\vec{F}(t\vec{r}) = g(tr)\,\hat{\theta} \Rightarrow \vec{F}(t\vec{r}) \times t\vec{r} = g(tr)\,\hat{\theta} \times tr\,\hat{r}$ $\qquad \hat{r}\,\hat{\varphi}\,\hat{\theta}\,\hat{r}\,\hat{\varphi}$

$$= g(tr)\cdot tr\,\hat{\theta}\times\hat{r}$$

$$= g(tr)\cdot tr\,\hat{\varphi}$$

Entonces $\operatorname{rot}(\vec{F}(t\vec{r})\times t\vec{r}) = \dfrac{1}{r^2\operatorname{sen}(\varphi)}\begin{vmatrix} \hat{r} & r\hat{\varphi} & r\operatorname{sen}(\varphi)\,\hat{\theta} \\ \partial_r & \partial_\varphi & \partial_\theta \\ 0 & r\times g(tr)\,tr & 0 \end{vmatrix}$

$$= \frac{1}{r^2\operatorname{sen}(\varphi)}\left(\hat{r}\left(-\partial_\theta\left(r\cdot g(tr)\,tr\right)\right) - r\hat{\varphi}\,\emptyset + r\operatorname{sen}\varphi\,\hat{\theta}\left(\partial_r\left(r\cdot g(tr)tr\right) - 0\right)\right)$$

$$= \frac{1}{r^{\cancel{2}}\cancel{\operatorname{sen}(\varphi)}}\left(\cancel{r\operatorname{sen}(\varphi)}\,\hat{\theta}\,\partial_r\left(r^2 t\,g(tr)\right)\right)$$

$$= \frac{t}{\cancel{r}}\,\hat{\theta}\left(2\cancel{r}\,g(rt) + r^{\cancel{2}} g'(tr)t\right) = 2tg(rt)\,\hat{\theta} + rt^2 g'(tr)\,\hat{\theta} = (*)$$

Por otro lado, $2t\,\vec{F}(t\vec{r}) + t^2\dfrac{d}{dt}\vec{F}(t\vec{r}) = 2tg(tr)\,\hat{\theta} + t^2\dfrac{d}{dt}g(tr)\,\hat{\theta}$

$$= 2tg(tr)\,\hat{\theta} + t^2 g'(tr)\,r\,\hat{\theta} = (\#)$$

Como $(\#) = (*)$, se tiene la igualdad

Verificar que $\operatorname{div}(\vec{F}) = 0$,

$$\operatorname{div}(\vec{F}) = \frac{1}{r^2\operatorname{sen}(\varphi)}\left(\partial_r\emptyset + \partial_\varphi\emptyset + \partial_\theta\left(1\cdot r\cdot g(r)\right)\right) = \frac{1}{r^2\operatorname{sen}(\varphi)}\partial_\theta\left(r\,g(r)\right) = 0. \quad //$$

c) Sea $\vec{F}$ campo tq $\operatorname{div}(\vec{F})=0$ en una bola $B \subseteq \mathbb{R}^3$ centrada en 0. Asumir que la fórmula anterior es valida en la bola B. Sea $\vec{G}(\vec{r}) = \int_0^1 \left(\vec{F}(t\vec{r}) \times t\vec{r}\right) dt$. Usando lo anterior concluya que $\operatorname{rot} \vec{G} = \vec{F}$ en B.

Sol Usando la parte (a) y la parte (b)

$$\operatorname{rot}(\vec{G}) = \int_0^1 \operatorname{rot}\left(\vec{F}(t\vec{r}) \times t\vec{r}\right) dt$$

$$= \int_0^1 \left(2t\,\vec{F}(t\vec{r}) + t^2 \frac{d}{dt}\vec{F}(t\vec{r})\right) dt$$

$$= \int_0^1 \frac{d}{dt}\left(t^2 \vec{F}(t\vec{r})\right) dt$$

$$\overset{TFC}{=} 1^2 \cdot \vec{F}(1 \cdot \vec{r}) - 0^2 \cdot \vec{F}(0\vec{r}) = \vec{F}(\vec{r}).$$

Problema 4 Consideremos el sistema de coordenadas dado por

$$\vec{r}(x,\rho,\theta) = \begin{pmatrix} x \\ \rho\cos\theta \\ \rho\,\mathrm{sen}\,\theta \end{pmatrix}, \quad x\in\mathbb{R}, \ \theta\in[0,2\pi), \ \rho\geq 0$$

(a) Determinar el triedro de vectores unitarios $\hat{x},\hat{\rho},\hat{\theta}$. ¿Son ortogonales?, Calcule $\hat{\theta}\times\hat{x}$, $\hat{\theta}\times\hat{\rho}$

sol

$$\hat{x} = \frac{\partial\vec{r}}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \hat{\rho} = \frac{\partial\vec{r}}{\partial\rho} \Big/ \left\|\frac{\partial\vec{r}}{\partial\rho}\right\| = \begin{pmatrix} 0 \\ \cos(\theta) \\ \mathrm{sen}(\theta) \end{pmatrix}$$

$$\hat{\theta} = \frac{\partial\vec{r}}{\partial\theta} \Big/ \left\|\frac{\partial\vec{r}}{\partial\theta}\right\| = \begin{pmatrix} 0 \\ -\rho\,\mathrm{sen}(\theta) \\ \rho\cos(\theta) \end{pmatrix} \cdot \frac{1}{\rho} = \begin{pmatrix} 0 \\ -\mathrm{sen}(\theta) \\ \cos(\theta) \end{pmatrix}$$

Es claro que $\hat{x}$ es ortogonal a $\hat{\rho}$ y $\hat{\theta}$. Ahora:

$$\hat{\rho}\cdot\hat{\theta} = 0\times 0 + \cos\theta(-\mathrm{sen}(\theta)) + \mathrm{sen}(\theta)\cos\theta = 0$$

$\Rightarrow$ $\hat{\rho}$ es ortogonal a $\hat{\theta}$

Ahora,

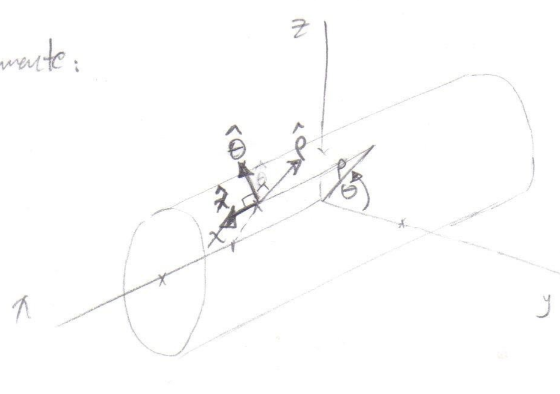
$$\cdot)\ \hat{\theta}\times\hat{x} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -\mathrm{sen}\,\theta & \cos\theta \\ 1 & 0 & 0 \end{vmatrix} = \hat{\imath}\times 0 - \hat{\jmath}(-\cos(\theta)) + \hat{k}(-\,-\mathrm{sen}(\theta))$$

$$= \cos(\theta)\,\hat{\jmath} + \mathrm{sen}(\theta)\,\hat{k} = \hat{\rho}$$

$$\cdot)\ \hat{\theta}\times\hat{\rho} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -\mathrm{sen}\,\theta & \cos\theta \\ 0 & \cos\theta & \mathrm{sen}\,\theta \end{vmatrix} = \hat{\imath}\left(-\mathrm{sen}^2(\theta) - \cos^2(\theta)\right) + 0\hat{\jmath} + 0\hat{k}$$

$$= -\hat{\imath} = -\hat{x}.$$

Gráficamente:



b) Encontrar expresiones para el gradiente, divergencia, laplaciano y rotor en este sistema de coordenadas.

Sol Recordar que si $\vec{F}(u,v,w) = F_u \hat{u} + F_v \hat{v} + F_w \hat{w}$; $f = f(u,v,w)$

Entonces: $\Rightarrow$ $div(\vec{F})(u,v,w) = \frac{1}{h_u h_v h_w}\left(\partial_u (F_u h_v h_w) + \partial_v (h_u F_v h_w) + \partial_w (h_u h_v F_w) \right)$

$$\Rightarrow \nabla f(u,v,w) = \frac{1}{h_u} \partial_u f \, \hat{u} + \frac{1}{h_v} \partial_v f \, \hat{v} + \frac{1}{h_w} \partial_w f \, \hat{w}$$

$$\Rightarrow rot(\vec{F})(u,v,w) = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{u} & h_v \hat{v} & h_w \hat{w} \\ \partial_u & \partial_v & \partial_w \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Entonces, notemos que en nuestro sistema, $h_x = 1$, $h_\rho = 1$, $h_\theta = \rho$. Así, si

$$\vec{F}(x,\rho,\theta) = F_x \hat{x} + F_\rho \hat{\rho} + F_\theta \hat{\theta}$$

$$\Rightarrow div(\vec{F}) = \frac{1}{\rho}\left(\partial_x (F_x \rho) + \partial_\rho (F_\rho \rho) + \partial_\theta (F_\theta) \right) = \partial_x F_x + \frac{1}{\rho} \partial_\rho (F_\rho \rho) + \frac{1}{\rho} \partial_\theta F_\theta$$

$$\Rightarrow rot(\vec{F}) = \frac{1}{\rho} \begin{vmatrix} \hat{x} & \hat{\rho} & \rho\hat{\theta} \\ \partial_x & \partial_\rho & \partial_\theta \\ F_x & F_\rho & \rho F_\theta \end{vmatrix} = \frac{1}{\rho}\Big(\hat{x}\left(\partial_\rho (\rho F_\theta) - \partial_\theta F_\rho \right) - \hat{\rho}\left(\partial_x (\rho F_\theta) - \partial_\theta F_x \right)$$
$$+ \rho\hat{\theta}\left(\partial_x F_\rho - \partial_\rho F_x \right) \Big)$$

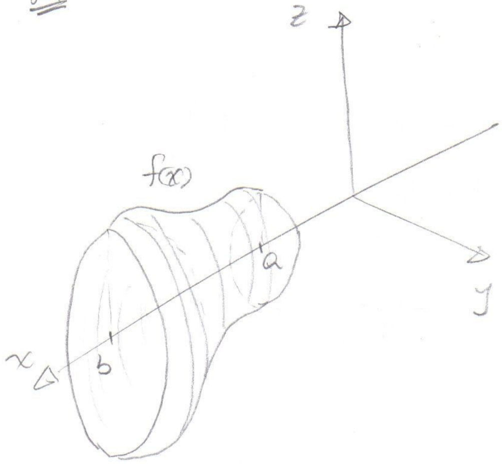
$$\Rightarrow \nabla f = \partial_x f \, \hat{x} + \partial_\rho f \, \hat{\rho} + \frac{1}{\rho} \partial_\theta f \, \hat{\theta}$$

$$\Rightarrow \Delta f = div(\nabla f) = \partial_x (\partial_x f) + \frac{1}{\rho} \partial_\rho \left(\partial_\rho (f \rho) \right) + \frac{1}{\rho} \partial_\theta \left(\frac{1}{\rho} \partial_\theta f \right)$$
$$= \partial^2_{xx} f + \frac{1}{\rho} \partial^2_{\rho\rho} (f \rho) + \frac{1}{\rho^2} \partial^2_{\theta\theta} f$$

c) Dada $f:[a,b]\to \mathbb{R}_+$ diferenciable, bosqueje la superficie de ecuación $y^2+z^2=f(x)^2$

Verifique que una parametrización de esta superficie es $\vec{r}_1(x,\theta)=x\hat{\imath}+f(x)\hat{\rho}$

sol



Veamos que si $x\in[a,b]$, $\theta\in[0,2\pi)$, entonces

$\vec{r}_1(x,\theta)$ satisface la ecuación de la superficie.

$$\vec{r}_1(x,\theta)=x\hat{\imath}+f(x)\hat{\rho}=\begin{pmatrix}x\\0\\0\end{pmatrix}+\begin{pmatrix}0\\f(x)\cos(\theta)\\f(x)\,\mathrm{sen}(\theta)\end{pmatrix}$$

$$=\begin{pmatrix}x\\f(x)\cos(\theta)\\f(x)\,\mathrm{sen}(\theta)\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Veamos si se cumple que $y^2+z^2=f(x)^2$

$$y^2+z^2=f(x)^2\cos^2(\theta)+f(x)^2\,\mathrm{sen}^2(\theta)=f(x)^2 \quad \checkmark$$

Además, $\vec{r}_1$ es continua.