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FIFTH EDITION

A First Course in PROBABILITY

Sheldon Ross

fifth edition

A First Course in Probability

Sheldon Ross

University of California, Berkeley

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For Rebecca

Contents

(optional sections are noted with an asterisk)

Preface

"We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it. . . . It is remarkable that this science, which originated in the consideration of games of chance, should have become the most important object of human knowledge. . . . The most important questions of life are, for the most part, really only problems of probability." So said the famous French mathematician and astronomer (the "Newton of France") Pierre Simon, Marquis de Laplace. Although many people might feel that the famous marquis, who was also one of the great contributors to the development of probability, might have exaggerated somewhat, it is nevertheless true that probability theory has become a tool of fundamental importance to nearly all scientists, engineers, medical practitioners, jurists, and industrialists. In fact, the enlightened individual had learned to ask not "Is it so?" but rather "What is the probability that it is so?"

This book is intended as an elementary introduction to the mathematical theory of probability for students in mathematics, engineering, and the sciences (including the social sciences and management science) who possess the prerequisite knowledge of elementary calculus. It attempts to present not only the mathematics of probability theory, but also, through numerous examples, the many diverse possible applications of this subject.

In Chapter 1 we present the basic principles of combinatorial analysis, which are most useful in computing probabilities.

In Chapter 2 we consider the axioms of probability theory and show how they can be applied to compute various probabilities of interest. This chapter

includes a proof of the important (and, unfortunately, often neglected) continuity property of probabilities, which is then used in the study of a "logical paradox."

Chapter 3 deals with the extremely important subjects of conditional probability and independence of events. By a series of examples we illustrate how conditional probabilities come into play not only when some partial information is available, but also as a tool to enable us to compute probabilities more easily, even when no partial information is present. This extremely important technique of obtaining probabilities by "conditioning" reappears in Chapter 7, where we use it to obtain expectations.

In Chapters 4, 5 and 6 we introduce the concept of random variables. Discrete random variables are dealt with in Chapter 4, continuous random variables in Chapter 5, and jointly distributed random variables in Chapter 6. The important concepts of the expected value and the variance of a random variable are introduced in Chapters 4 and 5. These quantities are then determined for many of the common types of random variables.

Additional properties of the expected value are considered in Chapter 7. Many examples illustrating the usefulness of the result that the expected value of a sum of random variables is equal to the sum of their expected values are presented. Sections on conditional expectation, including its use in prediction, and moment generating functions are contained in this chapter. In addition, the final section introduces the multivariate normal distribution and presents a simple proof concerning the joint distribution of the sample mean and sample variance of a sample from a normal distribution.

In Chapter 8 we present the major theoretical results of probability theory. In particular, we prove the strong law of large numbers and the central limit theorem. Our proof of the strong law is a relatively simple one which assumes that the random variables have a finite fourth moment, and our proof of the central limit theorem assumes Levy's continuity theorem. Also in this chapter we present such probability inequalities as Markov's inequality, Chebyshev's inequality, and Chernoff bounds. The final section of Chapter 8 gives a bound on the error involved when a probability concerning a sum of independent Bernoulli random variables is approximated by the corresponding probability for a Poisson random variable having the same expected value.

Chapter 9 presents some additional topics, such as Markov chains, the Poisson process, and an introduction to information and coding theory, and Chapter 10 considers simulation.

NEW TO THE FIFTH EDITION

Each chapter in the fifth edition has been updated in response to reviewers comments. Professors who wish to move through the first chapters quickly, will appreciate the addition of asterisks to denote optional sections that may safely be skipped. Among new text material included are discussions on the odds-ratio in Chapter 3, and two new discussions in Chapter 6: a new section on exchangeable random variables and a discussion of the fact that independence is a symmetric relation.

A goal of the Fifth Edition is to make the book more accessible to students. The examples are updated to include many interesting and practical examples including one dealing with the counterintuitive ace of spades versus the two of clubs problem (Example 5j in Chapter 2); the two girls problem (Example 3j in Chapter 3); the analysis of the quicksort algorithm (Example 2o of Chapter 7); and the best prize problem (Example 4I in Chapter 7). In addition, the problems are thoroughly revised with over 25% being new to this edition. The chapter exercises are reorganized to present the more mechanical **problems** before the **theoretical exercises.** Prose summaries now conclude each chapter and a new study tool is included in the book. The new **Self-Test Problems and Exercises** section is designed to help students test their comprehension and study for exams. After working through the problems and theoretical exercises in each chapter, students are encouraged to do the Self-test problems and to check their work against the complete solutions that appear in Appendix B.

Another new feature of the Fifth Edition, in the addition of the **Probability Models Disk.** This easy to use PC Disk is packaged in the back of each copy of the book. Referenced in text, this disk allows students to quickly and easily perform calculations and simulations in six key areas.

- Three of the modules derive probabilities for, respectively, binomial, Poisson, and normal random variables.
- Another illustrates the central limit theorem. It considers random variables that take on one of the values 0, 1, 2, 3, 4 and allows the user to enter the probabilities for these values along with a number n. The module then plots the probability mass function of the sum of n independent random variables of this type. By increasing n one can "see" the mass function coverage to the shape of a normal density function.
- The other two modules illustrate the strong law of large numbers. Again the user enters probabilities for the five possible values of the random variable along with an integer n. The program then uses random numbers to simulate n random variables having the prescribed distribution. The modules graph the number of times each outcome occurs along with the average of all outcomes. The modules differ in how they graph the results of the trials.

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S. R.

CHAPTER 1

Combinatorial Analysis

1.1 INTRODUCTION

Here is a typical problem of interest involving probability. A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals—and will be called *functional*—as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where $n = 4$ and $m = 2$ there are 6 possible system configurations—namely,

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

where 1 means that the antenna is working and 0 that it is defective. As the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability. In the case of general n and m, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system being functional and then divide by the total number of all possible configurations.

From the above we see that it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different

ways that a certain event can occur. The mathematical theory of counting is formally known as *combinatorial analysis.*

1.2 THE BASIC PRINCIPLE OF COUNTING

The following principle of counting will be basic to all our work. Loosely put, it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are mn possible outcomes of the two experiments.

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Proof of the Basic Principle: The basic principle may be proved by enumerating all the possible outcomes of the two experiments as follows:

$$\begin{array}{l}(1, 1), (1, 2), \ldots, (1, n)\\ (2, 1), (2, 2), \ldots, (2, n)\\ \quad\vdots\\ (m, 1), (m, 2), \ldots, (m, n)\end{array}$$

where we say that the outcome is (i, j) if experiment 1 results in its ith possible outcome and experiment 2 then results in the jth of its possible outcomes. Hence the set of possible outcomes consists of m rows, each row containing n elements, which proves the result.

Example 2a. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices. ∎

When there are more than two experiments to be performed, the basic principle can be generalized as follows.

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if . . . , then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 2b. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. Hence it follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees. ∎

Example 2c. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution By the generalized version of the basic principle the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$. ∎

Example 2d. How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions. ∎

Example 2e. In Example 2c, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution In this case there would be $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$ possible license plates. ∎

1.3 PERMUTATIONS

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6: namely, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Each arrangement is known as a *permutation.* Thus there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining

2, and the third object in the permutation is then chosen from the remaining 1. Thus there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters shows that there are

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Example 3a. How many different batting orders are possible for a baseball team consisting of 9 players?

Solution There are $9! = 362{,}880$ possible batting orders. ∎

Example 3b. A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.
(a) How many different rankings are possible?
(b) If the men are ranked just among themselves and the women among themselves, how many different rankings are possible?

Solution (a) As each ranking corresponds to a particular ordered arrangement of the 10 people, we see that the answer to this part is $10! = 3{,}628{,}800$.

(b) As there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in this case. ∎

Example 3c. Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is 4! 4! 3! 2! 1! = 6912. ∎

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other. To set this straight in our minds, consider the following example.

Example 3d. How many different letter arrangements can be formed using the letters $P\,E\,P\,P\,E\,R$?

Solution We first note that there are 6! permutations of the letters $P_1\ E_1\ P_2\ P_3\ E_2\ R$ when the 3 P's and the 2 E's are distinguished from each other. However, consider any one of these permutations—for instance, $P_1\ P_2\ E_1\ P_3\ E_2\ R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form $P\,P\,E\,P\,E\,R$. That is, all 3! 2! permutations

$$\begin{array}{ll}
P_1\ P_2\ E_1\ P_3\ E_2\ R & P_1\ P_2\ E_2\ P_3\ E_1\ R \\
P_1\ P_3\ E_1\ P_2\ E_2\ R & P_1\ P_3\ E_2\ P_2\ E_1\ R \\
P_2\ P_1\ E_1\ P_3\ E_2\ R & P_2\ P_1\ E_2\ P_3\ E_1\ R \\
P_2\ P_3\ E_1\ P_1\ E_2\ R & P_2\ P_3\ E_2\ P_1\ E_1\ R \\
P_3\ P_1\ E_1\ P_2\ E_2\ R & P_3\ P_1\ E_2\ P_2\ E_1\ R \\
P_3\ P_2\ E_1\ P_1\ E_2\ R & P_3\ P_2\ E_2\ P_1\ E_1\ R
\end{array}$$

are of the form $P\,P\,E\,P\,E\,R$. Hence there are 6!/3! 2! = 60 possible letter arrangements of the letters $P\,E\,P\,P\,E\,R$. ∎

In general, the same reasoning as that used in Example 3d shows that there are

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Example 3e. A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution There are

$$\frac{10!}{4!\,3!\,2!\,1!} = 12{,}600$$

possible outcomes. ∎

Example 3f. How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution There are

$$\frac{9!}{4!\,3!\,2!} = 1260$$

different signals. ∎

1.4 COMBINATIONS

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this, reason as follows: Since there are 5 ways to select the initial item, 4 ways

to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3, say, the group consisting of items A, B, and C, will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, and CBA will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \le n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.†

Thus $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

Example 4a. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees. ∎

† By convention, 0! is defined to be 1. Thus $\binom{n}{0} = \binom{n}{n} = 1$. We also take $\binom{n}{i}$ to be equal to 0 when either $i < 0$ or $i > n$.

Example 4b. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \left(\frac{5 \cdot 4}{2 \cdot 1}\right)\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$ possible committees consisting of 2 women and 3 men.

On the other hand, if 2 of the men refuse to serve on the committee together, then, as there are $\binom{2}{0}\binom{5}{3}$ possible groups of 3 men not containing either of the 2 feuding men and $\binom{2}{1}\binom{5}{2}$ groups of 3 men containing exactly 1 of the feuding men, it follows that there are $\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2} = 30$ groups of 3 men not containing both of the feuding men. Since there are $\binom{5}{2}$ ways to choose the 2 women, it follows that in this case there are $30\binom{5}{2} = 300$ possible committees. ∎

Example 4c. Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution Imagine that the $n-m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n-m+1$ possible positions—represented in Figure 1.1 by carets—between the $n-m$ functional antennas, we must select m of these in which to put the defective antennas. Hence there are

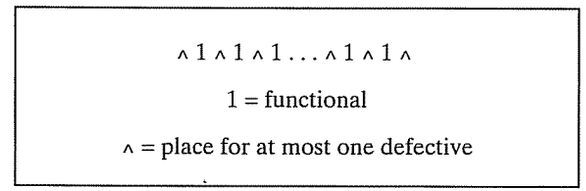


Figure 1.1

$\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones. ∎

A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \le r \le n \tag{4.1}$$

Equation (4.1) may be proved analytically or by the following combinatorial argument. Consider a group of n objects and fix attention on some particular one of these objects—call it object 1. Now, there are $\binom{n-1}{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r-1$ from the remaining $n-1$ objects). Also, there are $\binom{n-1}{r}$ groups of size r that do not contain object 1. As there is a total of $\binom{n}{r}$ groups of size r, Equation (4.1) follows.

The values $\binom{n}{r}$ are often referred to as *binomial coefficients*. This is so because of their prominence in the binomial theorem.

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4.2}$$

We shall present two proofs of the binomial theorem. The first is a proof by mathematical induction, and the second is a proof based on combinatorial considerations.

Proof of the Binomial Theorem by Induction: When $n = 1$, Equation (4.2) reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x$$

Assume Equation (4.2) for $n-1$. Now,

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} \\ &= (x+y) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}\end{aligned}$$

Letting $i = k+1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}(x+y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\ &= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\ &= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n \\ &= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\end{aligned}$$

where the next-to-last equality follows by Equation (4.1). By induction the theorem is now proved.

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, \ldots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have as factors k of the x_i's and $(n-k)$ of the y_i's? As each term consisting of k of the x_i's and $(n-k)$ of the y_i's corresponds to a choice of a group of k from the n values $x_1, x_2, \ldots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x$, $y_i = y$, $i = 1, \ldots, n$, we see that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 4d. Expand $(x+y)^3$.

Solution

$$\begin{aligned}(x+y)^3 &= \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^0 \\ &= y^3 + 3xy^2 + 3x^2 y + x^3\end{aligned}$$

∎

Example 4e. How many subsets are there of a set consisting of n elements?

Solution Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1 + 1)^n = 2^n$$

This result could also have been obtained by assigning to each element in the set either the number 0 or the number 1. To each assignment of numbers there corresponds, in a one-to-one fashion, a subset, namely, that subset consisting of all elements that were assigned the value 1. As there are 2^n possible assignments, the result follows.

Note that we have included as a subset the set consisting of 0 elements (that is, the null set). Hence the number of subsets that contain at least one element is $2^n - 1$. ∎

1.5 MULTINOMIAL COEFFICIENTS

In this section we consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible? To answer this, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group there are $\binom{n - n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups there are $\binom{n - n_1 - n_2}{n_3}$ possible choices for the third group; and so on. Hence it follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n - n_1}{n_2} \cdots \binom{n - n_1 - n_2 - \cdots - n_{r-1}}{n_r}$$

$$= \frac{n!}{(n - n_1)!\, n_1!} \frac{(n - n_1)!}{(n - n_1 - n_2)!\, n_2!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{r-1})!}{0!\, n_r!}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

possible divisions.

Notation

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Thus $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 5a. A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution There are $\frac{10!}{5!\, 2!\, 3!} = 2520$ possible divisions. ∎

Example 5b. Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution There are $\frac{10!}{5!\, 5!} = 252$ possible divisions. ∎

Example 5c. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution Note that this example is different from Example 5b because now the order of the two teams is irrelevant. That is, there is no A and B team but just a division consisting of 2 groups of 5 each. Hence the desired answer is

$$\frac{10!/5!\, 5!}{2!} = 126$$ ∎

The proof of the following theorem, which generalizes the binomial theorem, is left as an exercise.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, \ldots, n_r):\\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients*.

Example 5d

$$\begin{aligned}(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 \\ &\quad + \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 \\ &\quad + \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\end{aligned}$$

*1.6 ON THE DISTRIBUTION OF BALLS IN URNS

There are r^n possible outcomes when n distinguishable balls are to be distributed into r distinguishable urns. This follows because each ball may be distributed into any of r possible urns. Let us now, however, suppose that the n balls are indistinguishable from each other. In this case, how many different outcomes are possible? As the balls are indistinguishable, it follows that the outcome of the experiment of distributing the n balls into r urns can be described by a vector $(x_1, x_2, \ldots x_r)$, where x_i denotes the number of balls that are distributed into the ith urn. Hence the problem reduces to finding the number of distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = n$$

To compute this, let us start by considering the number of positive integer-valued solutions. Toward this end, imagine that we have n indistinguishable objects lined

* Note that asterisks denote material that is optional.

up and that we want to divide them into r nonempty groups. To do so, we can select $r - 1$ of the $n - 1$ spaces between adjacent objects as our dividing points (see Figure 1.2). For instance, if we have $n = 8$ and $r = 3$ and choose the 2 divisors as shown

ooo|ooo|oo

then the vector obtained is $x_1 = 3$, $x_2 = 3$, $x_3 = 2$. As there are $\binom{n-1}{r-1}$ possible selections, we obtain the following proposition.

Proposition 6.1

There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n \qquad x_i > 0,\ i = 1, \ldots, r$$

To obtain the number of nonnegative (as opposed to positive) solutions, note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive solutions of $y_1 + \cdots + y_r = n + r$ (seen by letting $y_i = x_i + 1$, $i = 1, \ldots, r$). Hence, from Proposition 6.1, we obtain the following proposition.

Proposition 6.2

There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n \tag{6.1}$$

$0 \wedge 0 \wedge 0 \wedge \ldots \wedge 0 \wedge 0$

n objects 0

Choose $r - 1$ of the spaces $\wedge$.

Figure 1.2

Example 6a. How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

Solution There are $\binom{3+2-1}{2-1} = 4$ such solutions: (0, 3), (1, 2), (2, 1), (3, 0). ∎

Example 6b. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

Solution If we let x_i, $i = 1, 2, 3, 4$, denote the number of thousands invested in investment number i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying

$$x_1 + x_2 + x_3 + x_4 = 20 \qquad x_i \geq 0$$

Hence, by Proposition 6.2, there are $\binom{23}{3} = 1771$ possible investment strategies. If not all of the money need be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector $(x_1, x_2, x_3, x_4, x_5)$ satisfying

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

Hence, by Proposition 6.2, there are now $\binom{24}{4} = 10{,}626$ possible strategies. ∎

Example 6c. How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$?

Solution

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all nonnegative integer-valued $(n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_r = n$. Hence, by Proposition 6.2, there are $\binom{n+r-1}{r-1}$ such terms. ∎

Example 6d. Let us reconsider Example 4c, in which we have a set of n items, of which m are (indistinguishable and) defective and the remaining $n - m$ are (also indistinguishable and) functional. Our objective is to determine the number of linear orderings in which no two defectives are next to each other. To determine this quantity, let us imagine that the defective items are lined up among themselves and the functional ones are now to be put in position. Let us denote x_1 as the number of functional items to the left of the first defective, x_2 as the number of functional items between the first two defectives, and so on. That is, schematically we have

$$x_1\ 0\ x_2\ 0 \cdots x_m\ 0\ x_{m+1}$$

Now there will be at least one functional item between any pair of defectives as long as $x_i > 0$, $i = 2, \ldots, m$. Hence the number of outcomes satisfying the condition is the number of vectors $x_1, \cdots, x_{m+1}$ that satisfy

$$x_1 + \cdots + x_{m+1} = n - m \qquad x_1 \geq 0, x_{m+1} \geq 0, x_i > 0, i = 2, \ldots, m$$

But on letting $y_1 = x_1 + 1$, $y_i = x_i$, $i = 2, \ldots, m$, $y_{m+1} = x_{m+1} + 1$, we see that this is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2$$

Hence, by Proposition 6.1, there are $\binom{n-m+1}{m}$ such outcomes, which is in agreement with the results of Example 4c.

Suppose now that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional ones. By the same reasoning as that applied above, this would equal the number of vectors satisfying

$$x_1 + \cdots + x_{m+1} = n - m \qquad x_1 \geq 0, x_{m+1} \geq 0, x_i \geq 2, i = 2, \ldots, m$$

Upon letting $y_1 = x_1 + 1$, $y_i = x_i - 1$, $i = 2, \ldots, m$, $y_{m+1} = x_{m+1} + 1$, we see that this is the same as the number of positive solutions of

$$y_1 + \cdots + y_{m+1} = n - 2m + 3$$

Hence, from Proposition 6.1, there are $\binom{n-2m+2}{m}$ such outcomes.

SUMMARY

The basic principle of counting states that if an experiment consisting of two phases is such that there are n possible outcomes of phase 1, and for each of these n outcomes there are m possible outcomes of phase 2, there are nm possible outcomes of the experiment.

There are $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ possible linear orderings of n items. The quantity 0! is defined to equal 1.

Let

$$\binom{n}{i} = \frac{n!}{(n-i)!\, i!}$$

when $0 \leq i \leq n$, and let it equal 0 otherwise. This quantity represents the number of different subgroups of size i that can be chosen from a set of size n. It is often called a *binomial coefficient* because of its prominence in the binomial theorem, which states that

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

For nonnegative integers $n_1, \ldots, n_r$ summing to n,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

is the number of ways of dividing up n items into r distinct nonoverlapping subgroups of sizes $n_1, n_2, \ldots, n_r$.

PROBLEMS

1. **(a)** How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9; the second digit was either 0 or 1; the third digit was any integer between 1 and 9. How many area codes were possible? How many area codes starting with a 4 were possible?

6. A well-known nursery rhyme starts as follows:

As I was going to St. Ives
I met a man with 7 wives.
Each wife had 7 sacks.
Each sack had 7 cats.
Each cat had 7 kittens.

How many kittens did the traveler meet?

7. **(a)** In how many ways can 3 boys and 3 girls sit in a row?
(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
(c) In how many ways if only the boys must sit together?
(d) In how many ways if no two people of the same sex are allowed to sit together?

8. How many different letter arrangements can be made from the letters
(a) FLUKE;
(b) PROPOSE;
(c) MISSISSIPPI;
(d) ARRANGE?

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

10. In how many ways can 8 people be seated in a row if
(a) there are no restrictions on the seating arrangement;
(b) persons A and B must sit next to each other;
(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other;
(d) there are 5 men and they must sit next to each other;
(e) there are 4 married couples and each couple must sit together?

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if
(a) the books can be arranged in any order;
(b) the mathematics books must be together and the novels must be together;
(c) the novels must be together but the other books can be arranged in any order?

12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if
(a) a student can receive any number of awards;
(b) each student can receive at most 1 award?

13. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

14. How many 5-card poker hands are there?

15. A dance class consists of 22 students, 10 women and 12 men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

16. A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if
(a) both books are to be on the same subject;
(b) the books are to be on different subjects?

17. A total of 7 different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

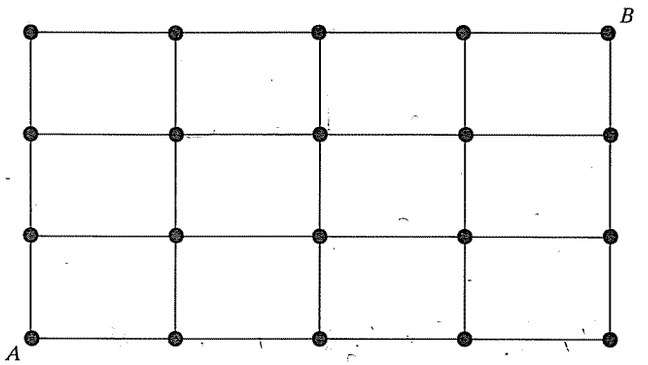
18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

19. From a group of 8 women and 6 men a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if
(a) 2 of the men refuse to serve together;
(b) 2 of the women refuse to serve together;
(c) 1 man and 1 woman refuse to serve together?

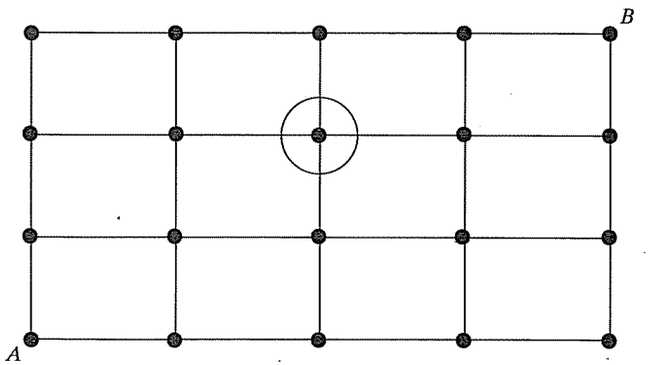
20. A person has 8 friends, of whom 5 will be invited to a party.
(a) How many choices are there if 2 of the friends are feuding and will not attend together?
(b) How many choices if 2 of the friends will only attend together?

21. Consider the grid of points shown below. Suppose that starting at the point labeled A you can go one step up or one step to the right at each move. This is continued until the point labeled B is reached. How many different paths from A to B are possible?

HINT: Note that to reach B from A you must take 4 steps to the right and 3 steps upward.



22. In Problem 21, how many different paths are there from A to B that go through the point circled below?



23. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

24. Expand $(3x^2 + y)^5$.

25. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

26. Expand $(x_1 + 2x_2 + 3x_3)^4$.

27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

29. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

30. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other, and the Russian and U.S. delegates are not to be next to each other?

***31.** If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many, if each school must receive at least 1 blackboard?

***32.** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

***33.** We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if
(a) an investment must be made in each opportunity;
(b) investments must be made in at least 3 of the 4 opportunities?

THEORETICAL EXERCISES

1. Prove the generalized version of the basic counting principle.

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome number i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n if the order of selection is considered relevant?

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n - r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

HINT: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n-k+1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

11. The following identity is known as Fermat's combinatorial identity.

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.

HINT: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for the above by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

HINT: (i) How many possible selections are there of a committee of size k and its chairperson?
(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2}n(n+1)$$

For a combinatorial proof of the above, consider a set of n people, and argue that both sides of the identity above represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).

HINT: (i) How many different selections result in the committee containing exactly k people?
(ii) How many different selections are there in which the chairperson and the secretary are the same? (ANSWER: $n2^{n-1}$.)
(iii) How many different selections result in the chairperson and the secretary being different?

(c) Now argue that

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3}n^2(n+3)$$

13. Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

HINT: Use the binomial theorem.

14. From a set of n people a committee of size j is to be chosen, and from this committee a subcommittee of size i, $i \leq j$, is also to be chosen.

(a) Derive a combinatorial identity by computing, in two ways, the number of possible choices of the committee and subcommittee—first by supposing that the committee is chosen first and then the subcommittee, and second by supposing that the subcommittee is chosen first and then the remaining members of the committee are chosen.

(b) Use part (a) to prove the following combinatorial identity:

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i} \qquad i \leq n$$

(c) Use part (a) and Theoretical Exercise 13 to show that

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} (-1)^{n-j} = 0 \qquad i \leq n$$

15. Let $H_k(n)$ be the number of vectors $x_1, \ldots, x_k$ for which each x_i is a positive integer satisfying $1 \leq x_i \leq n$ and $x_1 \leq x_2 \leq \cdots \leq x_k$.

(a) Without any computations, argue that

$$H_1(n) = n$$

$$H_k(n) = \sum_{j=1}^{n} H_{k-1}(j) \qquad k > 1$$

HINT: How many vectors are there in which $x_k = j$?

(b) Use the preceding recursion to compute $H_3(5)$.

HINT: First compute $H_2(n)$ for $n = 1, 2, 3, 4, 5$.

16. Consider a tournament of n contestants in which the outcome is an ordering of these contestants, with ties allowed. That is, the outcome partitions the players into groups, with the first group consisting of the players that tied for first place, the next group being those that tied for the next best position, and so on. Let $N(n)$ denote the number of different possible outcomes. For instance, $N(2) = 3$ since in a tournament with 2 contestants, player 1 could be uniquely first, player 2 could be uniquely first, or they could tie for first.

(a) List all the possible outcomes when $n = 3$.

(b) With $N(0)$ defined to equal 1, argue, without any computations, that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} N(n - i)$$

HINT: How many outcomes are there in which i players tie for last place?

(c) Show that the formula of part (b) is equivalent to the following:

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

(d) Use the recursion to find $N(3)$ and $N(4)$.

17. Present a combinatorial explanation of why $\binom{n}{r} = \binom{n}{r, n-r}$.

18. Argue that

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_r} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_r} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_r - 1}$$

HINT: Use an argument similar to the one used to establish Equation (4.1).

19. Prove the multinomial theorem.

***20.** In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

***21.** Argue that there are exactly $\binom{r}{k}\binom{n-1}{n-r+k}$ solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0.

***22.** Consider a function $f(x_1, \ldots, x_n)$ of n variables. How many different partial derivatives of order r does it possess?

***23.** Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \leq k$$

SELF-TEST PROBLEMS AND EXERCISES

1. How many different linear arrangements are there of the letters A, B, C, D, E, F for which

(a) A and B are next to each other;

(b) A is before B;

(c) A is before B and B is before C;

(d) A is before B and C is before D;

(e) A and B are next to each other and C and D are also next to each other;

(f) E is not last in line?

2. If 4 Americans, 3 Frenchmen, and 3 Englishmen are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if
   (a) there are no restrictions;
   (b) A and B will not serve together;
   (c) C and D will serve together or not at all;
   (d) E must be an officer;
   (e) F will serve only if he is president?
4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?
5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?
6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.
7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

8. Consider n-digit numbers where each digit is one of the 10 integers 0, 1, . . . , 9. How many such numbers are there for which
   (a) no two consecutive digits are equal;
   (b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$?
9. Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.
   (a) How many choices are possible?
   (b) How many choices are there in which all 3 students are in the same class?
   (c) How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?
   (d) How many choices are there in which all 3 students are in different classes?
   (e) Using the results of parts (a) through (d), write a combinatorial identity.

*10. An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all works were sold?

*11. Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq k$$

where $k \geq n$.

CHAPTER 2

Axioms of Probability

2.1 INTRODUCTION

In this chapter we introduce the concept of the probability of an event and then show how these probabilities can be computed in certain situations. As a preliminary, however, we need the concept of the sample space and the events of an experiment.

2.2 SAMPLE SPACE AND EVENTS

Consider an experiment whose outcome is not predictable with certainty in advance. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by S. Some examples follow.

1. If the outcome of an experiment consists in the determination of the sex of a newborn child, then

$$S = \{g, b\}$$

where the outcome g means that the child is a girl and b that it is a boy.

2. If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7, then

$$S = \{\text{all } 7! \text{ permutations of } (1, 2, 3, 4, 5, 6, 7)\}$$

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instance, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

3. If the experiment consists of flipping two coins, then the sample space consists of the following four points:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The outcome will be (H, H) if both coins are heads, (H, T) if the first coin is heads and the second tails, (T, H) if the first is tails and the second heads, and (T, T) if both coins are tails.

4. If the experiment consists of tossing two dice, then the sample space consists of the 36 points

$$S = \{(i, j): i, j = 1, 2, 3, 4, 5, 6\}$$

where the outcome (i, j) is said to occur if i appears on the leftmost die and j on the other die.

5. If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers. That is

$$S = \{x: 0 \leq x < \infty\}$$

Any subset E of the sample space is known as an *event*. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred. Some examples of events are the following.

In example 1 above, if $E = \{g\}$, then E is the event that the child is a girl. Similarly, if $F = \{b\}$, then F is the event that the child is a boy.

In example 2, if

$$E = \{\text{all outcomes in } S \text{ starting with a 3}\}$$

then E is the event that horse 3 wins the race.

In example 3, if $E = \{(H, H), (H, T)\}$, then E is the event that a head appears on the first coin.

In example 4, if $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, then E is the event that the sum of the dice equals 7.

In example 5, if $E = \{x: 0 \leq x \leq 5\}$, then E is the event that the transistor does not last longer than 5 hours.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consist of all points that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if *either* E or F occurs. For instance, in example 1 if event $E = \{g\}$ and $F = \{b\}$, then

$$E \cup F = \{g, b\}$$

That is, $E \cup F$ would be the whole sample space S. In example 3, if $E = \{(H, H), (H, T)\}$ and $F = \{(T, H)\}$, then

$$E \cup F = \{(H, H), (H, T), (T, H)\}$$

Thus $E \cup F$ would occur if a head appeared on either coin.

The event $E \cup F$ is called the *union* of the event E and the event F.

Similarly, for any two events E and F we may also define the new event EF, called the *intersection* of E and F, to consist of all outcomes that are both in E and in F. That is, the event EF (sometimes written $E \cap F$) will occur only if both E and F occur. For instance, in example 3 if $E = \{(H, H), (H, T), (T, H)\}$ is the event that at least 1 head occurs, and $F = \{(H, T), (T, H), (T, T)\}$ is the event that at least 1 tail occurs, then

$$EF = \{(H, T), (T, H)\}$$

is the event that exactly 1 head and 1 tail appear. In example 4 if $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$ is the event that the sum of the dice is 7 and $F = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$ is the event that the sum is 6, then the event EF does not contain any outcomes and hence could not occur. To give such an event a name, we shall refer to it as the null event and denote it by $\varnothing$ (that is, $\varnothing$ refers to the event consisting of no points). If $EF = \varnothing$, then E and F are said to be *mutually exclusive*.

We also define unions and intersections of more than two events in a similar manner. If $E_1, E_2, \ldots$ are events, the union of these events, denoted by $\bigcup_{n=1}^{\infty} E_n$, is defined to be that event which consists of all points that are in E_n for at least one value of $n = 1, 2, \ldots$. Similarly, the intersection of the events E_n, denoted by $\bigcap_{n=1}^{\infty} E_n$, is defined to be the event consisting of those points that are in all of the events E_n, $n = 1, 2, \ldots$.

Finally, for any event E we define the new event E^c, referred to as the complement of E, to consist of all points in the sample space S that are not in E. That is, E^c will occur if and only if E does not occur. In example 4, if event $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, then E^c will occur when the sum of the dice does not equal 7. Also note that because the experiment must result in some outcome, it follows that $S^c = \varnothing$.

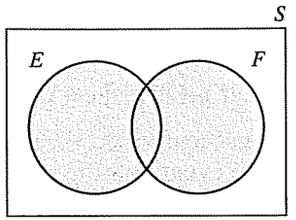
For any two events E and F, if all of the points in E are also in F, then we say that E is contained in F and write $E \subset F$ (or equivalently, $F \supset E$). Thus, if $E \subset F$, the occurrence of E necessarily implies the occurrence of F. If $E \subset F$ and $F \subset E$, we say that E and F are equal and write $E = F$.

A graphical representation that is very useful for illustrating logical relations among events is the Venn diagram. The sample space S is represented as consisting of all the points in a large rectangle, and the events $E, F, G, \ldots$ are represented as consisting of all the points in given circles within the rectangle. Events of interest can then be indicated by shading appropriate regions of the diagram. For instance, in the three Venn diagrams shown in Figure 2.1, the shaded areas represent, respectively, the events $E \cup F$, EF, and E^c. The Venn diagram in Figure 2.2 indicates that $E \subset F$.

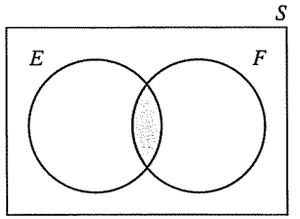
The operations of forming unions, intersections, and complements of events obey certain rules not dissimilar to the rules of algebra. We list a few of these rules.

Commutative laws	$E \cup F = F \cup E$	$EF = FE$
Associative laws	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive laws	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

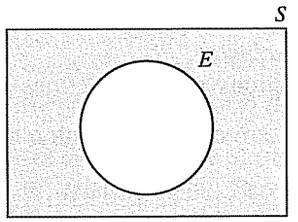
These relations are verified by showing that any outcome that is contained in the event on the left side of the equality sign is also contained in the event on the



(a) Shaded region: $E \cup F$.



(b) Shaded region: EF.



(c) Shaded region: E^c.

Figure 2.1

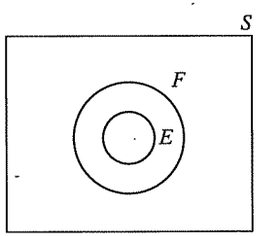
right side, and vice versa. One way of showing this is by means of Venn diagrams. For instance, the distributive law may be verified by the sequence of diagrams in Figure 2.3.

The following useful relationships between the three basic operations of forming unions, intersections, and complements are known as *DeMorgan's laws*:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

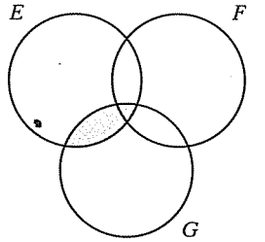
$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

To prove DeMorgan's laws, suppose first that x is a point of $\left(\bigcup_{i=1}^{n} E_i\right)^c$. Then x is not contained in $\bigcup_{i=1}^{n} E_i$, which means that x is not contained in any of the events E_i, $i = 1, 2, \ldots, n$, implying that x is contained in E_i^c for all $i = 1, 2, \ldots, n$ and thus is contained in $\bigcap_{i=1}^{n} E_i^c$. To go the other way, suppose that x is a point of $\bigcap_{i=1}^{n} E_i^c$. Then x is contained in E_i^c for all $i = 1, 2, \ldots, n$, which means that x is not contained in E_i for any $i = 1, 2, \ldots, n$, implying that x is not contained in $\bigcup_{i}^{n} E_i$, which yields that x is contained in $\left(\bigcup_{1}^{n} E_i\right)^c$. This proves the first of DeMorgan's laws.

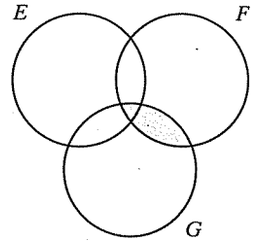


$E \subset F$

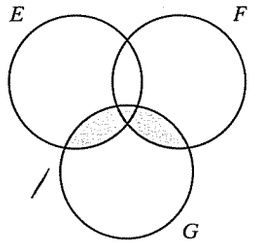
Figure 2.2



(a) Shaded region: EG.



(b) Shaded region: FG.



(c) Shaded region: $(E \cup F)G$.
$(E \cup F)G = EG \cup FG$

Figure 2.3

To prove the second of DeMorgan's laws, we use the first law to obtain

$$\left(\bigcup_{i=1}^{n} E_i^c\right)^c = \bigcap_{i=1}^{n} (E_i^c)^c$$

which, since $(E^c)^c = E$, is equivalent to

$$\left(\bigcup_{1}^{n} E_i^c\right)^c = \bigcap_{1}^{n} E_i$$

Taking complements of both sides of the above yields the result, namely,

$$\bigcup_{1}^{n} E_i^c = \left(\bigcap_{1}^{n} E_i\right)^c$$

2.3 AXIOMS OF PROBABILITY

One way of defining the probability of an event is in terms of its relative frequency. Such a definition usually goes as follows: We suppose that an experiment, whose sample space is S, is repeatedly performed under exactly the same conditions. For each event E of the sample space S, we define $n(E)$ to be the number of times in the first n repetitions of the experiment that the event E occurs. Then $P(E)$, the probability of the event E, is defined by

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

That is, $P(E)$ is defined as the (limiting) proportion of time that E occurs. It is thus the limiting frequency of E.

Although the preceding definition is certainly intuitively pleasing and should always be kept in mind by the reader, it possesses a serious drawback: How do we know that $n(E)/n$ will converge to some constant limiting value that will be the same for each possible sequence of repetitions of the experiment? For example, suppose that the experiment to be repeatedly performed consists of flipping a coin. How do we know that the proportion of heads obtained in the first n flips will converge to some value as n gets large? Also, even if it does converge to some value, how do we know that, if the experiment is repeatedly performed a second time, we shall again obtain the same limiting proportion of heads?

Proponents of the relative frequency definition of probability usually answer this objection by stating that the convergence of $n(E)/n$ to a constant limiting value is an assumption, or an *axiom*, of the system. However, to assume that $n(E)/n$ will necessarily converge to some constant value seems to be a very complex assumption. For, although we might indeed hope that such a constant limiting frequency exists, it does not at all seem to be a priori evident that this need be the case. In fact, would it not be more reasonable to assume a set of simpler and more self-evident axioms about probability and then attempt to prove that such a constant limiting frequency does in some sense exist? This latter approach is the modern axiomatic approach to probability theory that we shall adopt in this text. In particular, we shall assume that for each event E in the sample space S there exists a value $P(E)$, referred to as the probability of E. We shall then assume that the probabilities satisfy a certain set of axioms, which, we hope the reader will agree, is in accordance with our intuitive notion of probability.

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number $P(E)$ is defined and satisfies the following three axioms.

Axiom 1

$$0 \le P(E) \le 1$$

Axiom 2

$$P(S) = 1$$

Axiom 3

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_iE_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

We refer to $P(E)$ as the probability of the event E.

Thus Axiom 1 states that the probability that the outcome of the experiment is a point in E is some number between 0 and 1. Axiom 2 states that, with probability 1, the outcome will be a point in the sample space S. Axiom 3 states that for any sequence of mutually exclusive events the probability of at least one of these events occurring is just the sum of their respective probabilities.

If we consider a sequence of events $E_1, E_2, \ldots$, where $E_1 = S$, $E_i = \emptyset$ for $i > 1$, then, as the events are mutually exclusive and as $S = \bigcup_{i=1}^{\infty} E_i$, we have from Axiom 3 that

$$P(S) = \sum_{i=1}^{\infty} P(E_i) = P(S) + \sum_{i=2}^{\infty} P(\emptyset)$$

implying that

$$P(\emptyset) = 0$$

That is, the null event has probability 0 of occurring.

It should also be noted that it follows that for any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \tag{3.1}$$

This follows from Axiom 3 by defining E_i to be the null event for all values of i greater than n. Axiom 3 is equivalent to Equation (3.1) when the sample space is finite (why?). However, the added generality of Axiom 3 is necessary when the sample space consists of an infinite number of points.

Example 3a. If our experiment consists of tossing a coin and if we assume that a head is as likely to appear as a tail, then we would have

$$P(\{H\}) = P(\{T\}) = \tfrac{1}{2}$$

On the other hand, if the coin were biased and we felt that a head were twice as likely to appear as a tail, then we would have

$$P(\{H\}) = \frac{2}{3} \qquad P(\{T\}) = \frac{1}{3}$$

Example 3b. If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$. From Axiom 3 it would thus follow that the probability of rolling an even number would equal

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2}$$

The assumption of the existence of a set function P, defined on the events of a sample space S, and satisfying Axioms 1, 2, and 3, constitutes the modern mathematical approach to probability theory. Hopefully, the reader will agree that the axioms are natural and in accordance with our intuitive concept of probability as related to chance and randomness. Furthermore, using these axioms we shall be able to prove that if an experiment is repeated over and over again then, with probability 1, the proportion of time during which any specific event E occurs will equal $P(E)$. This result, known as the strong law of large numbers, is presented in Chapter 8. In addition, we present another possible interpretation of probability—as being a measure of belief—in Section 2.7.

TECHNICAL REMARK. We have supposed that $P(E)$ is defined for all the events E of the sample space. Actually, when the sample space is an uncountably infinite set $P(E)$ is defined only for a class of events called measurable. However, this restriction need not concern us as all events of any practical interest are measurable.

2.4 SOME SIMPLE PROPOSITIONS

In this section we prove some simple propositions regarding probabilities. We first note that as E and E^c are always mutually exclusive and since $E \cup E^c = S$, we have by Axioms 2 and 3 that

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

Or equivalently, we have the statement given in Proposition 4.1.

Proposition 4.1

$$P(E^c) = 1 - P(E)$$

In words, Proposition 4.1 states that the probability that an event does not occur is 1 minus the probability that it does occur. For instance, if the probability of obtaining a head on the toss of a coin is $\frac{3}{8}$, the probability of obtaining a tail must be $\frac{5}{8}$.

Our second proposition states that if the event E is contained in the event F, then the probability of E is no greater than the probability of F.

Proposition 4.2

If $E \subset F$, then $P(E) \leq P(F)$.

Proof: Since $E \subset F$, it follows that we can express F as

$$F = E \cup E^c F$$

Hence, as E and $E^c F$ are mutually exclusive, we obtain from Axiom 3 that

$$P(F) = P(E) + P(E^c F)$$

which proves the result, since $P(E^c F) \geq 0$.

Proposition 4.2 tells us, for instance, that the probability of rolling a 1 with a die is less than or equal to the probability of rolling an odd value with the die.

The next proposition gives the relationship between the probability of the union of two events in terms of the individual probabilities and the probability of the intersection.

Proposition 4.3

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Proof: To derive a formula for $P(E \cup F)$, we first note that $E \cup F$ can be written as the union of the two disjoint events E and $E^c F$. Thus from Axiom 3 we obtain that

$$\begin{aligned} P(E \cup F) &= P(E \cup E^c F) \\ &= P(E) + P(E^c F) \end{aligned}$$

Furthermore, since $F = EF \cup E^c F$, we again obtain from Axiom 3 that

$$P(F) = P(EF) + P(E^c F)$$

or, equivalently,

$$P(E^c F) = P(F) - P(EF)$$

thus completing the proof.

Proposition 4.3 could also have been proved by making use of the Venn diagram in Figure 2.4.

Let us divide $E \cup F$ into three mutually exclusive sections, as shown in Figure 2.5. In words, section I represents all the points in E that are not in F (that

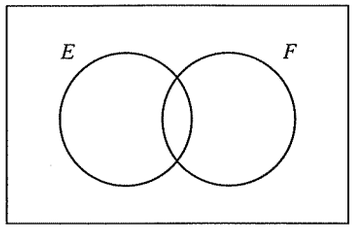


Figure 2.4 Venn diagram.

is, EF^c); section II represents all points both in E and in F (that is, EF); and section III represents all points in F that are not in E (that is, E^cF).

From Figure 2.5 we see that

$$E \cup F = \text{I} \cup \text{II} \cup \text{III}$$
$$E = \text{I} \cup \text{II}$$
$$F = \text{II} \cup \text{III}$$

As I, II, and III are mutually exclusive, it follows from Axiom 3 that

$$P(E \cup F) = P(\text{I}) + P(\text{II}) + P(\text{III})$$
$$P(E) = P(\text{I}) + P(\text{II})$$
$$P(F) = P(\text{II}) + P(\text{III})$$

which shows that

$$P(E \cup F) = P(E) + P(F) - P(\text{II})$$

and Proposition 4.3 is proved, since II $= EF$.

Example 4a. Suppose that we toss two coins and suppose that each of the four points in the sample space $S = \{(H, H), (H, T), (T, H), (T, T)\}$ is equally likely and hence has probability $\frac{1}{4}$. Let

$$E = \{(H, H), (H, T)\} \quad \text{and} \quad F = \{(H, H), (T, H)\}$$

That is, E is the event that the first coin falls heads, and F is the event that the second coin falls heads.

By Proposition 4.3 we have that $P(E \cup F)$, the probability that either the first or second coin falls heads, is given by

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(EF) \\ &= \tfrac{1}{2} + \tfrac{1}{2} - P(\{H, H)\}) \\ &= 1 - \tfrac{1}{4} \\ &= \tfrac{3}{4} \end{aligned}$$

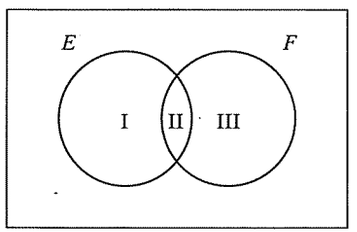


Figure 2.5 Venn diagram in sections.

This probability could, of course, have been computed directly because

$$P(E \cup F) = P(\{(H, H), (H, T), (T, H)\}) = \tfrac{3}{4}$$ ∎

We may also calculate the probability that any one of the three events E or F or G occurs:

$$P(E \cup F \cup G) = P[(E \cup F) \cup G]$$

which by Proposition 4.3 equals

$$P(E \cup F) + P(G) - P[(E \cup F)G]$$

Now, it follows from the distributive law that the events $(E \cup F)G$ and $EG \cup FG$ are equivalent, and hence we obtain from the preceding equations that

$$\begin{aligned} &P(E \cup F \cup G) \\ &\quad = P(E) + P(F) - P(EF) + P(G) - P(EG \cup FG) \\ &\quad = P(E) + P(F) - P(EF) + P(G) - P(EG) - P(FG) + P(EGFG) \\ &\quad = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \end{aligned}$$

In fact, the following proposition can be proved by induction.

Proposition 4.4

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r}) + \cdots + (-1)^{n+1} P(E_1E_2\cdots E_n)$$

The summation $\sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \ldots, n\}$.

In words, Proposition 4.4 states that the probability of the union of n events equals the sum of the probabilities of these events taken one at a time, minus the sum of the probabilities of these events taken two at a time, plus the sum of the probabilities of these events taken three at a time, and so on.

REMARK. For a noninductive argument for Proposition 4.4, note first that if a point of the sample space is not a member of any of the sets E_i then its probability does not contribute anything to either side of the equality. On the other hand, suppose that a point is in exactly m of the events E_i, where $m > 0$. Then since it is in $\bigcup_i E_i$ its probability is counted once in $P\left(\bigcup_i E_i\right)$; also as this point is

contained in $\binom{m}{k}$ subsets of the type $E_{i_1}E_{i_2}\cdots E_{i_k}$, its probability is counted

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots \pm \binom{m}{m}$$

times on the right of the equality sign in Proposition 4.4. Thus, for $m > 0$, we must show that

$$1 = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots \pm \binom{m}{m}$$

However, since $1 = \binom{m}{0}$, the preceding is equivalent to

$$\sum_{i=0}^{m} \binom{m}{i}(-1)^i = 0$$

and the latter equation follows from the binomial theorem since

$$0 = (-1 + 1)^m = \sum_{i=0}^{m} \binom{m}{i}(-1)^i(1)^{m-i}$$

2.5 SAMPLE SPACES HAVING EQUALLY LIKELY OUTCOMES

For many experiments it is natural to assume that all outcomes in the sample space are equally likely to occur. That is, consider an experiment whose sample space S is a finite set, say $S = \{1, 2, \ldots, N\}$. Then it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$$

which implies from Axioms 2 and 3 (why?) that

$$P(\{i\}) = \frac{1}{N} \qquad i = 1, 2, \ldots, N$$

From this it follows from Axiom 3 that for any event E

$$P(E) = \frac{\text{number of points in } E}{\text{number of points in } S}$$

In words, if we assume that all outcomes of an experiment are equally likely to occur, then the probability of any event E equals the proportion of points in the sample space that are contained in E.

Example 5a. If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomes, namely (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), that result in the sum of the dice being equal to 7, the desired probability is $\frac{6}{36} = \frac{1}{6}$. ∎

Example 5b. If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

Solution If we regard the order in which the balls are selected as being relevant, then the sample space consists of $11 \cdot 10 \cdot 9 = 990$ outcomes. Furthermore, there are $6 \cdot 5 \cdot 4 = 120$ outcomes in which the first ball selected is white and the other two black, $5 \cdot 6 \cdot 4 = 120$ outcomes in which the first is black, the second white, and the third black; and $5 \cdot 4 \cdot 6 = 120$ in which the first two are black and the third white. Hence, assuming that "randomly drawn" means that each outcome in the sample space is equally likely to occur, we see that the desired probability is

$$\frac{120 + 120 + 120}{990} = \frac{4}{11}$$

This problem could also have been solved by regarding the outcome of the experiment as the unordered set of drawn balls. From this point of view, there are $\binom{11}{3} = 165$ outcomes in the sample space. Now, each set of 3 balls corresponds to 3! outcomes when the order of selection is noted. As a result, if all outcomes are assumed equally likely when the order of selection is noted, then it follows that they remain equally likely when the outcome is taken to be the unordered set of selected balls. Hence, using the latter representation of the experiment, we see that the desired probability is

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

which, of course, agrees with the answer obtained previously. ∎

Example 5c. A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Solution Let us assume that *randomly selected* means that each of the $\binom{15}{5}$ possible combinations is equally likely to be selected. Hence the desired probability equals

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

∎

Example 5d. An urn contains n balls, of which one is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be

any of the balls that remain at the time, what is the probability that the special ball is chosen?

Solution Since all of the balls are treated in an identical manner, it follows that the set of k balls selected is equally likely to be any of the $\binom{n}{k}$ sets of k balls. Therefore,

$$P\{\text{special ball is selected}\} = \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

We could also have obtained the preceding result by letting A_i denote the event that the special ball is the ith ball to be chosen, $i = 1, \ldots, k$. Then, since each one of the n balls is equally likely to be the ith ball chosen, it follows that $P(A_i) = 1/n$. Hence, since these events are obviously mutually exclusive, we have that

$$P\{\text{special ball is selected}\} = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) = \frac{k}{n}$$

We could have argued that $P(A_i) = 1/n$, by noting that there are $n(n-1)\cdots(n-k+1) = n!/(n-k)!$ equally likely outcomes of the experiment, of which $(n-1)(n-2)\cdots(n-i+1)(1)(n-i)\cdots(n-k+1) = (n-1)!/(n-k)!$ result in the special ball being the ith one chosen. From this it follows that

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$ ∎

Example 5e. Suppose that $n + m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n+m)!$ possible orderings are equally likely. If we record the result of this experiment by only listing the colors of the successive balls, show that all the possible results remain equally likely.

Solution Consider any one of the $(n+m)!$ possible orderings and note that any permutation of the red balls among themselves and of the blue balls among themselves does not change the sequence of colors. As a result, every ordering of colorings corresponds to $n!\,m!$ different orderings of the $n + m$ balls, so every ordering of the colors has probability $\frac{n!\,m!}{(n+m)!}$ of occurring.

For example, suppose that there are 2 red balls, numbered r_1, r_2 and 2 blue balls, numbered b_1, b_2. Then, of the 4! possible orderings, there will be 2! 2! orderings that result in any specified color combination. For instance, the following orderings result in the successive balls alternating in color with a red ball first:

r_1, b_1, r_2, b_2 $\quad$ r_1, b_2, r_2, b_1 $\quad$ r_2, b_1, r_1, b_2 $\quad$ r_2, b_2, r_1, b_1

Hence each of the possible orderings of the colors has probability $\frac{4}{24} = \frac{1}{6}$ of occurring. ∎

Example 5f. A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

Solution We start by assuming that all $\binom{52}{5}$ possible poker hands are equally likely. To determine the number of outcomes that are straights, let us first determine the number of possible outcomes for which the poker hand consists of an ace, two, three, four, and five (the suits being irrelevant). Since the ace can be any 1 of the 4 possible aces, and similarly for the two, three, four, and five, it follows that there are 4^5 outcomes leading to exactly one ace, two, three, four, and five. Hence, since in 4 of these outcomes all the cards will be of the same suit (such a hand is called a straight flush), it follows that there are $4^5 - 4$ hands that make up a straight of the form ace, two, three, four, and five. Similarly, there are $4^5 - 4$ hands that make up a straight of the form ten, jack, queen, king, and ace. Hence there are $10(4^5 - 4)$ hands that are straights. Thus the desired probability is

$$\frac{10(4^5 - 4)}{\binom{52}{5}} \approx .0039$$ ∎

Example 5g. A 5-card poker hand is said to be a full house if it consists of 3 cards of the same denomination and 2 cards of the same denomination. (That is, a full house is three of a kind plus a pair.) What is the probability that one is dealt a full house?

Solution Again we assume that all $\binom{52}{5}$ possible hands are equally likely. To determine the number of possible full houses, we first note that there are $\binom{4}{2}\binom{4}{3}$ different combinations of, say, 2 tens and 3 jacks. Because there are 13 different choices for the kind of pair and, after a pair has been chosen, there are 12 other choices for the denomination of the remaining 3 cards, it follows that the probability of a full house is

$$\frac{13 \cdot 12 \cdot \binom{4}{2}\binom{4}{3}}{\binom{52}{5}} \approx .0014$$ ∎

Example 5h. In the game of bridge the entire deck of 52 cards is dealt out to 4 players. What is the probability that
(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?

Solution (a) There are $\binom{52}{13, 13, 13, 13}$ possible divisions of the cards among the 4 distinct players. As there are $\binom{39}{13, 13, 13}$ possible divisions of the cards leading to a fixed player having all 13 spades, it follows that the desired probability is given by

$$\frac{4\binom{39}{13, 13, 13}}{\binom{52}{13, 13, 13, 13}} \approx 6.3 \times 10^{-12}$$

(b) To determine the number of outcomes in which each of the distinct players receives exactly 1 ace, put aside the aces and note that there are $\binom{48}{12, 12, 12, 12}$ possible divisions of the other 48 cards when each player is to receive 12. As there are 4! ways of dividing the 4 aces so that each player receives 1, we see that the number of possible outcomes in which each player receives exactly 1 ace is $4!\binom{48}{12, 12, 12, 12}$. Hence the desired probability is

$$\frac{4!\binom{48}{12, 12, 12, 12}}{\binom{52}{13, 13, 13, 13}} \approx .105$$

∎

Some results in probability are quite surprising when initially encountered. Our next two examples illustrate this phenomenon.

Example 5i. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

Solution As each person can celebrate his or her birthday on any one of 365 days, there is a total of $(365)^n$ possible outcomes. (We are ignoring the possibility of someone's having been born on February 29.) Assuming that each outcome is equally likely, we see that the desired probability is $(365)(364)(363)\cdots(365 - n + 1)/(365)^n$. It is a rather surprising fact that when $n \geq 23$, this probability is less than $\frac{1}{2}$. That is, if there are 23 or more people in a room, then the probability that at least two of them have the same birthday exceeds $\frac{1}{2}$. Many people are initially surprised by this result, since 23 seems so small in relation to 365, the number of days of the year. However, every pair of individuals has probability $\frac{365}{(365)^2} = \frac{1}{365}$ of having the same birthday, and in a group of 23 people there are $\binom{23}{2} = 253$ different pairs of individuals. Looked at this way, the result no longer seems so surprising.

When there are 50 people in the room, the probability that at least two share the same birthday is approximately .970. And with 100 persons in the room, the odds are better than 3,000,000:1 (that is, the probability is greater than $\frac{3 \times 10^6}{3 \times 10^6 + 1}$) that at least two people have the same birthday. ∎

Example 5j. A deck of 52 playing cards is shuffled and the cards turned up one at a time until the first ace appears. Is the next card—that is, the card following the first ace—more likely to be the ace of spades or the two of clubs?

Solution To determine the probability that the card following the first ace is the ace of spades, we need to calculate how many of the (52)! possible orderings of the cards have the ace of spades immediately following the first ace. To begin, note that each ordering of the 52 cards can be obtained by first ordering the 51 cards different from the ace of spades and then inserting the ace of spades into that ordering. Furthermore, for each of the (51)! orderings of the other cards, there is only one place where the ace of spades can be placed so that it follows the first ace. For instance, if the ordering of the other 51 cards is

$$4c, 6h, Jd, 5s, Ac, 7d, \ldots, Kh$$

then the only insertion of the ace of spaces into this ordering that results in it following the first ace is

$$4c, 6h, Jd, 5s, Ac, As, 7d, \ldots, Kh$$

Therefore, we see that there are (51)! orderings that result in the ace of spades following the first ace, so

$$P\{\text{the ace of spades follows the first ace}\} = \frac{(51)!}{(52)!} = \frac{1}{52}$$

In fact, by exactly the same argument, it follows that the probability that the two of clubs (or any other specified card) follows the first ace is also $\frac{1}{52}$. In other words, each of the 52 cards of the deck is equally likely to be the one that follows the first ace!

Many people find this result rather surprising. Indeed, a common reaction is to suppose initially that it is more likely that the two of clubs (rather than the ace of spades) follows the first ace, since that first ace might itself

be the ace of spades. This reaction is often followed by the realization that the two of clubs might itself appear before the first ace, thus negating its chance of immediately following the first ace. However, as there is one chance in four that the ace of spades will be the first ace (because all 4 aces are equally likely to be first) and only one chance in five that the two of clubs will appear before the first ace (because each of the set of 5 cards consisting of the two of clubs and the 4 aces is equally likely to be the first of this set to appear), it again appears that the two of clubs is more likely. However, this is not the case and a more complete analysis shows that they are equally likely. ∎

Example 5k. A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs? What is the probability that there are $2i$ offensive–defensive roommate pairs, $i = 1, 2, \ldots, 10$?

Solution There are

$$\binom{40}{2, 2, \ldots, 2} = \frac{(40)!}{(2!)^{20}}$$

ways of dividing the 40 players into 20 *ordered* pairs of two each. [That is, there are $(40)!/2^{20}$ ways of dividing the players into a *first* pair, a *second* pair, and so on.] Hence there are $(40)!/2^{20}(20)!$ ways of dividing the players into (unordered) pairs of 2 each. Furthermore, since a division will result in no offensive–defensive pairs if the offensive (and defensive) players are paired among themselves, it follows that there are $[(20)!/2^{10}(10)!]^2$ such divisions. Hence the probability of no offensive–defensive roommate pairs, call it P_0, is given by

$$P_0 = \frac{\left(\frac{(20)!}{2^{10}(10)!}\right)^2}{\frac{(40)!}{2^{20}(20)!}} = \frac{[(20)!]^3}{[(10)!]^2(40)!}$$

To determine P_{2i}, the probability that there are $2i$ offensive–defensive pairs, we first note that there are $\binom{20}{2i}^2$ ways of selecting the $2i$ offensive players and the $2i$ defensive players who are to be in the offensive–defensive pairs. These $4i$ players can then be paired up into $(2i)!$ possible offensive–defensive pairs. (This is so because the first offensive can be paired with any of the $2i$ defensives, the second offensive with any of the remaining $2i - 1$ defensives, and so on.) As the remaining $20 - 2i$ offensives (and defensives) must be paired among themselves, it follows that there are

$$\binom{20}{2i}^2 (2i)! \left[\frac{(20 - 2i)!}{2^{10-i}(10 - i)!}\right]^2$$

divisions which lead to $2i$ offensive–defensive pairs. Hence

$$P_{2i} = \frac{\binom{20}{2i}^2 (2i)! \left[\frac{(20 - 2i)!}{2^{10-i}(10 - i)!}\right]^2}{\frac{(40)!}{2^{20}(20)!}} \qquad i = 0, 1, \ldots, 10$$

The $P_{2i}, i = 0, 1, \ldots, 10$, can now be computed or they can be approximated by making use of a result of Stirling which shows that $n!$ can be approximated by $n^{n+1/2}e^{-n}\sqrt{2\pi}$. For instance, we obtain that

$$P_0 \approx 1.3403 \times 10^{-6}$$
$$P_{10} \approx .345861$$
$$P_{20} \approx 7.6068 \times 10^{-6}$$

∎

Our next three examples illustrate the usefulness of Proposition 4.4. In Example 5l, the introduction of probability enables us to obtain a quick solution to a counting problem.

Example 5l. A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all three sports. How many members of this club play at least one of these sports?

Solution Let N denote the number of members of the club, and introduce probability by assuming that a member of the club is randomly selected. If for any subset C of members of the club, we let $P(C)$ denote the probability that the selected member is contained in C, then

$$P(C) = \frac{\text{number of members in } C}{N}$$

Now, with T being the set of members that plays tennis, S being the set that plays squash, and B being the set that plays badminton, we have from Proposition 4.4 that

$$\begin{aligned} P(T \cup S \cup B) &= P(T) + P(S) + P(B) - P(TS) - P(TB) - P(SB) + P(TSB) \\ &= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N} \\ &= \frac{43}{N} \end{aligned}$$

Hence we can conclude that 43 members play at least one of the sports. ∎

The next example in this section not only possesses the virtue of giving rise to a somewhat surprising answer but is also of theoretical interest.

Example 5m. ***The matching problem.*** Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that
(a) none of the men selects his own hat;
(b) exactly k of the men select their own hats?

Solution (a) We first calculate the complementary probability of at least one man's selecting his own hat. Let us denote by E_i, $i = 1, 2, \ldots, N$ the event that the ith man selects his own hat. Now, by Proposition 4.4 $P\left(\bigcup_{i=1}^{N} E_i\right)$, the probability that at least one of the men selects his own hat, is given by

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{n+1} \sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) + \cdots + (-1)^{N+1} P(E_1E_2\cdots E_N)$$

If we regard the outcome of this experiment as a vector of N numbers, where the ith element is the number of the hat drawn by the ith man, then there are $N!$ possible outcomes. [The outcome $(1, 2, 3, \ldots, N)$ means, for example, that each man selects his own hat.] Furthermore, $E_{i_1}E_{i_2}\ldots E_{i_n}$, the event that each of the n men $i_1, i_2, \ldots, i_n$ selects his own hat, can occur in any of $(N-n)(N-n-1)\cdots 3\cdot 2\cdot 1 = (N-n)!$ possible ways; for, of the remaining $N-n$ men, the first can select any of $N-n$ hats, the second can then select any of $N-n-1$ hats, and so on. Hence, assuming that all $N!$ possible outcomes are equally likely, we see that

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{(N-n)!}{N!}$$

Also, as there are $\binom{N}{n}$ terms in $\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n})$, we see that

$$\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{N!(N-n)!}{(N-n)!\,n!\,N!} = \frac{1}{n!}$$

and thus

$$P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{N+1}\frac{1}{N!}$$

Hence the probability that none of the men selects his own hat is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!}$$

which for N large is approximately equal to $e^{-1} \approx .36788$. In other words, for N large, the probability that none of the men selects his own hat is approximately .37. (How many readers would have incorrectly thought that this probability would go to 1 as $N \to \infty$?)

(b) To obtain the probability that exactly k of the N men select their own hats, we first fix attention on a particular set of k men. The number of ways in which these and only these k men can select their own hats is equal to the number of ways in which the other $N-k$ men can select among their hats in such a way that none of them selects his own hat. But, as

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}$$

is the probability that not one of $N-k$ men, selecting among their hats, selects his own, it follows that the number of ways in which the set of men selecting their own hats corresponds to the set of k men under consideration is

$$(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]$$

Hence, as there are $\binom{N}{k}$ possible selections of a group of k men, it follows that there are

$$\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]$$

ways in which exactly k of the men select their own hats. The desired probability is thus

$$\frac{\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]}{N!} = \frac{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}}{k!}$$

which for N large is approximately $e^{-1}/k!$. The values $e^{-1}/k!$, $k = 0, 1, \ldots$, are of some theoretical importance as they represent the values associated with the Poisson distribution. This is elaborated upon in Chapter 4.[†] ∎

For another illustration of the usefulness of Proposition 4.4, consider the following example.

[†] See Example 5c of Chapter 3 for another approach to this problem.

Example 5n. If 10 married couples are seated at random at a round table, compute the probability that no wife sits next to her husband.

Solution If we let E_i, $i = 1, 2, \ldots, 10$ denote the event that the ith couple sit next to each other, it follows that the desired probability is $1 - P\left(\bigcup_{i=1}^{10} E_i\right)$. Now, from Proposition 4.4,

$$P\left(\bigcup_{1}^{10} E_i\right) = \sum_{1}^{10} P(E_i) - \cdots + (-1)^{n+1} \sum_{i_1<i_2<\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) + \cdots - P(E_1E_2\cdots E_{10})$$

To compute $P(E_{i_1}E_{i_2}\cdots E_{i_n})$, we first note that there are 19! ways of arranging 20 people around a round table (why?). The number of arrangements that result in a specified set of n men sitting next to their wives can most easily be obtained by first thinking of each of the n married couples as being single entities. If this were the case, then we would need to arrange $20 - n - 1$)! such arrangements. Finally, since each of the n married couples can be arranged next to each other in one of two possible ways, it follows that there are $2^n(20 - n - 1)!$ arrangements that result in a specified set of n men each sitting next to their wives. Therefore,

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{2^n(19 - n)!}{(19)!}$$

Thus, from Proposition 4.4, we obtain that the probability that at least one married couple sits together equals

$$\binom{10}{1}2^1\frac{(18)!}{(19)!} - \binom{10}{2}2^2\frac{(17)!}{(19)!} + \binom{10}{3}2^3\frac{(16)!}{(19)!} - \cdots - \binom{10}{10}2^{10}\frac{9!}{(19)!} \approx .6605$$

and the desired probability is approximately .3395. ∎

Example 5o.** ***Runs. Consider an athletic team that had just finished its season with a final record of n wins and m losses. By examining the sequence of wins and losses, we are hoping to determine whether the team had stretches of games in which it was more likely to win than at other times. One way to gain some insight into this question is to count the number of runs of wins and then see how likely that result would be when all $(n + m)!/(n!\, m!)$ orderings of the n wins and m losses are assumed equally likely. By a run of wins we mean a consecutive sequence of wins. For instance, if $n = 10$, $m = 6$ and the sequence of outcomes was *WWLLWWWLWLLLWWWW*, then there would be 4 runs of wins—the first run being of size 2, the second of size 3, the third of size 1, and the fourth of size 4.

Suppose now that a team has n wins and m losses. Assuming that all $(n + m)!/(n!\, m!) = \binom{n+m}{n}$ orderings are equally likely, let us determine the probability that there will be exactly r runs of wins. To do so, consider first any vector of positive integers $x_1, x_2, \ldots, x_r$ with $x_1 + \cdots + x_r = n$, and let us see how many outcomes result in r runs of wins in which the ith run is of size x_i, $i = 1, \ldots, r$. For any such outcome, if we let y_1 denote the number of losses before the first run of wins, y_2 the number of losses between the first 2 runs of wins, $\ldots$, y_{r+1} the number of losses after the last run of wins, then the y_i satisfy

$$y_1 + y_2 + \cdots + y_{r+1} = m \qquad y_1 \geq 0, y_{r+1} \geq 0, y_i > 0, i = 2, \ldots, r$$

and the outcome can be represented schematically as

$$\underbrace{LL\ldots L}_{y_1}\ \underbrace{WW\ldots W}_{x_1}\ \underbrace{L\ldots L}_{y_2}\ \underbrace{WW\ldots W}_{x_2}\ \ldots\ \underbrace{WW}_{x_r}\ \underbrace{L\ldots L}_{y_{r+1}}$$

Hence the number of outcomes that result in r runs of wins—the ith of size x_i, $i = 1, \ldots r$—is equal to the number of integers $y_1, \ldots, y_{r+1}$ that satisfy the above, or equivalently, to the number of positive integers

$$\bar{y}_1 = y_1 + 1 \qquad \bar{y}_i = y_i, i = 2, \ldots, r, \quad \bar{y}_{r+1} = y_{r+1} + 1$$

that satisfy

$$\bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_{r+1} = m + 2$$

By Proposition 6.1 in Chapter 1 there are $\binom{m+1}{r}$ such outcomes. Hence the total number of outcomes that result in r runs of wins is $\binom{m+1}{r}$ multiplied by the number of positive integral solutions of $x_1 + \cdots + x_r = n$. Hence, again from Proposition 6.1, there are thus $\binom{m+1}{r}\binom{n-1}{r-1}$ outcomes resulting in r runs of wins. As there are $\binom{n+m}{n}$ equally likely outcomes, we thus see that

$$P(\{r \text{ runs of wins}\}) = \frac{\binom{m+1}{r}\binom{n-1}{r-1}}{\binom{m+n}{n}} \qquad r \geq 1$$

For example, if $n = 8$, $m = 6$, then the probability of 7 runs is $\binom{7}{7}\binom{7}{6}\Big/\binom{14}{8} = 1/429$ if all $\binom{14}{8}$ outcomes are equally likely. Hence,

if the outcome was *WLWLWLWLWWLWLW*, then we might suspect that the team's win probability was changing over time. (In particular, the probability that the team wins seems to be quite high when it lost its last game and quite low when it won its last game.) On the other extreme, if the outcome were *WWWWWWWWLLLLLL*, then there would have been only 1 run, and as $P(\{1 \text{ run}\}) = \binom{7}{1}\binom{7}{0}\Big/\binom{14}{8} = 1/429$, it would thus again seem unlikely that the team's win probability remained unchanged over its 14 games. ∎

*2.6 PROBABILITY AS A CONTINUOUS SET FUNCTION

A sequence of events $\{E_n, n \geq 1\}$ is said to be an increasing sequence if

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots$$

whereas it is said to be a decreasing sequence if

$$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots$$

If $\{E_n, n \geq 1\}$ is an increasing sequence of events, then we define a new event, denoted by $\lim_{n\to\infty} E_n$, by

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$$

when $E_n \subset E_{n+1}$ for all n. Similarly, if $\{E_n, n \geq 1\}$ is a decreasing sequence of events, we define $\lim_n E_n$ by

$$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$$

where $E_n \supset E_{n+1}$ for all n.

We now prove Proposition 6.1.

Proposition 6.1

If $\{E_n, n \geq 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$$

Proof: Suppose, first, that $\{E_n, n \geq 1\}$ is an increasing sequence and define the events F_n, $n \geq 1$ by

$$F_1 = E_1$$

$$F_n = E_n\left(\bigcup_1^{n-1} E_i\right)^c = E_n E_{n-1}^c \qquad n > 1$$

where we have used the fact that $\bigcup_1^{n-1} E_i = E_{n-1}$, since the events are increasing.

In words, F_n consists of those points in E_n that are not in any of the earlier E_i, $i < n$. It is easy to verify that the F_n are mutually exclusive events such that

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i \qquad \text{and} \qquad \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \qquad \text{for all } n \geq 1$$

Thus

$$\begin{aligned} P\left(\bigcup_1^{\infty} E_i\right) &= P\left(\bigcup_1^{\infty} F_i\right) \\ &= \sum_1^{\infty} P(F_i) \qquad \text{(by Axiom 3)} \\ &= \lim_{n\to\infty} \sum_1^{n} P(F_i) \\ &= \lim_{n\to\infty} P\left(\bigcup_1^{n} F_i\right) \\ &= \lim_{n\to\infty} P\left(\bigcup_1^{n} E_i\right) \\ &= \lim_{n\to\infty} P(E_n) \end{aligned}$$

which proves the result when $\{E_n, n \geq 1\}$ is increasing.

If $\{E_n, n \geq 1\}$ is a decreasing sequence, then $\{E_n^c, n \geq 1\}$ is an increasing sequence; hence, from the preceding equations,

$$P\left(\bigcup_1^{\infty} E_i^c\right) = \lim_{n\to\infty} P(E_n^c)$$

But as $\bigcup_1^{\infty} E_i^c = \left(\bigcap_1^{\infty} E_i\right)^c$, we see that

$$P\left(\left(\bigcap_1^{\infty} E_i\right)^c\right) = \lim_{n\to\infty} P(E_n^c)$$

or, equivalently,

$$1 - P\left(\bigcap_1^{\infty} E_i\right) = \lim_{n\to\infty} [1 - P(E_n)] = 1 - \lim_{n\to\infty} P(E_n)$$

or

$$P\left(\bigcap_1^{\infty} E_i\right) = \lim_{n\to\infty} P(E_n)$$

which proves the result.

Example 6a. ***Probability and a paradox.*** Suppose that we possess an infinitely large urn and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on. Consider an experiment performed as follows. At 1 minute to 12 P.M., balls numbered 1 through 10 are placed in the urn, and ball number 10 is withdrawn. (Assume the withdrawal takes no time.) At $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn, and ball number 20 is withdrawn. At $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn, and ball number 30 is withdrawn. At $\frac{1}{8}$ minute to 12 P.M., and so on. The question of interest is, how many balls are in the urn at 12 P.M.?

The answer to this question is clearly that there is an infinite number of balls in the urn at 12 P.M., since any ball whose number is not of the form $10n$, $n \geq 1$, will have been placed in the urn and will not have been withdrawn before 12 P.M. Hence the problem is solved when the experiment is performed as described.

However, let us now change the experiment and suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn, and ball number 1 is withdrawn; at $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn, and ball number 2 is withdrawn; at $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn, and ball number 3 is withdrawn; at $\frac{1}{8}$ minute to 12 P.M., balls numbered 31 through 40 are placed in the urn, and ball number 4 is withdrawn, and so on. For this new experiment how many balls are in the urn at 12 P.M.?

Surprisingly enough, the answer now is that the urn is *empty* at 12 P.M. For, consider any ball—say, ball number n. At some time prior to 12 P.M. [in particular, at $(\frac{1}{2})^{n-1}$ minutes to 12 P.M.], this ball would have been withdrawn from the urn. Hence for each n, ball number n is not in the urn at 12 P.M.; therefore, the urn must be empty at this time.

Thus we see from the preceding discussion that the manner in which the withdrawn balls are selected makes a difference. For, in the first case only balls numbered $10n$, $n \geq 1$, are ever withdrawn; whereas in the second case all of the balls are eventually withdrawn. Let us now suppose that whenever a ball is to be withdrawn that ball is randomly selected from among those present. That is, suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn, and a ball is randomly selected and withdrawn, and so on. In this case how many balls are in the urn at 12 P.M.?

Solution We shall show that, with probability 1, the urn is empty at 12 P.M. Let us first consider ball number 1. Define E_n to be the event that ball number 1 is still in the urn after the first n withdrawals have been made. Clearly,

$$P(E_n) = \frac{9 \cdot 18 \cdot 27 \cdots (9n)}{10 \cdot 19 \cdot 28 \cdots (9n+1)}$$

[To understand this equation, just note that if ball number 1 is still to be in the urn after the first n withdrawals, the first ball withdrawn can be any one of 9, the second any one of 18 (there are 19 balls in the urn at the time of the second withdrawal, one of which must be ball number 1), and so on. The denominator is similarly obtained.]

Now, the event that ball number 1 is in the urn at 12 P.M. is just the event $\bigcap_{n=1}^{\infty} E_n$. As the events E_n, $n \geq 1$, are decreasing events, it follows from Proposition 6.1 that

$$\begin{aligned} &P\{\text{ball number 1 is in the urn at 12 P.M.}\} \\ &\quad = P\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &\quad = \lim_{n\to\infty} P(E_n) \\ &\quad = \prod_{n=1}^{\infty} \left(\frac{9n}{9n+1}\right) \end{aligned}$$

We now show that

$$\prod_{n=1}^{\infty} \frac{9n}{9n+1} = 0$$

Since

$$\prod_{n=1}^{\infty} \left(\frac{9n}{9n+1}\right) = \left[\prod_{n=1}^{\infty} \left(\frac{9n+1}{9n}\right)\right]^{-1}$$

this is equivalent to showing that

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{9n}\right) = \infty$$

Now, for all $m \geq 1$,

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 + \frac{1}{9n}\right) &\geq \prod_{n=1}^{m} \left(1 + \frac{1}{9n}\right) \\ &= \left(1 + \frac{1}{9}\right)\left(1 + \frac{1}{18}\right)\left(1 + \frac{1}{27}\right) \cdots \left(1 + \frac{1}{9m}\right) \\ &> \frac{1}{9} + \frac{1}{18} + \frac{1}{27} + \cdots + \frac{1}{9m} \\ &= \frac{1}{9} \sum_{i=1}^{m} \frac{1}{i} \end{aligned}$$

Hence, letting $m \to \infty$ and using the fact that $\sum_{i=1}^{\infty} 1/i = \infty$ yields

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{9n}\right) = \infty$$

Hence, letting F_i denote the event that ball number i is in the urn at 12 P.M., we have shown that $P(F_1) = 0$. Similarly, we can show that $P(F_i) = 0$ for all i. (For instance, the same reasoning shows that $P(F_i) = \prod_{n=2}^{\infty} [9n/(9n + 1)]$ for $i = 11, 12, \ldots, 20$.) Therefore, the probability that the urn is not empty at 12 P.M., $P\left(\bigcup_1^\infty F_i\right)$, satisfies

$$P\left(\bigcup_1^\infty F_i\right) \leq \sum_1^\infty P(F_i) = 0$$

by Boole's inequality (see Self-Test Exercise 10).

Thus, with probability 1, the urn will be empty at 12 P.M. ∎

2.7 PROBABILITY AS A MEASURE OF BELIEF

Thus far we have interpreted the probability of an event of a given experiment as being a measure of how frequently the event will occur when the experiment is continually repeated. However, there are also other uses of the term *probability*. For instance, we have all heard such statements as, "it is 90 percent probable that Shakespeare actually wrote *Hamlet*," or "the probability that Oswald acted alone in assassinating Kennedy is .8." How are we to interpret these statements?

The most simple and natural interpretation is that the probabilities referred to are measures of the individual's belief in the statements that he or she is making. In other words, the individual making the foregoing statements is quite certain that Oswald acted alone and is even more certain that Shakespeare wrote *Hamlet*. This interpretation of probability as being a measure of one's belief is often referred to as the *personal* or *subjective* view of probability.

It seems logical to suppose that a "measure of belief" should satisfy all of the axioms of probability. For example, if we are 70 percent certain that Shakespeare wrote *Julius Caesar* and 10 percent certain that it was actually Marlowe, then it is logical to suppose that we are 80 percent certain that it was either Shakespeare or Marlowe. Hence, whether we interpret probability as a measure of belief or as a long-run frequency of occurrence, its mathematical properties remain unchanged.

Example 7a. Suppose that in a 7-horse race you feel that each of the first 2 horses has a 20 percent chance of winning, horses 3 and 4 each has a 15 percent chance, and the remaining 3 horses, a 10 percent chance each. Would it be better for you to wager at even money, that the winner will be one of the first three horses, or to wager, again at even money, that the winner will be one of the horses 1, 5, 6, 7?

Solution Based on your personal probabilities concerning the outcome of the race, your probability of winning the first bet is .2 + .2 + .15 = .55, whereas, it is .2 + .1 + .1 + .1 = .5 for the second. Hence the first wager is more attractive. ∎

It should be noted that in supposing that person's subjective probabilities are always consistent with the axioms of probability, we are dealing with an idealized rather than an actual person. For instance, if we were to ask someone what he or she thought the chances were of

(a) rain today,
(b) rain tomorrow,
(c) rain both today and tomorrow,
(d) rain either today or tomorrow,

it is quite possible that after some deliberation that this person might give 30 percent, 40 percent, 20 percent, and 60 percent as answers. Unfortunately, however, such answers (or such subjective probabilities) are not consistent with the axioms of probability (why not?). We would of course hope that after this was pointed out to the respondent, he or she would change the answers. (One possibility we could accept is 30 percent, 40 percent, 10 percent, and 60 percent.)

SUMMARY

Let S denote the set of all possible outcomes of an experiment. S is called the *sample space* of the experiment. An event is a subset of S. If A_i, $i = 1, \ldots, n$, are events, then $\bigcup_{i=1}^n A_i$, called the *union* of these events, consists of all outcomes that are in at least one of the events A_i, $i = 1, \ldots, n$. Similarly, $\bigcap_{i=1}^n A_i$, sometimes written as $A_1 \cdots A_n$, is called the *intersection* of the events A_i, and consists of all outcomes that are in all of the events A_i, $i = 1, \ldots, n$.

For any event A, we define A^c to consist of all outcomes in the sample space that are not in A. We call A^c the *complement* of the event A. The event S^c, which is empty of outcomes, is designated by $\emptyset$ and is called the *null* set. If $AB = \emptyset$, then we say that A and B are mutually exclusive.

For each event A of the sample space S we suppose that a number $P(A)$, called the probability of A, is defined and is such that

(i) $0 \leq P(A) \leq 1$
(ii) $P(S) = 1$
(iii) For mutually exclusive events A_i, $i \geq 1$,

$$P\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty P(A_i)$$

$P(A)$ represents the probability that the outcome of the experiment is in A.

It can be shown that

$$P(A^c) = 1 - P(A)$$

A useful result is that

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

which can be generalized to give

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{i<j} P(A_iA_j) + \sum\sum\sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n)$$

If S is finite and each one point set is assumed to have equal probability, then

$$P(A) = \frac{|A|}{|S|}$$

where $|E|$ denotes the number of points in the event E.

$P(A)$ can be interpreted either as a long-run relative frequency or as a measure of one's degree of belief.

PROBLEMS

1. A box contains 3 marbles, 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box, then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without first replacing the first marble.

2. A die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n? What is $\left(\bigcup_{1}^{\infty} E_n\right)^c$?

3. Two dice are thrown. Let E be the event that the sum of the dice is odd; let F be the event that at least one of the dice lands on 1; and let G be the event that the sum is 5. Describe the events EF, $E \cup F$, FG, EF^c, and EFG.

4. A, B, and C take turns in flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \{1, 01, 001, 0001, \ldots, 0000 \cdots\}$$

(a) Interpret the sample space.

(b) Define the following events in terms of S:

(i) A wins $= A$.

(ii) B wins $= B$.

(iii) $(A \cup B)^c$.

Assume that A flips first, then B, then C, then A, and so on.

5. A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

(d) Write out all the outcomes in the event AW.

6. A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

(a) Give the sample space of this experiment.

(b) Let A be the event that the patient is in serious condition. Specify the outcomes in A.

(c) Let B be the event that the patient is uninsured. Specify the outcomes in B.

(d) Give all the outcomes in the event $B^c \cup A$.

7. Consider an experiment that consists of determining the type of job—either blue collar or white collar—and the political affiliation—Republican, Democratic, or Independent—of the 15 members of an adult soccer team. How many outcomes are

(a) in the sample space;

(b) in the event that at least one of the team members is a blue-collar worker;

(c) in the event that none of the team members considers himself or herself an Independent?

8. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

(a) either A or B occurs;

(b) A occurs but B does not;

(c) both A and B occur?

9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both. What percentage of its customers carry a credit card that the establishment will accept?

10. Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) a ring or a necklace;

(b) a ring and a necklace?

over a 1-year span are, respectively, .05, .15, and .30. If 20 percent of the population are good risks, 50 percent are average risks, and 30 percent are bad risks, what proportion of people have accidents in a fixed year? If policyholder A had no accidents in 1997, what is the probability that he or she is a good (average) risk?

47. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, or weak are .7, .2, and .1, respectively.

(a) How certain is she that she will receive the new job offer?

(b) Given that she does receive the offer, how likely should she feel that she received a strong recommendation; a moderate recommendation; a weak recommendation?

(c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation; a moderate recommendation; a weak recommendation?

48. A high school student is anxiously waiting to receive mail telling her whether she has been accepted to a certain college. She estimates that the conditional probabilities, given that she is accepted and that she is rejected, of receiving notification on each day of next week are as follows:

Day	P(mail|accepted)	P(mail|rejected)
Monday	.15	.05
Tuesday	.20	.10
Wednesday	.25	.10
Thursday	.15	.15
Friday	.10	.20

She estimates that her probability of being accepted is .6.

(a) What is the probability that mail is received on Monday?

(b) What is the conditional probability that mail is received on Tuesday given that it is not received on Monday?

(c) If there is no mail through Wednesday, what is the conditional probability that she will be accepted?

(d) What is the conditional probability that she will be accepted if mail comes on Thursday?

(e) What is the conditional probability that she will be accepted if no mail arrives that week?

49. A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component independently works with probability $\frac{1}{2}$. Find the conditional probability that component 1 works given that the system is functioning.

50. If you had to construct a mathematical model for events E and F, as described in parts (a) through (e), would you assume that they were independent events? Explain your reasoning.

(a) E is the event that a businesswoman has blue eyes, and F is the event that her secretary has blue eyes.

(b) E is the event that a professor owns a car, and F is the event that he is listed in the telephone book.

(c) E is the event that a man is under 6 feet tall, and F is the event that he weighs over 200 pounds.

(d) E is the event that a woman lives in the United States, and F is the event that she lives in the western hemisphere.

(e) E is the event that it will rain tomorrow, and F is the event that it will rain the day after tomorrow.

51. In a class there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

52. Suppose that you continually collect coupons and that there are m different types. Suppose also that each time a new coupon is obtained it is a type i coupon with probability p_i, $i = 1, \ldots, m$. Suppose that you have just collected your nth coupon. What is the probability that it is a new type?

HINT: Condition on the type of this coupon.

53. A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or it moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

(a) What is the probability that after 2 days the stock will be at its original price?

(b) What is the probability that after 3 days the stock's price will have increased by 1 unit?

(c) Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

54. Suppose that we want to generate the outcome of the flip of a fair coin but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not be equal to $\frac{1}{2}$. Consider the following procedure for accomplishing our task.

1. Flip the coin.
2. Flip the coin again.
3. If both flips land heads or both land tails, return to step 1.
4. Let the result of the last flip be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails.

(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

55. Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first four outcomes are
(a) H, H, H, H;
(b) T, H, H, H?
(c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?

HINT FOR PART (C): How can the pattern H, H, H, H occur first?

56. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. (Because of the latter fact we say that the brown-eyed gene is *dominant* over the blue-eyed one.) A newborn child independently receives one eye gene from each of its parents and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.
(a) What is the probability that Smith possesses a blue-eyed gene?
Suppose that Smith's wife has blue eyes.
(b) What is the probability that their first child will have blue eyes?
(c) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

57. Genes relating to albinism are denoted by A and a. Only those people who receive the a gene from both parents will be albino. Persons having the gene pair A, a are normal in appearance and, because they can pass on the trait to their offspring, are called carriers. Suppose that a normal couple has two children, exactly one of whom is an albino. Suppose that the nonalbino child mates with a person who is known to be a carrier for albinism.
(a) What is the probability that their first offspring is an albino?
(b) What is the conditional probability that their second offspring is an albino given that their firstborn is not?

58. Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits the wooden duck target with probability p_1, while each shot of Dianne's hits it with probability p_2. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that
(a) both shots hit the duck;
(b) Barbara's shot hit the duck?
What independence assumptions have you made?

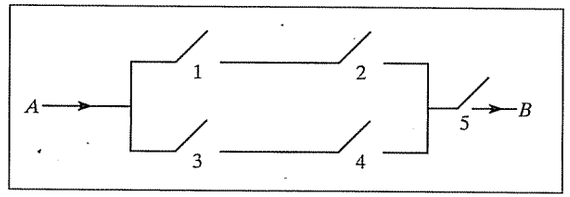
59. A and B are involved in a duel. The rules of the duel are that they are to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability p_A, and each shot of B will hit A with probability p_B. What is
(a) the probability that A is not hit;
(b) the probability that both duelists are hit;
(c) the probability that the duel ends after the nth round of shots;
(d) the conditional probability that the duel ends after the nth round of shots given that A is not hit;
(e) the conditional probability that the duel ends after the nth round of shots given that both duelists are hit?

60. A true–false question is to be posed to a husband and wife team on a quiz show. Both the husband and the wife will, independently, give the correct answer with probability p. Which of the following is a better strategy for this couple?
(a) Choose one of them and let that person answer the question; or
(b) have them both consider the question and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give?

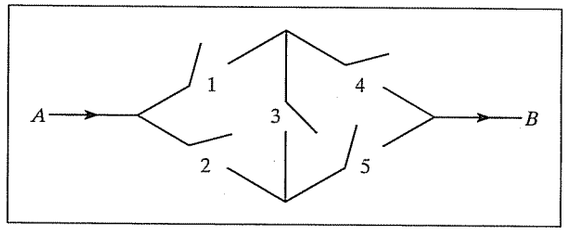
61. In Problem 60, if $p = .6$ and the couple uses the strategy in part (b), what is the conditional probability that the couple gives the correct answer given that they (a) agree; (b) disagree?

62. The probability of the closing of the ith relay in the circuits shown is given by p_i, $i = 1, 2, 3, 4, 5$. If all relays function independently, what is the probability that a current flows between A and B for the respective circuits?

(a)



(b)



HINT FOR (B): Condition on whether relay 3 closes.

63. An engineering system consisting of n components is said to be a k-out-of-n system ($k \leq n$) if the system functions if and only if at least k of the n

components function. Suppose that all components function independently of each other.

(a) If the ith component functions with probability P_i, $i = 1, 2, 3, 4$, compute the probability that a 2-out-of-4 system functions.

(b) Repeat part (a) for a 3-out-of-5 system.

(c) Repeat for a k-out-of-n system when all the P_i equal p (that is, $P_i = p$, $i = 1, 2, \ldots, n$).

64. In Problem 62a, find the conditional probability that relays 1 and 2 are both closed given that a current flows from A to B.

65. A certain organism possesses a pair of each of 5 different genes (which we will designate by the first 5 letters of the English alphabet). Each gene appears in 2 forms (which we designate by lowercase and capital letters). The capital letter will be assumed to be the dominant gene in the sense that if an organism possesses the gene pair xX, then it will outwardly have the appearance of the X gene. For instance, if X stands for brown eyes and x for blue eyes, then an individual having either gene pair XX or xX will have brown eyes, whereas one having gene pair xx will have blue eyes. The characteristic appearance of an organism is called its phenotype, whereas its genetic constitution is called its genotype. (Thus 2 organisms with respective genotypes aA, bB, cc, dD, ee and AA, BB, cc, DD, ee would have different genotypes but the same phenotype.) In a mating between 2 organisms each one contributes, at random, one of its gene pairs of each type. The 5 contributions of an organism (one of each of the 5 types) are assumed to be independent and are also independent of the contributions of its mate. In a mating between organisms having genotypes aA, bB, cC, dD, eE and aa, bB, cc, Dd, ee what is the probability that the progeny will (i) phenotypically and (ii) genotypically resemble

(a) the first parent;

(b) the second parent;

(c) either parent;

(d) neither parent?

66. There is a 50–50 chance that the queen carries the gene for hemophilia. If she is a carrier, then each prince has a 50–50 chance of having hemophilia. If the queen has had three princes without the disease, what is the probability the queen is a carrier? If there is a fourth prince, what is the probability that he will have hemophilia?

67. On the morning of September 31, 1982, the won–lost records of the three leading baseball teams in the western division of the National League of the United States were as follows:

Team	Won	Lost
Atlanta Braves	87	72
San Francisco Giants	86	73
Los Angeles Dodgers	86	73

Each team had 3 games remaining to be played. All 3 of the Giants games were with the Dodgers, and the 3 remaining games of the Braves were against the San Diego Padres. Suppose that the outcomes of all remaining games are independent and each game is equally likely to be won by either participant. What are the probabilities that each of the teams wins the division? If two teams tie for first place, they have a playoff game, which each team has an equal chance of winning.

68. A town council of 7 members contains a steering committee of size 3. New ideas for legislation go first to the steering committee and then on to the council as a whole if at least 2 of the 3 committee members approve the legislation. Once at the full council, the legislation requires a majority vote (of at least 4) to pass. Consider now a new piece of legislation and suppose that each town council member will approve it, independently, with probability p. What is the probability that a given steering committee member's vote is decisive in the sense that if that person's vote were reversed, then the final fate of the legislation would be reversed? What is the corresponding probability for a given council member not on the steering committee?

69. Suppose that each child born to a couple is equally likely to be a boy or a girl independent of the sex distribution of the other children in the family. For a couple having 5 children, compute the probabilities of the following events:

(a) All children are of the same sex.

(b) The 3 eldest are boys and the others girls.

(c) Exactly 3 are boys.

(d) The 2 oldest are girls.

(e) There is at least 1 girl.

70. The probability of winning on a single toss of some dice is p. A starts, and if he fails, he passes the dice to B, who then attempts to win on his roll. They continue to pass the dice back and forth until one of them wins. What are their respective probabilities of winning? Repeat if there are k players.

71. Repeat Problem 70 under the assumption that when A rolls the dice, she wins with probability P_1, and, when B rolls, B wins with probability P_2.

72. Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability $P(E)/[P(E) + P(F)]$.

73. When A and B flip coins, the one coming closest to a given line wins 1 penny from the other. If A starts with 3 and B with 7 pennies, what is the probability that A winds up with all of the money if both players are equally skilled? What if A were a better player who won 60 percent of the time?

74. In successive rolls of a pair of fair dice, what is the probability of getting 2 sevens before 6 even numbers?

75. Players are of equal skill, and in a contest the probability is $\frac{1}{2}$ that a specified one of the two contestants will be the victor. A group of 2^n players are paired off against each other at random. The 2^{n-1} winners are again paired off

randomly, and so on, until a single winner remains. Consider two specified contestants, A and B, and define the events A_i, $i \leq n$, E by

A_i: A plays in exactly i contests;
E: A and B ever play each other.

(a) Find $P(A_i)$, $i = 1, \ldots, n$.
(b) Find $P(E)$.
(c) Let $P_n = P(E)$. Show that

$$P_n = \frac{1}{2^n - 1} + \frac{2^n - 2}{2^n - 1}\left(\frac{1}{2}\right)^2 P_{n-1}$$

and use this to check your answer obtained in part (b).

HINT: Find $P(E)$ by conditioning on which of the events A_i, $i = 1, \ldots, n$ occur. In simplifying your answer use the algebraic identity

$$\sum_{i=1}^{n-1} ix^{i-1} = \frac{1 - nx^{n-1} + (n-1)x^n}{(1-x)^2}$$

For another approach for solving this problem, note that there are a total of $2^n - 1$ games played.

(d) Explain why a total of $2^n - 1$ games are played.
Number these games and let B_i denote the event that A and B play each other in game i, $i = 1, \ldots, 2^n - 1$.
(e) What is $P(B_i)$?
(f) Use part (e) to find $P(E)$.

76. A stock market investor owns shares in a stock whose present value is 25. She has decided that she must sell her stock if it either goes down to 10 or up to 40. If each change of price is either up 1 point with probability .55 or down 1 point with probability .45, and the successive changes are independent, what is the probability that the investor retires a winner?

77. A and B flip coins. A starts and continues flipping until a tail occurs. At this point B starts flipping and continues until there is a tail, then A takes over, and so on. Let P_1 be the probability of the coin's landing heads when A flips, and P_2 when B flips. The winner of the game is the first one to get
(a) 2 heads in a row;
(b) a total of 2 heads;
(c) 3 heads in a row;
(d) a total of 3 heads.
In each case, find the probability that A wins.

78. Die A has 4 red and 2 white faces, whereas die B has 2 red and 4 white faces. A fair coin is flipped once. If it lands on heads, the game continues with die A; if it lands tails, then die B is to be used.
(a) Show that the probability of red at any throw is $\frac{1}{2}$.

(b) If the first two throws result in red, what is the probability of red at the third throw?
(c) If red turns up at the first two throws, what is the probability that it is die A that is being used?

79. There are 12 balls, of which 4 are white, in an urn. Three players—A, B, C—successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the win probabilities for each player if
(a) each ball is replaced after it is drawn;
(b) the withdrawn balls are not replaced.

80. Repeat Problem 79 so that each of the 3 players selects from his own urn. That is, suppose that there are 3 different urns of 12 balls with 4 white in each.

81. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S.
(a) Show that

$$P\{A \subset B\} = \left(\tfrac{3}{4}\right)^n.$$

HINT: Let $N(B)$ denote the number of elements in B. Use

$$P\{A \subset B\} = \sum_{i=0}^{n} P\{A \subset B | N(B) = i\} P\{N(B) = i\}$$

(b) Show that $P\{AB = \varnothing\} = \left(\tfrac{3}{4}\right)^n$.

82. In Example 5d, what is the conditional probability that the ith coin was selected given that the first n trials all result in heads?

83. In Laplace's rule of succession, Example 5d, are the outcomes of the successive flips independent? Explain.

84. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability .7, whereas when the defendant is, in fact, innocent, this probability drops to .2. If 70 percent of defendants are guilty, compute the conditional probability that judge number 3 votes guilty given that
(a) judges 1 and 2 vote guilty;
(b) judges 1 and 2 cast 1 guilty and 1 not guilty vote;
(c) judges 1 and 2 both cast not guilty votes.
Let E_i, $i = 1, 2, 3$ denote the event that judge i casts a guilty vote. Are these events independent. Are they conditionally independent? Explain.

85. Suppose that n independent trials, each of which results in any of the outcomes 0, 1, or 2 with respective probabilities, p_0, p_1, and p_2, $\sum_{i=0}^{2} p_i = 1$, are performed. Find the probability that outcomes 1 and 2 both occur at least once.

As there is an equal probability that it is the left-hand box that is first discovered to be empty and there are k matches in the right-hand box at that time, the desired result is

$$2P(E) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}$$

■

Example 9f. Compute the expected value and the variance of a negative binomial random variable with parameters r and p.

Solution

$$\begin{aligned}
E[X^k] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \frac{r}{p}\sum_{n=r}^{\infty} n^{k-1}\binom{n}{r} p^{r+1}(1-p)^{n-r} \qquad \text{since} \quad n\binom{n-1}{r-1} = r\binom{n}{r} \\
&= \frac{r}{p}\sum_{m=r+1}^{\infty} (m-1)^{k-1}\binom{m-1}{r} p^{r+1}(1-p)^{m-(r+1)} \qquad \text{by setting } m = n+1 \\
&= \frac{r}{p}E[(Y-1)^{k-1}]
\end{aligned}$$

where Y is a negative binomial random variable with parameters $r + 1, p$. Setting $k = 1$ in the preceding equation yields

$$E[X] = \frac{r}{p}$$

Setting $k = 2$ in the preceding equation, and using the formula above for the expected value of a negative binomial random variable, gives that

$$\begin{aligned}
E[X^2] &= \frac{r}{p}E[Y-1] \\
&= \frac{r}{p}\left(\frac{r+1}{p} - 1\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{r}{p}\left(\frac{r+1}{p} - 1\right) - \left(\frac{r}{p}\right)^2 \\
&= \frac{r(1-p)}{p^2}
\end{aligned}$$

■

Thus we see from Example 9f that if independent trials, each of which is a success with probability p, are performed, then the expected value and variance of the number of trials that it takes to amass r successes is r/p and $r(1-p)/p^2$, respectively.

Since a geometric random variable is just a negative binomial with parameter $r = 1$, it follows from the preceding example that the variance of a geometric random variable with parameter p is equal to $(1-p)/p^2$, which checks with the result of Example 9c.

Example 9g. Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

Solution Since the random variable of interest is a negative binomial with parameter $r = 4$ and $p = \frac{1}{6}$, we see that

$$E[X] = 24$$

$$\mathrm{Var}(X) = \frac{4\left(\frac{5}{6}\right)}{\left(\frac{1}{6}\right)^2} = 120$$

■

4.9.3 The Hypergeometric Random Variable

Suppose that a sample of size n is to be chosen randomly (without replacement) from an urn containing N balls, of which m are white and $N - m$ are black. If we let X denote the number of white balls selected, then

$$P\{X = i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \qquad i = 0, 1, \ldots, n \tag{9.4}$$

A random variable X, whose probability mass function is given by Equation (9.4) for some values of n, N, m is said to be a *hypergeometric* random variable.

REMARK. Although we have written the hypergeometric probability mass function with i going from 0 to n, $P\{X = i\}$ will actually be 0 unless i satisfies the inequalities $n - (N - m) \le i \le \min(n, m)$. However, Equation (9.4) is always valid because of our convention that $\binom{r}{k}$ is equal to 0 when either $k < 0$ or $r < k$.

Example 9h. An unknown number, say N, of animals inhabit a certain region. To obtain some information about the population size, ecologists often perform the following experiment: They first catch a number, say m, of these animals, mark them in some manner, and release them. After allowing the marked animals time to disperse throughout the region, a new catch of size, say n, is made. Let X denote the number of marked animals in this second capture. If we assume that the population of animals in the region remained fixed between the time of the two catches and that each time an animal was caught it was equally likely to be any of the remaining uncaught animals, it follows that X is a hypergeometric random variable such that

$$P\{X = i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \equiv P_i(N)$$

Suppose now that X is observed to equal i. Then, as $P_i(N)$ represents the probability of the observed event when there are actually N animals present in the region, it would appear that a reasonable estimate of N would be the value of N that maximizes $P_i(N)$. Such an estimate is called a *maximum likelihood* estimate. (See Theoretical Exercises 13 and 18 for other examples of this type of estimation procedure.)

The maximization of $P_i(N)$ can most simply be done by first noting that

$$\frac{P_i(N)}{P_i(N-1)} = \frac{(N-m)(N-n)}{N(N-m-n+i)}$$

Now, the above ratio is greater than 1 if and only if

$$(N-m)(N-n) \geq N(N-m-n+i)$$

or, equivalently, if and only if

$$N \leq \frac{mn}{i}$$

Thus $P_i(N)$ is first increasing, and then decreasing, and reaches its maximum value at the largest integral value not exceeding mn/i. This value is thus the maximum likelihood estimate of N. For example, suppose that the initial catch consisted of $m = 50$ animals, which are marked and then released. If a subsequent catch consists of $n = 40$ animals of which $i = 4$ are marked, then we would estimate that there are some 500 animals in the region. (It should be noted that the above estimate could also have been obtained by assuming that the proportion of marked animals in the region, m/N, is approximately equal to the proportion of marked animals in our second catch, i/n.) ∎

Example 9i. A purchaser of electrical components buys them in lots of size 10. It is his policy to inspect 3 components randomly from a lot and to accept the lot only if all 3 are nondefective. If 30 percent of the lots have 4 defective components and 70 percent have only 1, what proportion of lots does the purchaser reject?

Solution Let A denote the event that the purchaser accepts a lot. Now,

$$\begin{aligned} P(A) &= P(A|\text{lot has 4 defectives})\frac{3}{10} + P(A|\text{lot has 1 defective})\frac{7}{10} \\ &= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}}\left(\frac{3}{10}\right) + \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}}\left(\frac{7}{10}\right) \\ &= \frac{54}{100} \end{aligned}$$

Hence 46 percent of the lots are rejected. ∎

If n balls are randomly chosen without replacement from a set of N balls, of which the fraction $p = m/N$ is white, then the number of white balls selected

is hypergeometric. Now, it would seem that when m and N are large in relation to n, it shouldn't make much difference whether the selection is being done with or without replacement. Because no matter which balls have previously been selected, each additional selection will, when m and N are large, be white with a probability approximately equal to p. In other words, it seems intuitive when m and N are large in relation to n that the probability mass function of X should approximately be that of a binomial random variable with parameters n and p. To verify this intuition note that if X is hypergeometric then, for $i \leq n$,

$$\begin{aligned} P\{X = i\} &= \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \\ &= \frac{m!}{(m-i)!\,i!}\,\frac{(N-m)!}{(N-m-n+i)!\,(n-i)!}\,\frac{(N-n)!\,n!}{N!} \\ &= \binom{n}{i}\frac{m}{N}\,\frac{m-1}{N-1}\cdots\frac{m-i+1}{N-i+1}\,\frac{N-m}{N-i}\,\frac{N-m-1}{N-i-1} \\ &\quad \cdots\frac{N-m-(n-i-1)}{N-i-(n-i-1)} \\ &\approx \binom{n}{i}p^i(1-p)^{n-i} \qquad \text{when } p = m/N \text{ and } m \text{ and } N \text{ are large in relation to } n \text{ and } i \end{aligned}$$

Example 9j. Determine the expected value and the variance of X, a hypergeometric random variable with parameters n, N, m.

Solution

$$\begin{aligned} E[X^k] &= \sum_{i=0}^{n} i^k P\{X = i\} \\ &= \sum_{i=1}^{n} i^k \binom{m}{i}\binom{N-m}{n-i}\Big/\binom{N}{n} \end{aligned}$$

Using the identities

$$i\binom{m}{i} = m\binom{m-1}{i-1} \qquad \text{and} \qquad n\binom{N}{n} = N\binom{N-1}{n-1}$$

we obtain that

$$\begin{aligned} E[X^k] &= \frac{nm}{N}\sum_{i=1}^{n} i^{k-1}\binom{m-1}{i-1}\binom{N-m}{n-i}\Big/\binom{N-1}{n-1} \\ &= \frac{nm}{N}\sum_{j=0}^{n-1}(j+1)^{k-1}\binom{m-1}{j}\binom{N-m}{n-1-j}\Big/\binom{N-1}{n-1} \\ &= \frac{nm}{N}E[(Y+1)^{k-1}] \end{aligned}$$

where Y is a hypergeometric random variable with parameters $n - 1$, $N - 1$, $m - 1$. Hence, upon setting $k = 1$ we see that

$$E[X] = \frac{nm}{N}$$

In words, if n balls are randomly selected from a set of N balls, of which m are white, then the expected number of white balls selected is nm/N.

Upon setting $k = 2$ in the equation for $E[X^k]$ we obtain that

$$\begin{aligned} E[X^2] &= \frac{nm}{N} E[Y + 1] \\ &= \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1} + 1\right] \end{aligned}$$

where the final equality uses our preceding result to compute the expected value of the hypergeometric random variable Y.

As $E[X] = nm/N$ we can conclude that

$$\text{Var}(X) = \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1} + 1 - \frac{nm}{N}\right] \tag{9.5}$$

If we let $p = m/N$ denote the fraction of balls that are white, then it follows from Equation (9.5), after a little algebra, that

$$\text{Var}(X) = \frac{N-n}{N-1} np(1-p) \tag{9.6}$$

■

REMARK. We have shown in Example 9j that if n balls are randomly selected without replacement from a set of N balls, of which the fraction p are white, then the expected number of white balls chosen is np. In addition, if N is large in relation to n [and so $(N - n)/(N - 1)$ is approximately equal to 1], then

$$\text{Var}(X) \approx np(1-p)$$

In other words, $E[X]$ is the same as when the selection of the balls is done with replacement (so the number of white balls is binomial with parameters n and p), and if the total collection of balls is large, then $\text{Var}(X)$ is approximately equal to what it would be if the selection were done with replacement. This is, of course, exactly what we would have guessed given our earlier result that when the number of balls in the urn is large, the number of white balls chosen approximately has the mass function of a binomial random variable.

4.9.4 The Zeta (or Zipf) Distribution

A random variable is said to have a zeta (sometimes called the Zipf) distribution if its probability mass function is given by

$$P\{X = k\} = \frac{C}{k^{\alpha+1}} \qquad k = 1, 2, \ldots$$

for some value of $\alpha > 0$. Since the sum of the foregoing probabilities must equal 1, it follows that

$$C = \left[\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{\alpha+1}\right]^{-1}$$

The zeta distribution owes its name to the fact that the function

$$\zeta(s) = 1 + \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s + \cdots + \left(\frac{1}{k}\right)^s + \cdots$$

is known in mathematical disciplines as the Riemann zeta function (after the German mathematician G. F. B. Riemann).

The zeta distribution was used by the Italian economist Pareto to describe the distribution of family incomes in a given country. However, it was G. K. Zipf who applied these distributions in a wide variety of different areas and, in doing so, popularized their use.

SUMMARY

A real-valued function defined on the outcome of a probability experiment is called a *random variable.*

If X is a random variable, then the function $F(x)$, defined by

$$F(x) = P\{X \leq x\}$$

is called the *distribution function* of X. All probabilities concerning X can be stated in terms of F.

A random variable whose set of possible values is either finite or countably infinite is called *discrete.* If X is a discrete random variable, then the function

$$p(x) = P\{X = x\}$$

is called the *probability mass function* of X. Also, the quantity $E[X]$, defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

is called the *expected value* of X. $E[X]$ is also commonly called the *mean* or the *expectation* of X.

A useful identity states that for a function g,

$$E[g(X)] = \sum_{x:p(x)>0} xg(x)p(x)$$

The *variance* of a random variable X, denoted by $\text{Var}(X)$, is defined by

$$\text{Var}(X) = E[(X - E[X])^2]$$

The variance, which is equal to the expected square of the difference between X and its expected value, is a measure of the spread of the possible values of X. A useful identity is that

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

The quantity $\sqrt{\text{Var}(X)}$ is called the *standard deviation* of X.

We now note some common types of discrete random variables. The random variable X whose probability mass function is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad i = 0, \ldots, n$$

is said to be a binomial random variable with parameters n and p. Such a random variable can be interpreted as being the number of successes that occur when n independent trials, each of which results in a success with probability p, are performed. Its mean and variance are given by

$$E[X] = np \qquad \text{Var}(X) = np(1-p)$$

The random variable X whose probability mass function is given by

$$p(i) = \frac{e^{-\lambda}\lambda^i}{i!} \qquad i \geq 0$$

is said to be a *Poisson* random variable with parameter λ. If a large number of (approximately) independent trials are performed, each having a small probability of being successful, then the number of successful trials that result will have a distribution that is approximately that of a Poisson random variable. The mean and variance of a Poisson random variable are both equal to its parameter λ. That is,

$$E[X] = \text{Var}(X) = \lambda$$

The random variable X whose probability mass function is given by

$$p(i) = p(1-p)^{i-1} \qquad i = 1, 2, \ldots$$

is said to be a *geometric* random variable with parameter p. Such a random variable represents the trial number of the first success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{1}{p} \qquad \text{Var}(X) = \frac{1-p}{p^2}$$

The random variable X whose probability mass function is given by

$$p(i) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \qquad i \geq r$$

is said to be a *negative binomial* random variable with parameters r and p. Such a random variable represents the trial number of the rth success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{r}{p} \qquad \text{Var}(X) = \frac{r(1-p)}{p^2}$$

A *hypergeometric* random variable with parameters n, N, m represents the number of white balls selected when n balls are randomly chosen from an urn that contains N balls, of which m are white. Its probability mass function is given by

$$p(i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \qquad i = 0, \ldots, m$$

With $p = m/N$, its mean and variance are

$$E[X] = np \qquad \text{Var}(X) = \frac{N-n}{N-1} np(1-p)$$

PROBLEMS

1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

2. Two fair dice are rolled. Let X equal the product of the 2 dice. Compute $P\{X = i\}$ for $i = 1, 2, \ldots$.

3. Three dice are rolled. By assuming that each of the $6^3 = 216$ possible outcomes is equally likely, find the probabilities attached to the possible values that X can take on, where X is the sum of the 3 dice.

4. Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 1$ if the top-ranked person is female). Find $P\{X = i\}$, $i = 1, 2, 3, \ldots, 8, 9, 10$.

5. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

6. In Problem 5, if the coin is assumed fair, for $n = 3$ what are the probabilities associated with the values that X can take on?

7. Suppose that a die is rolled twice. What are the possible values that the following random variables can take on

(a) the maximum value to appear in the two rolls;
(b) the minimum value to appear on the two rolls;
(c) the sum of the two rolls;
(d) the value of the first roll minus the value of the second roll?

8. If the die in Problem 7 is assumed fair, calculate the probabilities associated with the random variables in parts (a) through (d).

9. Repeat Example 1b when the balls are selected with replacement.

10. In Example 1d compute the conditional probability that we win i dollars, given that we win something; compute it for $i = 1, 2, 3$.

11. **(a)** An integer N is to be selected at random from $\{1, 2, \ldots, (10)^3\}$ in the sense that each integer has the same probability of being selected. What is the probability that N will be divisible by 3? by 5? by 7? by 15? by 105? How would your answer change if $(10)^3$ is replaced by $(10)^k$ as k became larger and larger?

(b) An important function in number theory—one whose properties can be shown to be related to what is probably the most important unsolved problem of mathematics, the Riemann hypothesis—is the Möbius function $\mu(n)$, defined for all positive integral values n as follows: Factor n into its prime factors. If there is a repeated prime factor, as in $12 = 2 \cdot 2 \cdot 3$ or $49 = 7 \cdot 7$, then $\mu(n)$ is defined to equal 0. Now let N be chosen at random from $\{1, 2, \ldots (10)^k\}$, where k is large. Determine $P\{\mu(N) = 0\}$ as $k \to \infty$.

HINT: To compute $P\{\mu(N) \neq 0\}$, use the identity

$$\prod_{i=1}^{\infty} \frac{P_i^2 - 1}{P_i^2} = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{24}{25}\right)\left(\frac{48}{49}\right) \cdots = \frac{6}{\pi^2}$$

where P_i is the ith smallest prime. (We do not include 1 as a prime.)

12. In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither guesses correctly, then no money is exchanged. Consider a specified player and denote by X the amount of money he wins in a single game of two-finger Morra.

(a) If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?

(b) Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?

13. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

14. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares with player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}$, $i = 0, 1, 2, 3, 4$.

15. The National Basketball Association (NBA) draft lottery involves the 11 teams that had the worst won–loss records during the year. A total of 66 balls are placed in an urn. Each of these balls is inscribed with the name of a team; 11 have the name of the team with the worst record, 10 have the name of the team with the second worst record, 9 have the name of the team with the third worst record, and so on (with 1 ball having the name of the team with the eleventh worst record). A ball is then chosen at random and the team whose name is on the ball is given the first pick in the draft of players about to enter the league. Another ball is then chosen and if it "belongs" to a different team than the one that received the first draft pick, then the team to which it belongs receives the second draft pick. (If the ball belongs to the team receiving the first pick, then it is discarded and another one is chosen; this continues until the ball of another team is chosen.) Finally, another ball is chosen and the team named on the ball (provided that it is different from the previous two teams) receives the third draft pick. The remaining draft picks 4 through 11 are then awarded to the 8 teams that did not "win the lottery" in inverse order of their won–loss records. For instance, if the team with the worst record did not receive any of the 3 lottery picks, then that team would receive the fourth draft pick. Let X denote the draft pick of the team with the worst record. Find the probability mass function of X.

16. In Problem 15, let team number 1 be the team with the worst record, let team number 2 be the team with the second worst record, and so on. Let Y_i denote the team that gets draft pick number i. Thus $Y_1 = 3$ if the first ball chosen belongs to team number 3. Find the probability mass function of (a) Y_1, (b) Y_2, and (c) Y_3.

17. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{4} & 0 \le b < 1 \\ \frac{1}{2} + \frac{b-1}{4} & 1 \le b < 2 \\ \frac{11}{12} & 2 \le b < 3 \\ 1 & 3 \le b \end{cases}$$

(a) Find $P\{X = i\}$, $i = 1, 2, 3$.

(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$.

18. Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

19. If the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{1}{2} & 0 \le b < 1 \\ \frac{3}{5} & 1 \le b < 2 \\ \frac{4}{5} & 2 \le b < 3 \\ \frac{9}{10} & 3 \le b < 3.5 \\ 1 & b \ge 3.5 \end{cases}$$

calculate the probability mass function of X.

20. A gambling book recommends the following "winning strategy" for the game of roulette. It recommends that a gambler bet \$1 on red. If red appears (which has probability $\frac{18}{38}$), then the gambler should take her \$1 profit and quit. If the gambler loses this bet (which has probability $\frac{20}{38}$ of occurring), she should make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote the gambler's winnings when she quits.

(a) Find $P\{X > 0\}$.

(b) Are you convinced that the strategy is indeed a "winning" strategy? Explain your answer!

(c) Find $E[X]$.

21. A total of 4 buses carrying 148 students from the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

(b) Compute $E[X]$ and $E[Y]$.

22. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by player A with probability p. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. Also show in both cases that this number is maximized when $p = \frac{1}{2}$.

23. A bin of 5 electrical components is known to contain 2 that are defective. If the components are to be tested one at a time, in random order, until the defectives are discovered, find the expected number of tests that are made.

24. A and B play the following game: A writes down either number 1 or number 2 and B must guess which one. If the number that A has written down is i and B has guessed correctly, B receives i units from A. If B makes a wrong guess, B pays $\frac{3}{4}$ unit to A. If B randomizes his decision by guessing 1 with probability p and 2 with probability $1 - p$, determine his expected gain if (a) A has written down number 1 and (b) A has written down number 2.

What value of p maximizes the minimum possible value of B's expected gain and what is this maximin value? (Note that B's expected gain depends not only on p but also on what A does.)

Consider now player A. Suppose that she also randomizes her decision, writing down number 1 with probability q. What is A's expected loss if (c) B chooses number 1 and (d) B chooses number 2?

What value of q minimizes A's maximum expected loss? Show that the minimum of A's maximum expected loss is equal to the maximum of B's minimum expected gain. This result, known as the minimax theorem, was first established in generality by the mathematician John von Neumann and is the fundamental result in the mathematical discipline known as the theory of games. The common value is called the value of the game to player B.

25. A typical slot machine has 3 dials, each with 20 symbols (cherries, lemons, plums, oranges, bells, and bars). A typical set of dials is set up as follows:

	Dial 1	Dial 2	Dial 3
Cherries	7	7	0
Oranges	3	7	6
Lemons	3	0	4
Plums	4	1	6
Bells	2	2	3
Bars	1	3	1
	20	20	20

According to this table, of the 20 slots on dial 1, 7 are cherries, 3 are oranges, and so on. A typical payoff on a 1-unit bet is as shown in the following table.

Dial 1	Dial 2	Dial 3	Payoff
Bar	Bar	Bar	60
Bell	Bell	Bell	20
Bell	Bell	Bar	18
Plum	Plum	Plum	14
Orange	Orange	Orange	10
Orange	Orange	Bar	8
Cherry	Cherry	Anything	4
Cherry	No cherry	Anything	2
	Anything else		−1

Compute the player's expected winnings on a single play of the slot machine. Assume that each dial acts independently.

26. One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes–no" answers. Compute the expected number of questions you will need to ask in each of the two cases:

(a) Your ith question is to be "Is it i?", $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible.

27. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

28. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

29. There are two possible causes for a breakdown of a machine. To check the first possibility would cost C_1 dollars, and, if that were the cause of the breakdown, the trouble could be repaired at a cost of R_1 dollars. Similarly, there are costs C_2 and R_2 associated with the second possibility. Let p and $1 - p$ denote, respectively, the probabilities that the breakdown is caused by the first and second possibilities. Under what conditions on p, C_i, R_i, $i = 1, 2$, should we check the first possible cause of breakdown and then the second, as opposed to reversing the checking order, so as to minimize the expected cost involved in returning the machine to working order?

NOTE: If the first check is negative, we must still check the other possibility.

30. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let X denote the player's winnings. Show that $E[X] = +\infty$. This problem is known as the St. Petersburg paradox.

(a) Would you be willing to pay \$1 million to play this game once?

(b) Would you be willing to pay \$1 million for each game if you could play for as long as you liked and only had to settle up when you stopped playing?

31. Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{cases} 1 - (1 - p)^2 & \text{if it does rain} \\ 1 - p^2 & \text{if it does not rain} \end{cases}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of this and so wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

32. To determine whether or not they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to group the people in groups of 10. The blood samples of the 10 people in each group will be pooled and anayzed together. If the test is negative, one test will suffice for the 10 people; whereas, if the test is positive each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume the probability that a person has the disease is .1 for all people, independently of each other, and compute the expected number of tests necessary for each group. (Note that we are assuming that the pooled test will be positive if at least one person in the pool has the disease.)

33. A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10, p = \frac{1}{3}$, approximately how many papers should he purchase so as to maximize his expected profit?

34. In Example 5b, suppose that the department store incurs an additional cost of c for each unit of unmet demand. (This is often referred to as a goodwill cost because the store loses the goodwill of those customers whose demands it cannot meet.) Compute the expected profit when the store stocks s units, and determine the value of s that maximizes the expected profit.

35. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win $-$\$1.00 (that is, you lose \$1.00). Calculate

(a) the expected value of the amount you win;

(b) the variance of the amount you win.

36. Consider Problem 22 with $i = 2$. Find the variance of the number of games played and show that this number is maximized when $p = \frac{1}{2}$.

37. Find $\text{Var}(X)$ and $\text{Var}(Y)$ for X and Y as given in Problem 21.

38. If $E[X] = 1$ and $\text{Var}(X) = 5$, find

(a) $E[(2 + X)^2]$;

(b) $\text{Var}(4 + 3X)$.

39. A ball is drawn from an urn containing 3 white and 3 black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first 4 balls drawn, exactly 2 are white?

40. On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student would get 4 or more correct answers just by guessing?

41. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times, and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he had no ESP?

42. Suppose that when in flight, airplane engines will fail with probability $1 - p$ independently from engine to engine. If an airplane needs a majority of its engines operative to make a successful flight, for what values of p is a 5-engine plane preferable to a 3-engine plane?

43. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability .2. Suppose that we want to transmit an important message consisting of one binary digit.

To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

44. A satellite system consists of n components and functions on any given day if at least k of the n components function on that day. On a rainy day each of the components independently functions with probability p_1, whereas on a dry day they each independently function with probability p_2. If the probability of rain tomorrow is α, what is the probability that the satellite system will function?

45. A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him independently of each other, with probability .8, whereas, if he has an off day, this probability will be reduced to .4. Suppose that the student will pass the examination if a majority of the examiners pass him. If the student feels that he is twice as likely to have an off day as he is to have an on day, should he request an examination with 3 examiners or with 5 examiners?

46. Suppose that it takes at least 9 votes from a 12-member jury to convict a defendant. Suppose that the probability that a juror votes a guilty person innocent is .2, whereas the probability that the juror votes an innocent person guilty is .1. If each juror acts independently and if 65 percent of the defendants are guilty, find the probability that the jury renders a correct decision. What percentage of defendants is convicted?

47. In some military courts, 9 judges are appointed. However, both the prosecution and the defense attorneys are entitled to a peremptory challenge of any judge, in which case that judge is removed from the case and is not replaced. A defendant is declared guilty if the majority of judges cast votes of guilty, and he or she is declared innocent otherwise. Suppose that when the defendant is, in fact, guilty, each judge will (independently) vote guilty with probability .7, whereas when the defendant is, in fact, innocent, this probability drops to .3.

(a) What is the probability that a guilty defendant is declared guilty when there are (i) 9, (ii) 8, and (iii) 7 judges?

(b) Repeat part (a) for an innocent defendant.

(c) If the prosecution attorney does not exercise the right to a peremptory challenge of a judge and if the defense is limited to at most two such challenges, how many challenges should the defense attorney make if he or she is 60 percent certain that the client is guilty?

48. It is known that diskettes produced by a certain company will be defective with probability .01, independently of each other. The company sells the diskettes in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 diskettes in the package will be defective. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them?

49. Suppose that 10 percent of the chips produced by a computer hardware manufacturer are defective. If we order 100 such chips, will the number of defective ones we receive be a binomial random variable?

50. Suppose that a biased coin that lands on heads with probability p is flipped 10 times. Given that a total of 6 heads result, find the conditional probability that the first 3 outcomes are

(a) H, T, T (meaning that the first flip is heads, the second is tails, and the third is tails);

(b) T, H, T.

51. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that the next page you read contains (a) 0 and (b) 2 or more typographical errors? Explain your reasoning!

52. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that there will be

(a) at least 2 such accidents in the next month;

(b) at most 1 accident in the next month?

Explain your reasoning!

53. Approximately 80,000 marriages took place in the state of New York last year. Estimate the probability that for at least one of these couples

(a) both partners were born on April 30;

(b) both partners celebrated their birthday on the same day of the year.

State your assumptions.

54. Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Approximate the probability that there will be

(a) no abandoned cars in the next week;

(b) at least 2 abandoned cars in the next week.

55. A certain typing agency employs 2 typists. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second. If your article is equally likely to be typed by either typist, approximate the probability that it will have no errors.

56. How many people are needed so that the probability that at least one of them has the same birthday as you is greater than $\frac{1}{2}$?

57. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

58. Compare the Poisson approximation with the correct binomial probability for the following cases:

(a) $P\{X = 2\}$ when $n = 8, p = .1$;

(b) $P\{X = 9\}$ when $n = 10, p = .95$;

(c) $P\{X = 0\}$ when $n = 10, p = .1$;

(d) $P\{X = 4\}$ when $n = 9, p = .2$.

59. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize
(a) at least once;
(b) exactly once;
(c) at least twice?

60. The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

61. The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that in 1000 hands of poker you will be dealt at least 2 full houses.

62. If n married couples are seated at random at a round table, approximately what is the probability that no wife sits next to her husband? When $n = 10$ compare your approximation with the exact answer given in Example 5n of Chapter 2.

63. People enter a gambling casino at a rate of 1 for every 2 minutes.
(a) What is the probability that no one enters between 12:00 and 12:05?
(b) What is the probability that at least 4 people enter the casino during that time?

64. The suicide rate in a certain state is 1 suicide per 100,000 inhabitants per month.
(a) Find the probability that in a city of 400,000 inhabitants within this state, there will be 8 or more suicides in a given month.
(b) What is the probability that there will be at least 2 months during the year that will have 8 or more suicides?
(c) Counting the present month as month number 1, what is the probability that the first month to have 8 or more suicides will be month number i, $i \geq 1$?
What assumptions are you making?

65. Each of 500 soldiers in an army company independently has a certain disease with probability $1/10^3$. This disease will show up in a blood test, and to facilitate matters blood samples from all 500 are pooled and tested.
(a) What is the (approximate) probability that the blood test will be positive (and so at least one person has the disease)?
Suppose now that the blood test yields a positive result.
(b) What is the probability, under this circumstance, that more than one person has the disease?
One of the 500 people is Jones, who knows that he has the disease.
(c) What does Jones think is the probability that more than one person has the disease?
As the pooled test was positive, the authorities have decided to test each individual separately. The first $i - 1$ of these tests were negative, and the ith one—which was on Jones—was positive.
(d) Given the above, as a function of i, what is the probability that any of the remaining people have the disease?

66. Consider a roulette wheel consisting of 38 numbers—1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that
(a) Smith will lose his first 5 bets;
(b) his first win will occur on his fourth bet?

67. Two athletic teams play a series of games; the first team to win 4 games is declared the overall winner. Suppose that one of the teams is stronger than the other and wins each game with probability .6, independent of the outcomes of the other games. Find the probability that the stronger team wins the series in exactly i games. Do it for $i = 4, 5, 6, 7$. Compare the probability that the stronger team wins with the probability that it would win a 2-out-of-3 series.

68. Suppose in Problem 67 that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

69. An interviewer is given a list of potential people she can interview. If the interviewer needs to interview 5 people and if each person (independently) agrees to be interviewed with probability $\frac{2}{3}$, what is the probability that her list of potential people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people? For part (b) what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

70. A fair coin is continually flipped until heads appears for the tenth time. Let X denote the number of tails that occur. Compute the probability mass function of X.

71. Solve the Banach match problem (Example 9e) when the left-hand matchbox originally contained N_1 matches and the right-hand box contained N_2 matches.

72. In Banach's matchbox problem find the probability that at the moment when the first box is emptied (as opposed to being found empty), the other box contains exactly k matches.

73. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

74. Suppose that a batch of 100 items contains 6 that are defective and 94 that are nondefective. If X is the number of defective items in a randomly drawn sample of 10 items from the batch, find (a) $P\{X = 0\}$ and (b) $P\{X > 2\}$.

75. A game popular in Nevada gambling casinos is Keno, which is played as follows: Twenty numbers are selected at random by the casino from the set of numbers 1 through 80. A player can select from 1 to 15 numbers; a win occurs if some fraction of the player's chosen subset matches with any of

the 20 numbers drawn by the house. The payoff is a function of the number of elements in the player's selection and the number of matches. For instance, if the player selects only 1 number, then he or she wins if this number is among the set of 20, and the payoff is \$2.2 won for every dollar bet. (As the player's probability of winning in this case is $\frac{1}{4}$, it is clear that the "fair" payoff should be \$3 won for every \$1 bet.) When the player selects 2 numbers, a payoff (of odds) of \$12 won for every \$1 bet is made when both numbers are among the 20,

(a) What would be the fair payoff in this case?
Let $P_{n,k}$ denote the probability that exactly k of the n numbers chosen by the player are among the 20 selected by the house.

(b) Compute $P_{n,k}$.

(c) The most typical wager at Keno consists of selecting 10 numbers. For such a bet the casino pays off as shown in the following table. Compute the expected payoff:

KENO PAYOFFS IN 10 NUMBER BETS

Number of matches	Dollars won for each \$1 bet
0–4	−1
5	1
6	17
7	179
8	1,299
9	2,599
10	24,999

76. In Example 9i, what percentage of i defective lots does the purchaser reject? Find it for $i = 1, 4$. Given that a lot is rejected, what is the conditional probability that it contained 4 defective components?

77. A purchaser of transistors buys them in lots of 20. It is his policy to randomly inspect 4 components from a lot and to accept the lot only if all 4 are nondefective. If each component in a lot is, independently, defective with probability .1, what proportion of lots is rejected?

THEORETICAL EXERCISES

1. There are N distinct types of coupons, and each time one is obtained it will, independently of past choices, be of type i with probability P_i, $i = 1, \ldots, N$. Let T denote the number one need select to obtain at least one of each type. Compute $P\{T = n\}$.

HINT: Use an argument similar to the one used in Example 1e.

2. Prove property 3 of a distribution function.

3. Express $P\{X \geq a\}$ in terms of the distribution function of X.

4. Prove or give a counterexample:

$$P\{X < b\} = \lim_{b_n \to b} P\{X < b_n\}$$

5. If X has distribution function F, what is the distribution function of the random variable $\alpha X + \beta$, where α and β are constants, $\alpha \neq 0$?

6. For a nonnegative integer-valued random variable N, show that

$$E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}$$

HINT: $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N = k\}$. Now interchange the order of summation.

7. For a nonnegative integer-valued random variable N, show that

$$\sum_{i=0}^{\infty} iP\{N > i\} = \tfrac{1}{2}(E[N^2] - E[N])$$

HINT: $\sum_{i=0}^{\infty} iP\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$. Now interchange the order of summation.

8. Let X be such that

$$P\{X = 1\} = p = 1 - P\{X = -1\}$$

Find $c \neq 1$ such that $E[c^X] = 1$.

9. Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of

$$Y = \frac{X - \mu}{\sigma}$$

10. Let X be a binomial random variable with parameters n and p. Show that

$$E\left[\frac{1}{X+1}\right] = \frac{1 - (1-p)^{n+1}}{(n+1)p}$$

11. Consider n independent sequential trials, each of which is successful with probability p. If there is a total of k successes, show that each of the $n!/[k!\,(n-k)!]$ possible arrangements of the k successes and $n - k$ failures is equally likely.

12. There are n components lined up in a linear arrangement. Suppose that each component independently functions with probability p. What is the probability that no 2 neighboring components are both nonfunctional?

HINT: Condition on the number of defective components and use the results of Example 4c of Chapter 1.

13. Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P\{X = k\}, k = 0, 1, \ldots, n$? This is an example of a statistical method used to estimate p when a binomial (n, p) random variable is observed to equal k. If we assume that n is known, then we estimate p by choosing that value of p that maximizes $P\{X = k\}$. This is known as the method of maximum likelihood estimation.

14. A family has n children with probability αp^n, $n \geq 1$, where $\alpha \leq (1 - p)/p$.
(a) What proportion of families has no children?
(b) If each child is equally likely to be a boy or a girl (independently of each other), what proportion of families consists of k boys (and any number of girls)?

15. Suppose that n independent tosses of a coin having probability p of coming up heads are made. Show that the probability that an even number of heads results is $\frac{1}{2}[1 + (q - p)^n]$, where $q = 1 - p$. Do this by proving and then utilizing the identity

$$\sum_{i=0}^{[n/2]} \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}[(p + q)^n + (q - p)^n]$$

where $[n/2]$ is the largest integer less than or equal to $n/2$. Compare this exercise with Theoretical Exercise 15 of Chapter 3.

16. Let X be a Poisson random variable with parameter λ. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.

HINT: Consider $P\{X = i\}/P\{X = i - 1\}$.

17. Let X be a Poisson random variable with parameter λ.
(a) Show that

$$P\{X \text{ is even}\} = \tfrac{1}{2}[1 + e^{-2\lambda}]$$

by using the results of Theoretical Exercise 15 and the relationship between Poisson and binomial random variables.
(b) Verify the above directly by making use of the expansion of $e^{-\lambda} + e^{\lambda}$.

18. Let X be a Poisson random variable with parameter λ. What value of λ maximizes $P\{X = k\}, k \geq 0$?

19. If X is a Poisson random variable with parameter λ, show that

$$E[X^n] = \lambda E[(X + 1)^{n-1}]$$

Now use this result to compute $E[X^3]$.

20. Let X be a Poisson random variable with parameter λ, where $0 < \lambda < 1$. Find $E[X!]$.

21. From a set of n randomly chosen people let E_{ij} denote the event that persons i and j have the same birthday. Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday. Find
(a) $P(E_{3,4}|E_{1,2})$;
(b) $P(E_{1,3}|E_{1,2})$;
(c) $P(E_{2,3}|E_{1,2} \cap E_{1,3})$.

What can you conclude from the above about the independence of the $\binom{n}{2}$ events E_{ij}?

22. An urn contains $2n$ balls, of which 2 are numbered 1, 2 are numbered 2, . . . , and 2 are numbered n. Balls are successively withdrawn 2 at a time without replacement. Let T denote the first selection in which the balls withdrawn have the same number (and let it equal infinity if none of the pairs withdrawn has the same number). For $0 < \alpha < 1$ we want to show that

$$\lim_n P\{T > \alpha n\} = e^{-\alpha/2}$$

To verify the above, let M_k denote the number of pairs withdrawn in the first k selections, $k = 1, \ldots, n$.
(a) Argue that when n is large, M_k can be regarded as the number of successes in k (approximately) independent trials.
(b) When n is large, approximate $P\{M_k = 0\}$.
(c) Write the event $\{T > \alpha n\}$ in terms of the value of one of the variables M_k.
(d) Verify the limiting probability above.

23. Suppose that the number of events that occur in a specified time is a Poisson random variable with parameter λ. If each event is counted with probability p, independently of every other event, show that the number of events that are counted is a Poisson random variable with parameter λp. Also, give an intuitive argument as to why this should be so.

As an application of the preceding paragraph, suppose that the number of distinct uranium deposits in a given area is a Poisson random variable with parameter $\lambda = 10$. If, in a fixed period of time, each deposit is discovered independently with probability $\frac{1}{50}$, find the probability that (a) exactly 1, (b) at least 1, and (c) at most 1 deposit is discovered during that time.

24. Prove

$$\sum_{i=0}^{n} e^{-\lambda} \frac{\lambda^i}{i!} = \frac{1}{n!} \int_{\lambda}^{\infty} e^{-x} x^n \, dx$$

HINT: Use integration by parts.

25. If X is a geometric random variable, show analytically that

$$P\{X = n + k | X > n\} = P\{X = k\}$$

Give a verbal argument using the interpretation of a geometric random variable as to why the equation above is true.

26. Let X be a negative binomial random variable with parameters r and p, and let Y be a binomial random variable with parameters n and p. Show that

$$P\{X > n\} = P\{Y < r\}$$

HINT: One could either attempt an analytical proof of the above, which is equivalent to proving the identity

$$\sum_{i=n+1}^{\infty} \binom{i-1}{r-1} p^r (1 - p)^{i-r} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1 - p)^{n-i}$$

or one could attempt a proof that uses the probabilistic interpretation of these random variables. That is, in the latter case start by considering a sequence of independent trials having a common success probability p. Then try to express the events $\{X > n\}$ and $\{Y < r\}$ in terms of the outcomes of this sequence.

27. For a hypergeometric random variable, determine

$$P\{X = k + 1\}/P\{X = k\}$$

28. Balls numbered 1 through N are in an urn. Suppose that n, $n \leq N$, of them are randomly selected without replacement. Let Y denote the largest number selected.

(a) Find the probability mass function of Y.

(b) Derive an expression for $E[Y]$ and then use Fermat's combinatorial identity (see Theoretical Exercise 11 of Chapter 1) to simplify.

29. A jar contains $m + n$ chips, numbered $1, 2, \ldots, n + m$. A set of size n is drawn. If we let X denote the number of chips drawn having numbers that exceed all the numbers of those remaining, compute the probability mass function of X.

30. A jar contains n chips. Suppose that a boy successively draws a chip from the jar, each time replacing the one drawn before drawing another. This continues until the boy draws a chip that he has previously drawn before. Let X denote the number of draws, and compute its probability mass function.

31. Show that Equation (9.6) follows from (9.5).

32. From a set of n elements a nonempty subset is chosen at random in the sense that all of the nonempty subsets are equally likely to be selected. Let X denote the number of elements in the chosen subset. Using the identities given in Theoretical Exercise 12 of Chapter 1, show that

$$E[X] = \frac{n}{2 - \left(\frac{1}{2}\right)^{n-1}}$$

$$\text{Var}(X) = \frac{n \cdot 2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

Show also that for n large,

$$\text{Var}(X) \sim \frac{n}{4}$$

in the sense that the ratio of the above approaches 1 as n approaches ∞. Compare this with the limiting form of $\text{Var}(Y)$ when $P\{Y = i\} = 1/n$, $i = 1, \ldots, n$.

33. An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen and then replaced along with another of the same color. Let X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2.

(a) Find $P\{X > i\}$, $i \geq 1$.

(b) Show that with probability 1, a blue ball is eventually chosen. (That is, show that $P\{X < \infty\} = 1$.)

(c) Find $E[X]$.

SELF-TEST PROBLEMS AND EXERCISES

1. Suppose that the random variable X is equal to the number of hits obtained by a certain baseball player in his next 3 at bats. If $P\{X = 1\} = .3$, $P\{X = 2\} = .2$, and $P\{X = 0\} = 3P\{X = 3\}$, find $E[X]$.

2. Suppose that X takes on one of the values 0, 1, 2. If for some constant c, $P\{X = i\} = cP\{X = i - 1\}$, $i = 1, 2$, find $E[X]$.

3. A coin that when flipped comes up heads with probability p is flipped until either heads or tails has occurred twice. Find the expected number of flips.

4. A certain community is composed of m families, n_i of which have i children, $\sum_{i=1}^{r} n_i = m$. If one of the families is randomly chosen, let X denote the number of children in that family. If one of the $\sum_{i=1}^{r} in_i$ children is randomly chosen, let Y denote the total number of children in the family of that child. Show that $E[Y] \geq E[X]$.

5. Suppose that $P\{X = 0\} = 1 - P\{X = 1\}$. If $E[X] = 3\text{Var}(X)$, find $P\{X = 0\}$.

6. There are 2 coins in a bin. When one of them is flipped it lands on heads with probability .6, and when the other is flipped it lands on heads with probability .3. One of these coins is to be randomly chosen and then flipped. Without knowing which coin is chosen, you can bet any amount up to 10 dollars and you then either win that amount if the coin comes up heads or lose it if it comes up tails. Suppose, however, that an insider is willing to sell you, for an amount C, the information as to which coin was selected. What is your expected payoff if you buy this information? Note that if you buy it and then bet x, then you will end up either winning $x - C$ or $-x - C$ (that is, losing $x + C$ in the latter case). Also, for what values of C does it pay to purchase the information?

7. A philanthropist writes a positive number x on a piece of red paper, shows it to an impartial observer, and then turns it face down on the table. The observer then flips a fair coin. If it shows heads, she writes the value $2x$, and, if tails, the value $x/2$, on a piece of blue paper which she then turns face down on the table. Without knowing either the value x or the result of the coin flip, you have the option of turning over either the red or the blue piece of paper. After doing so, and observing the number written on that paper, you may elect to receive as a reward either that amount or the (unknown) amount written on the other piece of paper. For instance, if you elect to turn over the blue paper and observe the value 100, then you can elect either to accept 100 as your reward or to take the amount (either 200 or 50) on the red paper. Suppose that you would like your expected reward to be large.

(a) Argue that there is no reason to turn over the red paper first because if you do so, then no matter what value you observe, it is always better to switch to the blue paper.

(b) Let y be a fixed nonnegative value, and consider the following strategy. Turn over the blue paper and if its value is at least y, then accept that amount. If it is less than y, then switch to the red paper. Let $R_y(x)$ denote the reward obtained if the philanthropist writes the amount x and you employ this strategy. Find $E[R_y(x)]$. Note that $E[R_0(x)]$ is the expected reward if the philanthropist writes the amount x when you employ the strategy of always choosing the blue paper.

8. Let $B(n, p)$ represent a binomial random variable with parameters n and p. Argue that

$$P\{B(n, p) \leq i\} = 1 - P\{B(n, 1 - p) \leq n - i - 1\}$$

HINT: The number of successes is less than or equal to i is equivalent to what statement about the number of failures?

9. If X is a binomial random variable with expected value 6 and variance 2.4, find $P\{X = 5\}$.

10. An urn contains n balls, numbered 1 through n. If m balls are randomly withdrawn in sequence, each time replacing the ball selected previously, find $P\{X = k\}, k = 1, \ldots, m$, where X is the maximum of the m chosen numbers.

HINT: First find $P\{X \leq k\}$.

11. Teams A and B play a series of games with the first team to win 3 games being declared the winner of the series. Suppose that team A independently wins each game with probability p. Find the conditional probability that team A wins

(a) the series given that it wins the first game;

(b) the first game given that it wins the series.

12. A local soccer team has 5 more games that it will play. If it wins its game this weekend, then it will play its final 4 games in the upper bracket of its league, and if it loses, then it will play its final games in the lower bracket. If it plays in the upper bracket, then it will independently win each of its games in this bracket with probability .4, and if it plays in the lower bracket, then it will independently win each of its games with probability .7. If the probability that it wins its game this weekend is .5, what is the probability that it wins at least 3 of its final 4 games?

13. On average, 5.2 hurricanes hit a certain region in a year. What is the probability that there will be 3 or fewer hurricanes hitting this year?

14. The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can only be observed if it is positive, since if it is 0, then we cannot know that such an insect was on the leaf. If we let Y denote the observed number of eggs, then

$$P\{Y = i\} = P\{X = i | X > 0\}$$

where X is Poisson with parameter λ. Find $E[Y]$.

15. A casino patron will continue to make \$5 bets on red in roulette until she has won 4 of these bets.

(a) What is the probability that she places a total of 9 bets?

(b) What is her expected winnings when she stops?

REMARK: On each bet she will either win \$5 with probability $\frac{18}{38}$ or lose \$5 with probability $\frac{20}{38}$.

16. When three friends go for coffee, they decide who will pay the check by each flipping a coin and then letting the "odd person" pay. If all three flips are the same (so there is no odd person), then they make a second round of flips, and continue to do so until there is an odd person. What is the probability that

(a) exactly 3 rounds of flips are made;

(b) more than 4 rounds are needed?

17. If X is a geometric random variable with parameter p, show that

$$E[1/X] = \frac{-p \log(p)}{1 - p}$$

HINT: You will need to evaluate an expression of the form $\sum_{i=1}^{\infty} a^i/i$. To do so, write $a^i/i = \int_0^a x^{i-1}\, dx$, and then interchange the sum and the integral.

CHAPTER 5

Continuous Random Variables

5.1 INTRODUCTION

In Chapter 4 we considered discrete random variables, that is, random variables whose set of possible values is either finite or countably infinite. However, there also exist random variables whose set of possible values is uncountable. Two examples would be the time that a train arrives at a specified stop and the lifetime of a transistor. Let X be such a random variable. We say that X is a *continuous*[†] random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers[‡]

$$P\{X \in B\} = \int_B f(x)\,dx \tag{1.1}$$

The function f is called the *probability density function* of the random variable X (see Figure 5.1).

In words, Equation (1.1) states that the probability that X will be in B may be obtained by integrating the probability density function over the set B. Since X must assume some value, f must satisfy

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\,dx$$

All probability statements about X can be answered in terms of f. For instance, letting $B = [a, b]$, we obtain from Equation (1.1) that

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx \tag{1.2}$$

[†] Sometimes called *absolutely continuous*.

[‡] Actually, for technical reasons Equation (1.1) is true only for the measurable sets B, which, fortunately, includes all sets of practical interest.

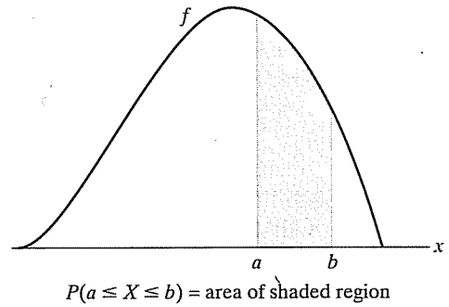


Figure 5.1 Probability density function f.

If we let $a = b$ in Equation (1.2), we obtain

$$P\{X = a\} = \int_a^a f(x)\,dx = 0$$

In words, this equation states that the probability that a continuous random variable will assume any fixed value is zero. Hence, for a continuous random variable,

$$P\{X < a\} = P\{X \le a\} = F(a) = \int_{-\infty}^a f(x)\,dx$$

Example 1a. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of C?
(b) Find $P\{X > 1\}$.

Solution (a) Since f is a probability density function, we must have that $\int_{-\infty}^{\infty} f(x)\,dx = 1$, implying that

$$C\int_0^2 (4x - 2x^2)\,dx = 1$$

or

$$C\left[2x^2 - \frac{2x^3}{3}\right]\Bigg|_{x=0}^{x=2} = 1$$

or

$$C = \frac{3}{8}$$

Hence

$$\text{(b) } P\{X > 1\} = \int_1^{\infty} f(x)\,dx = \tfrac{3}{8}\int_1^2 (4x - 2x^2)\,dx = \tfrac{1}{2}$$

■

Example 1b. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

What is the probability that
(a) a computer will function between 50 and 150 hours before breaking down;
(b) it will function less than 100 hours?

Solution (a) Since

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \lambda \int_0^{\infty} e^{-x/100}\,dx$$

we obtain

$$1 = -\lambda(100)e^{-x/100}\Big|_0^{\infty} = 100\lambda \qquad \text{or} \qquad \lambda = \frac{1}{100}$$

Hence the probability that a computer will function between 50 and 150 hours before breaking down is given by

$$P\{50 < X < 150\} = \int_{50}^{150} \frac{1}{100} e^{-x/100}\,dx = -e^{-x/100}\Big|_{50}^{150}$$
$$= e^{-1/2} - e^{-3/2} \approx .384$$

(b) Similarly,

$$P\{X < 100\} = \int_0^{100} \frac{1}{100} e^{-x/100}\,dx = -e^{-x/100}\Big|_0^{100} = 1 - e^{-1} \approx .633$$

In other words, approximately 63.3 percent of the time a computer will fail before registering 100 hours of use. ∎

Example 1c. The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(s) = \begin{cases} 0 & x \leq 100 \\ \dfrac{100}{x^2} & x > 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events E_i, $i = 1, 2, 3, 4, 5$, that the ith such tube will have to be replaced within this time, are independent.

Solution Now,

$$P(E_i) = \int_0^{150} f(x)\,dx$$
$$= 100 \int_{100}^{150} x^{-2}\,dx$$
$$= \frac{1}{3}$$

Hence, from the independence of the events E_i, it follows that the desired probability is

$$\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$$ ∎

The relationship between the cumulative distribution F and the probability density f is expressed by

$$F(a) = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)\,dx$$

Differentiating both sides of the above yields

$$\frac{d}{da}F(a) = f(a)$$

That is, the density is the derivative of the cumulative distribution function. A somewhat more intuitive interpretation of the density function may be obtained from Equation (1.2) as follows:

$$P\left\{a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\right\} = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)\,dx \approx \varepsilon f(a)$$

when ε is small and when $f(\cdot)$ is continuous at $x = a$. In other words, the probability that X will be contained in an interval of length ε around the point a is approximately $\varepsilon f(a)$. From this, we see that $f(a)$ is a measure of how likely it is that the random variable will be near a.

5.2 EXPECTATION AND VARIANCE OF CONTINUOUS RANDOM VARIABLES

In Chapter 4 we defined the expected value of a discrete random variable X by

$$E[X] = \sum_x xP\{X = x\}$$

If X is a continuous random variable having probability density function $f(x)$, then as

$$f(x)\,dx \approx P\{x \leq X \leq x + dx\} \qquad \text{for } dx \text{ small}$$

it is easy to see that the analogous definition is to define the expected value of X by

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Example 2a. Find $E[X]$ when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\begin{aligned} E[X] &= \int x f(x)\, dx \\ &= \int_0^1 2x^2\, dx \\ &= \tfrac{2}{3} \end{aligned}$$

Example 2b. The density function of X is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[e^X]$.

Solution Let $Y = e^X$. We start by determining F_Y, the probability distribution function of Y. Now, for $1 \le x \le e$,

$$\begin{aligned} F_Y(x) &= P\{Y \le x\} \\ &= P\{e^X \le x\} \\ &= P\{X \le \log(x)\} \\ &= \int_0^{\log(x)} f(y)\, dy \\ &= \log(x) \end{aligned}$$

By differentiating $F_Y(x)$, we obtain that the probability density function of Y is given by

$$f_Y(x) = \frac{1}{x} \qquad 1 \le x \le e$$

Hence

$$\begin{aligned} E[e^X] = E[Y] &= \int_{-\infty}^{\infty} x f_Y(x)\, dx \\ &= \int_1^e dx \\ &= e - 1 \end{aligned}$$

Although the method employed in Example 2b to compute the expected value of a function of X is always applicable, there is, as in the discrete case, an alternative way of proceeding. The following is a direct analog of Proposition 5.1 of Chapter 4.

Proposition 2.1

If X is a continuous random variable with probability density function $f(x)$, then for any real-valued function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

An application of Proposition 2.1 to Example 2b yields that

$$\begin{aligned} E[e^X] &= \int_0^1 e^x\, dx \qquad \text{since } f(x) = 1, \quad 0 < x < 1 \\ &= e - 1 \end{aligned}$$

which is in accord with the result of that example.

The proof of Proposition 2.1 is more involved than its discrete random variable analog and we will present one under the provision that the random variable $g(X)$ is nonnegative. (The general proof, which follows the argument in the case we present, is indicated in Theoretical Exercises 2 and 3.) We will need the following lemma, which is of independent interest.

Lemma 2.1

For a nonnegative random variable Y,

$$E[Y] = \int_0^{\infty} P\{Y > y\}\, dy$$

Proof: We present a proof when Y is a continuous random variable with probability density function f_Y. We have

$$\int_0^{\infty} P\{Y > y\}\, dy = \int_0^{\infty} \int_y^{\infty} f_Y(x)\, dx\, dy$$

where we have used the fact that $P\{Y > y\} = \int_y^{\infty} f_Y(x)\, dx$. Interchanging the order of integration in the preceding equation yields

$$\begin{aligned} \int_0^{\infty} P\{Y > y\}\, dy &= \int_0^{\infty} \left(\int_0^x dy \right) f_Y(x)\, dx \\ &= \int_0^{\infty} x f_Y(x)\, dx \\ &= E[Y] \end{aligned}$$

Proof of Proposition 2.1: For any function g for which $g(x) \geq 0$, we have from Lemma 2.1 that

$$\begin{aligned} E[g(X)] &= \int_0^\infty P\{g(X) > y\}\, dy \\ &= \int_0^\infty \int_{x:g(x)>y} f(x)\, dx\, dy \\ &= \int_{x:g(x)>0} \int_0^{g(x)} dy\, f(x)\, dx \\ &= \int_{x:g(x)>0} g(x) f(x)\, dx \end{aligned}$$

which completes the proof.

Example 2c. A stick of length 1 is split at a point U that is uniformly distributed over (0, 1). Determine the expected length of the piece that contains the point p, $0 \leq p \leq 1$.

Solution Let $L_p(U)$ denote the length of the substick that contains the point p, and note (see Figure 5.2) that

$$L_p(U) = \begin{cases} 1 - U & U < p \\ U & U > p \end{cases}$$

Hence from Proposition 2.1 we have that

$$\begin{aligned} E[L_p(U)] &= \int_0^1 L_p(u)\, du \\ &= \int_0^p (1 - u)\, du + \int_p^1 u\, du \\ &= \frac{1}{2} - \frac{(1-p)^2}{2} + \frac{1}{2} - \frac{p^2}{2} \\ &= \frac{1}{2} + p(1 - p) \end{aligned}$$

Since $p(1 - p)$ is maximized when $p = \frac{1}{2}$, it is interesting to note that the expected length of the substick containing the point p is maximized when p is the midpoint of the original stick. ∎

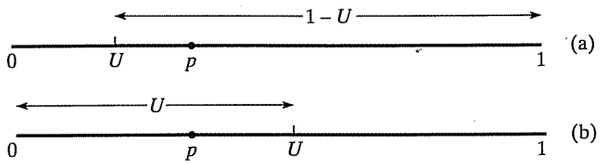


Figure 5.2 Substick containing point p: (a) $U < p$; (b) $U > p$.

Example 2d. Suppose that if you are s minutes early for an appointment, then you incur the cost cs, and if you are s minutes late, then you incur the cost ks. Suppose that the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function f. Determine the time at which you should depart if you want to minimize your expected cost.

Solution Let X denote the travel time. If you leave t minutes before your appointment, then your cost, call it $C_t(X)$, is given by

$$C_t(X) = \begin{cases} c(t - X) & \text{if } X \leq t \\ k(X - t) & \text{if } X \geq t \end{cases}$$

Therefore,

$$\begin{aligned} E[C_t(X)] &= \int_0^\infty C_t(x) f(x)\, dx \\ &= \int_0^t c(t - x) f(x)\, dx + \int_t^\infty k(x - t) f(x)\, dx \\ &= ct \int_0^t f(x)\, dx - c \int_0^t x f(x)\, dx + k \int_t^\infty x f(x)\, dx - kt \int_t^\infty f(x)\, dx \end{aligned}$$

The value of t that minimizes $E[C_t(X)]$ can now be obtained by calculus. Differentiation yields

$$\begin{aligned} \frac{d}{dt} E[C_t(X)] &= ct f(t) + cF(t) - ct f(t) - kt f(t) + kt f(t) - k[1 - F(t)] \\ &= (k + c)F(t) - k \end{aligned}$$

Equating to zero shows that the minimal expected cost is obtained when you leave t^* minutes before your appointment, where t^* satisfies

$$F(t^*) = \frac{k}{k + c}$$

∎

As in Chapter 4, we can use Proposition 2.1 to show the following.

Corollary 2.1

If a and b are constants, then

$$E[aX + b] = aE[X] + b$$

The proof of Corollary 2.1 for a continuous random variable X is the same as the one given for a discrete random variable. The only modification is that the sum is replaced by an integral and the probability mass function by a probability density function.

The variance of a continuous random variable is defined exactly as it is for a discrete one. Namely, if X is a random variable with expected value μ, then the variance of X is defined (for any type of random variable) by

$$\text{Var}(X) = E[(X - \mu)^2]$$

The alternative formula,

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

is established in a similar manner as in the discrete case.

Example 2e. Find Var(X) for X as given in Example 2a.

Solution We first compute $E[X^2]$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\ &= \int_0^1 2x^3\,dx \\ &= \tfrac{1}{2} \end{aligned}$$

Hence, since $E[X] = \frac{2}{3}$ we obtain that

$$\text{Var}(X) = \tfrac{1}{2} - \left(\tfrac{2}{3}\right)^2 = \tfrac{1}{18}$$

It can be shown, with the proof mimicking the one given for discrete random variables, that for constants a and b

$$\text{Var}(aX + b) = a^2\,\text{Var}(X)$$

There are several important classes of continuous random variables that appear frequently in applications of probability; the next few sections are devoted to a study of some of them.

5.3 THE UNIFORM RANDOM VARIABLE

A random variable is said to be *uniformly* distributed over the interval (0, 1) if its probability density function is given by

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Note that Equation (3.1) is a density function, since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 dx = 1$. Because $f(x) > 0$ only when $x \in (0, 1)$, it follows that X must assume a value in (0, 1). Also, since $f(x)$ is constant for $x \in (0, 1)$, X is just as likely to be near any value in (0, 1) as any other value. To check this, note that for any $0 < a < b < 1$,

$$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx = b - a$$

In other words, the probability that X is in any particular subinterval of (0, 1) equals the length of that subinterval.

In general, we say that X is a uniform random variable on the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Since $F(a) = \int_{-\infty}^{a} f(x)\,dx$, we obtain from Equation (3.2) that the distribution function of a uniform random variable on the interval (α, β) is given by

$$F(a) = \begin{cases} 0 & a \leq \alpha \\ \dfrac{a - \alpha}{\beta - \alpha} & \alpha < a < \beta \\ 1 & a \geq \beta \end{cases}$$

Figure 5.3 presents a graph of $f(a)$ and $F(a)$.

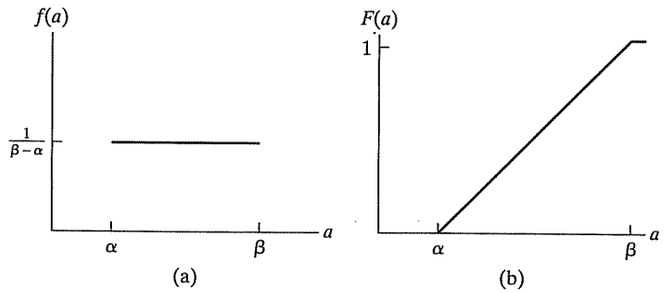


Figure 5.3 Graph of (a) $f(a)$ and (b) $F(a)$ for a uniform (α, β) random variable.

Example 3a. Let X be uniformly distributed over (α, β). Find (a) $E[X]$ and (b) Var(X).

Solution

$$\begin{aligned} \text{(a) } E[X] &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha}\,dx \\ &= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} \\ &= \frac{\beta + \alpha}{2} \end{aligned}$$

In words, the expected value of a random variable uniformly distributed over some interval is equal to the midpoint of that interval.

(b) To find $\text{Var}(X)$, we first calculate $E[X^2]$.

$$E[X^2] = \int_\alpha^\beta \frac{1}{\beta - \alpha} x^2 \, dx$$

$$= \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)}$$

$$= \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

Hence

$$\text{Var}(X) = \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \frac{(\alpha + \beta)^2}{4}$$

$$= \frac{(\beta - \alpha)^2}{12}$$

Therefore, the variance of a random variable that is uniformly distributed over some interval is the square of the length of that interval divided by 12. ∎

Example 3b. If X is uniformly distributed over (0, 10), calculate the probability that (a) $X < 3$, (b) $X > 6$, and (c) $3 < X < 8$.

Solution

(a) $P\{X < 3\} = \int_0^3 \frac{1}{10} dx = \frac{3}{10}$

(b) $P\{X > 6\} = \int_6^{10} \frac{1}{10} dx = \frac{4}{10}$

(c) $P\{3 < X < 8\} = \int_3^8 \frac{1}{10} dx = \frac{1}{2}$ ∎

Example 3c. Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus;

(b) more than 10 minutes for a bus.

Solution Let X denote the number of minutes past 7 that the passenger arrives at the stop. Since X is a uniform random variable over the interval (0, 30), it follows that the passenger will have to wait less than 5 minutes if (and only if) he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence the desired probability for part (a) is

$$P\{10 < X < 15\} + P\{25 < X < 30\} = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{3}$$

Similarly, he would have to wait more than 10 minutes if he arrives between 7 and 7:05 or between 7:15 and 7:20, and so the probability for part (b) is

$$P\{0 < X < 5\} + P\{15 < X < 20\} = \frac{1}{3}$$ ∎

The next example was first considered by the French mathematician L. F. Bertrand in 1889 and is often referred to as *Bertrand's paradox*. It represents our initial introduction to a subject commonly referred to as geometrical probability.

Example 3d. Consider a random chord of a circle. What is the probability that the length of the chord will be greater than the side of the equilateral triangle inscribed in that circle?

Solution The problem as stated is incapable of solution because it is not clear what is meant by a random chord. To give meaning to this phrase, we shall reformulate the problem in two distinct ways.

The first formulation is as follows: The position of the chord can be determined by its distance from the center of the circle. This distance can vary between 0 and r, the radius of the circle. Now, the length of the chord will be greater than the side of the equilateral triangle inscribed in the circle if its distance from the center is less than $r/2$. Hence, by assuming that a random chord is one whose distance D from the center is uniformly distributed between 0 and r, we see that the probability that it is greater than the side of an inscribed equilateral triangle is

$$P\left\{D < \frac{r}{2}\right\} = \frac{r/2}{r} = \frac{1}{2}$$

For our second formulation of the problem consider an arbitrary chord of the circle; through one end of the chord draw a tangent. The angle θ between the chord and the tangent, which can vary from 0° to 180°, determines the position of the chord (see Figure 5.4). Furthermore, the length of the chord will be greater than the side of the inscribed equilateral triangle if the angle θ is between 60° and 120°. Hence, assuming that a random chord is one whose angle θ is uniformly distributed between 0° and 180°, we see that the desired answer in this formulation is

$$P\{60 < \theta < 120\} = \frac{120 - 60}{180} = \frac{1}{3}$$

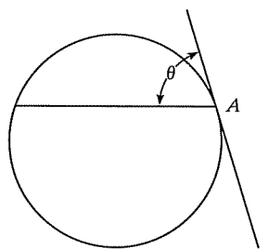


Figure 5.4

It should be noted that random experiments could be performed in such a way that $\frac{1}{2}$ or $\frac{1}{3}$ would be the correct probability. For instance, if a circular disk of radius r is thrown on a table ruled with parallel lines a distance $2r$ apart, then one and only one of these lines would cross the disk and form a chord. All distances from this chord to the center of the disk would be equally likely so that the desired probability that the chord's length will be greater than the side of an inscribed equilateral triangle is $\frac{1}{2}$. On the other hand, if the experiment consisted of rotating a needle freely about a point A on the edge (see Figure 5.4) of the circle, the desired answer would be $\frac{1}{3}$. ∎

5.4 NORMAL RANDOM VARIABLES

We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

This density function is a bell-shaped curve that is symmetric about μ (see Figure 5.5).

The normal distribution was introduced by the French mathematician Abraham de Moivre in 1733 and was used by him to approximate probabilities associated with binomial random variables when the binomial parameter n is large. This result was later extended by Laplace and others and is now encompassed in a probability theorem known as the central limit theorem, which is discussed in

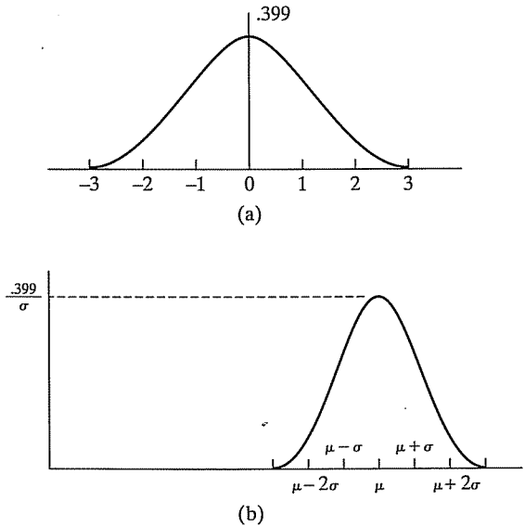


Figure 5.5 Normal density function: (a) $\mu = 0$, $\sigma = 1$; (b) arbitrary μ, σ^2.

Chapter 8. The central limit theorem, one of the two most important results in probability theory,† gives a theoretical base to the often noted empirical observation that, in practice, many random phenomena obey, at least approximately, a normal probability distribution. Some examples of this behavior are the height of a man, the velocity in any direction of a molecule in gas, and the error made in measuring a physical quantity.

To prove that $f(x)$ is indeed a probability density function, we need to show that

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = 1$$

By making the substitution $y = (x - \mu)/\sigma$, we see that

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

and hence we must show that

$$\int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$$

Toward this end, let $I = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$. Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(y^2+x^2)/2}\,dy\,dx \end{aligned}$$

We now evaluate the double integral by means of a change of variables to polar coordinates. (That is, let $x = r\cos\theta$, $y = r\sin\theta$, and $dy\,dx = r\,d\theta\,dr$.) Thus

$$\begin{aligned} I^2 &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\,r\,d\theta\,dr \\ &= 2\pi\int_0^{\infty} re^{-r^2/2}\,dr \\ &= -2\pi e^{-r^2/2}\Big|_0^{\infty} \\ &= 2\pi \end{aligned}$$

Hence $I = \sqrt{2\pi}$, and the result is proved.

We now show that the parameters μ and σ^2 of a normal random variable represent its expected value and variance.

† The other is the strong law of large numbers.

Example 4a. Find (a) $E[X]$ and (b) $\text{Var}(X)$ when X is a normal random variable with parameters μ and σ^2.

Solution

$$\text{(a) } E[X] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} xe^{-(x-\mu)^2/2\sigma^2}\,dx$$

Writing x as $(x - \mu) + \mu$ yields

$$E[X] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/2\sigma^2}\,dx + \mu\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Letting $y = x - \mu$ in the first integral yields

$$E[X] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} ye^{-y^2/2\sigma^2}\,dy + \mu\int_{-\infty}^{\infty} f(x)\,dx$$

where $f(x)$ is the normal density. By symmetry, the first integral must be 0, so

$$E[X] = \mu\int_{-\infty}^{\infty} f(x)\,dx = \mu$$

(b) Since $E[X] = \mu$, we have that

$$\text{Var}(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)^2e^{-(x-\mu)^2/2\sigma^2}\,dx \tag{4.1}$$

Substituting $y = (x - \mu)/\sigma$ in Equation (4.1) yields

$$\text{Var}(X) = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2e^{-y^2/2}\,dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left[-ye^{-y^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right] \text{ by integration by parts}$$

$$= \sigma^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \sigma^2$$

∎

An important fact about normal random variables is that if X is normally distributed with parameters μ and σ^2, then $Y = \alpha X + \beta$ is normally distributed with parameters $\alpha\mu + \beta$ and $\alpha^2\sigma^2$. To show this, suppose that $\alpha > 0$. (The verification when $\alpha < 0$ is similar.) Now, F_Y,[†] the cumulative distribution function

[†] When there is more than one random variable under consideration, we shall denote the cumulative distribution function of a random variable Z by F_Z. Similarly, we shall denote the density of Z by f_Z.

of the random variable Y, is given by

$$F_Y(a) = P\{\alpha X + \beta \le a\} = P\left\{X \le \frac{a-\beta}{\alpha}\right\} = F_X\left(\frac{a-\beta}{\alpha}\right)$$

Differentiation yields that the density function of Y is

$$f_Y(a) = \frac{1}{\alpha}f_X\left(\frac{a-\beta}{\alpha}\right) = \frac{1}{\sqrt{2\pi}\,\alpha\sigma}\exp\left\{-\left(\frac{a-\beta}{\alpha}-\mu\right)^2\Big/2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\alpha\sigma}\exp\{-(a-\beta-\alpha\mu)^2/2(\alpha\sigma)^2\} \tag{4.2}$$

which shows that Y is normal with mean $\alpha\mu + \beta$ and variance $\alpha^2\sigma^2$.

An important implication of the preceding result is that if X is normally distributed with parameters μ and σ^2, then $Z = (X - \mu)/\sigma$ is normally distributed with parameters 0 and 1. Such a random variable Z is said to have the *standard*, or *unit*, normal distribution.

It is traditional to denote the cumulative distribution function of a standard normal random variable by $\Phi(x)$. That is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

The values of $\Phi(x)$ for nonnegative x are given in Table 5.1. For negative values of x, $\Phi(x)$ can be obtained from Equation (4.3):

$$\Phi(-x) = 1 - \Phi(x) \qquad -\infty < x < \infty \tag{4.3}$$

[The values of $\Phi(x)$ can also be obtained from the text diskette.] The proof of Equation (4.3), which follows from the symmetry of the standard normal density, is left as an exercise. This equation states that if Z is a standard normal random variable, then

$$P\{Z \le -x\} = P\{Z > x\} \qquad -\infty < x < \infty$$

Since $Z = (X - \mu)/\sigma$ is a standard normal random variable whenever X is normally distributed with parameters μ and σ^2, it follows that the distribution

In this case, where the noise is Laplacian with parameter $\lambda = 1$, the 2 types of errors will have probabilities given by

$$\begin{aligned} P\{\text{error}|\text{message 1 is sent}\} &= P\{N < -1.5\} \\ &= \tfrac{1}{2}e^{-1.5} \\ &\approx .1116 \\ P\{\text{error}|\text{message 0 is sent}\} &= P\{N \geq 2.5\} \\ &= \tfrac{1}{2}e^{-2.5} \\ &\approx .041 \end{aligned}$$

On comparing this with the results of Example 4e, we see that the error probabilities are higher when the noise is Laplacian with $\lambda = 1$ than when it is a standard normal variable.

5.5.1 Hazard Rate Functions

Consider a positive continuous random variable X that we interpret as being the lifetime of some item, having distribution function F and density f. The *hazard rate* (sometimes called the *failure rate*) function $\lambda(t)$ of F is defined by

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} \qquad \bar{F} = 1 - F$$

To interpret $\lambda(t)$, suppose that the item has survived for a time t and we desire the probability that it will not survive for an additional time dt. That is, consider $P\{X \in (t, t + dt)|X > t\}$. Now

$$\begin{aligned} P\{X \in (t, t + dt)|X > t\} &= \frac{P\{X \in (t, t + dt), X > t\}}{P\{X > t\}} \\ &= \frac{P\{X \in (t, t + dt)\}}{P\{X > t\}} \\ &\approx \frac{f(t)}{\bar{F}(t)}\, dt \end{aligned}$$

That is, $\lambda(t)$ represents the conditional probability intensity that a t-unit-old item will fail.

Suppose now that the lifetime distribution is exponential. Then, by the memoryless property, it follows that the distribution of remaining life for a t-year-old item is the same as for a new item. Hence $\lambda(t)$ should be constant. This checks out, since

$$\begin{aligned} \lambda(t) &= \frac{f(t)}{\bar{F}(t)} \\ &= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} \\ &= \lambda \end{aligned}$$

Thus the failure rate function for the exponential distribution is constant. The parameter λ is often referred to as the *rate* of the distribution.

It turns out that the failure rate function $\lambda(t)$ uniquely determines the distribution F. To prove this, note that by definition

$$\lambda(t) = \frac{\frac{d}{dt}F(t)}{1 - F(t)}$$

Integrating both sides yields

$$\log(1 - F(t)) = -\int_0^t \lambda(t)\, dt + k$$

or

$$1 - F(t) = e^k \exp\left\{-\int_0^t \lambda(t)\, dt\right\}$$

Letting $t = 0$ shows that $k = 0$ and thus

$$F(t) = 1 - \exp\left\{-\int_0^t \lambda(t)\, dt\right\} \tag{5.4}$$

Hence a distribution function of a positive continuous random variable can be specified by giving its hazard rate function. For instance, if a random variable has a linear hazard rate function—that is, if

$$\lambda(t) = a + bt$$

then its distribution function is given by

$$F(t) = 1 - e^{-at - bt^2/2}$$

and differentiation yields that its density is

$$f(t) = (a + bt)e^{-(at + bt^2/2)} \qquad t \geq 0$$

When $a = 0$, the above is known as the Rayleigh density function.

Example 5f. One often hears that the death rate of a person who smokes is, at each age, twice that of a nonsmoker. What does this mean? Does it mean that a nonsmoker has twice the probability of surviving a given number of years as does a smoker of the same age?

Solution If $\lambda_s(t)$ denotes the hazard rate of a smoker of age t and $\lambda_n(t)$ that of a nonsmoker of age t, then the above is equivalent to the statement that

$$\lambda_s(t) = 2\lambda_n(t)$$

The probability that an A-year-old nonsmoker will survive until age B, $A < B$, is

$$
\begin{aligned}
&P\{A\text{-year-old nonsmoker reaches age } B\} \\
&\quad = P\{\text{nonsmoker's lifetime} > B \mid \text{nonsmoker's lifetime} > A\} \\
&\quad = \frac{1 - F_{\text{non}}(B)}{1 - F_{\text{non}}(A)} \\
&\quad = \frac{\exp\left\{-\int_0^B \lambda_n(t)\,dt\right\}}{\exp\left\{-\int_0^A \lambda_n(t)\,dt\right\}} \qquad \text{from (5.4)} \\
&\quad = \exp\left\{-\int_A^B \lambda_n(t)\,dt\right\}
\end{aligned}
$$

whereas the corresponding probability for a smoker is, by the same reasoning,

$$
\begin{aligned}
P\{A\text{-year-old smoker reaches age } B\} &= \exp\left\{-\int_A^B \lambda_s(t)\,dt\right\} \\
&= \exp\left\{-2\int_A^B \lambda_n(t)\,dt\right\} \\
&= \left[\exp\left\{-\int_A^B \lambda_n(t)\,dt\right\}\right]^2
\end{aligned}
$$

In other words, of two people of the same age, one of whom is a smoker and the other a nonsmoker, the probability that the smoker survives to any given age is the *square* (not one-half) of the corresponding probability for a nonsmoker. For instance, if $\lambda_n(t) = \frac{1}{30}$, $50 \le t \le 60$, then the probability that a 50-year-old nonsmoker reaches age 60 is $e^{-1/3} \approx .7165$, whereas the corresponding probability for a smoker is $e^{-2/3} \approx .5134$. ∎

5.6 OTHER CONTINUOUS DISTRIBUTIONS

5.6.1 The Gamma Distribution

A random variable is said to have a gamma distribution with parameters (t, λ), $\lambda > 0$, and $t > 0$ if its density function is given by

$$
f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(t)} & x \ge 0 \\ 0 & x < 0 \end{cases}
$$

where $\Gamma(t)$, called the gamma function, is defined as

$$
\Gamma(t) = \int_0^\infty e^{-y} y^{t-1}\,dy
$$

The integration by parts of $\Gamma(t)$ yields that

$$
\begin{aligned}
\Gamma(t) &= -e^{-y}y^{t-1}\Big|_0^\infty + \int_0^\infty e^{-y}(t-1)y^{t-2}\,dy \\
&= (t-1)\int_0^\infty e^{-y}y^{t-2}\,dy \qquad (6.1) \\
&= (t-1)\Gamma(t-1)
\end{aligned}
$$

For integral values of t, say $t = n$, we obtain by applying Equation (6.1) repeatedly that

$$
\begin{aligned}
\Gamma(n) &= (n-1)\Gamma(n-1) \\
&= (n-1)(n-2)\Gamma(n-2) \\
&= \cdots \\
&= (n-1)(n-2)\cdots 3 \cdot 2\Gamma(1)
\end{aligned}
$$

Since $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$, it follows that for integral values of n,

$$
\Gamma(n) = (n-1)!
$$

When t is a positive integer, say $t = n$, the gamma distribution with parameters (t, λ) often arises, in practice, as the distribution of the amount of time one has to wait until a total of n events has occurred. More specifically, if events are occurring randomly in time and in accordance with the three axioms of Section 4.8, then it turns out that the amount of time one has to wait until a total of n events has occurred will be a gamma random variable with parameters (n, λ). To prove this, let T_n denote the time at which the nth event occurs, and note that T_n is less than or equal to t if and only if the number of events that have occurred by time t is at least n. That is, with $N(t)$ equal to the number of events in $[0, t]$,

$$
\begin{aligned}
P\{T_n \le t\} &= P\{N(t) \ge n\} \\
&= \sum_{j=n}^\infty P\{N(t) = j\} \\
&= \sum_{j=n}^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!}
\end{aligned}
$$

where the final identity follows, since the number of events in $[0, t]$ has a Poisson distribution with parameter λt. Differentiation of the above yields that the density function of T_n is as follows:

$$
\begin{aligned}
f(t) &= \sum_{j=n}^\infty \frac{e^{-\lambda t} j(\lambda t)^{j-1}\lambda}{j!} - \sum_{j=n}^\infty \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} \\
&= \sum_{j=n}^\infty \frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^\infty \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} \\
&= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}
\end{aligned}
$$

Hence T_n is the gamma distribution with parameters (n, λ). (This distribution is often referred to in the literature as the n-Erlang distribution.) Note that when $n = 1$, this distribution reduces to the exponential.

The gamma distribution with $\lambda = \frac{1}{2}$ and $t = n/2$ (n being a positive integer) is called the χ^2_n (read "chi-squared") distribution with n degrees of freedom. The chi-squared distribution often arises in practice as being the distribution of the error involved in attempting to hit a target in n dimensional space when each coordinate error is normally distributed. This distribution will be studied in Chapter 6, where its relation to the normal distribution is detailed.

Example 6a. Let X be a gamma random variable with parameters t and λ. Calculate (a) $E[X]$ and (b) $\mathrm{Var}(X)$.

Solution

(a)
$$E[X] = \frac{1}{\Gamma(t)} \int_0^\infty \lambda x e^{-\lambda x} (\lambda x)^{t-1}\, dx$$
$$= \frac{1}{\lambda \Gamma(t)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{t}\, dx$$
$$= \frac{\Gamma(t+1)}{\lambda \Gamma(t)}$$
$$= \frac{t}{\lambda} \qquad \text{by Equation (6.1)}$$

(b) By first calculating $E[X^2]$, we can show that

$$\mathrm{Var}(X) = \frac{t}{\lambda^2}$$

The details are left as an exercise. ∎

5.6.2 The Weibull Distribution

The Weibull distribution is widely used in engineering practice due to its versatility. It was originally proposed for the interpretation of fatigue data, but now its use has extended to many other engineering problems. In particular, it is widely used, in the field of life phenomena, as the distribution of the lifetime of some object, particularly when the "weakest link" model is appropriate for the object. That is, consider an object consisting of many parts and suppose that the object experiences death (failure) when any of its parts fail. Under these conditions, it has been shown (both theoretically and empirically) that a Weibull distribution provides a close approximation to the distribution of the lifetime of the item.

The Weibull distribution function has the form

$$F(x) = \begin{cases} 0 & x \le \nu \\ 1 - \exp\left\{ -\left(\frac{x-\nu}{\alpha} \right)^{\beta} \right\} & x > \nu \end{cases} \tag{6.2}$$

A random variable whose cumulative distribution function is given by Equation (6.2) is said to be a Weibull random variable with parameters ν, α, and β. Differentiation yields that the density is

$$f(x) = \begin{cases} 0 & x \le \nu \\ \frac{\beta}{\alpha} \left(\frac{x-\nu}{\alpha} \right)^{\beta-1} \exp\left\{ -\left(\frac{x-\nu}{\alpha} \right)^{\beta} \right\} & x > \nu \end{cases}$$

5.6.3 The Cauchy Distribution

A random variable is said to have a Cauchy distribution with parameter θ, $-\infty < \theta < \infty$, if its density is given by

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x-\theta)^2} \qquad -\infty < x < \infty$$

Example 6b. Suppose that a narrow beam flashlight is spun around its center, which is located a unit distance from the x-axis (see Figure 5.7). When the flashlight has stopped spinning, consider the point X at which the beam intersects the x-axis. (If the beam is not pointing toward the x-axis, repeat the experiment.)

As indicated in Figure 5.7, the point X is determined by the angle θ between the flashlight and the y-axis, which from the physical situation appears to be uniformly distributed between $-\pi/2$ and $\pi/2$. The distribution function of X is thus given by

$$F(x) = P\{X \le x\}$$
$$= P\{\tan\theta \le x\}$$
$$= P\{\theta \le \tan^{-1} x\}$$
$$= \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x$$

where the last equality follows since θ, being uniform over $(-\pi/2, \pi/2)$, yields that

$$P\{\theta \le a\} = \frac{a - (-\pi/2)}{\pi} = \frac{1}{2} + \frac{a}{\pi} \qquad -\frac{\pi}{2} < a < \frac{\pi}{2}$$

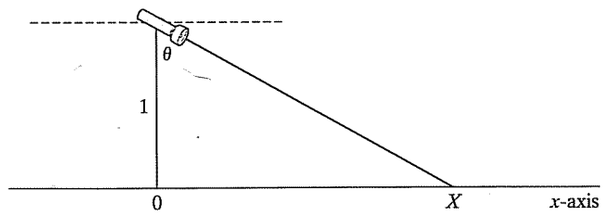


Figure 5.7

Hence the density function of X is given by

$$f(x) = \frac{d}{dx}F(x) = \frac{1}{\pi(1 + x^2)} \qquad -\infty < x < \infty$$

and we see that X has the Cauchy distribution.† ∎

5.6.4 The Beta Distribution

A random variable is said to have a beta distribution if its density is given by

$$f(x) = \begin{cases} \dfrac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\, dx$$

The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval $[c, d]$—which by letting c denote the origin and taking $d - c$ as a unit measurement can be transformed into the interval $[0, 1]$.

When $a = b$, the beta density is symmetric about $\frac{1}{2}$, giving more and more weight to regions about $\frac{1}{2}$ as the common value a increases (see Figure 5.8). When

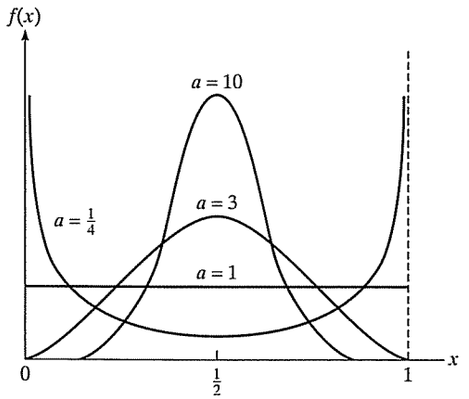


Figure 5.8 Beta densities with parameters (a, b) when $a = b$.

† That $d/dx \tan^{-1} x = 1/(1 + x^2)$ can be seen as follows: If $y = \tan^{-1} x$, then $\tan y = x$, so

$$1 = \frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y)\frac{dy}{dx} = \frac{d}{dy}\left(\frac{\sin y}{\cos y}\right)\frac{dy}{dx}$$
$$= \left(\frac{\cos^2 y + \sin^2 y}{\cos^2 y}\right)\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 y + \cos^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2 + 1}$$

$b > a$, the density is skewed to the left (in the sense that smaller values become more likely); and it is skewed to the right when $a > b$ (see Figure 5.9).

The following relationship can be shown to exist between

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\, dx$$

and the gamma function:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{6.3}$$

Upon using Equation (6.1) along with the identity (6.3), it is an easy matter to show that if X is a beta random variable with parameters a and b, then

$$E[X] = \frac{a}{a + b}$$

$$\text{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

REMARK. A verification of Equation (6.3) appears in Example 7c of Chapter 6.

5.7 THE DISTRIBUTION OF A FUNCTION OF A RANDOM VARIABLE

It is often the case that we know the probability distribution of a random variable and are interested in determining the distribution of some function of it. For instance, suppose that we know the distribution of X and want to find the distribu-

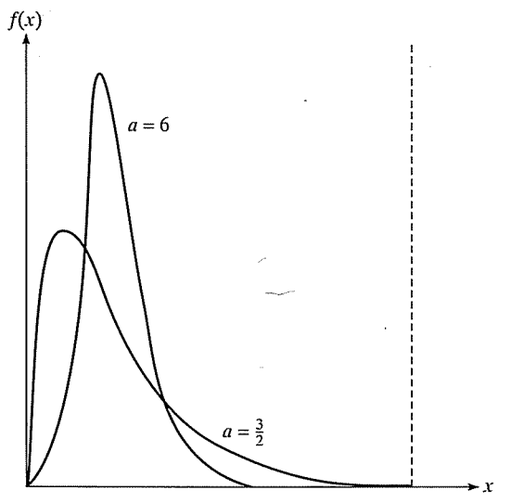


Figure 5.9 Beta densities with parameters (a, b) when $a/(a + b) = 1/20$.

tion of $g(X)$. To do so, it is necessary to express the event that $g(X) \leq y$ in terms of X being in some set. We illustrate by the following examples.

Example 7a. Let X be uniformly distributed over (0, 1). We obtain the distribution of the random variable Y, defined by $Y = X^n$, as follows: For $0 \leq y \leq 1$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{X^n \leq y\} \\ &= P\{X \leq y^{1/n}\} \\ &= F_X(y^{1/n}) \\ &= y^{1/n} \end{aligned}$$

For instance, the density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{1/n-1} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

■

Example 7b. If X is a continuous random variable with probability density f_X, then the distribution of $Y = X^2$ is obtained as follows: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{X^2 \leq y\} \\ &= P\{-\sqrt{y} \leq X \leq \sqrt{y}\} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

Differentiation yields

$$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$

■

Example 7c. If X has a probability density f_X, then $Y = |X|$ has a density function that is obtained as follows: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{|X| \leq y\} \\ &= P\{-y \leq X \leq y\} \\ &= F_X(y) - F_X(-y) \end{aligned}$$

Hence, on differentiation, we obtain

$$f_Y(y) = f_X(y) + f_X(-y) \qquad y \geq 0$$

■

The method employed in Examples 7a through 7c can be used to prove Theorem 7.1.

Theorem 7.1

Let X be a continuous random variable having probability density function f_X. Suppose that $g(x)$ is a strictly monotone (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a probability density function given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$.

We shall prove Theorem 7.1 when $g(x)$ is an increasing function.

Proof: Suppose that $y = g(x)$ for some x. Then, with $Y = g(X)$,

$$\begin{aligned} F_Y(y) &= P\{g(X) \leq y\} \\ &= P\{X \leq g^{-1}(y)\} \\ &= F_X(g^{-1}(y)) \end{aligned}$$

Differentiation gives that

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

which agrees with Theorem 7.1 since $g^{-1}(y)$ is nondecreasing, so its derivative is nonnegative.

When $y \neq g(x)$ for any x, then $F_Y(y)$ is either 0 or 1, and in either case $f_Y(y) = 0$. ■

Example 7d. Let X be a continuous nonnegative random variable with density function f, and let $Y = X^n$. Find f_Y, the probability density function of Y.

Solution If $g(x) = x^n$, then

$$g^{-1}(y) = y^{1/n}$$

and

$$\frac{d}{dy}\{g^{-1}(y)\} = \frac{1}{n} y^{1/n-1}$$

Hence, from Theorem 7.1, we obtain that

$$f_Y(y) = \frac{1}{n} y^{1/n-1} f(y^{1/n})$$

If $n = 2$, this gives

$$f_Y(y) = \frac{1}{2\sqrt{y}} f(\sqrt{y})$$

which (since $X \geq 0$) is in agreement with the result of Example 7b. ∎

SUMMARY

A random variable X is called *continuous* if there is a nonnegative function f, called the *probability density function* of X, such that for any set B

$$P\{X \in B\} = \int_B f(x)\,dx$$

If X is continuous, then its distribution function F will be differentiable and

$$\frac{d}{dx} F(x) = f(x)$$

The expected value of a continuous random variable X is defined by

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

A useful identity is that for any function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

As in the case of a discrete random variable, the variance of X is defined by

$$\text{Var}(X) = E[(X - E[X])^2]$$

A random variable X is said to be *uniform* over the interval (a, b) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Its expected value and variance are

$$E[X] = \frac{a+b}{2} \qquad \text{Var}(X) = \frac{(b-a)^2}{12}$$

A random variable X is said to be *normal* with parameters μ and σ^2 if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

It can be shown that

$$\mu = E[X] \qquad \sigma^2 = \text{Var}(X)$$

If X is normal with mean μ and variance σ^2, then Z, defined by

$$Z = \frac{X - \mu}{\sigma}$$

is normal with mean 0 and variance 1. Such a random variable is said to be a *standard* normal random variable. Probabilities about X can be expressed in terms of probabilities about the standard normal variable Z, whose probability distribution function can be obtained either from Table 5.1 or from the text diskette.

The probability distribution function of a binomial random variable with parameters n and p can, when n is large, be approximated by that of a normal random variable having mean np and variance $np(1 - p)$.

A random variable whose probability density function is of the form

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is said to be an *exponential* random variable with parameter λ. Its expected value and variance are

$$E[X] = \frac{1}{\lambda} \qquad \text{Var}(X) = \frac{1}{\lambda^2}$$

A key property, possessed only by exponential random variables, is that they are *memoryless* in the sense that for positive s and t,

$$P\{X > s + t | X > t\} = P\{X > s\}$$

If X represents the life of an item, then the memoryless property states that for any t, the remaining life of a t-year-old item has the same probability distribution as the life of a new item. Thus one need not remember the age of an item to know its distribution of remaining life.

Let X be a nonnegative continuous random variable with distribution function F and density function f. The function

$$\lambda(t) = \frac{f(t)}{1 - F(t)} \qquad t \geq 0$$

is called the *hazard rate,* or *failure rate,* function of F. If we interpret X as being the life of an item, then for small values of dt, $\lambda(t)\,dt$ is approximately the probability that a t-unit-old item will fail within an additional time dt. If F is the exponential distribution with parameter λ, then

$$\lambda(t) = \lambda \qquad t \geq 0$$

In addition, the exponential is the unique distribution having a constant failure rate.

A random variable is said to have a *gamma* distribution with parameters t and λ if its probability density function is equal to

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{t-1}}{\Gamma(t)} \qquad x \geq 0$$

and is 0 otherwise. The quantity $\Gamma(t)$ is called the gamma function and is defined by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx$$

The expected value and variance of a gamma random variable are

$$E[X] = \frac{t}{\lambda} \qquad \text{Var}(X) = \frac{t}{\lambda^2}$$

A random variable is said to have a *beta* distribution with parameters (a, b) if its probability density function is equal to

$$f(x) = \frac{1}{B(a, b)}x^{a-1}(1 - x)^{b-1} \qquad 0 \le x \le 1$$

and is equal to 0 otherwise. The constant $B(a, b)$ is given by

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\,dx$$

The mean and variance of such a random variable are

$$E[X] = \frac{a}{a + b} \qquad \text{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

PROBLEMS

1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?
(b) What is the cumulative distribution function of X?

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

3. Consider the function

$$f(x) = \begin{cases} C(2x - x^3) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

Could f be a probability density function? If so, determine C. Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

(a) Find $P\{X > 20\}$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that of 6 such types of devices at least 3 will function for at least 15 hours? What assumptions are you making?

5. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

what need the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

6. Compute $E[X]$ if X has a density function given by

(a) $f(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$;

(b) $f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$;

(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$.

7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \ge 0$$

Compute the expected lifetime of such a tube.

9. Consider Example 5b of Chapter 4, but now suppose that the seasonal demand is a continuous random variable having probability density function f. Show that the optimal amount to stock is the value s^* that satisfies

$$F(s^*) = \frac{b}{b + \ell}$$

where b is net profit per unit sale, ℓ is the net loss per unit unsold, and F is the cumulative distribution function of the seasonal demand.

10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

11. A point is chosen at random on a line segment of length L. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

12. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

13. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

14. Let X be a uniform (0, 1) random variable. Compute $E[X^n]$ by using Proposition 2.1 and then check the result by using the definition of expectation.

15. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$; **(b)** $P\{4 < X < 16\}$; **(c)** $P\{X < 8\}$;
(d) $P\{X < 20\}$; **(e)** $P\{X > 16\}$.

16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

17. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

18. Suppose that X is a normal random variable with mean 5. If $P\{X > 9\} = .2$, approximately what is $\mathrm{Var}(X)$?

19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P\{X > c\} = .10$.

20. If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

(a) at least 50 who are in favor of the proposition;

(b) between 60 and 70 inclusive who are in favor;

(c) fewer than 75 in favor.

21. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet 2 inches tall? What percentage of men in the 6-footer club are over 6 foot 5 inches?

22. The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = .9000$ and $\sigma = .0030$. The specification limits were given as $.9000 \pm .0050$.

(a) What percentage of forgings will be defective?

(b) What is the maximum allowable value of σ that will permit no more than 1 in 100 defectives when the widths are normally distributed with $\mu = .9000$ and σ?

23. One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that number 6 will appear between 150 and 200 times. If number 6 appears exactly 200 times, find the probability that number 5 will appear less than 150 times.

24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6?

25. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

26. Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

27. In 10,000 independent tosses of a coin, the coin landed heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.

28. An image is partitioned into 2 regions—one white and the other black. A reading taken from a randomly chosen point in the white section will give a reading that is normally distributed with $\mu = 4$ and $\sigma^2 = 4$, whereas one taken from a randomly chosen point in the black region will have a normally distributed reading with parameters (6, 9). A point is randomly chosen on the image and has a reading of 5. If the fraction of the image that is black is α, for what value of α would the probability of making an error be the same whether one concluded the point was in the black region or in the white region?

29. **(a)** A fire station is to be located along a road of length A, $A < \infty$. If fires will occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to

$$\text{minimize } E[|X - a|]$$

when X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length—stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$ where X is now exponential with rate λ.

30. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is

(a) the probability that a repair time exceeds 2 hours;

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

31. The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

32. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over (0, 40).

33. The lung cancer hazard rate of a t-year-old male smoker, $\lambda(t)$, is such that

$$\lambda(t) = .027 + .00025(t - 40)^2 \quad t \geq 40$$

Assuming that a 40-year-old male smoker survives all other hazards, what is the probability that he survives to (a) age 50 and (b) age 60 without contracting lung cancer?

34. Suppose that the life distribution of an item has hazard rate function $\lambda(t) = t^3$, $t > 0$. What is the probability that

(a) the item survives to age 2;

(b) the item's lifetime is between .4 and 1.4;

(c) a 1-year-old item will survive to age 2?

35. If X is uniformly distributed over $(-1, 1)$, find

(a) $P\{|X| > \frac{1}{2}\}$;

(b) the density function of the random variable $|X|$.

36. If Y is uniformly distributed over $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

37. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

38. If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

39. Find the distribution of $R = A \sin \theta$, where A is a fixed constant and θ is uniformly distributed on $(-\pi/2, \pi/2)$. Such a random variable R arises in the theory of ballistics. If a projectile is fired from the origin at an angle α from the earth with a speed v, then the point R at which it returns to the earth can be expressed as $R = (v^2/g) \sin 2\alpha$, where g is the gravitational constant, equal to 980 centimeters per second squared.

THEORETICAL EXERCISES

1. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f(x) = \begin{cases} ax^2 e^{-bx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $b = m/2kT$ and k, T, and m denote, respectively, Boltzmann's constant, the absolute temperature, and the mass of the molecule. Evaluate a in terms of b.

2. Show that

$$E[Y] = \int_0^\infty P\{Y > y\}\, dy - \int_0^\infty P\{Y < -y\}\, dy$$

HINT: Show that

$$\int_0^\infty P\{Y < -y\}\, dy = -\int_{-\infty}^0 x f_Y(x)\, dx$$

$$\int_0^\infty P\{Y > y\}\, dy = \int_0^\infty x f_Y(x)\, dx$$

3. If X has density function f, show that

$$E[g(X)] = \int_{-\infty}^\infty g(x) f(x)\, dx$$

HINT: Using Theoretical Exercise 2, start with

$$E[g(X)] = \int_0^\infty P\{g(X) > y\}\, dy - \int_0^\infty P\{g(X) < -y\}\, dy$$

and then proceed as in the proof given in the text when $g(X) \geq 0$.

4. Prove Corollary 2.1.

5. Use the result that for a nonnegative random variable Y,

$$E[Y] = \int_0^\infty P\{Y > t\}\, dt$$

to show that for a nonnegative random variable X,

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\}\, dx$$

HINT: Start with

$$E[X^n] = \int_0^\infty P\{X^n > t\}\, dt$$

and make the change of variables $t = xn$.

6. Define a collection of events E_a, $0 < a < 1$, having the property that $P(E_a) = 1$ for all a, but $P(\cap_a E_a) = 0$.

HINT: Let X be uniform over (0, 1) and define each E_a in terms of X.

7. The standard deviation of X, denoted $\mathrm{SD}(X)$, is given by

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

Find $\mathrm{SD}(aX + b)$ if X has variance σ^2.

8. Let X be a random variable that takes on values between 0 and c. That is, $P\{0 \le X \le c\} = 1$. Show that

$$\mathrm{Var}(X) \le \frac{c^2}{4}$$

HINT: One approach is to first argue that

$$E[X^2] \le cE[X]$$

Then use this to show that

$$\mathrm{Var}(X) \le c^2[\alpha(1 - \alpha)] \qquad \text{where} \quad \alpha = \frac{E[X]}{c}$$

9. If Z is a standard normal random variable, show that for $x > 0$,
(a) $P\{Z > x\} = P\{Z < -x\}$;
(b) $P\{|Z| > x\} = 2P\{Z > x\}$;
(c) $P\{|Z| < x\} = 2P\{Z < x\} - 1$.

10. Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$.

11. Use the identity of Theoretical Exercise 5 to derive $E[X^2]$ when X is an exponential random variable with parameter λ.

12. The median of a continuous random variable having distribution function F is that value m such that $F(m) = \frac{1}{2}$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if X is
(a) uniformly distributed over (a, b);
(b) normal with parameters μ, σ^2;
(c) exponential with rate λ.

13. The mode of a continuous random variable having density f is the value of x for which $f(x)$ attains its maximum. Compute the mode of X in cases (a), (b), and (c) of Theoretical Exercise 12.

14. If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ/c.

15. Compute the hazard rate function of X when X is uniformly distributed over $(0, a)$.

16. If X has hazard rate function $\lambda_X(t)$, compute the hazard rate function of aX where a is a positive constant.

17. Verify that the gamma density function integrates to 1.

18. If X is an exponential random variable with mean $1/\lambda$, show that

$$E[X^k] = \frac{k!}{\lambda^k} \qquad k = 1, 2, \ldots$$

HINT: Make use of the gamma density function to evaluate the above.

19. Verify that

$$\mathrm{Var}(X) = \frac{t}{\lambda^2}$$

when X is a gamma random variable with parameters t and λ.

20. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

HINT: $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}\, dx$. Make the change of variables $y = \sqrt{2x}$ and then relate the resulting expression to the normal distribution.

21. Compute the hazard rate function of a gamma random variable with parameters (t, λ) and show it is increasing when $t \ge 1$ and decreasing when $t \le 1$.

22. Compute the hazard rate function of a Weibull random variable and show it is increasing when $\beta \ge 1$ and decreasing when $\beta \le 1$.

23. Show that a plot of $\log(\log(1 - F(x))^{-1})$ against $\log x$ will be a straight line with slope β when $F(\cdot)$ is a Weibull distribution function. Show also that approximately 63.2 percent of all observations from such a distribution will be less than α. Assume that $\nu = 0$.

24. Let

$$Y = \left(\frac{X - \nu}{\alpha}\right)^\beta$$

Show that if X is a Weibull random variable with parameters ν, α, and β, then Y is an exponential random variable with parameter $\lambda = 1$ and vice versa.

25. If X is a beta random variable with parameters a and b show that

$$E[X] = \frac{a}{a + b} \qquad \mathrm{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

26. If X is uniformly distributed over (a, b), what random variable, having a linear relation with X, is uniformly distributed over $(0, 1)$?

27. Consider the beta distribution with parameters (a, b). Show that
(a) when $a > 1$ and $b > 1$, the density is unimodal (that is, it has a unique mode) with mode equal to $(a - 1)/(a + b - 2)$;
(b) when $a \leq 1$, $b \leq 1$, and $a + b < 2$, the density is either unimodal with mode at 0 or 1 or U-shaped with modes at both 0 and 1;
(c) when $a = 1 = b$, all points in $[0, 1]$ are modes.

28. Let X be a continuous random variable having cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.

29. Let X have probability density f_X. Find the probability density function of the random variable Y, defined by $Y = aX + b$.

30. Find the probability density function of $Y = e^X$ when X is normally distributed with parameters μ and σ^2. The random variable Y is said to have a lognormal distribution (since $\log Y$ has a normal distribution) with parameters μ and σ^2.

31. Let X and Y be independent random variables that are both equally likely to be either $1, 2, \ldots, (10)^N$, where N is very large. Let D denote the greatest common divisor of X and Y, and let $Q_k = P\{D = k\}$.

(a) Give a heuristic argument that $Q_k = \dfrac{1}{k^2} Q_1$.

HINT: Note that in order for D to equal k, k must divide both X and Y and also X/k and Y/k must be relatively prime. (That is, they must have a greatest common divisor equal to 1.)

(b) Use part (a) to show that

$$Q_1 = P\{X \text{ and } Y \text{ are relatively prime}\} = \frac{1}{\sum_{k=1}^{\infty} 1/k^2}$$

It is a well-known identity that $\sum_{1}^{\infty} 1/k^2 = \pi^2/6$, so $Q_1 = 6/\pi^2$ (In number theory this is known as the Legendre theorem.)

(c) Now argue that

$$Q_1 = \prod_{i=1}^{\infty} \left(\frac{P_i^2 - 1}{P_i^2}\right)$$

where P_i is the ith smallest prime greater than 1.

HINT: X and Y will be relatively prime if they have no common prime factors.

Hence, from part (b), we see that

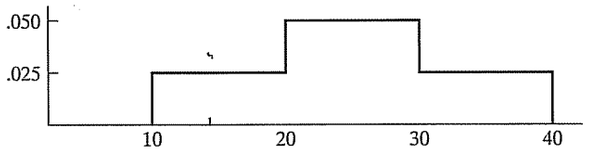
$$\prod_{i=1}^{\infty} \left(\frac{P_i^2 - 1}{P_i^2}\right) = \frac{6}{\pi^2}$$

which was noted without explanation in Problem 11 of Chapter 4. (The relationship between this problem and Problem 11 of Chapter 4 is that X and Y are relatively prime if XY has no multiple prime factors.)

32. Prove Theorem 7.1 when $g(x)$ is a decreasing function.

SELF-TEST PROBLEMS AND EXERCISES

1. The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given below.



Find the probability that the player plays
(a) over 15 minutes;
(b) between 20 and 35 minutes;
(c) less than 30 minutes;
(d) more than 36 minutes.

2. For some constant c, the random variable X has probability density function

$$f(x) = \begin{cases} cx^n & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) c and (b) $P\{X > x\}$, $0 < x < 1$.

3. For some constant c, the random variable X has probability density function

$$f(x) = \begin{cases} cx^4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) $E[X]$ and (b) $\text{Var}(X)$.

4. The random variable X has probability density function

$$f(x) = \begin{cases} ax + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = .6$, find (a) $P\{X < \frac{1}{2}\}$ and (b) $\text{Var}(X)$.

5. The random variable X is said to be a discrete uniform random variable on the integers $1, 2, \ldots, n$ if

$$P\{X = i\} = \frac{1}{n} \qquad i = 1, 2, \ldots, n$$

For any nonnegative real number x, let $\text{Int}(x)$ (sometimes written as $[x]$) be the largest integer that is less than or equal to x. If U is a uniform random

variable on (0, 1), show that $X = \text{Int}(nU) + 1$ is a discrete uniform random variable on $1, \ldots, n$.

6. Your company must make a sealed bid for a construction project. If you succeed in winning the contract (by having the lowest bid), then you plan to pay another firm 100 thousand dollars to do the work. If you believe that the maximum bid (in thousands of dollars) of the other participating companies can be modeled as being the value of a random variable that is uniformly distributed on (70, 140), how much should you bid to maximize your expected profit?

7. To be a winner in the following game, you must be successful in three successive rounds. The game depends on the value of U, a uniform random variable on (0, 1). If $U > .1$, then you are successful in round 1; if $U > .2$, then you are successful in round 2; and if $U > .3$, then you are successful in round 3.
(a) Find the probability that you are successful in round 1.
(b) Find the conditional probability that you are successful in round 2 given that you were successful in round 1.
(c) Find the conditional probability that you are successful in round 3 given that you were successful in rounds 1 and 2.
(d) Find the probability that you are a winner.

8. A randomly chosen IQ test taker obtains a score that is approximately a normal random variable with mean 100 and standard deviation 15. What is the probability that the test score of such a person is (a) above 125; (b) between 90 and 110?

9. Suppose that the travel time from your home to your office is normally distributed with mean 40 minutes and standard deviation 7 minutes. If you want to be 95 percent certain that you will not be late for an office appointment at 1 P.M., what is the latest time that you should leave home?

10. The life of a certain type of automobile tire is normally distributed with mean 34,000 miles and standard deviation 4000 miles.
(a) What is the probability that such a tire lasts over 40,000 miles?
(b) What is the probability that it lasts between 30,000 and 35,000 miles?
(c) Given that it has survived 30,000 miles, what is the conditional probability that it survives another 10,000 miles?

11. The annual rainfall in Cleveland, Ohio is approximately a normal random variable with mean 40.2 inches and standard deviation 8.4 inches. What is the probability that
(a) next year's rainfall will exceed 44 inches;
(b) the yearly rainfalls in exactly three of the next seven years will exceed 44 inches?
Assume that if A_i is the event that the rainfall exceeds 44 inches in year i (from now), then the events A_i, $i \geq 1$, are independent.

12. The following table uses 1992 data concerning the percentages of male and female fulltime workers whose annual salaries fall in different ranges.

Earnings range	Percentage of females	Percentage of males
≤9999	8.6	4.4
10,000–19,999	38.0	21.1
20,000–24,999	19.4	15.8
25,000–49,999	29.2	41.5
≥50,000	4.8	17.2

Suppose that random samples of 200 male and 200 female fulltime workers are chosen. Approximate the probability that
(a) at least 70 of the women earn $25,000 or more;
(b) at most 60 percent of the men earn $25,000 or more;
(c) at least three-fourths of the men and at least half the women earn $20,000 or more.

13. At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

14. Suppose that the cumulative distribution function of the random variable X is given by

$$F(x) = 1 - e^{-x^2} \qquad x > 0$$

Evaluate (a) $P\{X > 2\}$; (b) $P\{1 < X < 3\}$; (c) the hazard rate function of F; (d) $E[X]$; (e) $\text{Var}(X)$.

HINT: For parts (d) and (e) you might want to make use of the results of Theoretical Exercise 5.

15. The number of years that a washing machine functions is a random variable whose hazard rate function is given by

$$\lambda(t) = \begin{cases} .2 & 0 < t < 2 \\ .2 + .3(t-2) & 2 \leq t < 5 \\ 1.1 & t > 5 \end{cases}$$

(a) What is the probability that the machine will still be working six years after being purchased?
(b) If it is still working six years after being purchased, what is the conditional probability that it will fail within the succeeding two years?

16. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1 + x^2)} \qquad -\infty < x < \infty$$

If X is a standard Cauchy random variable, show that $1/X$ is also a standard Cauchy random variable.

CHAPTER 6

Jointly Distributed Random Variables

6.1 JOINT DISTRIBUTION FUNCTIONS

Thus far, we have only concerned ourselves with probability distributions for single random variables. However, we are often interested in probability statements concerning two or more random variables. In order to deal with such probabilities, we define, for any two random variables X and Y, the *joint cumulative probability distribution function* of X and Y by

$$F(a, b) = P\{X \leq a, Y \leq b\} \qquad -\infty < a, b < \infty$$

The distribution of X can be obtained from the joint distribution of X and Y as follows:

$$\begin{aligned} F_X(a) &= P\{X \leq a\} \\ &= P\{X \leq a, Y < \infty\} \\ &= P\left(\lim_{b\to\infty}\{X \leq a, Y \leq b\}\right) \\ &= \lim_{b\to\infty} P\{X \leq a, Y \leq b\} \\ &= \lim_{b\to\infty} F(a, b) \\ &\equiv F(a, \infty) \end{aligned}$$

The reader should note that we have, in the preceding set of equalities, once again made use of the fact that probability is a continuous set (that is, event) function. Similarly, the cumulative distribution function of Y is given by

$$\begin{aligned} F_Y(b) &= P\{Y \leq b\} \\ &= \lim_{a\to\infty} F(a, b) \\ &\equiv F(\infty, b) \end{aligned}$$

The distribution functions F_X and F_Y are sometimes referred to as the *marginal* distributions of X and Y.

All joint probability statements about X and Y can, in theory, be answered in terms of their joint distribution function. For instance, suppose we wanted to compute the joint probability that X is greater than a and Y is greater than b. This could be done as follows.

$$\begin{aligned} P\{X > a, Y > b\} &= 1 - P(\{X > a, Y > b\}^c) \\ &= 1 - P(\{X > a\}^c \cup \{Y > b\}^c) \\ &= 1 - P(\{X \leq a\} \cup \{Y \leq b\}) \\ &= 1 - [P\{X \leq a\} + P\{Y \leq b\} - P\{X \leq a, Y \leq b\}] \\ &= 1 - F_X(a) - F_Y(b) + F(a, b) \end{aligned} \tag{1.1}$$

Equation (1.1) is a special case of Equation (1.2), whose verification is left as an exercise.

$$\begin{aligned} &P\{a_1 < X \leq a_2, b_1 < Y \leq b_2\} \\ &\quad = F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1) \end{aligned} \tag{1.2}$$

whenever $a_1 < a_2$, $b_1 < b_2$.

In the case when X and Y are both discrete random variables, it is convenient to define the *joint probability mass function* of X and Y by

$$p(x, y) = P\{X = x, Y = y\}$$

The probability mass function of X can be obtained from $p(x, y)$ by

$$\begin{aligned} p_X(x) &= P\{X = x\} \\ &= \sum_{y:p(x,y)>0} p(x, y) \end{aligned}$$

Similarly,

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x, y)$$

Example 1a. Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote, respectively, the number of red and white balls chosen, then the joint probability mass function of X and Y, $p(i, j) = P\{X = i, Y = j\}$, is given by

$$p(0, 0) = \binom{5}{3}\Big/\binom{12}{3} = \frac{10}{220}$$

$$p(0, 1) = \binom{4}{1}\binom{5}{2}\Big/\binom{12}{3} = \frac{40}{220}$$

$$p(0, 2) = \binom{4}{2}\binom{5}{1}\Big/\binom{12}{3} = \frac{30}{220}$$

$$p(0, 3) = \binom{4}{3}\Big/\binom{12}{3} = \frac{4}{220}$$

6.4 CONDITIONAL DISTRIBUTIONS: DISCRETE CASE

Recall that for any two events E and F, the conditional probability of E given F is defined, provided that $P(F) > 0$, by

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Hence, if X and Y are discrete random variables, it is natural to define the conditional probability mass function of X given that $Y = y$, by

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x|Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{p(x, y)}{p_Y(y)} \end{aligned}$$

for all values of y such that $p_Y(y) > 0$. Similarly, the conditional probability distribution function of X given that $Y = y$ is defined, for all y such that $p_Y(y) > 0$, by

$$\begin{aligned} F_{X|Y}(x|y) &= P\{X \leq x|Y = y\} \\ &= \sum_{a \leq x} p_{X|Y}(a|y) \end{aligned}$$

In other words, the definitions are exactly the same as in the unconditional case except that everything is now conditional on the event that $Y = y$. If X is independent of Y, then the conditional mass function and distribution function are the same as the unconditional ones. This follows because if X is independent of Y, then

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x|Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{P\{X = x\}\, P\{Y = y\}}{P\{Y = y\}} \\ &= P\{X = x\} \end{aligned}$$

Example 4a. Suppose that $p(x, y)$, the joint probability mass function of X and Y, is given by

$$p(0, 0) = .4 \qquad p(0, 1) = .2 \qquad p(1, 0) = .1 \qquad p(1, 1) = .3$$

Calculate the conditional probability mass function of X, given that $Y = 1$.

Solution We first note that

$$p_Y(1) = \sum_x p(x, 1) = p(0, 1) + p(1, 1) = .5$$

Hence

$$p_{X|Y}(0|1) = \frac{p(0, 1)}{p_Y(1)} = \frac{2}{5}$$

and

$$p_{X|Y}(1|1) = \frac{p(1, 1)}{p_Y(1)} = \frac{3}{5}$$

■

Example 4b. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X, given that $X + Y = n$.

Solution We calculate the conditional probability mass function of X given that $X + Y = n$ as follows:

$$\begin{aligned} P\{X = k|X + Y = n\} &= \frac{P\{X = k, X + Y = n\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = k, Y = n - k\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = k\}P\{Y = n - k\}}{P\{X + Y = n\}} \end{aligned}$$

where the last equality follows from the assumed independence of X and Y. Recalling (Example 3d) that $X + Y$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$, we see that the above equals

$$\begin{aligned} P\{X = k|X + Y = n\} &= \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n - k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n}{n!}\right]^{-1} \\ &= \frac{n!}{(n - k)!\, k!}\frac{\lambda_1^k\, \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\ &= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k} \end{aligned}$$

In other words, the conditional distribution of X, given that $X + Y = n$, is the binomial distribution with parameters n and $\lambda_1/(\lambda_1 + \lambda_2)$. ■

6.5 CONDITIONAL DISTRIBUTIONS: CONTINUOUS CASE

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X, given that $Y = y$, is defined for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

To motivate this definition, multiply the left-hand side by dx and the right-hand side by $(dx\,dy)/dy$ to obtain

$$\begin{aligned} f_{X|Y}(x|y)\,dx &= \frac{f(x, y)\,dx\,dy}{f_Y(y)\,dy} \\ &\approx \frac{P\{x \le X \le x + dx, y \le Y \le y + dy\}}{P\{y \le Y \le y + dy\}} \\ &= P\{x \le X \le x + dx | y \le Y \le y + dy\} \end{aligned}$$

In other words, for small values of dx and dy, $f_{X|Y}(x|y)\,dx$ represents the conditional probability that X is between x and $x + dx$, given that Y is between y and $y + dy$.

The use of conditional densities allows us to define conditional probabilities of events associated with one random variable when we are given the value of a second random variable. That is, if X and Y are jointly continuous, then for any set A,

$$P\{X \in A | Y = y\} = \int_A f_{X|Y}(x|y)\,dx$$

In particular, by letting $A = (-\infty, a]$, we can define the conditional cumulative distribution function of X, given that $Y = y$, by

$$F_{X|Y}(a|y) \equiv P\{X \le a | Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx$$

The reader should note that, by using the ideas presented in the preceding discussion, we have been able to give workable expressions for conditional probabilities, even though the event on which we are conditioning (namely, the event $\{Y = y\}$) has probability 0.

Example 5a. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{15}{2}x(2 - x - y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of X, given that $Y = y$, where $0 < y < 1$.

Solution For $0 < x < 1$, $0 < y < 1$, we have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)\,dx} \\ &= \frac{x(2 - x - y)}{\int_0^1 x(2 - x - y)\,dx} \\ &= \frac{x(2 - x - y)}{\frac{2}{3} - y/2} \\ &= \frac{6x(2 - x - y)}{4 - 3y} \end{aligned}$$

Example 5b. Suppose that the joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{e^{-x/y}\,e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1 | Y = y\}$.

Solution We first obtain the conditional density of X, given that $Y = y$.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \frac{e^{-x/y}\,e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}\,dx} \\ &= \frac{1}{y}e^{-x/y} \end{aligned}$$

Hence

$$\begin{aligned} P\{X > 1 | Y = y\} &= \int_1^\infty \frac{1}{y}e^{-x/y}\,dx \\ &= -e^{-x/y}\Big|_1^\infty \\ &= e^{-1/y} \end{aligned}$$

If X and Y are independent continuous random variables, the conditional density of X, given $Y = y$, is just the unconditional density of X. This is so because, in the independent case,

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

We can also talk about conditional distributions when the random variables are neither jointly continuous nor jointly discrete. For example, suppose that X is a continuous random variable having probability density function f and N is a discrete random variable, and consider the conditional distribution of X given that $N = n$. Then

$$\frac{P\{x < X < x + dx | N = n\}}{dx} = \frac{P\{N = n | x < X < x + dx\}}{P\{N = n\}} \frac{P\{x < X < x + dx\}}{dx}$$

and letting dx approach 0 gives

$$\lim_{dx \to 0} \frac{P\{x < X < x + dx | N = n\}}{dx} = \frac{P\{N = n | X = x\}}{P\{N = n\}} f(x)$$

thus showing that the conditional density of X given that $N = n$ is given by

$$f_{X|N}(x|n) = \frac{P\{N = n | X = x\}}{P\{N = n\}} f(x)$$

Example 5c. Consider $n + m$ trials having a common probability of success. Suppose, however, that this success probability is not fixed in advance but is chosen from a uniform (0, 1) population. What is the conditional distribution of the success probability given that the $n + m$ trials result in n successes?

Solution If we let X denote the trial success probability, then X is a uniform (0, 1) random variable. Also, given that $X = x$, the $n + m$ trials are independent with common success probability x, and so N, the number of successes, is a binomial random variable with parameters $(n + m, x)$. Hence the conditional density of X given that $N = n$ is as follows:

$$\begin{aligned} f_{X|N}(x|n) &= \frac{P\{N = n|X = x\} f_X(x)}{P\{N = n\}} \\ &= \frac{\binom{n+m}{n} x^n (1-x)^m}{P\{N = n\}} \qquad 0 < x < 1 \\ &= cx^n(1-x)^m \end{aligned}$$

where c does not depend on x. Hence the conditional density is that of a beta random variable with parameters $n + 1, m + 1$.

The result above is quite interesting, for it states that if the original or *prior* (to the collection of data) distribution of a trial success probability is uniformly distributed over (0, 1) [or, equivalently, is beta with parameters (1, 1)] then the posterior (or conditional) distribution given a total of n successes in $n + m$ trials is beta with parameters $(1 + n, 1 + m)$. This is valuable, for it enhances our intuition as to what it means to assume that a random variable has a beta distribution. ∎

*6.6 ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed, continuous random variables having a common density f and distribution function F. Define

$$\begin{aligned} X_{(1)} &= \text{smallest of } X_1, X_2, \ldots, X_n \\ X_{(2)} &= \text{second smallest of } X_1, X_2, \ldots, X_n \\ &\vdots \\ X_{(j)} &= j\text{th smallest of } X_1, X_2, \ldots, X_n \\ &\vdots \\ X_{(n)} &= \text{largest of } X_1, X_2, \ldots, X_n \end{aligned}$$

The ordered values $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are known as the *order statistics* corresponding to the random variables $X_1, X_2, \ldots, X_n$. In other words, $X_{(1)}, \ldots, X_{(n)}$ are the ordered values of $X_1, \ldots, X_n$.

The joint density function of the order statistics is obtained by noting that the order statistics $X_{(1)}, \ldots, X_{(n)}$ will take on the values $x_1 \le x_2 \le \cdots \le x_n$ if and only if for some permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$

$$X_1 = x_{i_1}, X_2 = x_{i_2}, \ldots, X_n = x_{i_n}$$

Since, for any permutation $(i_1, \ldots, i_n)$ of $(1, 2, \ldots, n)$,

$$\begin{aligned} &P\left\{x_{i_1} - \frac{\varepsilon}{2} < X_1 < x_{i_1} + \frac{\varepsilon}{2}, \ldots, x_{i_n} - \frac{\varepsilon}{2} < X_n < x_{i_n} + \frac{\varepsilon}{2}\right\} \\ &\quad \approx \varepsilon^n f_{X_1, \ldots, X_n}(x_{i_1}, \ldots, x_{i_n}) \\ &\quad = \varepsilon^n f(x_{i_1}) \cdots f(x_{i_n}) \\ &\quad = \varepsilon^n f(x_1) \cdots f(x_n) \end{aligned}$$

we see that for $x_1 < x_2 < \cdots < x_n$,

$$\begin{aligned} &P\left\{x_1 - \frac{\varepsilon}{2} < X_{(1)} < x_1 + \frac{\varepsilon}{2}, \ldots, x_n - \frac{\varepsilon}{2} < X_{(n)} < x_n + \frac{\varepsilon}{2}\right\} \\ &\quad \approx n!\, \varepsilon^n f(x_1) \cdots f(x_n) \end{aligned}$$

Dividing by ε^n and letting $\varepsilon \to 0$ yields

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) \cdots f(x_n) \qquad x_1 < x_2 < \cdots < x_n \tag{6.1}$$

Equation (6.1) is most simply explained by arguing that in order for the vector $\langle X_{(1)}, \ldots, X_{(n)} \rangle$ to equal $\langle x_1, \ldots, x_n \rangle$, it is necessary and sufficient for $\langle X_1, \ldots, X_n \rangle$ to equal one of the $n!$ permutations of $\langle x_1, \ldots, x_n \rangle$. As the probability (density) that $\langle X_1, \ldots, X_n \rangle$ equals any given permutation of $\langle x_1, \ldots, x_n \rangle$ is just $f(x_1) \cdots f(x_n)$, Equation (6.1) follows.

Example 6a. Along a road 1 mile long are 3 people "distributed at random." Find the probability that no 2 people are less than a distance of d miles apart, when $d \le \frac{1}{2}$.

Solution Let us assume that "distributed at random" means that the positions of the 3 people are independent and uniformly distributed over the road. If X_i denotes the position of the ith person, the desired probability is $P\{X_{(i)} > X_{(i-1)} + d, i = 2, 3\}$. As

$$f_{X_{(1)}, X_{(2)}, X_{(3)}}(x_1, x_2, x_3) = 3! \qquad 0 < x_1 < x_2 < x_3 < 1$$

it follows that

$$\begin{aligned} P\{X_{(i)} > X_{(i-1)} + d, i = 2, 3\} &= \iiint_{\substack{x_i > x_{i-1} + d \\ i = 2,3}} f_{X_{(1)}, X_{(2)}, X_{(3)}}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \\ &= 3! \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} dx_3\, dx_2\, dx_1 \end{aligned}$$

$$= 6\int_0^{1-2d}\int_{x_1+d}^{1-d}(1-d-x_2)\,dx_2\,dx_1$$

$$= 6\int_0^{1-2d}\int_0^{1-2d-x_1} y_2\,dy_2\,dx_1$$

where we have made the change of variables $y_2 = 1 - d - x_2$. Hence continuing the string of equalities yields

$$= 3\int_0^{1-2d}(1-2d-x_1)^2\,dx_1$$

$$= 3\int_0^{1-2d} y_1^2\,dy_1$$

$$= (1-2d)^3$$

Hence the desired probability that no 2 people are within a distance d of each other when 3 people are uniformly and independently distributed over an interval of size 1 is $(1 - 2d)^3$ when $d \le \frac{1}{2}$. In fact, the same method can be used to prove that when there are n people distributed at random over the unit interval the desired probability is

$$[1-(n-1)d]^n \qquad \text{when } d \le \frac{1}{n-1}$$

The proof is left as an exercise. ∎

The density function of the jth-order statistic $X_{(j)}$ can be obtained either by integrating the joint density function (6.1) or by direct reasoning as follows: in order for $X_{(j)}$ to equal x, it is necessary for $j - 1$ of the n values $X_1, \ldots, X_n$ to be less than x, $n - j$ of them to be greater than x, and 1 of them to equal x. Now, the probability density that any given set of $j - 1$ of the X_i's are less than x, another given set of $n - j$ are all greater than x, and the remaining value is equal to x, equals

$$[F(x)]^{j-1}\,[1-F(x)]^{n-j}f(x)$$

Hence, as there are

$$\binom{n}{j-1, n-j, 1} = \frac{n!}{(n-j)!\,(j-1)!}$$

different partitions of the n random variables $X_1, \ldots, X_n$ into the three groups, we see that the density function of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(n-j)!\,(j-1)!}[F(x)]^{j-1}\,[1-F(x)]^{n-j}f(x) \tag{6.2}$$

Example 6b. When a sample of $2n + 1$ random variables (that is, when $2n + 1$ independent and identically distributed random variables) are observed, the $(n + 1)$st smallest is called the sample median. If a sample of size 3 from a uniform distribution over (0, 1) is observed, find the probability that the sample median is between $\frac{1}{4}$ and $\frac{3}{4}$.

Solution From Equation (6.2) the density of $X_{(2)}$ is given by

$$f_{X_{(2)}}(x) = \frac{3!}{1!\,1!}x(1-x) \qquad 0 < x < 1$$

Hence

$$P\{\tfrac{1}{4} < X_{(2)} < \tfrac{3}{4}\} = 6\int_{1/4}^{3/4} x(1-x)\,dx$$

$$= 6\left\{\frac{x^2}{2} - \frac{x^3}{3}\right\}\Bigg|_{x=1/4}^{x=3/4} = \frac{11}{16}$$

∎

The cumulative distribution function of $X_{(j)}$ can be obtained by integrating Equation (6.2). That is,

$$F_{X_{(j)}}(y) = \frac{n!}{(n-j)!\,(j-1)!}\int_{-\infty}^{y}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x)\,dx \tag{6.3}$$

However, $F_{X_{(j)}}(y)$ could also have been derived directly by noting that the jth order statistic is less than or equal to y if and only if there are j or more of the X_i's that are less than or equal to y. Hence, as the number of the X_i's that are less than or equal to y is a binomial random variable with parameters $[n, p = F(y)]$, it follows that

$$F_{X_{(j)}}(y) = P\{X_{(j)} \le y\} = P\{j \text{ or more of the } X_i\text{'s are} \le y\}$$

$$= \sum_{k=j}^{n}\binom{n}{k}[F(y)]^k[1-F(y)]^{n-k} \tag{6.4}$$

If, in Equations (6.3) and (6.4), we take F to be the uniform (0, 1) distribution [that is, $f(x) = 1$, $0 < x < 1$], then we obtain the interesting analytical identity

$$\sum_{k=j}^{n}\binom{n}{k}y^k(1-y)^{n-k} = \frac{n!}{(n-j)!\,(j-1)!}\int_0^y x^{j-1}(1-x)^{n-j}\,dx$$

$$0 \le y \le 1 \tag{6.5}$$

By employing the same type of argument that we used in establishing Equation (6.2), we can show that the joint density function of the order statistics $X_{(i)}$ and $X_{(j)}$, when $i < j$, is

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}[F(x_i)]^{i-1} \times [F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{n-j}f(x_i)\,f(x_j) \tag{6.6}$$

for all $x_i < x_j$.

Example 6c. ***Distribution of the range of a random sample.*** Suppose that n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ are

observed. The random variable R, defined by $R = X_{(n)} - X_{(1)}$, is called the *range* of the observed random variables. If the random variables X_i have distribution function F and density function f, then the distribution of R can be obtained from Equation (6.6) as follows: for $a \geq 0$,

$$\begin{aligned} P\{R \leq a\} &= P\{X_{(n)} - X_{(1)} \leq a\} \\ &= \iint_{x_n - x_1 \leq a} f_{X_{(1)},X_{(n)}}(x_1, x_n)\, dx_1\, dx_n \\ &= \int_{-\infty}^{\infty} \int_{x_1}^{x_1 + a} \frac{n!}{(n-2)!} [F(x_n) - F(x_1)]^{n-2} f(x_1) f(x_n)\, dx_n\, dx_1 \end{aligned}$$

Making the change of variable $y = F(x_n) - F(x_1)$, $dy = f(x_n)\, dx_n$ yields

$$\begin{aligned} \int_{x_1}^{x_1 + a} [F(x_n) - F(x_1)]^{n-2} f(x_n)\, dx_n &= \int_0^{F(x_1 + a) - F(x_1)} y^{n-2}\, dy \\ &= \frac{1}{n-1} [F(x_1 + a) - F(x_1)]^{n-1} \end{aligned}$$

and thus

$$P\{R \leq a\} = n \int_{-\infty}^{\infty} [F(x_1 + a) - F(x_1)]^{n-1} f(x_1)\, dx_1 \tag{6.7}$$

Equation (6.7) can be explicitly evaluated only in a few special cases. One such case is when the X_i's are all uniformly distributed on (0, 1). In this case we obtain from Equation (6.7) that for $0 < a < 1$,

$$\begin{aligned} P\{R < a\} &= n \int_0^1 [F(x_1 + a) - F(x_1)]^{n-1} f(x_1)\, dx_1 \\ &= n \int_0^{1-a} a^{n-1}\, dx_1 + n \int_{1-a}^1 (1 - x_1)^{n-1}\, dx_1 \\ &= n(1 - a)a^{n-1} + a^n \end{aligned}$$

Differentiation yields that the density function of the range is given, in this case, by

$$f_R(a) = \begin{cases} n(n-1)a^{n-2}(1-a) & 0 \leq a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

That is, the range of n independent uniform (0, 1) random variables is a beta random variable with parameters $n - 1, 2$. ∎

6.7 JOINT PROBABILITY DISTRIBUTION OF FUNCTIONS OF RANDOM VARIABLES

Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1,X_2}. It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2.

Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2.

Assume that the functions g_1 and g_2 satisfy the following conditions:

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2 with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the following 2×2 determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

at all points (x_1, x_2).

Under these two conditions it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) |J(x_1, x_2)|^{-1} \tag{7.1}$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

A proof of Equation (7.1) would proceed along the following lines:

$$P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \iint_{\substack{(x_1,x_2): \\ g_1(x_1,x_2) \leq y_1 \\ g_2(x_1,x_2) \leq y_2}} f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 \tag{7.2}$$

The joint density function can now be obtained by differentiating Equation (7.2) with respect to y_1 and y_2. That the result of this differentiation will be equal to the right-hand side of Equation (7.1) is an exercise in advanced calculus whose proof will not be presented in this book.

Example 7a. Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.

Solution Let $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Then

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Also, as the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have as their solution $x_1 = (y_1 + y_2)/2$, $x_2 = (y_1 - y_2)/2$, it follows from Equation (7.1) that the desired density is

$$f_{Y_1,Y_2}(y_1, y_2) = \tfrac{1}{2} f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

For instance, if X_1 and X_2 are independent, uniform (0, 1) random variables, then

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 \le y_1 + y_2 \le 2,\ 0 \le y_1 - y_2 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

or if X_1 and X_2 were independent, exponential random variables with respective parameters λ_1 and λ_2, then

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1\lambda_2}{2} \exp\left\{-\lambda_1\left(\frac{y_1+y_2}{2}\right) - \lambda_2\left(\frac{y_1-y_2}{2}\right)\right\} & y_1 + y_2 \ge 0,\ y_1 - y_2 \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally, if X_1 and X_2 are independent unit normal random variables,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{1}{4\pi} e^{-[(y_1+y_2)^2/8 + (y_1-y_2)^2/8]} \\ &= \frac{1}{4\pi} e^{-(y_1^2+y_2^2)/4} \\ &= \frac{1}{\sqrt{4\pi}} e^{-y_1^2/4} \frac{1}{\sqrt{4\pi}} e^{-y_2^2/4} \end{aligned}$$

Thus, not only do we obtain (in agreement with Proposition 3.2) that both $X_1 + X_2$ and $X_1 - X_2$ are normal with mean 0 and variance 2, we also obtain the interesting result that these two random variables are independent. (In fact, it can be shown that if X_1 and X_2 are independent random variables having a common distribution function F, then $X_1 + X_2$ will be independent of $X_1 - X_2$ if and only if F is a normal distribution function.) ∎

Example 7b. Let (X, Y) denote a random point in the plane and assume that the rectangular coordinates X and Y are independent unit normal random variables. We are interested in the joint distribution of R, Θ, the polar coordinate representation of this point (see Figure 6.4).

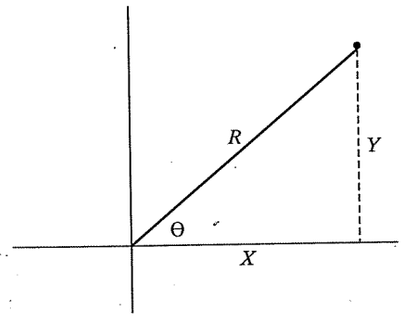


Figure 6.4 • = Random point. $(X, Y) = R, \Theta$.

Letting $r = g_1(x, y) = \sqrt{x^2 + y^2}$ and $\theta = g_2(x, y) = \tan^{-1} y/x$, we see that

$$\frac{\partial g_1}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial g_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial g_2}{\partial x} = \frac{1}{1 + (y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \qquad \frac{\partial g_2}{\partial y} = \frac{1}{x[1 + (y/x)^2]} = \frac{x}{x^2+y^2}$$

Hence

$$J(x, y) = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

As the joint density function of X and Y is

$$f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

we see that the joint density function of $R = \sqrt{x^2+y^2}$, $\Theta = \tan^{-1} y/x$, is given by

$$f(r, \theta) = \frac{1}{2\pi} r e^{-r^2/2} \qquad 0 < \theta < 2\pi, \qquad 0 < r < \infty$$

As this joint density factors into the marginal densities for R and Θ, we obtain that R and Θ are independent random variables, with Θ being uniformly distributed over $(0, 2\pi)$ and R having the Rayleigh distribution with density

$$f(r) = re^{-r^2/2} \qquad 0 < r < \infty$$

(Thus, for instance, when one is aiming at a target in the plane, if the horizontal and vertical miss distances are independent unit normals, then the absolute value of the error has the above Rayleigh distribution.)

The above result is quite interesting, for it certainly is not evident a priori that a random vector whose coordinates are independent unit normal random variables will have an angle of orientation that is not only uniformly distributed, but is also independent of the vector's distance from the origin.

If we wanted the joint distribution of R^2 and Θ, then, as the transformation $d = g_1(x, y) = x^2 + y^2$ and $\theta = g_2(x, y) = \tan^{-1} y/x$ has a Jacobian

$$J = \begin{vmatrix} 2x & 2y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = 2$$

we see that

$$f(d, \theta) = \tfrac{1}{2} e^{-d/2} \frac{1}{2\pi} \qquad 0 < d < \infty, \qquad 0 < \theta < 2\pi$$

Therefore, R^2 and Θ are independent, with R^2 having an exponential distribution with parameter $\frac{1}{2}$. But as $R^2 = X^2 + Y^2$, it follows, by definition, that

R^2 has a chi-squared distribution with 2 degrees of freedom. Hence we have a verification of the result that the exponential distribution with parameter $\frac{1}{2}$ is the same as the chi-squared distribution with 2 degrees of freedom.

The above result can be used to simulate (or generate) normal random variables by making a suitable transformation on uniform random variables. Let U_1 and U_2 be independent random variables each uniformly distributed over (0, 1). We will transform U_1, U_2 into two independent unit normal random variables X_1 and X_2 by first considering the polar coordinate representation (R, Θ) of the random vector (X_1, X_2). From the above, R^2 and Θ will be independent, and, in addition, $R^2 = X_1^2 + X_2^2$ will have an exponential distribution with parameter $\lambda = \frac{1}{2}$. But $-2 \log U_1$ has such a distribution since, for $x > 0$,

$$\begin{aligned} P\{-2 \log U_1 < x\} &= P\left\{\log U_1 > -\frac{x}{2}\right\} \\ &= P\{U_1 > e^{-x/2}\} \\ &= 1 - e^{-x/2} \end{aligned}$$

Also, as $2\pi U_2$ is a uniform $(0, 2\pi)$ random variable, we can use it to generate Θ. That is, if we let

$$\begin{aligned} R^2 &= -2 \log U_1 \\ \Theta &= 2\pi U_2 \end{aligned}$$

then R^2 can be taken to be the square of the distance from the origin and θ as the angle of orientation of (X_1, X_2). As $X_1 = R \cos \Theta$, $X_2 = R \sin \Theta$, we obtain that

$$\begin{aligned} X_1 &= \sqrt{-2 \log U_1} \cos(2\pi U_2) \\ X_2 &= \sqrt{-2 \log U_1} \sin(2\pi U_2) \end{aligned}$$

are independent unit normal random variables. ∎

Example 7c. If X and Y are independent gamma random variables with parameters (α, λ) and (β, λ), respectively, compute the joint density of $U = X + Y$ and $V = X/(X + Y)$.

Solution The joint density of X and Y is given by

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)} \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda(x+y)} x^{\alpha-1} y^{\beta-1} \end{aligned}$$

Now, if $g_1(x, y) = x + y$, $g_2(x, y) = x/(x + y)$, then

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_1}{\partial y} = 1 \qquad \frac{\partial g_2}{\partial x} = \frac{y}{(x + y)^2} \qquad \frac{\partial g_2}{\partial y} = -\frac{x}{(x + y)^2}$$

and so

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ \dfrac{y}{(x + y)^2} & \dfrac{-x}{(x + y)^2} \end{vmatrix} = -\frac{1}{x + y}$$

Finally, as the equations $u = x + y$, $v = x/(x + y)$ have as their solutions $x = uv$, $y = u(1 - v)$, we see that

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}[uv, u(1 - v)]u \\ &= \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \frac{v^{\alpha-1}(1 - v)^{\beta-1}\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

Hence $X + Y$ and $X/(X + Y)$ are independent, with $X + Y$ having a gamma distribution with parameters $(\alpha + \beta, \lambda)$ and $X/(X + Y)$ having a beta distribution with parameters (α, β). The above also shows that $B(\alpha, \beta)$, the normalizing factor in the beta density, is such that

$$\begin{aligned} B(\alpha, \beta) &\equiv \int_0^1 v^{\alpha-1}(1 - v)^{\beta-1}\, dv \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \end{aligned}$$

The result above is quite interesting. For suppose there are $n + m$ jobs to be performed, with each (independently) taking an exponential amount of time with rate λ for performance, and suppose that we have two workers to perform these jobs. Worker I will do jobs $1, 2, \ldots, n$, and worker II will do the remaining m jobs. If we let X and Y denote the total working times of workers I and II, respectively, then (either from the above result or from Example 3b) X and Y will be independent gamma random variables having parameters (n, λ) and (m, λ), respectively. Then the above result yields that independently of the working time needed to complete all $n + m$ jobs (that is, of $X + Y$), the proportion of this work that will be performed by worker I has a beta distribution with parameters (n, m). ∎

When the joint density function of the n random variables $X_1, X_2, \ldots, X_n$ is given and we want to compute the joint density function of $Y_1, Y_2, \ldots, Y_n$, where

$$Y_1 = g_1(X_1, \ldots, X_n) \qquad Y_2 = g_2(X_1, \ldots, X_n), \ldots$$
$$Y_n = g_n(X_1, \ldots, X_n)$$

the approach is the same. Namely, we assume that the functions g_i have continuous partial derivatives and that the Jacobian determinant $J(x_1, \ldots, x_n) \neq 0$ at all points $(x_1, \ldots, x_n)$, where

$$J(x_1, \ldots, x_n) = \begin{vmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_2}{\partial x_n} \\ \dfrac{\partial g_n}{\partial x_1} & \dfrac{\partial g_n}{\partial x_2} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{vmatrix}$$

Furthermore, we suppose that the equations $y_1 = g_1(x_1, \ldots, x_n)$, $y_2 = g_2(x_1, \ldots, x_n)$, $\ldots$, $y_n = g_n(x_1, \ldots, x_n)$ have a unique solution, say, $x_1 = h_1(y_1, \ldots, y_n)$, $\ldots$, $x_n = h_n(y_1, \ldots, y_n)$. Under these assumptions, the joint density function of the random variables Y_i is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\,|J(x_1, \ldots, x_n)|^{-1} \quad (7.3)$$

where $x_i = h_i(y_1, \ldots, y_n)$, $i = 1, 2, \ldots, n$.

Example 7d. Let X_1, X_2, and X_3 be independent unit normal random variables. If $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, $Y_3 = X_1 - X_3$, compute the joint density function of Y_1, Y_2, Y_3.

Solution Letting $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, $Y_3 = X_1 - X_3$, the Jacobian of these transformations is given by

$$J = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

As the transformations above yield that

$$X_1 = \frac{Y_1 + Y_2 + Y_3}{3} \qquad X_2 = \frac{Y_1 - 2Y_2 + Y_3}{3} \qquad X_3 = \frac{Y_1 + Y_2 - 2Y_3}{3}$$

we see from Equation (7.3) that

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3} f_{X_1,X_2,X_3}\left(\frac{y_1 + y_2 + y_3}{3}, \frac{y_1 - 2y_2 + y_3}{3}, \frac{y_1 + y_2 - 2y_3}{3}\right)$$

Hence, as

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} e^{-\sum_{i=1}^{3} x_i^2/2}$$

we see that

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3(2\pi)^{3/2}} e^{-Q(y_1,y_2,y_3)/2}$$

where

$$Q(y_1, y_2, y_3) = \left(\frac{y_1 + y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - 2y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 + y_2 - 2y_3}{3}\right)^2$$

$$= \frac{y_1^2}{3} + \frac{2}{3}y_2^2 + \frac{2}{3}y_3^2 - \frac{2}{3}y_2 y_3$$

Example 7e. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with rate λ. Let

$$Y_i = X_1 + \cdots + X_i \qquad i = 1, \ldots, n$$

(a) Find the joint density function of $Y_1, \ldots, Y_n$.
(b) Use the result of part (a) to find the density of Y_n.

Solution (a) The Jacobian of the transformations $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $\ldots$, $Y_n = X_1 + \cdots + X_n$ is

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & & & & \\ \cdots & \cdots & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

Since only the first term of the determinant will be nonzero, we have that $J = 1$. Now the joint density function of $X_1, \ldots, X_n$ is given by

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \qquad 0 < x_i < \infty,\ i = 1, \ldots, n$$

Hence, as the preceding transformations yield that

$$X_1 = Y_1, X_2 = Y_2 - Y_1, \ldots, X_i = Y_i - Y_{i-1}, \ldots, X_n = Y_n - Y_{n-1}$$

we obtain from Equation (7.3) that the joint density function of $Y_1, \ldots, Y_n$ is

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) &= f_{X_1,\ldots,X_n}(y_1, y_2 - y_1, \ldots, y_i - y_{i-1}, \ldots, y_n - y_{n-1}) \\ &= \lambda^n \exp\left\{-\lambda\left[y_1 + \sum_{i=2}^{n}(y_i - y_{i-1})\right]\right\} \\ &= \lambda^n e^{-\lambda y_n} \qquad 0 < y_1, 0 < y_i - y_{i-1}, i = 2, \ldots, n \\ &= \lambda^n e^{-\lambda y_n} \qquad 0 < y_1 < y_2 < \cdots < y_n \end{aligned}$$

(b) To obtain the marginal density of Y_n, let us integrate out the other variables one at a time. This gives

$$\begin{aligned} f_{Y_2,\ldots,Y_n}(y_2, \ldots, y_n) &= \int_0^{y_2} \lambda^n e^{-\lambda y_n}\, dy_1 \\ &= \lambda^n y_2 e^{-\lambda y_n} \qquad 0 < y_2 < y_3 < \cdots < y_n \end{aligned}$$

Continuing gives that

$$\begin{aligned} f_{Y_3,\ldots,Y_n}(y_3, \ldots, y_n) &= \int_0^{y_3} \lambda^n y_2 e^{-\lambda y_n}\, dy_2 \\ &= \lambda^n \frac{y_3^2}{2} e^{-\lambda y_n} \qquad 0 < y_3 < y_4 < \cdots < y_n \end{aligned}$$

The next integration yields that

$$f_{Y_4,\ldots,Y_n}(y_4, \ldots, y_n) = \lambda^n \frac{y_4^3}{3!} e^{-\lambda y_n} \qquad 0 < y_4 < \cdots < y_n$$

Continuing in this fashion yields

$$f_{Y_n}(y_n) = \lambda^n \frac{y_n^{n-1}}{(n-1)!} e^{-\lambda y_n} \qquad 0 < y_n$$

which, in agreement with the result obtained in Example 3b, shows that $X_1 + \cdots + X_n$ is a gamma random variable with parameters n and λ. ∎

*6.8 EXCHANGEABLE RANDOM VARIABLES

The random variables $X_1, X_2, \ldots, X_n$ are said to be *exchangeable* if for every permutation $i_1, \ldots, i_n$ of the integers $1, \ldots, n$

$$P\{X_{i_1} \le x_1, X_{i_2} \le x_2, \ldots, X_{i_n} \le x_n\} = P\{X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n\}$$

for all $x_1, \ldots, x_n$. That is, the n random variables are exchangeable if their joint distribution is the same no matter in which order they are observed.

Discrete random variables will be exchangeable if

$$P\{X_{i_1} = x_1, X_{i_2} = x_2, \ldots, X_{i_n} = x_n\} = P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$$

for all permutations $i_1, \ldots, i_n$, and all values $x_1, \ldots, x_n$. This is equivalent to stating that $p(x_1, x_2, \ldots, x_n) = P\{X_1 = x_1, \ldots, X_n = x_n\}$ is a symmetric function of the vector $(x_1, \ldots, x_n)$, which means that its value does not change when the values of the vector are permuted.

Example 8a. Suppose that balls are withdrawn one at a time and without replacement from an urn that initially contains n balls, of which k are considered special, in such a manner that each withdrawal is equally likely to be any of the balls that remain in the urn at the time. Let $X_i = 1$ if the ith ball withdrawn is special and let it be 0 otherwise. We will show that the random variables $X_1, \ldots, X_n$ are exchangeable. To do so, let $(x_1, \ldots, x_n)$ be a vector consisting of k ones and $n - k$ zeros. However, before considering the joint mass function evaluated at $(x_1, \ldots, x_n)$, let us try to gain some insight by considering a fixed such vector—for instance, consider the vector $(1, 1, 0, 1, 0, \ldots, 0, 1)$, which is assumed to have k ones and $n - k$ zeros. Then

$$p(1, 1, 0, 1, 0, \ldots, 0, 1) = \frac{k}{n}\frac{k-1}{n-1}\frac{n-k}{n-2}\frac{k-2}{n-3}\frac{n-k-1}{n-4}\cdots\frac{1}{2}\frac{1}{1}$$

which follows since the probability that the first ball is special is k/n, the conditional probability that the next one is special is $(k-1)/(n-1)$, the conditional probability that the next one is not special is $(n-k)/(n-2)$, and so on. By the same argument, it follows that $p(x_1, \ldots, x_n)$ can be expressed as the product of n fractions. The successive denominator terms of these fractions will go from n down to 1. The numerator term at the location where the vector $(x_1, \ldots, x_n)$ is 1 for the ith time is $k - (i - 1)$,

and where it is 0 for the ith time it is $n - k - (i - 1)$. Hence, since the vector $(x_1, \ldots, x_n)$ consists of k ones and $n - k$ zeros, we obtain that

$$p(x_1, \ldots, x_n) = \frac{k!\,(n-k)!}{n!} \qquad x_i = 0, 1, \sum_{i=1}^{n} x_i = k$$

Since this is a symmetric function of $(x_1, \ldots, x_n)$, it follows that the random variables are exchangeable.

REMARK. Another way to obtain the preceding formula for the joint probability mass function is to regard all the n balls as distinguishable from each other. Then, as the outcome of the experiment is an ordering of these balls, it follows that there are $n!$ equally likely outcomes. As the number of outcomes having special and nonspecial balls in specified places is equal to the number of ways of permuting the special and the nonspecial balls among themselves, namely $k!\,(n-k)!$, we obtain the preceding density function. ∎

If $X_1, X_2, \ldots, X_n$ are exchangeable, it easily follows that each X_i has the same probability distribution. For instance, if X and Y are exchangeable discrete random variables, then

$$P\{X = x\} = \sum_y P\{X = x, Y = y\} = \sum_y P\{X = y, Y = x\} = P\{Y = x\}$$

For instance, it follows from Example 8*a* that the ith ball withdrawn will be special with probability k/n, which is intuitively clear since each of the n balls is equally likely to be the ith one selected.

Example 8b. In Example 8a, let Y_i denote the selection number of the first special ball withdrawn, let Y_2 denote the additional number that are then withdrawn until the second special ball appears, and in general, let Y_i denote the additional number of balls withdrawn after the $(i-1)$st special ball is selected until the ith is selected, $i = 1, \ldots, k$. For instance, if $n = 4$, $k = 2$ and $X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1$, then $Y_1 = 1, Y_2 = 3$. Since $Y_1 = i_1, Y_2 = i_2, \ldots, Y_k = i_k \Leftrightarrow X_{i_1} = X_{i_1+i_2} = \cdots = X_{i_1+\cdots+i_k} = 1$, $X_j = 0$, otherwise; we obtain from the joint mass function of the X_i that

$$P\{Y_1 = i_1, Y_2 = i_2, \ldots, Y_k = i_k\} = \frac{k!\,(n-k)!}{n!} \qquad i_1 + \cdots + i_k \le n$$

Hence we see that the random variables $Y_1, \ldots, Y_k$ are exchangeable. For instance, it follows from this that the number of cards one must select from a well-shuffled deck until an ace appears has the same distribution as the number of additional cards one must select after the first ace appears until the next one does, and so on.

Example 8c. The following is known as Polya's urn model. Suppose that an urn initially contains n red and m blue balls. At each stage a ball is randomly chosen, its color is noted, and it is then replaced along with another ball of the same color. Let $X_i = 1$ if the ith ball selected is red and let it equal 0

if the ith ball is blue, $i \geq 1$. To obtain a feeling for the joint probabilities of these X_i, note the following special cases.

$$P\{X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 0\}$$
$$= \frac{n}{n+m}\frac{n+1}{n+m+1}\frac{m}{n+m+2}\frac{n+2}{n+m+3}\frac{m+1}{n+m+4}$$
$$= \frac{n(n+1)(n+2)m(m+1)}{(n+m)(n+m+1)(n+m+2)(n+m+3)(n+m+4)}$$

and

$$P\{X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 1\}$$
$$= \frac{m}{n+m}\frac{n}{n+m+1}\frac{m+1}{n+m+2}\frac{n+1}{n+m+3}\frac{n+2}{n+m+4}$$
$$= \frac{n(n+1)(n+2)m(m+1)}{(n+m)(n+m+1)(n+m+2)(n+m+3)(n+m+4)}$$

By the same reasoning it follows that for any sequence $x_1, \ldots, x_k$ that contains r ones and $k - r$ zeros, we have

$$P\{X_1 = x_1, \ldots, X_k = x_k\}$$
$$= \frac{n(n+1)\cdots(n+r-1)m(m+1)\cdots(m+k-r-1)}{(n+m)\cdots(n+m+k-1)}$$

Therefore, we see that for any value of k, the random variables $X_1, \ldots, X_k$ are exchangeable.

Our final example deals with continuous random variables that are exchangeable.

Example 8d. Let $X_1, X_2, \ldots, X_n$ be independent uniform (0, 1) random variables, and let $X_{(1)}, \ldots, X_{(n)}$ denote their order statistics. That is, $X_{(j)}$ is the jth smallest of $X_1, X_2, \ldots, X_n$. Also, let

$$Y_1 = X_{(1)},$$
$$Y_i = X_{(i)} - X_{(i-1)}, \qquad i = 2, \ldots, n$$

Show that $Y_1, \ldots, Y_n$ are exchangeable.

Solution The transformations

$$y_1 = x_1, \ldots, y_i = x_i - x_{i-1} \qquad i = 2, \ldots, n$$

yield that

$$x_i = y_1 + \cdots + y_i \qquad i = 1, \ldots, n$$

As it is easy to see that the Jacobian of the preceding transformations is equal to 1, we obtain from Equation (7.3) that

$$f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = f(y_1, y_1 + y_2, \ldots, y_1 + \cdots + y_n)$$

where f is the joint density function of the order statistics. Hence from Equation (6.1) we obtain that

$$f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = n!$$
$$0 < y_1 < y_1 + y_2 < \cdots < y_1 + \cdots + y_n < 1$$

or, equivalently,

$$f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = n!$$
$$0 < y_i < 1, i = 1, \ldots, n, \qquad y_1 + \cdots + y_n < 1$$

As the preceding joint density is a symmetric function of $y_1, \ldots, y_n$, we see that the random variables $Y_1, \ldots, Y_n$ are exchangeable. ∎

SUMMARY

The *joint cumulative probability distribution function* of the pair of random variables X and Y is defined by

$$F(x, y) = P\{X \leq x, Y \leq y\} \qquad -\infty < x, y < \infty$$

All probabilities regarding the pair can be obtained from F. To obtain the individual probability distribution functions of X and Y, use

$$F_X(x) = \lim_{y\to\infty} F(x, y) \qquad F_Y(y) = \lim_{x\to\infty} F(x, y)$$

If X and Y are both discrete random variables, then their *joint probability mass function* is defined by

$$p(i, j) = P\{X = i, Y = j\}$$

The individual mass functions are

$$P\{X = i\} = \sum_j p(i, j) \qquad P\{Y = j\} = \sum_i p(i, j)$$

The random variables X and Y are said to be *jointly continuous* if there is a function $f(x, y)$, called the *joint probability density function,* such that for any two-dimensional set C,

$$P\{(X, Y) \in C\} = \iint_C f(x, y)\, dx\, dy$$

It follows from the preceding that

$$P\{x < X < x + dx, y < Y < y + dy\} \approx f(x, y)\, dx\, dy$$

If X and Y are jointly continuous, then they are individually continuous with density functions

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

The random variables X and Y are *independent* if for all sets A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

If the joint distribution function (or the joint probability mass function in the discrete case, or the joint density function in the continuous case) factors into a part depending only on x and a part depending only on y, then X and Y are independent.

In general, the random variables $X_1, \ldots, X_n$ are independent if for all sets of real numbers $A_1, \ldots, A_n$,

$$P\{X_1 \in A_1, \ldots, X_n \in A_n\} = P\{X_1 \in A_1\} \cdots P\{X_n \in A_n\}$$

If X and Y are independent continuous random variables, then the distribution function of their sum can be obtained from the identity

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy$$

If X_i, $i = 1, \ldots, n$, are independent normal random variables with respective parameters μ_i and σ_i^2, $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i$ is normal with parameters $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$.

If X_i, $i = 1, \ldots, n$, are independent Poisson random variables with respective parameters λ_i, $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i$ is Poisson with parameter $\sum_{i=1}^{n} \lambda_i$.

If X and Y are discrete random variables, then the *conditional probability mass function* of X given that $Y = y$ is defined by

$$P\{X = x | Y = y\} = \frac{p(x, y)}{p_Y(y)}$$

where p is their joint probability mass function. Also, if X and Y are jointly continuous with joint density function f, then the *conditional probability density function* of X given that $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

The ordered values $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ of a set of independent and identically distributed random variables are called the *order statistics* of that set. If the random variables are continuous with density function f, then the joint density function of the order statistics is

$$f(x_1, \ldots, x_n) = n!\, f(x_1) \cdots f(x_n) \qquad x_1 \leq x_2 \leq \cdots \leq x_n$$

The random variables $X_1, \ldots, X_n$ are *exchangeable* if the joint distribution of $X_{i_1}, \ldots, X_{i_n}$ is the same for every permutation $i_1, \ldots, i_n$ of $1, \ldots, n$.

PROBLEMS

1. Two fair dice are rolled. Find the joint probability mass function of X and Y when
   (a) X is the largest value obtained on any die and Y is the sum of the values;
   (b) X is the value on the first die and Y is the larger of the two values;
   (c) X is the smallest and Y is the largest value obtained on the dice.
2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
   (a) X_1, X_2;
   (b) X_1, X_2, X_3.
3. In Problem 2, suppose that the white balls are numbered, and let Y_i equal 1 if the ith white ball is selected and 0 otherwise. Find the joint probability mass function of
   (a) Y_1, Y_2;
   (b) Y_1, Y_2, Y_3.
4. Repeat Problem 2 when the ball selected is replaced in the urn before the next selection.
5. Repeat Problem 3a when the ball selected is replaced in the urn before the next selection.
6. A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by N_1 the number of tests made until the first defective is spotted and by N_2 the number of additional tests until the second defective is spotted; find the joint probability mass function of N_1 and N_2.
7. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.
8. The joint probability density function of X and Y is given by
$$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \leq x \leq y,\ 0 < y < \infty$$
   (a) Find c.
   (b) Find the marginal densities of X and Y.
   (c) Find $E[X]$.
9. The joint probability density function of X and Y is given by
$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$
   (a) Verify that this is indeed a joint density function.
   (b) Compute the density function of X.
   (c) Find $P\{X > Y\}$.

(d) Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.
(e) Find $E[X]$.
(f) Find $E[Y]$.

10. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 \le x < \infty, \ 0 \le y < \infty$$

Find (a) $P\{X < Y\}$ and (b) $P\{X < a\}$.

11. A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a color television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 color set on that day?

12. The number of people that enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

13. A man and a woman agree to meet at a certain location about 12:30 P.M. If the man arrives at a time uniformly distributed between 12:15 and 12:45 and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 P.M., find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

14. An ambulance travels back and forth, at a constant speed, along a road of length L. At a certain moment of time an accident occurs at a point uniformly distributed on the road. [That is, its distance from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, compute, assuming independence, the distribution of its distance from the accident.

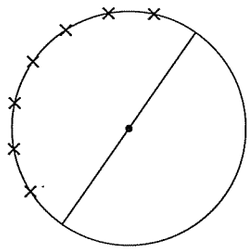
15. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

$$f(x, y) = \begin{cases} c & \text{if } (x, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $1/c =$ area of region R.
Suppose that (X, Y) is uniformly distributed over the square centered at $(0, 0)$, whose sides are of length 2.
(b) Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.
(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \le 1\}$.

16. Suppose that n points are independently chosen at random on the perimeter of a circle, and we want the probability that they all lie in some semicircle.

(That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line.)



Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for 180°, $i = 1, \ldots, n$.
(a) Express A in terms of the A_i.
(b) Are the A_i mutually exclusive?
(c) Find $P(A)$.

17. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3?

18. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

19. In Problem 18 find the probability that the 3 line segments from 0 to X, from X to Y, and from Y to L could be made to form the three sides of a triangle. (Note that three line segments can be made to form a triangle if the length of each of them is less than the sum of the lengths of the others.)

20. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, \ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? What if $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y, \ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

21. Let

$$f(x, y) = 24xy \qquad 0 \le x \le 1, 0 \le y \le 1, 0 \le x + y \le 1$$

and let it equal 0 otherwise.

(a) Show that $f(x, y)$ is a joint probability density function.
(b) Find $E[X]$.
(c) Find $E[Y]$.

22. The joint density function of X and Y is

$$f(x, y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?
(b) Find the density function of X.
(c) Find $P\{X + Y < 1\}$.

23. The random variables X and Y have joint density function.

$$f(x, y) = 12xy(1 - x) \qquad 0 < x < 1, 0 < y < 1$$

and equal to 0 otherwise.
(a) Are X and Y independent?
(b) Find $E[X]$.
(c) Find $E[Y]$.
(d) Find $\text{Var}(X)$.
(e) Find $\text{Var}(Y)$.

24. Consider independent trials each of which results in outcome i, $i = 0, 1, \ldots, k$ with probability p_i, $\sum_{i=0}^{k} p_i = 1$. Let N denote the number of trials needed to obtain an outcome that is not equal to 0, and let X be that outcome.
(a) Find $P\{N = n\}, n \geq 1$.
(b) Find $P\{X = j\}, j = 1, \ldots, k$.
(c) Show that $P\{N = n, X = j\} = P\{N = n\}P\{X = j\}$.
(d) Is it intuitive to you that N is independent of X?
(e) Is it intuitive to you that X is independent of N?

25. Suppose that 10^6 people arrive at a service station at times that are independent random variables, each of which is uniformly distributed over $(0, 10^6)$. Let N denote the number that arrive in the first hour. Find an approximation for $P\{N = i\}$.

26. Suppose that A, B, C, are independent random variables, each being uniformly distributed over (0, 1).
(a) What is the joint cumulative distribution function of A, B, C?
(b) What is the probability that all of the roots of the equation $Ax^2 + Bx + C = 0$ are real?

27. If X is uniformly distributed over (0, 1) and Y is exponentially distributed with parameter $\lambda = 1$, find the distribution of (a) $Z = X + Y$ and (b) $Z = X/Y$. Assume independence.

28. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $P\{X_1 < X_2\}$.

29. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by $W = I^2R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1 - x) \qquad 0 \leq x \leq 1$$
$$f_R(x) = 2x \qquad 0 \leq x \leq 1$$

Determine the density function of W.

30. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that an article of 10 pages contains (a) 0, and (b) 2 or more typographical errors? Explain your reasoning!

31. The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be
(a) more than 2 such accidents in the next month;
(b) more than 4 such accidents in the next 2 months;
(c) more than 5 such accidents in the next 3 months?
Explain your reasoning!

32. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that
(a) the total gross sales over the next 2 weeks exceeds \$5000;
(b) weekly sales exceed \$2000 in at least 2 of the next 3 weeks?
What independence assumptions have you made?

33. Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, then assuming that their scores are independent random variables, approximate the probability that
(a) Jack's score is higher;
(b) the total of their scores is above 350.

34. According to the U.S. National Center for Health Statistics, 25.2 percent of males and 23.6 percent of females never eat breakfast. Suppose that random samples of 200 men and 200 women are chosen. Approximate the probability that
(a) at least 110 of these 400 people never eat breakfast;
(b) the number of the women who never eat breakfast is at least as large as the number of the men who never eat breakfast.

35. In Problem 2, calculate the conditional probability mass function of X_1 given that
(a) $X_2 = 1$;
(b) $X_2 = 0$.

36. In Problem 4, calculate the conditional probability mass function of X_1 given that
(a) $X_2 = 1$;
(b) $X_2 = 0$.

37. In Problem 3, calculate the conditional probability mass function of Y_1 given that
(a) $Y_2 = 1$;
(b) $Y_2 = 0$.

38. In Problem 5, calculate the conditional probability mass function of Y_1 given that
(a) $Y_2 = 1$;
(b) $Y_2 = 0$.

39. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, \ldots, X\}$. Call this second number Y.
(a) Find the joint mass function of X and Y.
(b) Find the conditional mass function of X given that $Y = i$. Do it for $i = 1, 2, 3, 4, 5$.
(c) Are X and Y independent? Why?

40. Two dice are rolled. Let X and Y denote, respectively, the largest and smallest values obtained. Compute the conditional mass function of Y given $X = i$, for $i = 1, 2, \ldots, 6$. Are X and Y independent? Why?

41. The joint probability mass function of X and Y is given by

$$p(1, 1) = \tfrac{1}{8} \qquad p(1, 2) = \tfrac{1}{4}$$
$$p(2, 1) = \tfrac{1}{8} \qquad p(2, 2) = \tfrac{1}{2}$$

(a) Compute the conditional mass function of X given $Y = i$, $i = 1, 2$.
(b) Are X and Y independent?
(c) Compute $P\{XY \leq 3\}$, $P\{X + Y > 2\}$, $P\{X/Y > 1\}$.

42. The joint density function of X and Y is given by

$$f(x, y) = xe^{-x(y+1)} \qquad x > 0,\ y > 0$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.
(b) Find the density function of $Z = XY$.

43. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x} \qquad 0 \leq x < \infty,\ -x \leq y \leq x$$

Find the conditional distribution of Y, given $X = x$.

44. An insurance company supposes that each person has an accident parameter and that the yearly number of accidents of someone whose accident parameter is λ is Poisson distributed with mean λ. They also suppose that the parameter value of a newly insured person can be assumed to be the value of a gamma random variable with parameters s and α. If a newly insured person has n accidents in her first year, find the conditional density of her accident parameter. Also, determine the expected number of accidents that she will have in the following year.

45. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over (a, b), compute the probability that the largest of the three is greater than the sum of the other two.

46. A complex machine is able to operate effectively as long as at least 3 of its 5 motors are functioning. If each motor independently functions for a random amount of time with density function $f(x) = xe^{-x}$, $x > 0$, compute the density function of the length of time that the machine functions.

47. If 3 trucks break down at points randomly distributed on a road of length L, find the probability that no 2 of the trucks are within a distance d of each other when $d \leq L/2$.

48. Consider a sample of size 5 from a uniform distribution over (0, 1). Compute the probability that the median is in the interval $(\frac{1}{4}, \frac{3}{4})$.

49. If X_1, X_2, X_3, X_4, X_5 are independent and identically distributed exponential random variables with the parameter λ, compute
(a) $P\{\min(X_1, \ldots, X_5) \leq a\}$;
(b) $P\{\max(X_1, \ldots, X_5) \leq a\}$.

50. Derive the distribution of the range of a sample of size 2 from a distribution having density function $f(x) = 2x$, $0 < x < 1$.

51. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 \leq 1$$

Find the joint density function of the polar coordinates $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1} Y/X$.

52. If X and Y are independent random variables both uniformly distributed over (0, 1), find the joint density function of $R = \sqrt{X^2 + Y^2}$, $\Theta = \tan^{-1} Y/X$.

53. If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with rate 1, show directly (without using the results of Example 7b) that X and Y defined by

$$X = \sqrt{2Z}\cos U$$
$$Y = \sqrt{2Z}\sin U$$

are independent unit normal random variables.

54. If X and Y have joint density function

$$f(x, y) = \frac{1}{x^2y^2} \qquad x \geq 1,\ y \geq 1$$

(a) Compute the joint density function of $U = XY$, $V = X/Y$.
(b) What are the marginal densities?

55. If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of
(a) $U = X + Y$, $V = X/Y$;

(b) $U = X, V = X/Y$;
(c) $U = X + Y, V = X/(X + Y)$.

56. Repeat Problem 55 when X and Y are independent exponential random variables, each with parameter $\lambda = 1$.

57. If X_1 and X_2 are independent exponential random variables each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

58. If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}$, $0 < x < \infty$, derive the joint distribution of $U = X + Y, V = X + Z, W = Y + Z$.

59. In Example 8b, let $Y_{k+1} = n + 1 - \sum_{i=1}^{k} Y_i$. Show that $Y_1, \ldots, Y_k, Y_{k+1}$ are exchangeable. Note that Y_{k+1} is the number of balls one must observe to obtain a special ball if one considers the balls in their reverse order of withdrawal.

60. Consider an urn containing n balls, numbered $1, \ldots, n$, and suppose that k of them are randomly withdrawn. Let X_i equal 1 if ball numbered i is removed and let it be 0 otherwise. Show that $X_1, \ldots, X_n$ are exchangeable.

THEORETICAL EXERCISES

1. Verify Equation (1.2).

2. Suppose that the number of events that occur in a given time period is a Poisson random variable with parameter λ. If each event is classified as a type i event with probability p_i, $i = 1, \ldots, n$, $\sum p_i = 1$, independently of other events, show that the numbers of type i events that occur, $i = 1, \ldots, n$, are independent Poisson random variables with respective parameters λp_i, $i = 1, \ldots, n$.

3. Suggest a procedure for using Buffon's needle problem to estimate π. Surprisingly enough, this was once a common method of evaluating π.

4. Solve Buffon's needle problem when $L > D$.

ANSWER: $\frac{2L}{\pi D}(1 - \sin\theta) + 2\theta/\pi$, where θ is such that $\cos\theta = D/L$.

5. If X and Y are independent continuous positive random variables, express the density function of (a) $Z = X/Y$ and (b) $Z = XY$ in terms of the density functions of X and Y. Evaluate these expressions in the special case where X and Y are both exponential random variables.

6. Show analytically (by induction) that $X_1 + \cdots + X_n$ has a negative binomial distribution when the X_i, $i = 1, \ldots, n$ are independent and identically distributed geometric random variables. Also, give a second argument that verifies the above without any need for computations.

7. (a) If X has a gamma distribution with parameters (t, λ), what is the distribution of cX, $c > 0$?
(b) Show that

$$\frac{1}{2\lambda}\chi^2_{2n}$$

has a gamma distribution with parameters n, λ when n is a positive integer and χ^2_{2n} is a chi-squared random variable with $2n$ degrees of freedom.

8. Let X and Y be independent continuous random variables with respective hazard rate functions $\lambda_X(t)$ and $\lambda_Y(t)$, and set $W = \min(X, Y)$.
(a) Determine the distribution function of W in terms of those of X and Y.
(b) Show that $\lambda_W(t)$, the hazard rate function of W, is given by

$$\lambda_W(t) = \lambda_X(t) + \lambda_Y(t)$$

9. Let $X_1, \ldots, X_n$ be independent exponential random variables having a common parameter λ. Determine the distribution of $\min(X_1, \ldots, X_n)$.

10. The lifetimes of batteries are independent exponential random variables, each having parameter λ. A flashlight needs 2 batteries to work. If one has a flashlight and a stockpile of n batteries, what is the distribution of time that the flashlight can operate?

11. Let X_1, X_2, X_3, X_4, X_5, be independent continuous random variables having a common distribution function F and density function f, and set

$$I = P\{X_1 < X_2 > X_3 < X_4 > X_5\}$$

(a) Show that I does not depend on F.

HINT: Write I as a five-dimensional integral and make the change of variables $u_i = F(x_i)$, $i = 1, \ldots, 5$.

(b) Evaluate I.

12. Show that the jointly continuous (discrete) random variables $X_1, \ldots, X_n$ are independent if and only if their joint probability density (mass) function $f(x_1, \ldots, x_n)$ can be written as

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} g_i(x_i)$$

for nonnegative functions $g_i(x)$, $i = 1, \ldots, n$.

13. In Example 5c we computed the conditional density of a success probability for a sequence of trials when the first $n + m$ trials resulted in n successes. Would the conditional density change if we actually specified which n of these trials resulted in successes?

14. Suppose that X and Y are independent geometric random variables with the same parameter p.
(a) Without any computations, what do you think is the value of

$$P\{X = i | X + Y = n\}?$$

HINT: Imagine that you continually flip a coin having probability p of coming up heads. If the second head occurs on the nth flip, what is the probability mass function of the time of the first head?

(b) Verify your conjecture in part (a).

15. If X and Y are independent binomial random variables with identical parameters n and p, show analytically that the conditional distribution of X, given that $X + Y = m$, is the hypergeometric distribution. Also, give a second argument that yields the result without any computations.

HINT: Suppose that $2n$ coins are flipped. Let X denote the number of heads in the first n flips and Y the number in the second n flips. Argue that given a total of m heads, the number of heads in the first n flips has the same distribution as the number of white balls selected when a sample of size m is chosen from n white and n black balls.

16. Consider an experiment that results in one of three possible outcomes, outcome i occurring with probability p_i, $i = 1, 2, 3$. Suppose that n independent replications of this experiment are performed and let X_i, $i = 1, 2, 3$ denote the number of times that outcome i occurs. Determine the conditional probability mass function of X_1, given that $X_2 = m$.

17. Let X_1, X_2, X_3 be independent and identically distributed continuous random variables. Compute

(a) $P\{X_1 > X_2 | X_1 > X_3\}$;

(b) $P\{X_1 > X_2 | X_1 < X_3\}$;

(c) $P\{X_1 > X_2 | X_2 > X_3\}$;

(d) $P\{X_1 > X_2 | X_2 < X_3\}$.

18. Let U denote a random variable uniformly distributed over (0, 1). Compute the conditional distribution of U given that

(a) $U > a$;

(b) $U < a$;

where $0 < a < 1$.

19. Suppose that W, the amount of moisture in the air on a given day, is a gamma random variable with parameters (t, β). That is, its density is $f(w) = \beta e^{-\beta w} (\beta w)^{t-1}/\Gamma(t)$, $w > 0$. Suppose also that given that $W = w$, the number of accidents during that day—call it N—has a Poisson distribution with mean w. Show that the conditional distribution of W given that $N = n$ is the gamma distribution with parameters $(t + n, \beta + 1)$.

20. Let W be a gamma random variable with parameters (t, β), and suppose that conditional on $W = w$, $X_1, X_2, \ldots, X_n$ are independent exponential random variables with rate w. Show that the conditional distribution of W given that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is gamma with parameters $\left(t + n, \beta + \sum_{i=1}^{n} x_i\right)$.

21. A rectangular array of mn numbers arranged in n rows, each consisting of m columns, is said to contain a *saddlepoint* if there is a number that is both the minimum of its row and the maximum of its column. For instance, in the array

$$\begin{matrix} 1 & 3 & 2 \\ 0 & -2 & 6 \\ .5 & 12 & 3 \end{matrix}$$

the number 1 in the first row, first column is a saddlepoint. The existence of a saddlepoint is of significance in the theory of games. Consider a rectangular array of numbers as described above and suppose that there are two individuals—A and B—that are playing the following game: A is to choose one of the numbers $1, 2, \ldots, n$ and B one of the numbers $1, 2, \ldots, m$. These choices are announced simultaneously, and if A chose i and B chose j, then A wins from B the amount specified by the number in the ith row, jth column of the array. Now suppose that the array contains a saddlepoint—say the number in the row r and column k—call this number x_{rk}. Now if player A chooses row r, then that player can guarantee herself a win at least x_{rk} (since x_{rk} is the minimum number in the row r). On the other hand, if player B chooses column k, then he can guarantee that he will lose no more than x_{rk} (since x_{rk} is the maximum number in the column k). Hence, as A has a way of playing that guarantees her a win of x_{rk} and as B has a way of playing that guarantees he will lose no more than x_{rk}, it seems reasonable to take these two strategies as being optimal and declare that the value of the game to player A is x_{rk}.

If the nm numbers in the rectangular array described above are independently chosen from an arbitrary continuous distribution, what is the probability that the resulting array will contain a saddlepoint?

22. The random variables X and Y are said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

(a) Show that the conditional density of X, given that $Y = y$, is the normal density with parameters

$$\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y) \quad \text{and} \quad \sigma_x^2(1-\rho^2)$$

(b) Show that X and Y are both normal random variables with respective parameters μ_x, σ_x^2 and μ_y, σ_y^2.

(c) Show that X and Y are independent when $\rho = 0$.

23. Suppose that $F(x)$ is a cumulative distribution function. Show that (a) $F^n(x)$ and (b) $1 - [1 - F(x)]^n$ are also cumulative distribution functions when n is a positive integer.

HINT: Let $X_1, \ldots, X_n$ be independent random variables having the common distribution function F. Define random variables Y and Z in terms of the X_i so that $P\{Y \leq x\} = F^n(x)$, and $P\{Z \leq x\} = 1 - [1 - F(x)]^n$.

24. Show that if n people are distributed at random along a road L miles long, then the probability that no 2 people are less than a distance of D miles apart is, when $D \leq L/(n-1)$, $[1 - (n-1)D/L]^n$. What if $D > L/(n-1)$?

25. Establish Equation (6.2) by differentiating Equation (6.4).

26. Show that the median of a sample of size $2n + 1$ from a uniform distribution on (0, 1) has a beta distribution with parameters $(n + 1, n + 1)$.

27. Verify Equation (6.6), which gives the joint density of $X_{(i)}$ and $X_{(j)}$.

28. Compute the density of the range of a sample of size n from a continuous distribution having density function f.

29. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the ordered values of n independent uniform (0, 1) random variables. Prove that for $1 \leq k \leq n + 1$,

$$P\{X_{(k)} - X_{(k-1)} > t\} = (1 - t)^n$$

where $X_0 \equiv 0, X_{n+1} \equiv t$.

30. Let $X_1, \ldots, X_n$ be a set of independent and identically distributed continuous random variables having distribution function F, and let $X_{(i)}$, $i = 1, \ldots, n$ denote their ordered values. If X, independent of the X_i, $i = 1, \ldots, n$, also has distribution F, determine
(a) $P\{X > X_{(n)}\}$;
(b) $P\{X > X_{(1)}\}$;
(c) $P\{X_{(i)} < X < X_{(j)}\}$, $1 \leq i < j \leq n$.

31. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function F and density f. The quantity $M \equiv [X_{(1)} + X_{(n)}]/2$, defined to be the average of the smallest and largest value, is called the midrange. Show that its distribution function is

$$F_M(m) = n \int_{-\infty}^{m} [F(2m - x) - F(x)]^{n-1} f(x)\, dx$$

32. Let $X_1, \ldots, X_n$ be independent uniform (0, 1) random variables. Let $R = X_{(n)} - X_{(1)}$ denote the range and $M = [X_{(n)} + X_{(1)}]/2$ the midrange. Compute the joint density function of R and M.

33. If X and Y are independent standard normal random variables, determine the joint density function of

$$U = X \qquad V = \frac{X}{Y}$$

Then use your result to show that X/Y has a Cauchy distribution.

SELF-TEST PROBLEMS AND EXERCISES

1. Each throw of a unfair die lands on each of the odd numbers 1, 3, 5 with probability C and on each of the even numbers with probability $2C$.
(a) Find C.
(b) Suppose that the die is tossed. Let X equal 1 if the result is an even number, and let it be 0 otherwise. Also, let Y equal 1 if the result is a number greater than three and let it be 0 otherwise. Find the joint probability mass function of X and Y.
Suppose now that 12 independent tosses of the die are made.
(c) Find the probability that each of the six outcomes occurs exactly twice.
(d) Find the probability that 4 of the outcomes are either one or two, 4 are either three or four, and 4 are either five or six.
(e) Find the probability that at least 8 of the tosses land on even numbers.

2. The joint probability mass function of the random variables X, Y, Z is

$$p(1, 2, 3) = p(2, 1, 1) = p(2, 2, 1) = p(2, 3, 2) = \tfrac{1}{4}$$

Find (a) $E[XYZ]$, and (b) $E[XY + XZ + YZ]$.

3. The joint density of X and Y is given by

$$f(x, y) = C(y - x)e^{-y} \qquad -y < x < y, \qquad 0 < y < \infty$$

(a) Find C.
(b) Find the density function of X.
(c) Find the density function of Y.
(d) Find $E[X]$.
(e) Find $E[Y]$.

4. Suppose that X, Y, and Z are independent random variables that are each equally likely to be either 1 or 2. Find the probability mass function of (a) XYZ, (b) $XY + XZ + YZ$, and (c) $X^2 + YZ$.

5. Let X and Y be continuous random variables with joint density function

$$f(x) = \begin{cases} \frac{x}{5} + cy & 0 < x < 1, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.
(a) What is the value of c?
(b) Are X and Y independent?
(c) Find $P\{X + Y > 3\}$.

6. The joint density function of X and Y is

$$f(x, y) = \begin{cases} xy & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

(b) Find the density function of X.
(c) Find the density function of Y.
(d) Find the joint distribution function.
(e) Find $E[Y]$.
(f) Find $P\{X + Y < 1\}$.

7. Consider two components and three types of shocks. A type 1 shock causes component 1 to fail, a type 2 shock causes component 2 to fail, and a type 3 shock causes both components 1 and 2 to fail. The times until shocks 1, 2, and 3 occur are independent exponential random variables with respective rates λ_1, λ_2, and λ_3. Let X_i denote the time at which component i fails, $i = 1, 2$. The random variables X_1, X_2 are said to have a joint bivariate exponential distribution. Find $P\{X_1 > s, X_2 > t\}$.

8. Consider a directory of classified advertisements that consists of m pages, where m is very large. Suppose that the number of advertisements per page varies and that your only method of finding out how many advertisements there are on a specified page is to count them. In addition, suppose that there are too many pages for it to be feasible to make a complete count of the total number of advertisements and that your objective is to choose a directory advertisement in such a way that each of them has an equal chance of being selected.

(a) If you randomly choose a page and then randomly choose an advertisement from that page, would that satisfy your objective? Why or why not?

Let $n(i)$ denote the number of advertisements on page i, $i = 1, \ldots, m$, and suppose that whereas these quantities are unknown, we can assume that they are all less than or equal to some specified value n. Consider the following algorithm for choosing an advertisement.

Step 1. Choose a page at random. Suppose it is page X. Determine $n(X)$ by counting the number of advertisements on page X.
Step 2. "Accept" page X with probability $n(X)/n$. If page X is accepted, go to step 3. Otherwise, return to step 1.
Step 3. Randomly choose one of the advertisements on page X.

Call each pass of the algorithm through step 1 an iteration. For instance, if the first randomly chosen page is rejected and the second accepted, than we would have needed 2 iterations of the algorithm to obtain an advertisement.

(b) What is the probability that a single iteration of the algorithm results in the acceptance of an advertisement on page i?
(c) What is the probability that a single iteration of the algorithm results in the acceptance of an advertisement?
(d) What is the probability that the algorithm goes through k iterations, accepting the jth advertisement on page i on the final iteration?
(e) What is the probability that the jth advertisement on page i is the advertisement obtained from the algorithm?
(f) What is the expected number of iterations taken by the algorithm?

9. The "random" parts of the algorithm in Self-Test Problem 8 can be written in terms of the generated values of a sequence of independent uniform (0, 1) random variables, known as random numbers. With $[x]$ defined as the largest integer less than or equal to x, the first step can be written as follows:

Step 1. Generate a uniform (0, 1) random variable U. Let $X = [mU] + 1$, and determine the value of $n(X)$.

(a) Explain why the above is equivalent to step 1 of Problem 8.

HINT: What is the probability mass function of X?

(b) Write the remaining steps of the algorithm in a similar style.

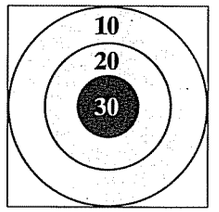
10. Let $X_1, X_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. For a fixed constant c, define the random variable N by

$$N = \min\{n : X_n > c\}$$

Is N independent of X_N? That is, does knowing the value of the first random variable that is greater than c affect the probability distribution of when this random variable occurs? Give an intuitive explanation for your answer.

11. The following dartboard is a square whose sides are of length 6. The three circles are all centered at the center of the board and are of radii 1, 2, and 3. Darts landing within the circle of radius 1 score 30 points, those landing outside this circle but within the circle of radius 2 are worth 20 points, and those landing outside the circle of radius 2 but within the circle of radius 3 are worth 10 points. Darts that do not land within the circle of radius 3 do not score any points. Assuming that each dart that you throw will, independent of what occurred on your previous throws, land on a point uniformly distributed in the square, find the probabilities of the following events.

(a) You score 20 on a throw of the dart.
(b) You score at least 20 on a throw of the dart.
(c) You score 0 on a throw of the dart.
(d) The expected value of your score on a throw of the dart.
(e) Both of your first two throws score at least 10.
(f) Your total score after two throws is 30.



12. A model proposed for NBA basketball supposes that when two teams with roughly the same record play each other, then the number of points scored in a quarter by the home team minus the number scored by the visiting team

Thus

$$\begin{aligned} E[X] &\approx \sum_{i=1}^{n-1} 2\log(n-i+1) \\ &\approx 2\int_1^{n-1} \log(n-x+1)\,dx \\ &= 2\int_2^n \log(y)\,dy \\ &= 2(y\log(y)-y)\Big|_2^n \\ &\approx 2n\log(n) \end{aligned}$$

Thus we see that when n is large, the quicksort algorithm requires, on average, approximately $2n\log(n)$ comparisons to sort n distinct values. ∎

Example 2p. ***The probability of a union of events.*** Let $A_1, \ldots A_n$ denote events and define the indicator variables X_i, $i = 1, \ldots, n$, by

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Now, note that

$$1 - \prod_{i=1}^{n}(1-X_i) = \begin{cases} 1 & \text{if } \cup A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$E\left[1 - \prod_{i=1}^{n}(1-X_i)\right] = P\left(\bigcup_{i=1}^{n} A_i\right)$$

Expanding the left side of the above yields that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = E\left[\sum_{i=1}^{n} X_i - \sum\sum_{i<j} X_iX_j + \sum\sum\sum_{i<j<k} X_iX_jX_k - \cdots + (-1)^{n+1}X_1\cdots X_n\right] \tag{2.3}$$

However, as

$$X_{i_1}X_{i_2}\cdots X_{i_k} = \begin{cases} 1 & \text{if } A_{i_1}A_{i_2}\cdots A_{i_k} \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

we see that

$$E[X_{i_1}\cdots X_{i_k}] = P(A_{i_1}\cdots A_{i_k})$$

and thus (2.3) is just a statement of the well-known formula for the union of events

$$P(\cup A_i) = \sum_i P(A_i) - \sum\sum_{i<j} P(A_iA_j) + \sum\sum\sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1}P(A_1\cdots A_n)$$

∎

Our next example gives another illustration of how the introduction of randomness can sometimes be employed advantageously.

Example 2q. A round-robin tournament of n contestants is one in which each of the $\binom{n}{2}$ pairs of contestants play each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. Suppose that the n players are initially numbered as player 1, player 2, and so on. The permutation $i_1, i_2, \ldots, i_n$ is said to be a *Hamiltonian permutation* if i_1 beats i_2, i_2 beats $i_3, \ldots,$ and i_{n-1} beats i_n. A problem of some interest is to determine the largest possible number of Hamiltonian permutations.

For instance, suppose that there are 3 players. Then it is easy to see that if one of the players wins twice, then there is a single Hamiltonian permutation (for instance, if 1 wins twice and 2 beats 3 then the only Hamiltonian is 1, 2, 3); and if each of the players wins once, then there will be three Hamiltonians (for instance, if 1 beats 2, 2 beats 3, and 3 beats 1, then 1,2,3, 2,3,1, and 3,1,2 are all Hamiltonians). Thus, when $n = 3$ the largest possible number of Hamiltonian permutations is 3.

Although the Hamiltonian permutation problem does not involve probability, we will introduce randomnesses to show that in a round-robin tournament of n players, $n > 2$, there is an outcome for which the number of Hamiltonian permutations is greater than $n!/2^{n-1}$.

To verify the above, let us suppose that the results of the $\binom{n}{2}$ games are independent and that either of the two contestants is equally likely to win each encounter. If we let X denote the number of Hamiltonians that result, then X is a random variable whose set of possible values is all the possible numbers of Hamiltonian permutations that can result from a round-robin tournament of n contestants. Since at least one of the possible values of a nonconstant random variable must exceed its mean, it follows that there must be at least one possible tournament result which has more than $E[X]$ Hamiltonian permutations. To determine $E[X]$, number the $n!$ permutations and let, for $i = 1, \ldots, n!$,

$$X_i = \begin{cases} 1 & \text{if permutation } i \text{ is a Hamiltonian} \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$X = \sum_i X_i$$

so

$$E[X] = \sum_i E[X_i]$$

But

$$\begin{aligned} E[X_i] &= P\{\text{permutation } i \text{ is a Hamiltonian}\} \\ &=^{1} \left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

The preceding equality being true because the probability that any permutation, say $i_1, i_2, \ldots, i_n$, is a Hamiltonian is, by independence, the probability that i_1 beats i_2, multiplied by the probability that i_2 beats i_3, and so on. Hence we obtain from the preceding that

$$E[X] = \frac{n!}{2^{n-1}}$$

Since, for $n > 2$, X is not a constant random variable, it thus follows that at least one of its possible values exceeds $n!/2^{n-1}$. ∎

When one is dealing with an infinite collection of random variables X_i, $i \geq 1$, each having a finite expectation, it is not necessarily true that

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i] \tag{2.4}$$

To determine when (2.4) is valid, we note that $\sum_{i=1}^{\infty} X_i = \lim_{n\to\infty} \sum_{i=1}^{n} X_i$ and thus

$$\begin{aligned} E\left[\sum_{i=1}^{\infty} X_i\right] &= E\left[\lim_{n\to\infty} \sum_{i=1}^{n} X_i\right] \\ &\overset{?}{=} \lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i\right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} E[X_i] \\ &= \sum_{i=1}^{\infty} E[X_i] \end{aligned} \tag{2.5}$$

Hence Equation (2.4) is valid whenever we are justified in interchanging the expectation and limit operations in Equation (2.5). Although, in general, this interchange is not justified, it can be shown to be valid in two important special cases:

1. The X_i are all nonnegative random variables (that is, $P\{X_i \geq 0\} = 1$ for all i).
2. $\sum_{i=1}^{\infty} E[|X_i|] < \infty$.

Example 2r. Consider any nonnegative, integer-valued random variable X. If for each $i \geq 1$, we define

$$X_i = \begin{cases} 1 & \text{if } X \geq i \\ 0 & \text{if } X < i \end{cases}$$

then

$$\begin{aligned} \sum_{i=1}^{\infty} X_i &= \sum_{i=1}^{X} X_i + \sum_{i=X+1}^{\infty} X_i \\ &= \sum_{i=1}^{X} 1 + \sum_{i=X+1}^{\infty} 0 \\ &= X \end{aligned}$$

Hence, since the X_i are all nonnegative,

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} E(X_i) \\ &= \sum_{i=1}^{\infty} P\{X \geq i\} \end{aligned} \tag{2.6}$$

a useful identity. ∎

Example 2s. Suppose that n elements—call them $1, 2, \ldots, n$—must be stored in a computer in the form of an ordered list. Each unit of time a request will be made for one of these elements—i being requested, independently of the past, with probability $P(i)$, $i \geq 1$, $\sum_i P(i) = 1$. Assuming these probabilities are known, what ordering minimizes the average position on the line of the element requested?

Solution Suppose that the elements are numbered so that $P(1) \geq P(2) \geq \cdots \geq P(n)$. To show that $1, 2, \ldots, n$ is the optimal ordering, let X denote the position of the requested element. Now under any ordering—say, $O = i_1, i_2, \ldots, i_n$,

$$\begin{aligned} P_O\{X \geq k\} &= \sum_{j=k}^{n} P(i_j) \\ &\geq \sum_{j=k}^{n} P(j) \\ &= P_{1,2,\ldots,n}\{X \geq k\} \end{aligned}$$

Summing over k and using Equation (2.6) yields

$$E_O[X] \geq E_{1,2,\ldots,n}[X]$$

thus showing that ordering the elements in decreasing order of their request probabilities minimizes the expected position of the element requested. ∎

7.3 COVARIANCE, VARIANCE OF SUMS, AND CORRELATIONS

We start with the following proposition, which shows that the expectation of a product of independent random variables is equal to the product of their expectations.

Proposition 3.1

If X and Y are independent, then for any functions h and g,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof: Suppose that X and Y are jointly continuous with joint density $f(x, y)$. Then

$$\begin{aligned} E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x, y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \\ &= E[h(Y)]E[g(X)] \end{aligned}$$

The proof in the discrete case is similar.

Just as the expected value and the variance of a single random variable give us information about this random variable, so does the covariance between two random variables give us information about the relationship between the random variables.

Definition

The covariance between X and Y, denoted by $\text{Cov}(X, Y)$, is defined by

$$\text{Cov}(X, Y) = E[(X - E[X])\,(Y - E[Y])]$$

Upon expanding the right side of the definition above, we see that

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY - E[X]Y - XE[Y] + E[Y]E[X]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Note that if X and Y are independent then, by Proposition 3.1, it follows that $\text{Cov}(X, Y) = 0$. However, the converse is not true. A simple example of two dependent random variables X and Y having zero covariance is obtained by letting X be a random variable such that

$$P\{X = 0\} = P\{X = 1\} = P\{X = -1\} = \tfrac{1}{3}$$

and define

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

now, $XY = 0$, so $E[XY] = 0$. Also, $E[X] = 0$ and thus

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0$$

However, X and Y are clearly not independent.

The following proposition lists some of the properties of covariance.

Proposition 3.2

(i) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

(ii) $\text{Cov}(X, X) = \text{Var}(X)$

(iii) $\text{Cov}(aX, Y) = a\,\text{Cov}(X, Y)$

(iv) $\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \text{Cov}(X_i, Y_j)$

Proof of Proposition 3.2: Parts (i) and (ii) follow immediately from the definition of covariance, and part (iii) is left as an exercise for the reader. To prove (iv), which states that the covariance operation is additive (as is the operation of taking expectations), let $\mu_i = E[X_i]$ and $\nu_j = E[Y_j]$. Then

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mu_i, \qquad E\left[\sum_{j=1}^{m} Y_j\right] = \sum_{j=1}^{m} \nu_j$$

and

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) &= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} \nu_j\right)\right] \\ &= E\left[\sum_{i=1}^{n} (X_i - \mu_i) \sum_{j=1}^{m} (Y_j - \nu_j)\right] \\ &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m} (X_i - \mu_i)(Y_j - \nu_j)\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} E[(X_i - \mu_i)(Y_j - \nu_j)] \end{aligned}$$

where the last equality follows because the expected value of a sum of random variables is equal to the sum of the expected values. ∎

It follows from parts (ii) and (iv) of Proposition 3.2, upon taking $Y_j = X_j$, $j = 1, \ldots, n$, that

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^{n} X_i\right) &= \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \text{Var}(X_i) + \sum\sum_{i \neq j} \text{Cov}(X_i, X_j) \end{aligned}$$

Since each pair of indices i, j, $i \neq j$, appears twice in the double summation, the above is equivalent to the following:

$$\boxed{\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j)} \tag{3.1}$$

If $X_1, \ldots, X_n$ are pairwise independent, in that X_i and X_j are independent for $i \neq j$, then Equation (3.1) reduces to

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$

The following examples illustrate the use of Equation (3.1).

Example 3a. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and as in Example 2c, let $\overline{X} = \sum_{i=1}^{n} X_i/n$ be the sample mean. The quantities $X_i - \overline{X}$, $i = 1, \ldots, n$, are called *deviations,* as they equal the differences between the individual data and the sample mean. The random variable

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

is called the *sample variance.* Find (a) $\text{Var}(\overline{X})$ and (b) $E[S^2]$.

Solution

(a)
$$\begin{aligned}\text{Var}(\overline{X}) &= \left(\frac{1}{n}\right)^2 \text{Var}\left(\sum_{i=1}^{n} X_i\right) \\ &= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i) \qquad \text{by independence} \\ &= \frac{\sigma^2}{n}\end{aligned}$$

(b) We start with the following algebraic identity

$$\begin{aligned}(n-1)S^2 &= \sum_{i=1}^{n} (X_i - \mu + \mu - \overline{X})^2 \\ &= \sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\overline{X} - \mu)^2 - 2(\overline{X} - \mu)\sum_{i=1}^{n} (X_i - \mu) \\ &= \sum_{i=1}^{n} (X_i - \mu)^2 + n(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)\, n\, (\overline{X} - \mu) \\ &= \sum_{i=1}^{n} (X_i - \mu)^2 - n(\overline{X} - \mu)^2\end{aligned}$$

Taking expectations of the above yields that

$$\begin{aligned}(n-1)E[S^2] &= \sum_{i=1}^{n} E[(X_i - \mu)^2] - nE[(\overline{X} - \mu)^2] \\ &= n\sigma^2 - n\,\text{Var}(\overline{X}) \\ &= (n-1)\sigma^2\end{aligned}$$

where the final equality made use of part (a), and the one preceding it made use of the result, of Example 2c, that $E[\overline{X}] = \mu$. Dividing through by $n-1$ shows that the expected value of the sample variance is the distribution variance σ^2. ∎

Our next example presents an alternative method (to the one used in Chapter 4) for obtaining the variance of a binomial random variable.

Example 3b. ***Variance of a binomial random variable.*** Compute the variance of a binomial random variable X with parameters n and p.

Solution Since such a random variable represents the number of successes in n independent trials when each trial has a common probability p of being a success, we may write

$$X = X_1 + \cdots + X_n$$

where the X_i are independent Bernoulli random variables such that

$$X_i = \begin{cases} 1 & \text{if the } i\text{th trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

Hence, from Equation (3.1) we obtain

$$\text{Var}(X) = \text{Var}(X_1) + \cdots + \text{Var}(X_n)$$

But

$$\begin{aligned}\text{Var}(X_i) &= E[X_i^2] - (E[X_i])^2 \\ &= E[X_i] - (E[X_i])^2 \qquad \text{since } X_i^2 = X_i \\ &= p - p^2\end{aligned}$$

and thus

$$\text{Var}(X) = np(1-p)$$ ∎

Example 3c. ***Variance of the number of matches.*** Compute the variance of X, the number of people that select their own hats in Example 2h.

Solution Using the same representation for X as we do in Example 2h, namely,

$$X = X_1 + \cdots + X_N$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th man selects his own hat} \\ 0 & \text{otherwise} \end{cases}$$

we obtain from Equation (3.1) that

$$\text{Var}(X) = \sum_{i=1}^{N} \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j) \tag{3.2}$$

Since $P\{X_i = 1\} = 1/N$, we see from the preceding example that

$$\text{Var}(X_i) = \frac{1}{N}\left(1 - \frac{1}{N}\right) = \frac{N-1}{N^2}$$

Also,

$$\text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j]$$

Now,

$$X_iX_j = \begin{cases} 1 & \text{if the } i\text{th and } j\text{th men both select their own hats} \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$\begin{aligned} E[X_iX_j] &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\} \\ &= \frac{1}{N}\,\frac{1}{N-1} \end{aligned}$$

Hence

$$\text{Cov}(X_i, X_j) = \frac{1}{N(N-1)} - \left(\frac{1}{N}\right)^2 = \frac{1}{N^2(N-1)}$$

and from Equation (3.2),

$$\begin{aligned} \text{Var}(X) &= \frac{N-1}{N} + 2\binom{N}{2}\frac{1}{N^2(N-1)} \\ &= \frac{N-1}{N} + \frac{1}{N} \\ &= 1 \end{aligned}$$

Thus both the mean and variance of the number of matches are equal to 1. In a way, this result is not unexpected because, as shown in Example 5m of Chapter 2, when N is large, the probability of i matches is approximately $e^{-1}/i!$. That is, when N is large, the number of matches is approximately distributed as a Poisson random variable with mean 1. Hence, as the mean and variance of a Poisson random variable are equal, the result obtained in this example is not surprising. ∎

Example 3d. ***Sampling from a finite population.*** Consider a set of N people each of whom has an opinion about a certain subject that is measured by a real number v, which represents the person's "strength of feeling" about the subject. Let v_i represent the strength of feeling of person i, $i = 1, \ldots, N$.

Suppose that these quantities v_i, $i = 1, \ldots, N$ are unknown and to gather information a group of n of the N people is "randomly chosen" in the sense that all of the $\binom{N}{n}$ subsets of size n are equally likely to be chosen. These n people are then questioned and their feelings determined. If S denotes the sum of the n sampled values, determine its mean and variance.

An important application of the above is to a forthcoming election in which each person in the population is either for or against a certain candidate or proposition. If we take v_i to equal 1 if person i is in favor and 0 if he or she is against, then $\bar{v} = \sum_{i=1}^{N} v_i/N$ represents the proportion of the population that is in favor. To estimate $\bar{v}$, a random sample of n people is chosen, and these people are polled. The proportion of those polled that are in favor—that is, S/n—is often used as an estimate of $\bar{v}$.

Solution For each person i, $i = 1, \ldots, N$, define an indicator variable I_i to indicate whether or not that person is included in the sample. That is,

$$I_i = \begin{cases} 1 & \text{if person } i \text{ is in the random sample} \\ 0 & \text{otherwise} \end{cases}$$

Now S can be expressed by

$$S = \sum_{i=1}^{N} v_iI_i$$

so

$$\begin{aligned} E[S] &= \sum_{i=1}^{N} v_iE[I_i] \\ \text{Var}(S) &= \sum_{i=1}^{N} \text{Var}(v_iI_i) + 2\sum\sum_{i<j} \text{Cov}(v_iI_i, v_jI_j) \\ &= \sum_{i=1}^{N} v_i^2 \text{Var}(I_i) + 2\sum\sum_{i<j} v_iv_j \text{Cov}(I_i, I_j) \end{aligned}$$

As

$$\begin{aligned} E[I_i] &= \frac{n}{N} \\ E[I_iI_j] &= \frac{n}{N}\,\frac{n-1}{N-1} \end{aligned}$$

we see that

$$\begin{aligned} \text{Var}(I_i) &= \frac{n}{N}\left(1 - \frac{n}{N}\right) \\ \text{Cov}(I_i, I_j) &= \frac{n(n-1)}{N(N-1)} - \left(\frac{n}{N}\right)^2 \\ &= \frac{-n(N-n)}{N^2(N-1)} \end{aligned}$$

Hence

$$E[S] = n\sum_{i=1}^{N}\frac{v_i}{N} = n\bar{v}$$

$$\text{Var}(S) = \frac{n}{N}\left(\frac{N-n}{N}\right)\sum_{i=1}^{N} v_i^2 - \frac{2n(N-n)}{N^2(N-1)}\sum\sum_{i<j} v_i v_j$$

The expression for $\text{Var}(S)$ can be simplified somewhat by using the identity $(v_1 + \cdots + v_N)^2 = \sum_{i=1}^{N} v_i^2 + 2\sum\sum_{i<j} v_i v_j$ to give, after some simplification,

$$\text{Var}(S) = \frac{n(N-n)}{N-1}\left(\frac{\sum_{i=1}^{N} v_i^2}{N} - \bar{v}^2\right)$$

Consider now the special case in which Np of the v's are equal to 1 and the remainder equal to 0. Then in this case S is a hypergeometric random variable and has mean and variance given by

$$E[S] = n\bar{v} = np \qquad \text{since } \bar{v} = \frac{Np}{N} = p$$

$$\text{Var}(S) = \frac{n(N-n)}{N-1}\left(\frac{Np}{N} - p^2\right)$$

$$= \frac{n(N-n)}{N-1}p(1-p)$$

The quantity S/n, equal to the proportion of those sampled that have values equal to 1, is such that

$$E\left[\frac{S}{n}\right] = p$$

$$\text{Var}\left(\frac{S}{n}\right) = \frac{N-n}{n(N-1)}p(1-p)$$ ∎

The correlation of two random variables X and Y, denoted by $\rho(X, Y)$, is defined, as long as $\text{Var}(X)\,\text{Var}(Y)$ is positive, by

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

It can be shown that

$$-1 \le \rho(X, Y) \le 1 \tag{3.3}$$

To prove Equation (3.3), suppose that X and Y have variances given by σ_x^2 and σ_y^2, respectively. Then

$$0 \le \text{Var}\left(\frac{X}{\sigma_x} + \frac{Y}{\sigma_y}\right)$$

$$= \frac{\text{Var}(X)}{\sigma_x^2} + \frac{\text{Var}(Y)}{\sigma_y^2} + \frac{2\,\text{Cov}(X, Y)}{\sigma_x\sigma_y}$$

$$= 2[1 + \rho(X, Y)]$$

implying that

$$-1 \le \rho(X, Y)$$

On the other hand,

$$0 \le \text{Var}\left(\frac{X}{\sigma_x} - \frac{Y}{\sigma_y}\right)$$

$$= \frac{\text{Var}(X)}{\sigma_x^2} + \frac{\text{Var}\,Y}{(-\sigma_y)^2} - \frac{2\,\text{Cov}(X, Y)}{\sigma_x\sigma_y}$$

$$= 2[1 - \rho(X, Y)]$$

implying that

$$\rho(X, Y) \le 1$$

which completes the proof of Equation (3.3).

In fact, since $\text{Var}(Z) = 0$ implies that Z is constant with probability 1 (this intuitive fact will be rigorously proved in Chapter 8), we see from the proof of (3.3) that $\rho(X, Y) = 1$ implies that $Y = a + bX$, where $b = \sigma_y/\sigma_x > 0$ and $\rho(X, Y) = -1$ implies that $Y = a + bX$, where $b = -\sigma_y/\sigma_x < 0$. We leave it as an exercise for the reader to show that the reverse is also true: that if $Y = a + bX$, then $\rho(X, Y)$ is either $+1$ or -1, depending on the sign of b.

The correlation coefficient is a measure of the degree of linearity between X and Y. A value of $\rho(X, Y)$ near $+1$ or -1 indicates a high degree of linearity between X and Y, whereas a value near 0 indicates a lack of such linearity. A positive value of $\rho(X, Y)$ indicates that Y tends to increase when X does, whereas a negative value indicates that Y tends to decrease when X increases. If $\rho(X, Y) = 0$, then X and Y are said to be *uncorrelated*.

Example 3e. Let I_A and I_B be indicator variables for the events A and B. That is,

$$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$I_B = \begin{cases} 1 & \text{if } B \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[I_A] = P(A)$$
$$E[I_B] = P(B)$$
$$E[I_A I_B] = P(AB)$$

so

$$\text{Cov}(I_A, I_B) = P(AB) - P(A)P(B)$$
$$= P(B)[P(A|B) - P(A)]$$

Thus we obtain the quite intuitive result that the indicator variables for A and B are either positively correlated, uncorrelated, or negatively correlated depending on whether $P(A|B)$ is greater than, equal to, or less than $P(A)$. ∎

Our next example shows that the sample mean and a deviation from the sample mean are uncorrelated.

Example 3f. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having variance σ^2. Show that

$$\text{Cov}(X_i - \bar{X}, \bar{X}) = 0$$

Solution

$$\begin{aligned}\text{Cov}(X_i - \bar{X}, \bar{X}) &= \text{Cov}(X_i, \bar{X}) - \text{Cov}(\bar{X}, \bar{X}) \\ &= \text{Cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) - \text{Var}(\bar{X}) \\ &= \frac{1}{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j) - \frac{\sigma^2}{n} \\ &= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0\end{aligned}$$

where the next-to-last equality uses the result of Example 3a, and the final equality follows since

$$\text{Cov}(X_i, X_j) = \begin{cases} 0 & \text{if } j \neq i \quad \text{by independence} \\ \sigma^2 & \text{if } j = i \quad \text{since } \text{Var}(X_i) = \sigma^2 \end{cases}$$

Although $\bar{X}$ and the deviation $X_i - \bar{X}$ are uncorrelated, they are not, in general, independent. However, in the special case where the X_i are normal random variables it turns out that not only is $\bar{X}$ independent of a single deviation but it is independent of the entire sequence of deviations $X_j - \bar{X}$, $j = 1, \ldots, n$. This result will be established in Section 9, where we will also show that, in this case, the sample mean $\bar{X}$ and the sample variance $S^2/(n-1)$ are independent, with S^2/σ^2 having a chi-squared distribution with $n-1$ degrees of freedom. (See Example 3a for the definition of S^2.) ∎

Example 3g. Consider m independent trials, each of which results in any of r possible outcomes with probabilities $P_1, P_2, \ldots, P_r$, $\sum_1^r P_i = 1$. If we let N_i, $i = 1, \ldots, r$, denote the number of the m trials that result in outcome i, then $N_1, N_2, \ldots, N_r$ have the multinomial distribution

$$\begin{aligned}&P\{N_1 = n_1, N_2 = n_2, \ldots, N_r = n_r\} \\ &\quad = \frac{m!}{n_1!\, n_2! \ldots n_r!} P_1^{n_1} P_2^{n_2} \cdots P_r^{n_r} \qquad \sum_{i=1}^{r} n_i = m\end{aligned}$$

For $i \neq j$ it seems likely that when N_i is large N_j would tend to be small, and hence it is intuitive that they should be negatively correlated. Let us compute their covariance by using Proposition 3.2(iv) and the representation

$$N_i = \sum_{k=1}^{m} I_i(k) \qquad \text{and} \qquad N_j = \sum_{k=1}^{m} I_j(k)$$

where

$$I_i(k) = \begin{cases} 1 & \text{if trial } k \text{ results in outcome } i \\ 0 & \text{otherwise} \end{cases}$$

$$I_j(k) = \begin{cases} 1 & \text{if trial } k \text{ results in outcome } j \\ 0 & \text{otherwise} \end{cases}$$

From Proposition 3.2(iv) we have

$$\text{Cov}(N_i, N_j) = \sum_{\ell=1}^{m}\sum_{k=1}^{m} \text{Cov}(I_i(k), I_j(\ell))$$

Now, when $k \neq \ell$,

$$\text{Cov}(I_i(k), I_j(\ell)) = 0$$

since the outcome of trial k is independent of the outcome of trial ℓ. On the other hand,

$$\begin{aligned}\text{Cov}(I_i(\ell), I_j(\ell)) &= E[I_i(\ell)I_j(\ell)] - E[I_i(\ell)]E[I_j(\ell)] \\ &= 0 - P_iP_j = -P_iP_j\end{aligned}$$

where the above uses that $I_i(\ell)I_j(\ell) = 0$ since trial ℓ cannot result in both outcome i and outcome j. Hence we obtain that

$$\text{Cov}(N_i, N_j) = -mP_iP_j$$

which is in accord with our intuition that N_i and N_j are negatively correlated.

7.4 CONDITIONAL EXPECTATION

7.4.1 Definitions

Recall that if X and Y are jointly discrete random variables, the conditional probability mass function of X, given that $Y = y$, is defined for all y such that $P\{Y = y\} > 0$, by

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{p(x, y)}{p_Y(y)}$$

It is therefore natural to define, in this case, the conditional expectation of X, given that $Y = y$, for all values of y such that $p_Y(y) > 0$ by

$$\begin{aligned}E[X|Y = y] &= \sum_x xP\{X = x|Y = y\} \\ &= \sum_x xp_{X|Y}(x|y)\end{aligned}$$

Example 4a. If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X, given that $X + Y = m$.

Solution Let us first calculate the conditional probability mass function of X, given that $X + Y = m$. For $k \leq \min(n, m)$,

$$\begin{aligned} P\{X = k|X + Y = m\} &= \frac{P\{X = k, X + Y = m\}}{P\{X + Y = m\}} \\ &= \frac{P\{X = k, Y = m - k\}}{P\{X + Y = m\}} \\ &= \frac{P\{X = k\}\,P\{Y = m - k\}}{P\{X + Y = m\}} \\ &= \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}} \\ &= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} \end{aligned}$$

where we have used the fact (see Example 3e of Chapter 6) that $X + Y$ is a binomial random variable with parameters $2n$ and p. Hence the conditional distribution of X, given that $X + Y = m$, is the hypergeometric distribution; thus, from Example 2g, we obtain

$$E[X|X + Y = m] = \frac{m}{2}$$ ∎

Similarly, let us recall that if X and Y are jointly continuous, with a joint probability density function $f(x, y)$, the conditional probability density of X, given that $Y = y$, is defined for all values of y such that $f_Y(y) > 0$ by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

It is natural, in this case, to define the conditional expectation of X, given that $Y = y$, by

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

provided that $f_Y(y) > 0$.

Example 4b. Suppose that the joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty, 0 < y < \infty$$

Compute $E[X|Y = y]$.

Solution We start by computing the conditional density

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)\,dx} \\ &= \frac{(1/y)e^{-x/y}e^{-y}}{\int_0^{\infty}(1/y)e^{-x/y}e^{-y}\,dx} \\ &= \frac{(1/y)e^{-x/y}}{\int_0^{\infty}(1/y)e^{-x/y}\,dx} \\ &= \frac{1}{y}e^{-x/y} \end{aligned}$$

Hence the conditional distribution of X, given that $Y = y$, is just the exponential distribution with mean y. Thus

$$E[X|Y = y] = \int_0^{\infty} \frac{x}{y}e^{-x/y}\,dx = y$$ ∎

REMARK. Just as conditional probabilities satisfy all of the properties of ordinary probabilities, so do conditional expectations satisfy the properties of ordinary expectations. For instance, such formulas as

$$E[g(X)|Y = y] = \begin{cases} \sum_x g(x)p_{X|Y}(x|y) & \text{in the discrete case} \\ \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)\,dx & \text{in the continuous case} \end{cases}$$

and

$$E\left[\sum_{i=1}^{n} X_i|Y = y\right] = \sum_{i=1}^{n} E[X_i|Y = y]$$

remain valid. As a matter of fact, conditional expectation given $Y = y$ can be thought of as being an ordinary expectation on a reduced sample space consisting only of outcomes for which $Y = y$.

7.4.2 Computing Expectations by Conditioning

Let us denote by $E[X|Y]$ that function of the random variable Y whose value at $Y = y$ is $E[X|Y = y]$. Note that $E[X|Y]$ is itself a random variable. An extremely important property of conditional expectation is given by the following proposition.

Proposition 4.1

$$E[X] = E[E[X|Y]] \tag{4.1}$$

If Y is a discrete random variable, then Equation (4.1) states that

$$E[X] = \sum_y E[X|Y = y]P\{Y = y\} \tag{4.1a}$$

whereas if Y is continuous with density $f_Y(y)$, then Equation (4.1) states

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)\,dy \tag{4.1b}$$

We now give a proof of Equation (4.1) in the case where X and Y are both discrete random variables.

Proof of Equation (4.1) When X and Y Are Discrete: We must show that

$$E[X] = \sum_y E[X|Y = y]P\{Y = y\} \tag{4.2}$$

Now, the right-hand side of Equation (4.2) can be written as

$$\begin{aligned}
\sum_y E[X|Y = y]P\{Y = y\} &= \sum_y \sum_x xP\{X = x|Y = y\}P\{Y = y\} \\
&= \sum_y \sum_x x\frac{P\{X = x, Y = y\}}{P\{Y = y\}}P\{Y = y\} \\
&= \sum_y \sum_x xP\{X = x, Y = y\} \\
&= \sum_x x\sum_y P\{X = x, Y = y\} \\
&= \sum_x xP\{X = x\} \\
&= E[X]
\end{aligned}$$

and the result is proved.

One way to understand Equation (4.2) is to interpret it as follows: To calculate $E[X]$, we may take a weighted average of the conditional expected value of X, given that $Y = y$, each of the terms $E[X|Y = y]$ being weighted by the probability of the event on which it is conditioned. (Of what does this remind you?) This is an extremely useful result that often enables us to easily compute expectations by first conditioning on some appropriate random variable. The following examples illustrate its use.

Example 4c. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

Solution Let X denote the amount of time (in hours) until the miner reaches safety, and let Y denote the door he initially chooses. Now

$$\begin{aligned}
E[X] &= E[X|Y = 1]P\{Y = 1\} + E[X|Y = 2]P\{Y = 2\} \\
&\quad + E[X|Y = 3]P\{Y = 3\} \\
&= \tfrac{1}{3}(E[X|Y = 1] + E[X|Y = 2] + E[X|Y = 3])
\end{aligned}$$

However,

$$\begin{aligned}
E[X|Y = 1] &= 3 \\
E[X|Y = 2] &= 5 + E[X] \\
E[X|Y = 3] &= 7 + E[X]
\end{aligned} \tag{4.3}$$

To understand why Equation (4.3) is correct, consider, for instance, $E[X|Y = 2]$ and reason as follows: If the miner chooses the second door, he spends 5 hours in the tunnel and then returns to his cell. But once he returns to his cell the problem is as before; thus his expected additional time until safety is just $E[X]$. Hence $E[X|Y = 2] = 5 + E[X]$. The argument behind the other equalities in Equation (4.3) is similar. Hence

$$E[X] = \tfrac{1}{3}(3 + 5 + E[X] + 7 + E[X])$$

or

$$E[X] = 15$$

∎

Example 4d. ***Expectation of a random number of random variables.*** Suppose that the number of people entering a department store on a given day is a random variable with mean 50. Suppose further that the amounts of money spent by these customers are independent random variables having a common mean of \$8. Assume also that the amount of money spent by a customer is also independent of the total number of customers to enter the store. What is the expected amount of money spent in the store on a given day?

Solution If we let N denote the number of customers that enter the store and X_i the amount spent by the ith such customer, then the total amount of money spent can be expressed as $\sum_{i=1}^{N} X_i$. Now,

$$E\left[\sum_1^N X_i\right] = E\left[E\left[\sum_1^N X_i|N\right]\right]$$

But

$$\begin{aligned}
E\left[\sum_1^N X_i|N = n\right] &= E\left[\sum_1^n X_i|N = n\right] \\
&= E\left[\sum_1^n X_i\right] \qquad \text{by the independence of the } X_i \text{ and } N \\
&= nE[X] \qquad \text{where } E[X] = E[X_i]
\end{aligned}$$

which implies that

$$E\left[\sum_1^N X_i | N\right] = NE[X]$$

and thus

$$E\left[\sum_{i=1}^N X_i\right] = E[NE[X]] = E[N]E[X]$$

Hence, in our example, the expected amount of money spent in the store is 50×8, or \$400. ∎

Example 4e. Consider n points that are independently and uniformly distributed on the interval (0, 1). Say that any one of these points is "isolated" if there are no other points within a distance d of it, where d is a specified constant such that $0 < d < \frac{1}{2}$. Compute the expected number of the n points that are isolated from the others.

Solution Let the points be $U_1, \ldots, U_n$, and define I_j as the indicator variable for the event that U_j is an isolated point, $j = 1, \ldots, n$. That is, I_j is 1 if U_j is an isolated point, and is 0 otherwise. Then $\sum_{j=1}^n I_j$ represents the number of isolated points. Now

$$E\left[\sum_{j=1}^n I_j\right] = \sum_{j=1}^n E[I_j]$$

To compute $E[I_j]$ we condition on U_j:

$$E[I_j] = \int_0^1 E[I_j | U_j = x]\, dx$$

Now, if $x \leq d$, then a point at location x will be isolated if none of the other $n - 1$ points occur within the interval from 0 to $x + d$; if $d < x \leq 1 - d$, then it will be isolated if none of the other points occur within the interval from $x - d$ to $x + d$; if $x > 1 - d$, then it will be isolated if none of the other points occur within the interval from $x - d$ to 1 (see Figure 7.2). Hence we see that

$$\begin{aligned}
E[I_j] &= \int_0^d E[I_j | U_j = x]\, dx + \int_d^{1-d} E[I_j | U_j = x]\, dx + \int_{1-d}^1 E[I_j | U_j = x]\, dx \\
&= \int_0^d (1 - d - x)^{n-1}\, dx + \int_d^{1-d} (1 - 2d)^{n-1}\, dx \\
&\quad + \int_{1-d}^1 (1 - x + d)^{n-1}\, dx \\
&= \int_{1-2d}^{1-d} y^{n-1}\, dy + (1 - 2d)(1 - 2d)^{n-1} + \int_d^{2d} y^{n-1}\, dy \\
&= \frac{(1-d)^n}{n} - \frac{(1-2d)^n}{n} + (1 - 2d)^n + \frac{(2d)^n}{n} - \frac{d^n}{n}
\end{aligned}$$

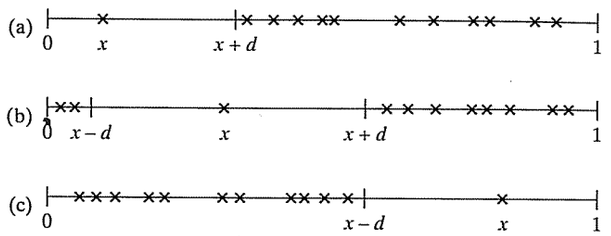


Figure 7.2 x is isolated point. (a) $x \leq d$; (b) $d < x \leq 1 - d$; (c) $1 - d < x$.

Therefore,

$$E\left[\sum_{j=1}^n I_j\right] = (1 - d)^n + (n - 1)(1 - 2d)^n + (2^n - 1)d^n$$

For instance, if $d = c/n$, then for n large it follows from the preceding that

$$E\left[\sum_{j=1}^n I_j\right] \approx e^{-c} + (n - 1)e^{-2c}$$ ∎

Example 4f. An urn contains a white and b black balls. One ball at a time is randomly withdrawn until the first white ball is drawn. Find the expected number of black balls that are withdrawn.

Solution This problem was previously treated in Example 2m. Here we present a solution using conditioning. Let X denote the number of black balls withdrawn and, to make explicit the dependence on a and b, let $M_{a,b} = E[X]$. We obtain an expression for $M_{a,b}$ by conditioning on the initial ball that is withdrawn. That is, define

$$Y = \begin{cases} 1 & \text{if the first ball selected is white} \\ 0 & \text{if the first ball selected is black} \end{cases}$$

Conditioning on Y yields

$$M_{a,b} = E[X] = E[X|Y = 1]P\{Y = 1\} + E[X|Y = 0]P\{Y = 0\}$$

However,

$$E[X|Y = 1] = 0 \tag{4.4}$$
$$E[X|Y = 0] = 1 + M_{a,b-1} \tag{4.5}$$

To understand Equations (4.4) and (4.5), suppose, for instance, that the first ball withdrawn is black. Then, after the first withdrawal, the situation is exactly the same as if we had started with a white balls and $b - 1$ black balls, which establishes Equation (4.5).

Since $P\{Y = 0\} = b/(a + b)$, we see that

$$M_{a,b} = \frac{b}{a + b}[1 + M_{a,b-1}]$$

Now, $M_{a,0}$ is clearly equal to 0, and we obtain

$$M_{a,1} = \frac{1}{a+1}[1 + M_{a,0}] = \frac{1}{a+1}$$

$$M_{a,2} = \frac{2}{a+2}[1 + M_{a,1}] = \frac{2}{a+2}\left[1 + \frac{1}{a+1}\right] = \frac{2}{a+1}$$

$$M_{a,3} = \frac{3}{a+3}[1 + M_{a,2}] = \frac{3}{a+3}\left[1 + \frac{2}{a+1}\right] = \frac{3}{a+1}$$

By using induction, one can easily verify that

$$M_{a,b} = \frac{b}{a+1}$$

∎

It is also possible to obtain the variance of a random variable by conditioning. We illustrate this by the following example.

Example 4g. ***Variance of the geometric distribution.*** Independent trials each resulting in a success with probability p are successively performed. Let N be the time of the first success. Find $\mathrm{Var}(N)$.

Solution Let $Y = 1$ if the first trial results in a success and $Y = 0$ otherwise. Now,

$$\mathrm{Var}(N) = E[N^2] - (E[N])^2$$

To calculate $E[N^2]$, we condition on Y as follows:

$$E[N^2] = E[E[N^2|Y]]$$

However,

$$E[N^2|Y = 1] = 1$$
$$E[N^2|Y = 0] = E[(1 + N)^2]$$

These two equations follow because, if the first trial results in a success, then clearly $N = 1$; thus $N^2 = 1$. On the other hand, if the first trial results in a failure, then the total number of trials necessary for the first success will have the same distribution as one (the first trial that results in failure) plus the necessary number of additional trials. Since the latter quantity has the same distribution as N, we obtain that $E[N^2|Y = 0] = E[(1 + N)^2]$. Hence we see that

$$\begin{aligned} E[N^2] &= E[N^2|Y = 1]P\{Y = 1\} + E[N^2|Y = 0]P\{Y = 0\} \\ &= p + (1 - p)E[(1 + N)^2] \\ &= 1 + (1 - p)E[2N + N^2] \end{aligned}$$

However, as was shown in Example 9b of Chapter 4, $E[N] = 1/p$; therefore,

$$E[N^2] = 1 + \frac{2(1-p)}{p} + (1 - p)E[N^2]$$

or

$$E[N^2] = \frac{2 - p}{p^2}$$

Therefore,

$$\begin{aligned} \mathrm{Var}(N) &= E[N^2] - (E[N])^2 \\ &= \frac{2 - p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1 - p}{p^2} \end{aligned}$$

∎

In our next example we determine the expected number of uniform (0, 1) random variables that need to be added for their sum to exceed 1. The answer, surprisingly, is e.

Example 4h. Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. Find $E[N]$ when

$$N = \min\left\{n: \sum_{i=1}^{n} U_i > 1\right\}$$

Solution We will solve the above by obtaining a more general result. For $x \in [0, 1]$, let

$$N(x) = \min\left\{n: \sum_{i=1}^{n} U_i > x\right\}$$

and set

$$m(x) = E[N(x)]$$

That is, $N(x)$ is the number of uniform (0, 1) random variables we need add until their sum exceeds x, and $m(x)$ is its expected value. We will now derive an equation for $m(x)$ by conditioning on U_1. This gives, from Equation (4.1b),

$$m(x) = \int_0^1 E[N(x)|U_1 = y]\,dy \tag{4.6}$$

Now,

$$E[N(x)|U_1 = y] = \begin{cases} 1 & \text{if } y > x \\ 1 + m(x - y) & \text{if } y \le x \end{cases} \tag{4.7}$$

The preceding formula is obvious when $y > x$. It is true when $y \le x$ since if the first uniform value is y, then at that point the remaining number of uniforms needed is the same as if we were just starting and were going to add uniforms until their sum exceeded $x - y$. Substituting (4.7) into (4.6) gives that

$$\begin{aligned} m(x) &= 1 + \int_0^x m(x - y)\,dy \\ &= 1 + \int_0^x m(u)\,du \qquad \text{by letting } u = x - y \end{aligned}$$

Differentiating the preceding equation yields that

$$m'(x) = m(x)$$

or, equivalently,

$$\frac{m'(x)}{m(x)} = 1$$

Integrating this gives

$$\log[m(x)] = x + c$$

or

$$m(x) = ke^x$$

Since $m(0) = 1$ we see that $k = 1$, so we obtain that

$$m(x) = e^x$$

Therefore, $m(1)$, the expected number of uniform (0, 1) random variables that need to be added until their sum exceeds 1 is equal to e. ∎

7.4.3 Computing Probabilities by Conditioning

Not only can we obtain expectations by first conditioning on an appropriate random variable, but we may also use this approach to compute probabilities. To see this, let E denote an arbitrary event and define the indicator random variable X by

$$X = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E \text{ does not occur} \end{cases}$$

It follows from the definition of X that

$$E[X] = P(E)$$
$$E[X|Y = y] = P(E|Y = y) \qquad \text{for any random variable } Y$$

Therefore, from Equations (4.1a) and (4.1b) we obtain

$$\begin{aligned} P(E) &= \sum_y P(E|Y = y)P(Y = y) && \text{if } Y \text{ is discrete} \\ &= \int_{-\infty}^{\infty} P(E|Y = y)f_Y(y)\,dy && \text{if } Y \text{ is continuous} \end{aligned} \tag{4.8}$$

Note that if Y is a discrete random variable taking on one of the values $y_1, \ldots, y_n$, then, by defining the events F_i, $i = 1, \ldots, n$ by $F_i = \{Y = y_i\}$, Equation (4.8) reduces to the familiar equation

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

where $F_1, \ldots, F_n$ are mutually exclusive events whose union is the sample space.

Example 4i. ***The best prize problem.*** Suppose that we are to be presented with n distinct prizes in sequence. After being presented with a prize we must immediately decide whether to accept it or to reject it and consider the next prize. The only information we are given when deciding whether to accept a prize is the relative rank of that prize compared to ones already seen. That is, for instance, when the fifth prize is presented, we learn how it compares with the four prizes already seen. Suppose that once a prize is rejected it is lost, and that our objective is to maximize the probability of obtaining the best prize. Assuming that all $n!$ orderings of the prizes are equally likely, how well can we do?

Solution Rather surprisingly, we can do quite well. To see this, fix a value k, $0 \leq k < n$, and consider the strategy that rejects the first k prizes and then accepts the first one that is better than all of those first k. Let $P_k(\text{best})$ denote the probability that the best prize is selected when this strategy is employed. To compute this probability, condition on X, the position of the best prize. This gives

$$\begin{aligned} P_k(\text{best}) &= \sum_{i=1}^{n} P_k(\text{best}|X = i)P(X = i) \\ &= \frac{1}{n}\sum_{i=1}^{n} P_k(\text{best}|X = i) \end{aligned}$$

Now, if the overall best prize is among the first k, then no prize is ever selected under the strategy considered. That is,

$$P_k(\text{best}|X = i) = 0 \qquad \text{if } i \leq k$$

On the other hand, if the best prize is in position i, where $i > k$, then the best prize will be selected if the best of the first $i - 1$ prizes is among the first k (for then none of the prizes in positions $k + 1, k + 2, \ldots, i - 1$ would be selected). But conditional on the best prize being in position i, it is easy to verify that all possible orderings of the other prizes remain equally likely, which implies that each of the first $i - 1$ prizes is equally likely to be the best of that batch. Hence we see that

$$\begin{aligned} P_k(\text{best}|X = i) &= P\{\text{best of first } i - 1 \text{ is among the first } k|X = i\} \\ &= \frac{k}{i - 1} \qquad \text{if } i > k \end{aligned}$$

From the preceding, we obtain that

$$\begin{aligned} P_k(\text{best}) &= \frac{k}{n}\sum_{i=k+1}^{n} \frac{1}{i - 1} \\ &\approx \frac{k}{n}\int_{k+1}^{n} \frac{1}{x - 1}\,dx \\ &= \frac{k}{n}\log\left(\frac{n - 1}{k}\right) \\ &\approx \frac{k}{n}\log\left(\frac{n}{k}\right) \end{aligned}$$

Now, if we consider the function

$$g(x) = \frac{x}{n}\log\left(\frac{n}{x}\right)$$

then

$$g'(x) = \frac{1}{n}\log\left(\frac{n}{x}\right) - \frac{1}{n}$$

so

$$g'(x) = 0 \Rightarrow \log\left(\frac{n}{x}\right) = 1 \Rightarrow x = \frac{n}{e}$$

Thus, since $P_k(\text{best}) \approx g(k)$, we see that the best strategy of the type considered is to let the first n/e prizes go by and then accept the first one to appear that is better than all of those. In addition, since $g(n/e) = 1/e$, the probability that this strategy selects the best prize is approximately $1/e \approx .36788$.

REMARK. Most people are quite surprised by the size of the probability of obtaining the best prize, thinking that this probability would be close to 0 when n is large. However, even without going through the calculations, a little thought reveals that the probability of obtaining the best prize can be made reasonably large. For consider the strategy of letting half of the prizes go by and then selecting the first one to appear that is better than all of those. The probability that a prize is actually selected is the probability that the overall best is among the second half, and this is $\frac{1}{2}$. In addition, given that a prize is selected, at the time of selection that prize would have been the best of more than $n/2$ prizes to have appeared and would thus have probability of at least $\frac{1}{2}$ of being the overall best. Hence, the strategy of letting the first half of all prizes go by and then accepting the first one that is better than all of those prizes has a probability greater than $\frac{1}{4}$ of obtaining the best prize. ∎

Example 4j. Let U be a uniform random variable on (0, 1), and suppose that the conditional distribution of X, given that $U = p$, is binomial with parameters n and p. Find the probability mass function of X.

Solution Conditioning on the value of U gives

$$\begin{aligned} P\{X = i\} &= \int_0^1 P\{X = i | U = p\} f_U(p)\, dp \\ &= \int_0^1 P\{X = i | U = p\}\, dp \\ &= \frac{n!}{i!\,(n-i)!}\int_0^1 p^i(1-p)^{n-i}\, dp \end{aligned}$$

Now it can be shown (a probabilistic proof is given in Section 6.6) that

$$\int_0^1 p^i(1-p)^{n-i}\, dp = \frac{i!\,(n-i)!}{(n+1)!}$$

Hence we obtain that

$$P\{X = i\} = \frac{1}{n+1} \qquad i = 0, \ldots, n$$

That is, we obtain the surprising result that if a coin whose probability of coming up heads is uniformly distributed over (0, 1) is flipped n times, then the number of heads occurring is equally likely to be any the values $0, \ldots, n$.

Because the preceding conditional distribution has such a nice form, it is worth trying to find another argument to enhance our intuition as to why such a result is true. To do so, let $U, U_1, \ldots, U_n$ be $n + 1$ independent uniform (0, 1) random variables, and let X denote the number of the random variables $U_1, \ldots, U_n$ that are smaller than U. Since all the random variables $U, U_1, \ldots, U_n$ have the same distribution, it follows that U is equally likely to be the smallest, or second smallest, or largest of them; so X is equally likely to be any of the values $0, 1, \ldots, n$. However, given that $U = p$, the number of the U_i that are less than U is a binomial random variable with parameters n and p, thus establishing our previous result. ∎

Example 4k. Suppose that X and Y are independent continuous random variables having densities f_X and f_Y, respectively. Compute $P\{X < Y\}$.

Solution Conditioning on the value of Y yields

$$\begin{aligned} P\{X < Y\} &= \int_{-\infty}^{\infty} P\{X < Y | Y = y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} P\{X < y | Y = y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} P\{X < y\} f_Y(y)\, dy \qquad \text{by independence} \\ &= \int_{-\infty}^{\infty} F_X(y) f_Y(y)\, dy \end{aligned}$$

where

$$F_X(y) = \int_{-\infty}^{y} f_X(x)\, dx$$

∎

Example 4l. Suppose that X and Y are independent continuous random variables. Find the distribution of $X + Y$.

Solution By conditioning on the value of Y, we obtain

$$\begin{aligned} P\{X + Y < a\} &= \int_{-\infty}^{\infty} P\{X + Y < a | Y = y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} P\{X + y < a | Y = y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} P\{X < a - y\} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy \end{aligned}$$

∎

7.4.4 Conditional Variance

Just as we have defined the conditional expectation of X given the value of Y, we can also define the conditional variance of X given that $Y = y$, which is defined as follows:

$$\text{Var}(X|Y) \equiv E[(X - E[X|Y])^2|Y]$$

That is, $\text{Var}(X|Y)$ is equal to the (conditional) expected square of the difference between X and its (conditional) mean when the value of Y is given. Or, in other words, $\text{Var}(X|Y)$ is exactly analogous to the usual definition of variance, but now all expectations are conditional on the fact that Y is known.

There is a very useful relationship between $\text{Var}(X)$, the unconditional variance of X, and $\text{Var}(X|Y)$, the conditional variance of X given Y, that can often be applied to compute $\text{Var}(X)$. To obtain this relationship, note first that by the same reasoning that yields $\text{Var}(X) = E[X^2] - (E[X])^2$ we have that

$$\text{Var}(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

so

$$\begin{aligned} E[\text{Var}(X|Y)] &= E[E[X^2|Y]] - E[(E[X|Y])^2] \\ &= E[X^2] - E[(E[X|Y])^2] \end{aligned} \tag{4.9}$$

Also, as $E[E[X|Y]] = E[X]$, we have that

$$\text{Var}(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2 \tag{4.10}$$

Hence, by adding Equations (4.9) and (4.10), we arrive at the following proposition.

Proposition 4.2 The conditional variance formula

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

Example 4m. Suppose that by any time t the number of people that have arrived at a train depot is a Poisson random variable with mean λt. If the initial train arrives at the depot at a time (independent of when the passengers arrive) that is uniformly distributed over $(0, T)$, what is the mean and variance of the number of passengers that enter the train?

Solution Let, for each $t \geq 0$, $N(t)$ denote the number of arrivals by t, and let Y denote the time at which the train arrives. The random variable of interest is then $N(Y)$. Conditioning on Y gives:

$$\begin{aligned} E[N(Y)|Y = t] &= E[N(t)|Y = t] \\ &= E[N(t)] \qquad \text{by the independence of } Y \text{ and } N(t) \\ &= \lambda t \qquad \text{since } N(t) \text{ is Poisson with mean } \lambda t \end{aligned}$$

Hence

$$E[N(Y)|Y] = \lambda Y$$

so taking expectations gives

$$E[N(Y)] = \lambda E[Y] = \frac{\lambda T}{2}$$

To obtain $\text{Var}(N(Y))$, we use the conditional variance formula:

$$\begin{aligned} \text{Var}(N(Y)|Y = t) &= \text{Var}(N(t)|Y = t) \\ &= \text{Var}(N(t)) \qquad \text{by independence} \\ &= \lambda t \end{aligned}$$

so

$$\begin{aligned} \text{Var}(N(Y)|Y) &= \lambda Y \\ E[N(Y)|Y] &= \lambda Y \end{aligned}$$

Hence, from the conditional variance formula,

$$\begin{aligned} \text{Var}(N(Y)) &= E[\lambda Y] + \text{Var}(\lambda Y) \\ &= \lambda \frac{T}{2} + \lambda^2 \frac{T^2}{12} \end{aligned}$$

where the above uses that $\text{Var}(Y) = T^2/12$. ∎

Example 4n. ***Variance of a random number of random variables.*** Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables and let N be a nonnegative integer-valued random variable that is independent of the sequence X_i, $i \geq 1$. To compute $\text{Var}\left(\sum_{i=1}^{N} X_i\right)$, we condition on N:

$$E\left[\sum_{i=1}^{N} X_i|N\right] = NE[X]$$

$$\text{Var}\left(\sum_{i=1}^{N} X_i|N\right) = N\,\text{Var}(X)$$

The result above follows, since given N, $\sum_{i=1}^{N} X_i$ is just the sum of a fixed number of independent random variables, so its expectation and variance is just the sum of the individual means and variances. Hence, from the conditional variance formula,

$$\text{Var}\left(\sum_{i=1}^{N} X_i\right) = E[N]\,\text{Var}(X) + (E[X])^2\,\text{Var}(N)$$

7.5 CONDITIONAL EXPECTATION AND PREDICTION

Sometimes a situation arises where the value of a random variable X is observed and then, based on the observed value, an attempt is made to predict the value of a second random variable Y. Let $g(X)$ denote the predictor, that is, if X is observed to equal x, then $g(x)$ is our prediction for the value of Y. Clearly, we would like to choose g so that $g(X)$ tends to be close to Y. One possible criterion for closeness is to choose g so as to minimize $E[(Y - g(X))^2]$. We now show that under this criterion, the best possible predictor of Y is $g(X) = E[Y|X]$.

Proposition 5.1

$$E[(Y - g(X))^2] \geq E[(Y - E[Y|X])^2]$$

Proof

$$\begin{aligned} E[(Y - g(X))^2|X] &= E[(Y - E[Y|X] + E[Y|X] - g(X))^2|X] \\ &= E[(Y - E[Y|X])^2|X] \\ &\quad + E[(E[Y|X] - g(X))^2|X] \\ &\quad + 2E[(Y - E[Y|X])(E[Y|X] - g(X))|X] \end{aligned} \tag{5.1}$$

However, given X, $E[Y|X] - g(X)$, being a function of X, can be treated as a constant. Thus

$$\begin{aligned} &E[(Y - E[Y|X])(E[Y|X] - g(X))|X] \\ &= (E[Y|X] - g(X))E[Y - E[Y|X]|X] \\ &= (E[Y|X] - g(X))(E[Y|X] - E[Y|X]) \\ &= 0 \end{aligned} \tag{5.2}$$

Thus, from Equations (5.1) and (5.2), we obtain

$$E[(Y - g(X))^2|X] \geq E[(Y - E[Y|X])^2|X]$$

and the result follows by taking expectations of both sides of the expression above.

REMARK. A second, more intuitive although less rigorous argument verifying Proposition 5.1 is as follows. It is straightforward to verify that $E[(Y - c)^2]$ is minimized at $c = E[Y]$ (see Theoretical Exercise 1). Thus if we want to predict the value of Y when there are no data available to use, the best possible prediction, in the sense of minimizing the mean square error, is to predict that Y will equal its mean. On the other hand, if the value of the random variable X is observed to be x, then the prediction problem remains exactly as in the previous (no data) case with the exception that all probabilities and expectations are now conditional on the event that $X = x$. Hence it follows that the best prediction in this situation is to predict that Y will equal its conditional expected value given that $X = x$, thus establishing Proposition 5.1.

Example 5a. Suppose that the son of a man of height x (in inches) attains a height that is normally distributed with mean $x + 1$ and variance 4. What is the best prediction of the height at full growth of the son of a man who is 6 feet tall?

Solution Formally, this model can be written as

$$Y = X + 1 + e$$

where e is a normal random variable, independent of X, having mean 0 and variance 4. The X and Y, of course, represent the heights of the man and his son, respectively. The best prediction $E[Y|X = 72]$ is thus equal to

$$\begin{aligned} E[Y|X = 72] &= E[X + 1 + e|X = 72] \\ &= 73 + E[e|X = 72] \\ &= 73 + E(e) \qquad \text{by independence} \\ &= 73 \end{aligned}$$

Example 5b. Suppose that if a signal value s is sent from location A, then the signal value received at location B is normally distributed with parameters $(s, 1)$. If S, the value of the signal sent at A, is normally distributed with parameters (μ, σ^2), what is the best estimate of the signal sent if R, the value received at B, is equal to r?

Solution Let us start by computing the conditional density of S given R as follows:

$$\begin{aligned} f_{S|R}(s|r) &= \frac{f_{S,R}(s,r)}{f_R(r)} \\ &= \frac{f_S(s)f_{R|S}(r|s)}{f_R(r)} \\ &= Ke^{-(s-\mu)^2/2\sigma^2} e^{-(r-s)^2/2} \end{aligned}$$

where K does not depend on s. Now,

$$\begin{aligned} \frac{(s-\mu)^2}{2\sigma^2} + \frac{(r-s)^2}{2} &= s^2\left(\frac{1}{2\sigma^2} + \frac{1}{2}\right) - \left(\frac{\mu}{\sigma^2} + r\right)s + C_1 \\ &= \frac{1+\sigma^2}{2\sigma^2}\left[s^2 - 2\left(\frac{\mu + r\sigma^2}{1+\sigma^2}\right)s\right] + C_1 \\ &= \frac{1+\sigma^2}{2\sigma^2}\left(s - \frac{(\mu + r\sigma^2)}{1+\sigma^2}\right)^2 + C_2 \end{aligned}$$

where C_1 and C_2 do not depend on s. Hence

$$f_{S|R}(s|r) = C\exp\left\{\frac{-\left[s - \frac{(\mu + r\sigma^2)}{1+\sigma^2}\right]^2}{2\left(\frac{\sigma^2}{1+\sigma^2}\right)}\right\}$$

where C does not depend on s. Hence we may conclude that the conditional distribution of S, the signal sent, given that r is received, is normal with mean and variance now given by

$$E[S|R = r] = \frac{\mu + r\sigma^2}{1 + \sigma^2}$$

$$\text{Var}(S|R = r) = \frac{\sigma^2}{1 + \sigma^2}$$

Hence, from Proposition 5.1, given that the value received is r, the best estimate, in the sense of minimizing the mean square error, for the signal sent is

$$E[S|R = r] = \frac{1}{1 + \sigma^2}\mu + \frac{\sigma^2}{1 + \sigma^2}r$$

Writing the conditional mean as we did above is informative, for it shows that it equals a weighted average of μ, the a priori expected value of the signal and r, the value received. The relative weights given to μ and r are in the same proportion to each other as 1 (the conditional variance of the received signal when s is sent) is to σ^2 (the variance of the signal to be sent). ∎

Example 5c. In digital signal processing raw continuous analog data X must be quantized, or discretized, in order to obtain a digital representation. In order to quantize the raw data X, an increasing set of numbers a_i, $i = 0, \pm 1, \pm 2, \ldots$, such that $\lim_{i \to +\infty} a_i = \infty$, $\lim_{i \to -\infty} a_i = -\infty$, is fixed and the raw data are then quantized according to the interval $(a_i, a_{i+1}]$ in which X lies. Let us denote by y_i the discretized value when $X \in (a_i, a_{i+1}]$, and let Y denote the observed discretized value—that is,

$$Y = y_i \qquad \text{if } a_i < X \leq a_{i+1}$$

The distribution of Y is given by

$$P\{Y = y_i\} = F_X(a_{i+1}) - F_X(a_i)$$

Suppose now that we want to choose the values y_i, $i = 0, \pm 1, \pm 2, \ldots$ so as to minimize $E[(X - Y)^2]$, the expected mean square difference between the raw data and their quantized version.

(a) Find the optimal values y_i, $i = 0, \pm 1, \ldots$.

For the optimal quantizer Y show that:

(b) $E[Y] = E[X]$, so the mean square error quantizer preserves the input mean;

(c) $\text{Var}(Y) = \text{Var}(X) - E[(X - Y)^2]$.

Solution (a) For any quantizer Y, upon conditioning on the value of Y we obtain

$$E[(X - Y)^2] = \sum_i E[(X - y_i)^2 | a_i < X \leq a_{i+1}]P\{a_i < X \leq a_{i+1}\}$$

Now, if we let

$$I = i \qquad \text{if } a_i < X \leq a_{i+1}$$

then

$$E[(X - y_i)^2 | a_i < X \leq a_{i+1}] = E[(X - y_i)^2 | I = i]$$

and by Proposition 5.1 this quantity is minimized when

$$\begin{aligned} y_i &= E[X|I = i] \\ &= E[X|a_i < X \leq a_{i+1}] \\ &= \int_{a_i}^{a_{i+1}} \frac{x f_X(s)\, dx}{F_X(a_{i+1}) - F_X(a_i)} \end{aligned}$$

Now, since the optimal quantizer is given by $Y = E[X|I]$, it follows that

(b) $E[Y] = E[X]$

(c) $\begin{aligned} \text{Var}(X) &= E[\text{Var}(X|I)] + \text{Var}(E[X|I]) \\ &= E[E[(X - Y)^2|I]] + \text{Var}(Y) \\ &= E[(X - Y)^2] + \text{Var}(Y) \end{aligned}$ ∎

It sometimes happens that the joint probability distribution of X and Y is not completely known; or if it is known, it is such that the calculation of $E[Y|X = x]$ is mathematically intractable. If, however, the means and variances of X and Y and the correlation of X and Y are known, then we can at least determine the best *linear* predictor of Y with respect to X.

To obtain the best linear predictor of Y with respect to X, we need to choose a and b so as to minimize $E[(Y - (a + bX))^2]$. Now,

$$\begin{aligned} E[(Y - (a + bX))^2] &= E[Y^2 - 2aY - 2bXY + a^2 + 2abX + b^2X^2] \\ &= E[Y^2] - 2aE[Y] - 2bE[XY] + a^2 \\ &\quad + 2abE[X] + b^2E[X^2] \end{aligned}$$

Taking partial derivatives, we obtain

$$\begin{aligned} \frac{\partial}{\partial a} E[(Y - a - bX)^2] &= -2E[Y] + 2a + 2bE[X] \\ \frac{\partial}{\partial b} E[(Y - a - bX)^2] &= -2E[XY] + 2aE[X] + 2bE[X^2] \end{aligned} \tag{5.3}$$

Equating Equations (5.3) to 0 and solving for a and b yields the solutions

$$\begin{aligned} b &= \frac{E[XY] - E[X]E[Y]}{E[X^2] - (E[X])^2} = \frac{\text{Cov}(X, Y)}{\sigma_x^2} = \rho\frac{\sigma_y}{\sigma_x} \\ a &= E[Y] - bE[X] = E[Y] - \frac{\rho\sigma_y E[X]}{\sigma_x} \end{aligned} \tag{5.4}$$

where $\rho = \text{Correlation}(X, Y)$, $\sigma_y^2 = \text{Var}(Y)$, and $\sigma_x^2 = \text{Var}(X)$. It is easy to verify that the values of a and b from Equation (5.4) minimize

$E[(Y - a - bX)^2]$, and thus the best (in the sense of mean square error) linear predictor Y with respect to X is

$$\mu_y + \frac{\rho\sigma_y}{\sigma_x}(X - \mu_x)$$

where $\mu_y = E[Y]$ and $\mu_x = E[X]$.

The mean square error of this predictor is given by

$$E\left[\left(Y - \mu_y - \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)\right)^2\right]$$

$$= E[(Y - \mu_y)^2] + \rho^2\frac{\sigma_y^2}{\sigma_x^2}E[(X - \mu_x)^2] - 2\rho\frac{\sigma_y}{\sigma_x}E[(Y - \mu_y)(X - \mu_x)]$$

$$= \sigma_y^2 + \rho^2\sigma_y^2 - 2\rho^2\sigma_y^2$$

$$= \sigma_y^2(1 - \rho^2) \tag{5.5}$$

We note from Equation (5.5) that if ρ is near $+1$ or -1, then the mean square error of the best linear predictor is near zero.

Example 5d. An example in which the conditional expectation of Y given X is linear in X, and hence the best linear predictor of Y with respect to X is the best overall predictor, is when X and Y have a bivariate normal distribution. In this case their joint density is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}}\exp\left\{-\frac{1}{2(1 - \rho^2)}\left[\left(\frac{x - \mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} + \left(\frac{y - \mu_y}{\sigma_y}\right)^2\right]\right\}$$

We leave it for the reader to verify that the conditional density of Y, given $X = x$, is given by

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_y\sqrt{1 - \rho^2}} \cdot \exp\left\{-\frac{1}{2\sigma_y^2(1 - \rho^2)}\left(y - \mu_y - \frac{\rho\sigma_y}{\sigma_x}(x - \mu_x)\right)^2\right\}$$

Hence the conditional distribution of Y, given $X = x$, is the normal distribution with mean

$$E[Y|X = x] = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$$

and variance $\sigma_y^2(1 - \rho^2)$.

7.6 MOMENT GENERATING FUNCTIONS

The moment generating function $M(t)$ of the random variable X is defined for all real values of t by

$$M(t) = E[e^{tX}]$$

$$= \begin{cases} \sum_x e^{tx}p(x) & \text{if } X \text{ is discrete with mass function } p(x) \\ \int_{-\infty}^{\infty} e^{tx}f(x)\,dx & \text{if } X \text{ is continuous with density } f(x) \end{cases}$$

We call $M(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$. For example,

$$M'(t) = \frac{d}{dt}E[e^{tX}]$$

$$= E\left[\frac{d}{dt}(e^{tX})\right] \tag{6.1}$$

$$= E[Xe^{tX}]$$

where we have assumed that the interchange of the differentiation and expectation operators is legitimate. That is, we have assumed that

$$\frac{d}{dt}\left[\sum_x e^{tx}p(x)\right] = \sum_x \frac{d}{dt}[e^{tx}p(x)]$$

in the discrete case, and

$$\frac{d}{dt}\left[\int e^{tx}f(x)\,dx\right] = \int \frac{d}{dt}[e^{tx}f(x)]\,dx$$

in the continuous case. This assumption can almost always be justified and, indeed, is valid for all of the distributions considered in this book. Hence, from Equation (6.1) we obtain, by evaluating at $t = 0$, that

$$M'(0) = E[X]$$

Similarly,

$$M''(t) = \frac{d}{dt}M'(t)$$

$$= \frac{d}{dt}E[Xe^{tX}]$$

$$= E\left[\frac{d}{dt}(Xe^{tX})\right]$$

$$= E[X^2e^{tX}]$$

and thus

$$M''(0) = E[X^2]$$

In general, the nth derivative of $M(t)$ is given by

$$M^n(t) = E[X^n e^{tX}] \qquad n \geq 1$$

implying that

$$M^n(0) = E[X^n] \qquad n \geq 1$$

We now compute $M(t)$ for some common distributions.

Example 6a. ***Binomial distribution with parameters n and p.*** If X is a binomial random variable with parameters n and p, then

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\ &= (pe^t + 1 - p)^n \end{aligned}$$

where the last equality follows from the binomial theorem.

$$M'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

and thus

$$E[X] = M'(0) = np$$

which checks with the result obtained in Example 1c. Differentiating a second time yields

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

so

$$E[X^2] = M''(0) = n(n-1)p^2 + np$$

The variance of X is given by

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p) \end{aligned}$$

verifying the result of Example 3b. ∎

Example 6b. ***Poisson distribution with mean λ.*** If X is a Poisson random variable with parameter λ, then

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= \exp\{\lambda(e^t - 1)\} \end{aligned}$$

Differentiation yields

$$\begin{aligned} M'(t) &= \lambda e^t \exp\{\lambda(e^t - 1)\} \\ M''(t) &= (\lambda e^t)^2 \exp\{\lambda(e^t - 1)\} + \lambda e^t \exp\{\lambda(e^t - 1)\} \end{aligned}$$

and thus

$$\begin{aligned} E[X] &= M'(0) = \lambda \\ E[X^2] &= M''(0) = \lambda^2 + \lambda \\ \mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \lambda \end{aligned}$$

Hence both the mean and the variance of the Poisson random variable equal λ. ∎

Example 6c. ***Exponential distribution with parameter λ***

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx \\ &= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx \\ &= \frac{\lambda}{\lambda - t} \qquad \text{for } t < \lambda \end{aligned}$$

We note from this derivation that for the exponential distribution, $M(t)$ is only defined for values of t less than λ. Differentiation of $M(t)$ yields

$$M'(t) = \frac{\lambda}{(\lambda - t)^2} \qquad M''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

Hence

$$E[X] = M'(0) = \frac{1}{\lambda} \qquad E[X^2] = M''(0) = \frac{2}{\lambda^2}$$

The variance of X is given by

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{1}{\lambda^2} \end{aligned}$$

∎

Example 6d. ***Normal distribution.*** We first compute the moment generating function of a unit normal random variable with parameters 0 and 1. Letting Z be such a random variable, we have

$$\begin{aligned} M_Z(t) &= E[e^{tZ}] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x^2 - 2tx)}{2}\right\} dx \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-t)^2}{2} + \frac{t^2}{2}\right\} dx$$
$$= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, dx$$
$$= e^{t^2/2}$$

Hence the moment generating function of the unit normal random variable Z is given by $M_Z(t) = e^{t^2/2}$. To obtain the moment generating function of an arbitrary normal random variable, we recall (see Section 5.4) that $X = \mu + \sigma Z$ will have a normal distribution with parameters μ and σ^2 whenever Z is a unit normal random variable. Hence the moment generating function of such a random variable is given by

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= E[e^{t(\mu+\sigma Z)}] \\ &= E[e^{t\mu} e^{t\sigma Z}] \\ &= e^{t\mu} E[e^{t\sigma Z}] \\ &= e^{t\mu} M_Z(t\sigma) \\ &= e^{t\mu} e^{(t\sigma)^2/2} \\ &= \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\} \end{aligned}$$

By differentiating, we obtain

$$M'_X(t) = (\mu + t\sigma^2) \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}$$
$$M''_X(t) = (\mu + t\sigma^2)^2 \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\} + \sigma^2 \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}$$

and thus

$$E[X] = M'(0) = \mu$$
$$E[X^2] = M''(0) = \mu^2 + \sigma^2$$

implying that

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2] - E([X])^2 \\ &= \sigma^2 \end{aligned}$$

∎

Tables 7.1 and 7.2 give the moment generating function for some common discrete and continuous distributions.

An important property of moment generating functions is that the moment generating function of the sum of independent random variables equals the product of the individual moment generating functions. To prove this, suppose that X and Y are independent and have moment generating functions $M_X(t)$ and

TABLE 7.1 DISCRETE PROBABILITY DISTRIBUTION

	Probability mass function, $p(x)$	Moment generating function, $M(t)$	Mean	Variance
Binomial with parameters n, p; $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, \ldots, n$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$ $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative binomial with parameters r, p; $0 \le p \le 1$	$\binom{n-1}{r-1} p^r (1-p)^{n-r}$ $n = r, r+1, \ldots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

$M_Y(t)$, respectively. Then $M_{X+Y}(t)$, the moment generating function of $X + Y$, is given by

$$\begin{aligned} M_{X+Y}(t) &= E[e^{t(X+Y)}] \\ &= E[e^{tX} e^{tY}] \\ &= E[e^{tX}] E[e^{tY}] \\ &= M_X(t) M_Y(t) \end{aligned}$$

where the next-to-last equality follows from Proposition 3.1, since X and Y are independent.

Another important result is that the moment generating function uniquely determines the distribution. That is, if $M_X(t)$ exists and is finite in some region about $t = 0$, then the distribution of X is uniquely determined. For instance, if $M_X(t) = \left(\frac{1}{2}\right)^{10}(e^t + 1)^{10}$, then it follows from Table 7.1 that X is a binomial random variable with parameters 10 and $\frac{1}{2}$.

Example 6e. Suppose that the moment generating function of a random variable X is given by $M(t) = e^{3(e^t - 1)}$. What is $P\{X = 0\}$?

Solution We see from Table 7.1 that $M(t) = e^{3(e^t-1)}$ is the moment generating function of a Poisson random variable with mean 3. Hence, by the one-to-one correspondence between moment generating functions and distribution functions, it follows that X must be a Poisson random variable with mean 3. Thus $P\{X = 0\} = e^{-3}$. ∎

Example 6f. ***Sums of independent binomial random variables.*** If X and Y are independent binomial random variables with parameters (n, p) and (m, p), respectively, what is the distribution of $X + Y$?

TABLE 7.2 CONTINUOUS PROBABILITY DISTRIBUTION

	Probability mass function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma with parameters (s, λ), $\lambda > 0$	$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\left(\dfrac{\lambda}{\lambda - t}\right)^s$	$\dfrac{s}{\lambda}$	$\dfrac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$	$\exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Solution The moment generating function of $X + Y$ is given by

$$M_{X+Y}(t) = M_X(t)M_Y(t) = (pe^t + 1 - p)^n(pe^t + 1 - p)^m$$
$$= (pe^t + 1 - p)^{m+n}$$

However, $(pe^t + 1 - p)^{m+n}$ is the moment generating function of a binomial random variable having parameters $m + n$ and p. Thus this must be the distribution of $X + Y$. ∎

Example 6g. ***Sums of independent Poisson random variables.*** Calculate the distribution of $X + Y$ when X and Y are independent Poisson random variables with means λ_1 and λ_2, respectively.

Solution

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$
$$= \exp\{\lambda_1(e^t - 1)\} \exp\{\lambda_2(e^t - 1)\}$$
$$= \exp\{(\lambda_1 + \lambda_2)(e^t - 1)\}$$

Hence $X + Y$ is Poisson distributed with mean $\lambda_1 + \lambda_2$, verifying the result given in Example 3d of Chapter 6. ∎

Example 6h. ***Sums of independent normal random variables.*** Show that if X and Y are independent normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively, then $X + Y$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

Solution

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$
$$= \exp\left\{\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right\} \exp\left\{\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right\}$$
$$= \exp\left\{\frac{(\sigma_1^2 + \sigma_2^2)t^2}{2} + (\mu_1 + \mu_2)t\right\}$$

which is the moment generating function of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Hence the result follows because the moment generating function uniquely determines the distribution. ∎

Example 6i. Compute the moment generating function of a chi-squared random variable with n degrees of freedom.

Solution We can represent such a random variable as

$$Z_1^2 + \cdots + Z_n^2$$

where $Z_1, \ldots, Z_n$ are independent standard normal random variables. Let $M(t)$ be its moment generating function. By the above,

$$M(t) = (E[e^{tZ^2}])^n$$

where Z is a standard normal. Now,

$$\begin{aligned} E[e^{tZ^2}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx^2}e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx \qquad \text{where } \sigma^2 = (1-2t)^{-1} \\ &= \sigma \\ &= (1-2t)^{-1/2} \end{aligned}$$

where the next-to-last equality uses that the normal density with mean 0 and variance σ^2 integrates to 1. Therefore,

$$M(t) = (1-2t)^{-n/2}$$ ∎

Example 6j. ***Moment generating function of the sum of a random number of random variables.*** Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, and let N be a nonnegative, integer-valued random variable that is independent of the sequence X, $i \geq 1$. We want to compute the moment generating function of

$$Y = \sum_{i=1}^{N} X_i$$

(In Example 4d, Y was interpreted as the amount of money spent in a store on a given day when both the amount spent by a customer and the number of such customers are random variables.)

To compute the moment generating function of Y, we first condition on N as follows:

$$\begin{aligned} E[\exp\{t\Sigma_1^N X_i\}|N=n] &= E[\exp\{t\Sigma_1^n X_i\}|N=n] \\ &= E[\exp\{t\Sigma_1^n X_i\}] \\ &= [M_X(t)]^n \end{aligned}$$

where

$$M_X(t) = E[e^{tX_i}]$$

Hence

$$E[e^{tY}|N] = (M_X(t))^N$$

and thus

$$M_Y(t) = E[(M_X(t))^N]$$

The moments of Y can now be obtained upon differentiation, as follows:

$$M_Y'(t) = E[N(M_X(t))^{N-1}M_X'(t)]$$

so

$$\begin{aligned} E[Y] &= M_Y'(0) \\ &= E[N(M_X(0))^{N-1}M_X'(0)] \\ &= E[NEX] \\ &= E[N]E[X] \end{aligned} \tag{6.2}$$

verifying the result of Example 4d. (In this last set of equalities we have used the fact that $M_X(0) = E[e^{0X}] = 1$.)

Also,

$$M_Y''(t) = E[N(N-1)(M_X(t))^{N-2}(M_X'(t))^2 + N(M_X(t))^{N-1}M_X''(t)]$$

so

$$\begin{aligned} E[Y^2] &= M_Y''(0) \\ &= E[N(N-1)(E[X])^2 + NE[X^2]] \\ &= (E[X])^2(E[N^2] - E[N]) + E[N]E[X^2] \\ &= E[N](E[X^2] - (E[X])^2) + (E[X])^2E[N^2] \\ &= E[N]\,\mathrm{Var}(X) + (E[X])^2E[N^2] \end{aligned} \tag{6.3}$$

Hence, from Equations (6.2) and (6.3), we see that

$$\begin{aligned} \mathrm{Var}(Y) &= E[N]\,\mathrm{Var}(X) + (E[X])^2(E[N^2] - (E[N])^2) \\ &= E[N]\,\mathrm{Var}(X) + (E[X])^2\,\mathrm{Var}(N) \end{aligned}$$

Example 6k. Let Y denote a uniform random variable on (0, 1), and suppose that conditional on $Y = p$, the random variable X has a binomial distribution with parameters n and p. In Example 4j we showed that X is equally likely to take on any of the values $0, 1, \ldots, n$. Establish this result by using moment generating functions.

Solution To compute the moment generating function of X, start by conditioning on the value of Y. Using the formula for the binomial moment generating function gives

$$E[e^{tX}|Y=p] = (pe^t + 1 - p)^n$$

Hence, since Y is uniform on (0, 1), we obtain upon taking expectations,

$$\begin{aligned} E[e^{tX}] &= \int_0^1 (pe^t + 1 - p)^n\,dp \\ &= \frac{1}{e^t - 1}\int_1^{e^t} y^n\,dy \qquad \text{(by the substitution } y = pe^t + 1 - p) \\ &= \frac{1}{n+1}\,\frac{e^{t(n+1)} - 1}{e^t - 1} \\ &= \frac{1}{n+1}(1 + e^t + e^{2t} + \cdots + e^{nt}) \end{aligned}$$

As the preceding is the moment generating function of a random variable that is equally likely to be any of the values $0, 1, \ldots, n$, the result follows from the fact that the moment generating function of a random variable uniquely determines its distribution. ∎

7.6.1 Joint Moment Generating Functions

It is also possible to define the joint moment generating function of two or more random variables. This is done as follows. For any n random variables $X_1, \ldots, X_n$, the joint moment generating function, $M(t_1, \ldots, t_n)$, is defined for all real values of $t_1, \ldots, t_n$ by

$$M(t_1, \ldots, t_n) = E[e^{t_1X_1 + \cdots + t_nX_n}]$$

The individual moment generating functions can be obtained from $M(t_1, \ldots, t_n)$ by letting all but one of the t_j be 0. That is,

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, \ldots, 0, t, 0, \ldots, 0)$$

where the t is in the ith place.

It can be proved (although the proof is too advanced for this text) that $M(t_1, \ldots, t_n)$ uniquely determines the joint distribution of $X_1, \ldots, X_n$. This result can then be used to prove that the n random variables $X_1, \ldots, X_n$ are independent if and only if

$$M(t_1, \ldots, t_n) = M_{X_1}(t_1) \cdots M_{X_n}(t_n) \tag{6.4}$$

This follows because, if the n random variables are independent, then

$$\begin{aligned} M(t_1, \ldots, t_n) &= E[e^{(t_1X_1 + \cdots + t_nX_n)}] \\ &= E[e^{t_1X_1} \cdots e^{t_nX_n}] \\ &= E[e^{t_1X_1}] \cdots E[e^{t_nX_n}] \qquad \text{by independence} \\ &= M_{X_1}(t_1) \cdots M_{X_n}(t_n) \end{aligned}$$

On the other hand, if Equation (6.4) is satisfied, then the joint moment generating function $M(t_1, \ldots, t_n)$ is the same as the joint moment generating function of n independent random variables, the ith of which has the same distribution as X_i. As the joint moment generating function uniquely determines the joint distribution, this must be the joint distribution; hence the random variables are independent.

Example 6l. Let X and Y be independent normal random variables, each with mean μ and variance σ^2. In Example 7a of Chapter 6 we showed that $X + Y$ and $X - Y$ are independent. Let us now establish this result by computing their joint moment generating function.

$$\begin{aligned} E[e^{t(X+Y)+s(X-Y)}] &= E[e^{(t+s)X+(t-s)Y}] \\ &= E[e^{(t+s)X}]E[e^{(t-s)Y}] \\ &= e^{\mu(t+s)+\sigma^2(t+s)^2/2}e^{\mu(t-s)+\sigma^2(t-s)^2/2} \\ &= e^{2\mu t+\sigma^2t^2}e^{\sigma^2s^2} \end{aligned}$$

But we recognize the preceding as the joint moment generating function of the sum of a normal random variable with mean 2μ and variance $2\sigma^2$ and an independent normal random variable with mean 0 and variance $2\sigma^2$. As the joint moment generating function uniquely determines the joint distribution, it thus follows that $X + Y$ and $X - Y$ are independent normal random variables. ∎

In the next example we use the joint moment generating function to verify a result that was established in Example 2b of Chapter 6.

Example 6m. Suppose that the number of events that occur is a Poisson random variable with mean λ, and that each event is independently counted with probability p. Show that the number of counted events and the number of uncounted events are independent Poisson random variables with respective means λp and $\lambda(1 - p)$.

Solution Let X denote the total number of events, and let X_c denote the number of them that are counted. To compute the joint moment generating function of X_c, the number of events that are counted, and $X - X_c$, the number that are uncounted, start by conditioning on X to obtain

$$\begin{aligned} E[e^{sX_c+t(X-X_c)}|X = n] &= e^{tn}E[e^{(s-t)X_c}|X = n] \\ &= e^{tn}(pe^{s-t} + 1 - p)^n \\ &= (pe^s + (1 - p)e^t)^n \end{aligned}$$

where the preceding equation follows since conditional on $X = n$, X_c is a binomial random variable with parameters n and p. Hence

$$E[e^{sX_c+t(X-X_c)}|X] = (pe^s + (1 - p)e^t)^X$$

Taking expectations of both sides of the preceding yields that

$$E[e^{sX_c+t(X-X_c)}] = E[(pe^s + (1 - p)e^t)^X]$$

Now, since X is Poisson with mean λ, it follows that $E[e^{tX}] = e^{\lambda(e^t-1)}$. Therefore, for any positive value a we see (by letting $a = e^t$) that $E[a^X] = e^{\lambda(a-1)}$. Thus

$$\begin{aligned} E[e^{sX_c+t(X-X_c)}] &= e^{\lambda(pe^s+(1-p)e^t-1)} \\ &= e^{\lambda p(e^s-1)}e^{\lambda(1-p)(e^t-1)} \end{aligned}$$

As the preceding is the joint moment generating function of independent Poisson random variables with respective means λp and $\lambda(1 - p)$, the result is proven. ∎

7.7 ADDITIONAL PROPERTIES OF NORMAL RANDOM VARIABLES

7.7.1 The Multivariate Normal Distribution

Let $Z_1, \ldots, Z_n$ be a set of n independent unit normal random variables. If, for some constants a_{ij}, $1 \le i \le m$, $1 \le j \le n$, and μ_i, $1 \le i \le m$,

$$\begin{aligned} X_1 &= a_{11}Z_1 + \cdots + a_{1n}Z_n + \mu_1 \\ X_2 &= a_{21}Z_1 + \cdots + a_{2n}Z_n + \mu_2 \\ &\vdots \\ X_i &= a_{i1}Z_1 + \cdots + a_{in}Z_n + \mu_i \\ &\vdots \\ X_m &= a_{m1}Z_1 + \cdots + a_{mn}Z_n + \mu_m \end{aligned}$$

then the random variables $X_1, \ldots, X_m$ are said to have a multivariate normal distribution.

It follows from the fact that the sum of independent normal random variables is itself a normal random variable that each X_i is a normal random variable with mean and variance given by

$$E[X_i] = \mu_i$$
$$\text{Var}(X_i) = \sum_{j=1}^{n} a_{ij}^2$$

Let us now consider

$$M(t_1, \ldots, t_m) = E[\exp\{t_1X_1 + \cdots + t_mX_m\}]$$

the joint moment generating function of $X_1, \ldots, X_m$. The first thing to note is that since $\sum_{i=1}^{m} t_iX_i$ is itself a linear combination of the independent normal random variables $Z_1, \ldots, Z_n$, it is also normally distributed. Its mean and variance are

$$E\left[\sum_{i=1}^{m} t_iX_i\right] = \sum_{i=1}^{m} t_i\mu_i$$

and

$$\text{Var}\left(\sum_{i=1}^{m} t_iX_i\right) = \text{Cov}\left(\sum_{i=1}^{m} t_iX_i, \sum_{j=1}^{m} t_jX_j\right)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m} t_it_j \,\text{Cov}(X_i, X_j)$$

Now, if Y is a normal random variable with mean μ and variance σ^2, then

$$E[e^Y] = M_Y(t)\big|_{t=1} = e^{\mu+\sigma^2/2}$$

Thus we see that

$$M(t_1, \ldots, t_m) = \exp\left\{\sum_{i=1}^{m} t_i\mu_i + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} t_it_j \,\text{Cov}(X_i, X_j)\right\}$$

which shows that the joint distribution of $X_1, \ldots, X_m$ is completely determined from a knowledge of the values of $E[X_i]$ and $\text{Cov}(X_i, X_j)$, $i, j = 1, \ldots, m$.

7.7.2 The Joint Distribution of the Sample Mean and Sample Variance

Let $X_1, \ldots, X_n$ be independent normal random variables, each with mean μ and variance σ^2. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ denote their sample mean. Since the sum of independent normal random variables is also a normal random variable, it follows

that $\bar{X}$ is a normal random variable with (from Examples 2c and 3a) expected value μ and variance σ^2/n.

Now, recall from Example 3f that

$$\text{Cov}(\bar{X}, X_i - \bar{X}) = 0, \qquad i = 1, \ldots, n \tag{7.1}$$

Also, note that since $\bar{X}, X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X}$ are all linear combinations of the independent standard normals $(X_i - \mu)/\sigma$, $i = 1, \ldots, n$, it follows that $\bar{X}, X_i - \bar{X}$, $i = 1, \ldots, n$ has a joint distribution that is multivariate normal. If we let Y be a normal random variable with mean μ and variance σ^2/n that is independent of the X_i, $i = 1, \ldots, n$, then $Y, X_i - \bar{X}$, $i = 1, \ldots, n$ also has a multivariate normal distribution and indeed, because of (7.1), has the same expected values and covariances as the random variables $\bar{X}, X_i - \bar{X}$, $i = 1, \ldots, n$. But since a multivariate normal distribution is determined completely by its expected values and covariances, we can conclude that $Y, X_i - \bar{X}$, $i = 1, \ldots, n$ and $\bar{X}, X_i - \bar{X}$, $i = 1, \ldots, n$ have the same joint distribution, thus showing that $\bar{X}$ is independent of the sequence of deviations $X_i - \bar{X}$, $i = 1, \ldots, n$.

Since $\bar{X}$ is independent of the sequence of deviations $X_i - \bar{X}$, $i = 1, \ldots, n$, it follows that it is also independent of the sample variance $S^2 \equiv \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$.

Since we already know that $\bar{X}$ is normal with mean μ and variance σ^2/n, it remains only to determine the distribution of S^2. To accomplish this, recall from Example 3a the algebraic identity

$$(n-1)S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$$
$$= \sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2$$

Upon dividing the equation above by σ^2, we obtain that

$$\frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \tag{7.2}$$

Now,

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

is the sum of the squares of n independent standard normal random variables, and so is a chi-squared random variable with n degrees of freedom. Hence, from Example 6i, its moment generating function is $(1 - 2t)^{-n/2}$. Also,

$$\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

is the square of a standard normal and so is a chi-squared random variable with 1 degree of freedom, and so has moment generating function $(1 - 2t)^{-1/2}$. Now, we have seen previously that the two random variables on the left side of Equation (7.2) are independent. Hence, as the moment generating function of the sum of independent random variables is equal to the product of their individual moment generating functions, we see that

$$E[e^{t(n-1)S^2/\sigma^2}](1 - 2t)^{-1/2} = (1 - 2t)^{-n/2}$$

or

$$E[e^{t(n-1)S^2/\sigma^2}] = (1 - 2t)^{-(n-1)/2}$$

But as $(1 - 2t)^{-(n-1)/2}$ is the moment generating function of a chi-squared random variable with $n - 1$ degrees of freedom, we can conclude, since the moment generating function uniquely determines the distribution of the random variable, that this is the distribution of $(n - 1)S^2/\sigma^2$.

Summing up, we have shown the following.

Proposition 7.1

If $X_1, \ldots, X_n$ are independent and identically distributed normal random variables with mean μ and variance σ^2, then the sample mean $\bar{X}$ and the sample variance S^2 are independent. $\bar{X}$ is a normal random variable with mean μ and variance σ^2/n; $(n - 1)S^2/\sigma^2$ is a chi-squared random variable with $n - 1$ degrees of freedom.

*7.8 GENERAL DEFINITION OF EXPECTATION

Up to this point we have defined expectations only for discrete and continuous random variables. However, there also exist random variables that are neither discrete nor continuous, and they too may possess an expectation. As an example of such a random variable, let X be a Bernoulli random variable with parameter $p = \frac{1}{2}$, and let Y be a uniformly distributed random variable over the interval $[0, 1]$. Furthermore, suppose that X and Y are independent and define the new random variable W by

$$W = \begin{cases} X & \text{if } X = 1 \\ Y & \text{if } X \neq 1 \end{cases}$$

Clearly, W is neither a discrete (since its set of possible values $[0, 1]$ is uncountable) nor a continuous $\left(\text{since } P\{W = 1\} = \frac{1}{2}\right)$ random variable.

In order to define the expectation of an arbitrary random variable, we require the notion of a Stieltjes integral. Before defining the Stieltjes integral, let us recall that for any function g, $\int_a^b g(x)\,dx$ is defined by

$$\int_a^b g(x)\,dx = \lim \sum_{i=1}^{n} g(x_i)(x_i - x_{i-1})$$

where the limit is taken over all $a = x_0 < x_1 < x_2 \cdots < x_n = b$ as $n \to \infty$ and $\max_{i=1,\ldots,n} (x_i - x_{i-1}) \to 0$.

For any distribution function F, we define the Stieltjes integral of the nonnegative function g over the interval $[a, b]$ by

$$\int_a^b g(x)\,dF(x) = \lim \sum_{i=1}^{n} g(x_i)[F(x_i) - F(x_{i-1})]$$

where, as before, the limit is taken over all $a = x_0 < x_1 < \cdots < x_n = b$ as $n \to \infty$ and $\max_{i=1,\ldots,n} (x_i - x_{i-1}) \to 0$. Further, we define the Stieltjes integral over the whole real line by

$$\int_{-\infty}^{\infty} g(x)\,dF(x) = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b g(x)\,dF(x)$$

Finally, if g is not a nonnegative function, we define g^+ and g^- by

$$g^+(x) = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ 0 & \text{if } g(x) < 0 \end{cases}$$

$$g^-(x) = \begin{cases} 0 & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) < 0 \end{cases}$$

As $g(x) = g^+(x) - g^-(x)$ and g^+ and g^- are both nonnegative functions, it is natural to define

$$\int_{-\infty}^{\infty} g(x)\,dF(x) = \int_{-\infty}^{\infty} g^+(x)\,dF(x) - \int_{-\infty}^{\infty} g^-(x)\,dF(x)$$

and we say that $\int_{-\infty}^{\infty} g(x)\,dF(x)$ exists as long as $\int_{-\infty}^{\infty} g^+(x)\,dF(x)$ and $\int_{-\infty}^{\infty} g^-(x)\,dF(x)$ are not both equal to $+\infty$.

If X is an arbitrary random variable having cumulative distribution F, we define the expected value of X by

$$E[X] = \int_{-\infty}^{\infty} x\,dF(x) \tag{8.1}$$

It can be shown that if X is a discrete random variable with mass function $p(x)$, then

$$\int_{-\infty}^{\infty} x\,dF(x) = \sum_{x:p(x)>0} xp(x)$$

whereas, if X is a continuous random variable with density function $f(x)$, then

$$\int_{-\infty}^{\infty} x\,dF(x) = \int_{-\infty}^{\infty} xf(x)\,dx$$

The reader should note that Equation (8.1) yields an intuitive definition of $E[X]$; for consider the approximating sum

$$\sum_{i=1}^{n} x_i[F(x_i) - F(x_{i-1})]$$

of $E[X]$. As $F(x_i) - F(x_{i-1})$ is just the probability that X will be in the interval $(x_{i-1}, x_i]$, the approximating sum multiplies the approximate value of X when it is in the interval $(x_{i-1}, x_i]$ by the probability that it will be in that interval and then sums over all the intervals. Clearly, as these intervals get smaller and smaller in length, we obtain the "expected value" of X.

Stieltjes integrals are mainly of theoretical interest because they yield a compact way of defining and dealing with the properties of expectation. For instance, use of Stieltjes integrals avoids the necessity of having to give separate statements and proofs of theorems for the continuous and the discrete cases. However, their properties are very much the same as those of ordinary integrals, and all of the proofs presented in this chapter can easily be translated into proofs in the general case.

SUMMARY

If X and Y have a joint probability mass function $p(x, y)$, then

$$E[g(X, Y)] = \sum_y \sum_x g(x, y)p(x, y)$$

whereas if they have a joint density function $f(x, y)$, then

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

A consequence of the preceding is that

$$E[X + Y] = E[X] + E[Y]$$

which generalizes to

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

The *covariance* between random variables X and Y is given by

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

A useful identity is that

$$\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j)$$

When $n = m$, and $Y_i = X_i$, $i = 1, \ldots, n$, the preceding gives that

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum\sum_{i<j} \text{Cov}(X_i, Y_j)$$

The correlation between X and Y, denoted by $\rho(X, Y)$, is defined by

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

If X and Y are jointly discrete random variables, then the conditional expected value of X given that $Y = y$ is defined by

$$E[X|Y = y] = \sum_x xP\{X = x|Y = y\}$$

If they are jointly continuous random variables, then

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)$$

where

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

is the conditional probability density of X given that $Y = y$. Conditional expectations, which are similar to ordinary expectations except that all probabilities are now computed conditional on the event that $Y = y$, satisfy all the properties of ordinary expectations.

Let $E[X|Y]$ denote that function of Y whose value at $Y = y$ is $E[X|Y = y]$. A very useful identity is that

$$E[X] = E[E[X|Y]]$$

In the case of discrete random variables, this reduces to the identity

$$E[X] = \sum_y E[X|Y = y]P\{Y = y\}$$

and, in the continuous case, to

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y)$$

The preceding equations can often be applied to obtain $E[X]$ by first "conditioning" on the value of some other random variable Y. In addition, since for any event A, $P(A) = E[I_A]$, where I_A is 1 if A occurs and 0 otherwise, we can also use them to compute probabilities.

The conditional variance of X given that $Y = y$ is defined by

$$\text{Var}(X|Y = y) = E[(X - E[X|Y = y])^2|Y = y]$$

Let $\text{Var}(X|Y)$ be that function of Y whose value at $Y = y$ is $\text{Var}(X|Y = y)$. The following is known as the *conditional variance formula:*

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

Suppose that the random variable X is to be observed and, based on its value, one must then predict the value of the random variable Y. In such a situation, it turns out that, among all predictors, $E[Y|X]$ has the smallest expectation of the square of the difference between it and Y.

The *moment generating function* of the random variable X is defined by

$$M(t) = E[e^{tX}]$$

The moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the resulting quantity at $t = 0$. Specifically, we have that

$$E[X^n] = \frac{d^n}{dt^n} M(t)\Big|_{t=0} \quad n = 1, 2, \ldots$$

Two useful results concerning moment generating functions are, first, that the moment generating function uniquely determines the distribution function of the random variable, and second, that the moment generating function of the sum of independent random variables is equal to the product of their moment generating functions. These results lead to simple proofs that the sum of independent normal (Poisson) [gamma] random variables remains a normal (Poisson) [gamma] random variable.

If $X_1, \ldots, X_m$ are all linear combinations of a finite set of independent standard normal random variables, then they are said to have a *multivariate normal distribution.* Their joint distribution is specified by the values of $E[X_i]$, $\text{Cov}(X_i, X_j)$, $i, j = 1, \ldots, m$.

If $X_1, \ldots, X_n$ are independent and identically distributed normal random variables, then their *sample mean*

$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

and their *sample variance*

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n - 1}$$

are independent. The sample mean $\bar{X}$ is a normal random variable with mean μ and variance σ^2/n; the random variable $(n - 1)S^2/\sigma^2$ is a chi-squared random variable with $n - 1$ degrees of freedom.

PROBLEMS

1. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings.

2. The game of Clue involves 6 suspects, 6 weapons, and 9 rooms. One of each is randomly chosen and the object of the game is to guess the chosen three.
(a) How many solutions are possible?
In one version of the game, after the selection is made each of the players is then randomly given three of the remaining cards. Let S, W, and R be, respectively, the numbers of suspects, weapons, and rooms in the set of three cards given to a specified player. Also, let X denote the number of solutions that are possible after that player observes his or her three cards.
(b) Express X in terms of S, W, and R.
(c) Find $E[X]$.

3. If X and Y are independent uniform (0, 1) random variables, show that

$$E[|X - Y|^\alpha] = \frac{2}{(\alpha + 1)(\alpha + 2)} \quad \text{for} \quad \alpha > 0$$

4. Let X and Y be independent random variables, both being equally likely to be any of the values $1, 2, \ldots, m$. Show that

$$E[|X - Y|] = \frac{(m - 1)(m + 1)}{3m}$$

5. The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are (0, 0), to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

6. A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

7. Suppose that A and B each randomly, and independently, choose 3 of 10 objects. Find the expected number of objects
(a) chosen by both A and B;
(b) not chosen by either A or B;
(c) chosen by exactly one of A and B.

8. N people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then either sits at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the $\binom{N}{2}$ pairs of people are, independently, friends with probability p, find the expected number of occupied tables.

HINT: Let X_i equal 1 or 0 depending on whether the ith arrival sits at a previously unoccupied table.

9. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find
(a) the expected number of urns that are empty;
(b) the probability that none of the urns is empty.

10. Consider 3 trials, each having the same probability of success. Let X denote the total number of successes in these trials. If $E[X] = 1.8$, what is
(a) the largest possible value of $P\{X = 3\}$;
(b) the smallest possible value of $P\{X = 3\}$?
In both cases construct a probability scenario that results in $P\{X = 3\}$ having the stated value.

HINT: For part (b) you might start by letting U be a uniform random variable on (0, 1) and then defining the trials in terms of the value of U.

11. Consider n independent flips of a coin having probability p of landing heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $H\ H\ T\ H\ T$, then there is a total of 3 changeovers. Find the expected number of changeovers.

HINT: Express the number of changeovers as the sum of $n - 1$ Bernoulli random variables.

12. A group of n men and m women are lined up at random. Determine the expected number of men that have a woman on at least one side of them.

HINT: Define an indicator random variable for each man.

13. Repeat Problem 12 when the group is seated at a round table.

14. An urn has m black balls. At each stage a black ball is removed and a new ball, that is black with probability p and white with probability $1 - p$, is put in its place. Find the expected number of stages needed until there are no more black balls in the urn.

NOTE: The above has possible applications to understanding the AIDS disease. Part of the body's immune system consists of a certain class of cells, known as T-cells. There are 2 types of T-cells, called CD4 and CD8. Now while the total number of T-cells of AIDS sufferers is (at least in the early stages of the disease) the same as that of healthy individuals, it has recently been discovered that the mix of CD4 and CD8 T-cells is different. Roughly 60 percent of the T-cells of a healthy person are of the CD4 type, whereas for AIDS sufferers the percentage of the T-cells that are of CD4 type appears to decrease continually. A recent model proposes that the HIV virus (the virus that causes AIDS) attacks CD4 cells, and that the body's mechanism for replacing killed T-cells does not differentiate between whether the killed T-cell was CD4 or CD8. Instead, it just produces a new T-cell that is CD4 with probability .6 and CD8 with probability .4. However, while this would seem to be a very efficient way of replacing killed T-cells when each one killed is equally likely to be any of the body's T-cells (and thus has probability .6 of being CD4), it has dangerous consequences when facing a virus that targets only the CD4 T-cells.

15. A ball is chosen, at random, from each of 5 urns. The urns contain, respectively, 1 white, 5 black; 3 white, 3 black; 6 white, 4 black; 2 white, 6 black; and 3 white, 7 black balls. Compute the expected number of white balls selected.

16. Let Z be a unit normal random variable, and for a fixed x, set

$$X = \begin{cases} Z & \text{if } Z > x \\ 0 & \text{otherwise} \end{cases}$$

Show that $E[X] = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

17. A deck of n cards, numbered 1 through n, is thoroughly shuffled so that all possible $n!$ orderings can be assumed to be equally likely. Suppose you are to make n guesses sequentially, where the ith one is a guess of the card in position i. Let N denote the number of correct guesses.

(a) If you are not given any information about your earlier guesses show that, for any strategy, $E[N] = 1$.

(b) Suppose that after each guess you are shown the card that was in the position in question. What do you think is the best strategy? Show that under this strategy

$$E[N] = \frac{1}{n} + \frac{1}{n-1} + \cdots + 1$$
$$\approx \int_1^n \frac{1}{x}\,dx = \log n$$

(c) Suppose that you are told after each guess whether you are right or wrong. In this case it can be shown that the strategy that maximizes $E[N]$ is one which keeps on guessing the same card until you are told you are correct and then changes to a new card. For this strategy show that

$$E[N] = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$
$$\approx e - 1$$

HINT: For all parts, express N as the sum of indicator (that is, Bernoulli) random variables.

18. Cards from an ordinary deck of 52 playing cards are turned face up one at a time. If the first card is an ace, or the second a deuce, or the third a three, or . . . , or the thirteenth a king, or the fourteenth an ace, and so on, we say that a match occurs. Note that we do not require that the $(13n + 1)$th card be any particular ace for a match to occur but only that it be an ace. Compute the expected number of matches that occur.

19. A certain region is inhabited by r distinct types of a certain kind of insect species, and each insect caught will, independently of the types of the previous catches, be of type i with probability

$$P_i,\ i = 1, \ldots, r \qquad \sum_1^r P_i = 1$$

(a) Compute the mean number of insects that are caught before the first type 1 catch.

(b) Compute the mean number of types of insects that are caught before the first type 1 catch.

20. An urn contains n balls—the ith having weight $W(i)$, $i = 1, \ldots, n$. The balls are removed without replacement one at a time according to the following rule: At each selection, the probability that a given ball in the urn is chosen is equal to its weight divided by the sum of the weights remaining in the urn. For instance, if at some time $i_1, \ldots, i_r$ is the set of balls remaining in the urn, then the next selection will be i_j with probability $W(i_j) \Big/ \sum_{k=1}^{r} W(i_k)$,

$j = 1, \ldots, r$. Compute the expected number of balls that are withdrawn before ball number 1.

21. For a group of 100 people compute
(a) the expected number of days of the year that are birthdays of exactly 3 people;
(b) the expected number of distinct birthdays.

22. How many times would you expect to roll a fair die before all 6 sides appeared at least once?

23. Urn 1 contains 5 white and 6 black balls, while urn 2 contains 8 white and 10 black balls. Two balls are randomly selected from urn 1 and are then put in urn 2. If 3 balls are then randomly selected from urn 2, compute the expected number of white balls in the trio.

HINT: Let $X_i = 1$ if the ith white ball initially in urn 1 is one of the three selected, and let $X_i = 0$ otherwise. Similarly, let $Y_i = 1$ if the ith white ball from urn 2 is one of the three selected, and let $Y_i = 0$ otherwise. The number of white balls in the trio can now be written as $\sum_1^5 X_i + \sum_1^8 Y_i$.

24. A bottle initially contains m large pills and n small pills. Each day a patient randomly chooses one of the pills. If a small pill is chosen, then that pill is eaten. If a large pill is chosen, then the pill is broken in two; one part is returned to the bottle (and is now considered a small pill) and the other part is then eaten.
(a) Let X denote the number of small pills in the bottle after the last large pill has been chosen and its smaller half returned. Find $E[X]$.

HINT: Define $n + m$ indicator variables, one for each of the small pills initially present and one for each of the m small pills created when a large one is split in two. Now use the argument of Example 2m.

(b) Let Y denote the day on which the last large pill is chosen. Find $E[Y]$.

HINT: What is the relationship between X and Y?

25. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N$$

That is, N is the point at which the sequence stops decreasing. Show that $E[N] = e$.

HINT: First find $P\{N \geq n\}$.

26. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having uniform distributions over (0, 1), find
(a) $E[\max(X_1, \ldots, X_n)]$;
(b) $E[\min(X_1, \ldots, X_n)]$.

27. In Problem 6, calculate the variance of the sum of the rolls.

28. In Problem 9, compute the variance of the number of empty urns.

29. If $E[X] = 1$ and $\text{Var}(X) = 5$ find
(a) $E[(2 + X)^2]$;
(b) $\text{Var}(4 + 3X)$.

30. If 10 married couples are randomly seated at a round table, compute (a) the expected number and (b) the variance of the number of wives who are seated next to their husbands.

31. Cards from an ordinary deck are turned face up one at a time. Compute the expected number of cards that need be turned face up in order to obtain
(a) 2 aces;
(b) 5 spades;
(c) all 13 hearts.

32. Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X, Y)$.

33. A die is rolled twice. Let X equal the sum of the outcomes, and let Y equal the first outcome minus the second. Compute $\text{Cov}(X, Y)$.

34. The random variables X and Y have a joint density function given by

$$f(x, y) = \begin{cases} 2e^{-2x}/x & 0 \leq x < \infty, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Cov}(X, Y)$.

35. Let $X_1, \ldots$ be independent with common mean μ and common variance σ^2, and set $Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \geq 0$, find $\text{Cov}(Y_n, Y_{n+j})$.

36. The joint density function of X and Y is given by

$$f(x, y) = \frac{1}{y} e^{-(y + x/y)}, \quad x > 0, \ y > 0$$

Find $E[X]$, $E[Y]$, and show that $\text{Cov}(X, Y) = 1$.

37. A pond contains 100 fish, of which 30 are carp. If 20 fish are caught, what are the mean and variance of the number of carp among these 20? What assumptions are you making?

38. A group of 20 people—consisting of 10 men and 10 women—are randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman. Now suppose the 20 people consisted of 10 married couples. Compute the mean and variance of the number of married couples that are paired together.

39. Let $X_1, X_2, \ldots, X_n$ be independent random variables having an unknown continuous distribution function F, and let $Y_1, Y_2, \ldots, Y_m$ be independent random variables having an unknown continuous distribution function G. Now order those $n + m$ variables and let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th smallest of the } n + m \text{ variables is from the } X \text{ sample} \\ 0 & \text{otherwise} \end{cases}$$

The random variable $R = \sum_{i=1}^{n+m} iI_i$ is the sum of the ranks of the X sample and is the basis of a standard statistical procedure (called the Wilcoxon sum of ranks test) for testing whether F and G are identical distributions. This test accepts the hypothesis that $F = G$ when R is neither too large nor too small. Assuming that the hypothesis of equality is in fact correct, compute the mean and variance of R.

HINT: Use the results of Example 3d.

40. There are two distinct methods for manufacturing certain goods, the quality of goods produced by method i being a continuous random variable having distribution F_i, $i = 1, 2$. Suppose that n goods are produced by method 1 and m by method 2. Rank the $n + m$ goods according to quality and let

$$X_j = \begin{cases} 1 & \text{if the } j\text{th best was produced from method 1} \\ 2 & \text{otherwise} \end{cases}$$

For the vector $X_1, X_2, \ldots, X_{n+m}$, which consists of n 1's and m 2's, let R denote the number of runs of 1. For instance, if $n = 5$, $m = 2$, and $\boldsymbol{X} = 1, 2, 1, 1, 1, 1, 2$, then $R = 2$. If $F_1 = F_2$ (that is, if the two methods produce identically distributed goods), what are the mean and variance of R?

41. If X_1, X_2, X_3, X_4 are (pairwise) uncorrelated random variables each having mean 0 and variance 1, compute the correlations of
(a) $X_1 + X_2$ and $X_2 + X_3$;
(b) $X_1 + X_2$ and $X_3 + X_4$.

42. Consider the following dice game, as played at a certain gambling casino: Players 1 and 2 roll in turn a pair of dice. The bank then rolls the dice to determine the outcome according to the following: player i, $i = 1, 2$, wins if his roll is strictly greater than the bank's. Let for $i = 1, 2$,

$$I_i = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$$

and show that I_1 and I_2 are positively correlated. Explain why this result was to be expected.

43. Consider a graph having n vertices labeled $1, 2, \ldots, n$, and suppose that between each of the $\binom{n}{2}$ pairs of distinct vertices an edge is, independently, present with probability p. The degree of vertex i, designated as D_i, is the number of edges that have vertex i as one of its vertices.
(a) What is the distribution of D_i?
(b) Find $\rho(D_i, D_j)$, the correlation between D_i and D_j.

44. A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5. Find
(a) $E[X]$;
(b) $E[X|Y = 1]$;
(c) $E[X|Y = 5]$.

45. An urn contains 4 white and 6 black balls. Two successive random samples of sizes 3 and 5, respectively, are drawn from the urn without replacement. Let X and Y denote the number of white balls in the two samples, and compute $E[X|Y = i]$, for $i = 1, 2, 3, 4$.

46. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty, 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

47. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y} \qquad 0 < x < y, 0 < y < \infty$$

Compute $E[X^3|Y = y]$.

48. A population is made up of r disjoint subgroups. Let p_i denote the proportion of the population that is in subgroup i, $i = 1, \ldots, r$. If the average weight of the members of subgroup i is w_i, $i = 1, \ldots, r$, what is the average weight of the members of the population?

49. A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 2 days travel. The second leads to a tunnel that returns him to his cell after 4 days travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities .5, .3, and .2, what is the expected number of days until the prisoner reaches freedom?

50. Consider the following dice game. A pair of dice are rolled. If the sum is 7, then the game ends and you win 0. If the sum is not 7, then you have the option of either stopping the game and receiving an amount equal to that sum or starting over again. For each value of i, $i = 2, \ldots, 12$, find your expected return if you employ the strategy of stopping the first time that a value at least as large as i appears. What value of i leads to the largest expected return?

HINT: Let X_i denote the return when you use the critical value i. To compute $E[X_i]$, condition on the initial sum.

51. Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability .6, compute the expected number of ducks that are hit. Assume that the number of ducks in a flock is a Poisson random variable with mean 6.

52. The number of people that enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor and if each person is equally likely to get off at any one of these N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

53. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2.5. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

54. A coin having probability p of coming up heads is continually flipped until both heads and tails have appeared. Find
(a) the expected number of flips;
(b) the probability that the last flip lands heads.

55. A person continually flips a coin until a run of 3 consecutive heads appears. Assuming that each flip independently lands heads with probability p, determine the expected number of flips required.

HINT: Let T denote the first flip that lands on tails and let it be 0 if all flips land on heads, and then condition on T.

56. There are $n + 1$ participants in a game. Each person, independently, is a winner with probability p. The winners share a total prize of 1 unit. (For instance, if 4 people win, then each of them receives $\frac{1}{4}$, whereas if there are no winners, then none of the participants receive anything.) Let A denote a specified one of the players, and let X denote the amount that is received by A.
(a) Compute the expected total prize shared by the players.
(b) Argue that $E[X] = \dfrac{1 - (1 - p)^{n+1}}{n + 1}$.
(c) Compute $E[X]$ by conditioning on whether A is a winner, and conclude that

$$E[(1 + B)^{-1}] = \frac{1 - (1 - p)^{n+1}}{(n + 1)p}$$

when B is a binomial random variable with parameters n and p.

57. Each of $m + 2$ players pays 1 unit to a kitty in order to play the following game. A fair coin is to be flipped successively n times, where n is an odd number, and the successive outcomes noted. Each player writes down, before the flips, a prediction of the outcomes. For instance, if $n = 3$, then a player might write down (H, H, T), which means that he or she predicts that the first flip will land heads, the second heads, and the third tails. After the coins are flipped, the players count their total number of correct predictions. Thus, if the actual outcomes are all heads, then the player who wrote (H, H, T) would have 2 correct predictions. The total kitty of $m + 2$ is then evenly split up among those players having the largest number of correct predictions.

Since each of the coin flips is equally likely to land on either heads or tails, m of the players have decided to make their predictions in a totally random fashion. Specifically, they will each flip one of their own fair coins n times and then use the result as their prediction. However, the final 2 of the players have formed a syndicate and will use the following strategy. One of them will make predictions in the same random fashion as the other m players, but the other one will then predict exactly the opposite of the first. That is, when the randomizing member of the syndicate predicts an H, the other member predicts a T. For instance, if the randomizing member of the syndicate predicts (H, H, T), then the other one predicts (T, T, H).
(a) Argue that exactly one of the syndicate members will have more than $n/2$ correct predictions. (Remember, n is odd.)
(b) Let X denote the number of the m nonsyndicate players that have more than $n/2$ correct predictions. What is the distribution of X?
(c) With X as defined in part (b), argue that

$$E[\text{payoff to the syndicate}] = (m + 2)E\left[\frac{1}{X + 1}\right]$$

(d) Use part (c) of Problem 56 to conclude that

$$E[\text{payoff to the syndicate}] = \frac{2(m + 2)}{m + 1}\left[1 - \left(\frac{1}{2}\right)^{m+1}\right]$$

and explicitly compute this when $m = 1$, 2, and 3.
As it can be shown that

$$\frac{2(m + 2)}{m + 1}\left[1 - \left(\frac{1}{2}\right)^{m+1}\right] > 2$$

it follows that the syndicate's strategy always gives it a positive expected profit.

58. Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. In Example 4h we showed that for $0 \le x \le 1$, $E[N(x)] = e^x$, where

$$N(x) = \min\left\{n: \sum_{i=1}^{n} U_i > x\right\}$$

This problem gives another approach to establishing this result.
(a) Show by induction on n that for $0 < x \le 1$ and all $n \ge 0$,

$$P\{N(x) \ge n + 1\} = \frac{x^n}{n!}$$

HINT: First condition on U_1 and then use the induction hypothesis.

(b) Use part (a) to conclude that

$$E[N(x)] = e^x$$

59. An urn contains 30 balls, of which 10 are red and 8 are blue. From this urn, 12 balls are randomly withdrawn. Let X denote the number of red, and Y the number of blue, balls that are withdrawn. Find $\text{Cov}(X, Y)$
(a) by defining appropriate indicator (that is, Bernoulli) random variables X_i, Y_j such that $X = \sum_{i=1}^{10} X_i$, $Y = \sum_{j=1}^{8} Y_j$;
(b) by conditioning (on either X or Y) to determine $E[XY]$.

60. Type i light bulbs function for a random amount of time having mean μ_i and standard deviation σ_i, $i = 1, 2$. A light bulb randomly chosen from a bin of bulbs is a type 1 bulb with probability p, and a type 2 bulb with probability $1 - p$. Let X denote the lifetime of this bulb. Find
(a) $E[X]$
(b) $\text{Var}(X)$.

61. In Example 4c compute the variance of the length of time until the miner reaches safety.

62. The dice game of craps was defined in Problem 26 of Chapter 2. Compute (a) the mean and (b) the variance of the number of rolls of the dice that it takes to complete one game of craps.

63. Consider a gambler who at each gamble either wins or loses her bet with probabilities p and $1 - p$. When $p > \frac{1}{2}$, a popular gambling system, known as the Kelley strategy, is to always bet the fraction $2p - 1$ of your current fortune. Compute the expected fortune after n gambles of a gambler who starts with x units and employs the Kelley strategy.

64. The number of accidents that a person has in a given year is a Poisson random variable with mean λ. However, suppose that the value of λ changes from person to person, being equal to 2 for 60 percent of the population and 3 for the other 40 percent. If a person is chosen at random, what is the probability that he will have (a) 0 accidents and (b) exactly 3 accidents in a year? What is the conditional probability that he will have 3 accidents in a given year, given that he had no accidents the preceding year?

65. Repeat Problem 64 when the proportion of the population having a value of λ less than x is equal to $1 - e^{-x}$.

66. Consider an urn containing a large number of coins and suppose that each of the coins has some probability p of turning up heads when it is flipped. However, this value of p varies from coin to coin. Suppose that the composition of the urn is such that if a coin is selected at random from the urn, then its p-value can be regarded as being the value of a random variable that is uniformly distributed over [0, 1]. If a coin is selected at random from the urn and flipped twice, compute the probability that
(a) the first flip is a head;
(b) both flips are heads.

67. In Problem 66, suppose that the coin is tossed n times. Let X denote the number of heads that occur. Show that

$$P\{X = i\} = \frac{1}{n+1} \qquad i = 0, 1, \ldots, n$$

HINT: Make use of the fact that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!\,(b-1)!}{(a+b-1)!}$$

when a and b are positive integers.

68. Suppose that in Problem 66 we continue to flip the coin until a head appears. Let N denote the number of flips needed. Find
(a) $P\{N \geq i\}$, $i \geq 0$;
(b) $P\{N = i\}$;
(c) $E[N]$.

69. In Example 5b let S denote the signal sent and R the signal received.
(a) Compute $E[R]$.
(b) Compute $\text{Var}(R)$.
(c) Is R normally distributed?
(d) Compute $\text{Cov}(R, S)$.

70. In Example 5c, suppose that X is uniformly distributed over (0, 1). If the discretized regions are determined by $a_0 = 0$, $a_1 = \frac{1}{2}$, $a_2 = 1$, determine the optimal quantizer Y and compute $E[(X - Y)^2]$.

71. The moment generating function of X is given by $M_X(t) = \exp\{2e^t - 2\}$ and that of Y by $M_Y(t) = \left(\frac{1}{4}\right)^{10}$. If X and Y are independent, what are
(a) $P\{X + Y = 2\}$;
(b) $P\{XY = 0\}$;
(c) $E[XY]$?

72. Let X be the value of the first die and Y the sum of the values when two dice are rolled. Compute the joint moment generating function of X and Y.

73. The joint density of X and Y is given by

$$f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2} \qquad 0 < y < \infty,\ -\infty < x < \infty$$

(a) Compute the joint moment generating function of X and Y.
(b) Compute the individual moment generating functions.

74. Two envelopes, each containing a check, are placed in front of you. You are to choose one of the envelopes, open it, and see the amount of the check. At this point you can either accept that amount or you can exchange it for the check in the unopened envelope. What should you do? Is it possible to devise a strategy that does better than just accepting the first envelope?

Let A and B, $A < B$, denote the (unknown) amounts of the checks, and note that the strategy that randomly selects an envelope and always accepts its check has an expected return of $(A + B)/2$. Consider the following strategy: Let $F(\cdot)$ be any strictly increasing (that is, continuous) distribution function. Randomly choose an envelope and open it. If the discovered check has value x then accept it with probability $F(x)$, and with probability $1 - F(x)$ exchange it.

(a) Show that if you employ the latter strategy, then your expected return is greater than $(A + B)/2$.

HINT: Condition on whether the first envelope has value A or B.

Now consider the strategy that fixes a value x, and then accepts the first check if its value is greater than x and exchanges it otherwise.

(b) Show that for any x, the expected return under the x-strategy is always at least $(A + B)/2$, and that it is strictly larger than $(A + B)/2$ if x lies between A and B.

(c) Let X be a continuous random variable on the whole line, and consider the following strategy: Generate the value of X, and if $X = x$ then employ the x-strategy of part (b). Show that the expected return under this strategy is greater than $(A + B)/2$.

THEORETICAL EXERCISES

1. Show that $E[(X - a)^2]$ is minimized at $a = E[X]$.

2. Suppose that X is a continuous random variable with density function f. Show that $E[|X - a|]$ is minimized when a is equal to the median of F.

HINT: Write

$$E[|X - a|] = \int |x - a| f(x)\, dx$$

Now break up the integral into the regions where $x < a$ and where $x > a$ and differentiate.

3. Prove Proposition 2.1 when

(a) X and Y have a joint probability mass function;

(b) X and Y have a joint probability density function and $g(x, y) \geq 0$ for all x, y.

4. Let X be a random variable having finite expectation μ and variance σ^2, and let $g(\cdot)$ be a twice differentiable function. Show that

$$E[g(X)] \approx g(\mu) + \frac{g''(\mu)}{2}\sigma^2$$

HINT: Expand $g(\cdot)$ in a Taylor series about μ. Use the first three terms and ignore the remainder.

5. Let $A_1, A_2, \ldots, A_n$ be arbitrary events, and define $C_k = \{\text{at least } k \text{ of the } A_i \text{ occur}\}$. Show that

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k)$$

HINT: Let X denote the number of the A_i that occur. Show that both sides of the above are equal to $E[X]$.

6. In the text we noted that

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i]$$

when the X_i are all nonnegative random variables. Since an integral is a limit of sums, one might expect that

$$E\left[\int_0^{\infty} X(t)\, dt\right] = \int_0^{\infty} E[X(t)]\, dt$$

whenever $X(t)$, $0 \leq t < \infty$, are all nonnegative random variables; and this result is indeed true. Use it to give another proof of the result that, for a nonnegative random variable X,

$$E[X] = \int_0^{\infty} P\{X > t\}\, dt$$

HINT: Define, for each nonnegative t, the random variable $X(t)$ by

$$X(t) = \begin{cases} 1 & \text{if } t < X \\ 0 & \text{if } t \geq X \end{cases}$$

Now relate $\int_0^{\infty} X(t)\, dt$ to X.

7. We say that X is *stochastically larger* than Y, written $X \geq_{st} Y$, if for all t,

$$P\{X > t\} \geq P\{Y > t\}$$

Show that if $X \geq_{st} Y$, then $E[X] \geq E[Y]$ when

(a) X and Y are nonnegative random variables;

(b) X and Y are arbitrary random variables.

HINT: Write X as

$$X = X^+ - X^-$$

where

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0, \end{cases} \qquad X^- = \begin{cases} 0 & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$$

Similarly, represent Y as $Y^+ - Y^-$. Then make use of part (a).

8. Show that X is stochastically larger than Y if and only if

$$E[f(X)] \geq E[f(Y)]$$

for all increasing functions f.

HINT: If $X \geq_{st} Y$, show that $E[f(X)] \geq E[f(Y)]$ by showing that $f(X) \geq_{st} f(Y)$ and then using Theoretical Exercise 7. To show that $E[f(X)] \geq E[f(Y)]$ for all increasing functions f implies that $P\{X > t\} \geq P\{Y > t\}$, define an appropriate increasing function f.

9. A coin having probability p of landing heads is flipped n times. Compute the expected number of runs of heads of size 1, of size 2, of size k, $1 \leq k \leq n$.

10. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed positive random variables. Find, for $k \leq n$,

$$E\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i}\right]$$

11. Consider n independent trials each resulting in any one of r possible outcomes with probabilities $P_1, P_2, \ldots, P_r$. Let X denote the number of outcomes that never occur in any of the trials. Find $E[X]$ and show that among all probability vectors $P_1, \ldots, P_r$, $E[X]$ is minimized when $P_i = 1/r$, $i = 1, \ldots, r$.

12. Independent trials are performed. If the ith such trial results in a success with probability P_i, compute (a) the expected number, and (b) the variance, of the number of successes that occur in the first n trials. Does independence make a difference in part (a)? In part (b)?

13. Let $X_1, \ldots, X_n$ be independent and identically distributed continuous random variables. We say that a record value occurs at time j, $j \leq n$, if $X_j \geq X_i$ for all $1 \leq i \leq j$. Show that

(a) $E[\text{number of record values}] = \sum_{j=1}^{n} 1/j$;

(b) $\text{Var}(\text{number of record values}) = \sum_{j=1}^{n} (j - 1)/j^2$.

14. For Example 2j show that the variance of the number of coupons needed to amass a full set is equal to

$$\sum_{i=1}^{N-1} \frac{iN}{(N-i)^2}$$

When N is large, this can be shown to be approximately equal (in the sense that their ratio approaches 1 as $N \to \infty$) to $N^2(\pi^2/6)$.

15. Consider n independent trials, the ith of which results in a success with probability P_i.

(a) Compute the expected number of successes in the n trials—call it μ.

(b) For fixed value of μ, what choice of $P_1, \ldots, P_n$ maximizes the variance of the number of successes?

(c) What choice minimizes the variance?

16. Suppose that balls are randomly removed from an urn initially containing n white and m black balls. It was shown in Example 2m that $E[X] = 1 + m/(n + 1)$, when X is the number of draws needed to obtain a white ball.

(a) Compute $\text{Var}(X)$.

(b) Show that the expected number of balls that need be drawn to amass a total of k white balls is $k[1 + m/(n + 1)]$.

HINT: Let Y_i, $i = 1, \ldots, n + 1$, denote the number of black balls withdrawn after the $(i - 1)$st white ball and before the ith white ball. Argue that the Y_i, $i = 1, \ldots, n + 1$, are identically distributed.

17. Suppose that X_1 and X_2 are independent random variables having a common mean μ. Suppose also that $\text{Var}(X_1) = \sigma_1^2$ and $\text{Var}(X_2) = \sigma_2^2$. The value of μ is unknown and it is proposed to estimate μ by a weighted average of X_1 and X_2. That is, $\lambda X_1 + (1 - \lambda)X_2$ will be used as an estimate of μ, for some appropriate value of λ. Which value of λ yields the estimate having the lowest possible variance? Explain why it is desirable to use this value of λ.

18. In Example 3g we showed that the covariance of the multinomial random variables N_i and N_j is equal to $-mP_iP_j$ by expressing N_i and N_j as the sum of indicator variables. This result could also have been obtained by using the formula

$$\text{Var}(N_i + N_j) = \text{Var}(N_i) + \text{Var}(N_j) + 2\,\text{Cov}(N_i, N_j)$$

(a) What is the distribution of $N_i + N_j$?

(b) Use the identity above to show that $\text{Cov}(N_i, N_j) = -mP_iP_j$.

19. If X and Y are identically distributed, not necessarily independent, show that

$$\text{Cov}(X + Y, X - Y) = 0$$

20. *The Conditional Covariance Formula.* The conditional covariance of X and Y, given Z, is defined by

$$\text{Cov}(X, Y|Z) \equiv E[(X - E[X|Z])(Y - E[Y|Z])|Z]$$

(a) Show that

$$\text{Cov}(X, Y|Z) = E[XY|Z] - E[X|Z]E[Y|Z]$$

(b) Prove the conditional covariance formula

$$\text{Cov}(X, Y) = E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z])$$

(c) Set $X = Y$ in part (b) and obtain the conditional variance formula.

21. Let $X_{(i)}$, $i = 1, \ldots, n$, denote the order statistics from a set of n uniform $(0, 1)$ random variables and note that the density function of $X_{(i)}$ is given by

$$f(x) = \frac{n!}{(i-1)!\,(n-i)!} x^{i-1}(1-x)^{n-i} \qquad 0 < x < 1$$

(a) Compute $\text{Var}(X_{(i)})$, $i = 1, \ldots, n$.

(b) Which value of i minimizes and which value maximizes $\text{Var}(X_{(i)})$?

22. If $Y = a + bX$, show that

$$\rho(X, Y) = \begin{cases} +1 & \text{if } b > 0 \\ -1 & \text{if } b < 0 \end{cases}$$

23. If Z is a unit normal random variable and if Y is defined by $Y = a + bZ + cZ^2$, show that

$$\rho(Y, Z) = \frac{b}{\sqrt{b^2 + 2c^2}}$$

24. Prove the Cauchy–Schwarz inequality, namely, that

$$(E[XY])^2 \leq E[X^2]E[Y^2]$$

HINT: Unless $Y = -tX$ for some constant, in which case this inequality holds with equality, if follows that for all t,

$$0 < E[(tX + Y)^2] = E[X^2]t^2 + 2E[XY]t + E[Y^2]$$

Hence the roots of the quadratic equation

$$E[X^2]t^2 + 2E[XY]t + E[Y^2] = 0$$

must be imaginary, which implies that the discriminant of this quadratic equation must be negative.

25. Show that if X and Y are independent, then

$$E[X|Y = y] = E[X] \qquad \text{for all } y$$

(a) in the discrete case;
(b) in the continuous case.

26. Prove that $E[g(X)Y|X] = g(X)E[Y|X]$.

27. Prove that if $E[Y|X = x] = E[Y]$ for all x, then X and Y are uncorrelated, and give a counterexample to show that the converse is not true.

HINT: Prove and use the fact that $E[XY] = E[XE[Y|X]]$.

28. Show $\text{Cov}(X, E[Y|X]) = \text{Cov}(X, Y)$.

29. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. Find

$$E[X_1|X_1 + \cdots + X_n = x]$$

30. Consider Example 3g, which is concerned with the multinomial distribution. Use conditional expectation to compute $E[N_iN_j]$ and then use this to verify the formula for $\text{Cov}(N_i, N_j)$ given in Example 3g.

31. An urn initially contains b black and w white balls. At each stage we add r black balls and then withdraw, at random, r from the $b + w + r$. Show that

$$E[\text{number of white balls after stage } t] = \left(\frac{b + w}{b + w + r}\right)^t w$$

32. Prove Equation (6.1b).

33. A coin, which lands on heads with probability p, is continually flipped. Compute the expected number of flips that are made until a string of r heads in a row is obtained.

HINT: Condition on the time of the first occurrence of tails, to obtain the equation

$$E[X] = (1 - p)\sum_{i=1}^{r} p^{i-1}(i + E[X]) + (1 - p)\sum_{i=r+1}^{\infty} p^{i-1}\, r$$

Simplify and solve for $E[X]$.

34. For another approach to Theoretical Exercise 33, let T_r denote the number of flips required to obtain a run of r consecutive heads.
(a) Determine $E[T_r|T_{r-1}]$.
(b) Determine $E[T_r]$ in terms of $E[T_{r-1}]$.

(c) What is $E[T_1]$?
(d) What is $E[T_r]$?

35. **(a)** Prove that

$$E[X] = E[X|X < a]P\{X < a\} + E[X|X \geq a]P\{X \geq a\}$$

HINT: Define an appropriate random variable and then compute $E[X]$ by conditioning on it.

(b) Use part (a) to prove Markov's inequality, which states that if $P\{X \geq 0\} = 1$, then for $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

36. One ball at a time is randomly selected from an urn containing a white and b black balls until all of the remaining balls are of the same color. Let $M_{a,b}$ denote the expected number of balls left in the urn when the experiment ends. Compute a recursive formula for $M_{a,b}$ and solve when $a = 3$, $b = 5$.

37. An urn contains a white and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.
(a) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a + b}\right)M_n + 1$$

(b) Use part (a) to prove that

$$M_n = a + b - b\left(1 - \frac{1}{a + b}\right)^n$$

(c) What is the probability that the $(n + 1)$st ball drawn is white?

38. The best linear predictor of Y with respect to X_1 and X_2 is equal to $a + bX_1 + cX_2$, where a, b, and c are chosen to minimize

$$E[(Y - (a + bX_1 + cX_2))^2]$$

Determine a, b, and c.

39. The best quadratic predictor of Y with respect to X is $a + bX + cX^2$, where a, b, and c are chosen to minimize $E[(Y - (a + bX + cX^2))^2]$. Determine a, b, and c.

40. X and Y are jointly normally distributed with joint density function given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \times \exp\left\{-\frac{1}{2(1 - \rho^2)}\left[\left(\frac{x - \mu_x}{\sigma_x}\right)^2 + \left(\frac{y - \mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

(a) Show that the conditional distribution of Y, given $X = x$, is normal with mean $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$ and variance $\sigma_2^2(1 - \rho^2)$.

(b) Show that $\text{Corr}(X, Y) = \rho$.

(c) Argue that X and Y are independent if and only if $\rho = 0$.

41. Let X be a normal random variable with parameters $\mu = 0$ and $\sigma^2 = 1$ and let I, independent of X, be such that $P\{I = 1\} = \frac{1}{2} = P\{I = 0\}$. Now define Y by

$$Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases}$$

In words, Y is equally likely to equal either X or $-X$.

(a) Are X and Y independent?

(b) Are I and Y independent?

(c) Show that Y is normal with mean 0 and variance 1.

(d) Show that $\text{Cov}(X, Y) = 0$.

42. It follows from Proposition 5.1 and the fact that the best linear predictor of Y with respect to X is $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)$ that if

$$E[Y|X] = a + bX$$

then

$$a = \mu_y - \rho\frac{\sigma_y}{\sigma_x}\mu_x \qquad b = \rho\frac{\sigma_y}{\sigma_x}$$

(Why?) Verify this directly.

43. For random variables X and Z show that

$$E[(X - Y)^2] = E[X^2] - E[Y^2]$$

where

$$Y = E[X|Z]$$

44. Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual will, by the end of its lifetime, have produced j new offspring with probability P_j, $j \geq 0$, independently of the number produced by any other individual. The number of individuals initially present, denoted by X_0, is called the size of the zeroth generation. All offspring of the zeroth generation constitute the first generation, and their number is denoted by X_1. In general, let X_n denote the size of the nth generation. Let $\mu = \sum_{j=0}^{\infty} jP_j$ and $\sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j$ denote, respectively, the mean and the variance of the number of offspring produced by a single individual. Suppose that $X_0 = 1$—that is, initially there is a single individual in the population.

(a) Show that

$$E[X_n] = \mu E[X_{n-1}]$$

(b) Use part (a) to conclude that

$$E[X_n] = \mu^n$$

(c) Show that

$$\text{Var}(X_n) = \sigma^2 \mu^{n-1} + \mu^2 \text{Var}(X_{n-1})$$

(d) Use part (c) to conclude that

$$\text{Var}(X_n) = \begin{cases} \sigma^2 \mu^{n-1}\left(\frac{\mu^n - 1}{\mu - 1}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

The case described above is known as a *branching process,* and an important question for a population that evolves along such lines is the probability that the population will eventually die out. Let π denote this probability when the population starts with a single individual. That is,

$$\pi = P\{\text{population eventually dies out}|X_0 = 1\}$$

(e) Argue that π satisfies

$$\pi = \sum_{j=0}^{\infty} P_j \pi^j$$

HINT: Condition on the number of offspring of the initial member of the population.

45. Verify the formula for the moment generating function of a uniform random variable that is given in Table 7.2. Also, differentiate to verify the formulas for the mean and variance.

46. For a standard normal random variable Z, let $\mu_n = E[Z^n]$. Show that

$$\mu_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \frac{(2j)!}{2^j j!} & \text{when } n = 2j \end{cases}$$

HINT: Start by expanding the moment generating function of Z into a Taylor series about 0 to obtain

$$E[e^{tZ}] = e^{t^2/2} = \sum_{j=0}^{\infty} \frac{(t^2/2)^j}{j!}$$

47. Let X be a normal random variable with mean μ and variance σ^2. Use the results of Theoretical Exercise 46 to show that

$$E[X^n] = \sum_{j=0}^{[n/2]} \frac{\binom{n}{2j} \mu^{n-2j}\sigma^{2j}(2j)!}{2^j j!}$$

In the equation above, $[n/2]$ is the largest integer less than or equal to $n/2$. Check your answer by letting $n = 1$ and $n = 2$.

48. If $Y = aX + b$, where a and b are constants, express the moment generating function of Y in terms of the moment generating function of X.

49. The positive random variable X is said to be a *lognormal* random variable with parameters μ and σ^2 if $\log(X)$ is a normal random variable with mean μ and variance σ^2. Use the normal moment generating function to find the mean and variance of a lognormal random variable.

50. Let X have moment generating function $M(t)$, and define $\Psi(t) = \log M(t)$. Show that

$$\Psi''(t)\big|_{t=0} = \text{Var}(X)$$

51. Use Table 7.2 to determine the distribution of $\sum_{i=1}^{n} X_i$ when $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables, each having mean $1/\lambda$.

52. Show how to compute $\text{Cov}(X, Y)$ from the joint moment generating function of X and Y.

53. Suppose that $X_1, \ldots, X_n$ have a multivariate normal distribution. Show that $X_1, \ldots, X_n$ are independent random variables if and only if

$$\text{Cov}(X_i, X_j) = 0 \qquad \text{when } i \neq j$$

54. If Z is a unit normal random variable, what is $\text{Cov}(Z, Z^2)$?

SELF-TEST PROBLEMS AND EXERCISES

1. Consider a list of m names, where the same name may appear more than once on the list. Let $n(i)$ denote the number of times that the name in position i appears on the list, $i = 1, \ldots, m$, and let d denote the number of distinct names on the list.
(a) Express d in terms of the variables m, n_i, $i = 1, \ldots, m$.
Let U be a uniform (0, 1) random variable, and let $X = [mU] + 1$.
(b) What is the probability mass function of X?
(c) Argue that $E[m/n(X)] = d$.

2. An urn has n white and m black balls which are removed one at a time in a randomly chosen order. Find the expected number of instances in which a white ball is immediately followed by a black one.

3. Twenty individuals, consisting of 10 married couples, are to be seated at five different tables, with four people at each table.
(a) If the seating is done "at random," what is the expected number of married couples that are seated at the same table?
(b) If two men and two women are randomly chosen to be seated at each table, what is the expected number of married couples that are seated at the same table?

4. If a die is to be rolled until all sides have appeared at least once, find the expected number of times that outcome 1 appears.

5. A deck of $2n$ cards consists of n red and n black cards. These cards are shuffled and then turned over one at a time. Suppose that each time a red card is turned over we win 1 unit if more red cards than black cards have been turned over by that time. (For instance, if $n = 2$ and the result is r b r b, then we would win a total of 2 units.) Find the expected amount that we win.

6. Let $A_1, A_2, \ldots, A_n$ be events, and let N denote the number of them that occur. Also, let $I = 1$ if all of these events occur, and let it be 0 otherwise. Prove Bonferroni's inequality, namely that

$$P(A_1 \cdots A_n) \geq \sum_{i=1}^{n} P(A_i) - (n - 1)$$

HINT: Argue first that $N \leq n - 1 + I$.

7. Suppose that k of the balls numbered $1, 2, \ldots, n$, where $n > k$, are randomly chosen. Let X denote the maximum numbered ball chosen. Also, let R denote the number of the $n - k$ unchosen balls that have higher numbers than all the chosen balls.
(a) What is the relationship between X and R?
(b) Express R as the sum of $n - k$ suitably defined Bernoulli random variables.
(c) Use parts (a) and (b) to find $E[X]$.
(Note that $E[X]$ was obtained previously in Theoretical Exercise 28 of Chapter 4.)

8. Let X be a Poisson random variable with mean λ. Show that if λ is not too small, then

$$\text{Var}(\sqrt{X}) \approx .25$$

HINT: Use the result of Theoretical Exercise 4 to approximate $E[\sqrt{X}]$.

9. Suppose in Self-Test Problem 3 that the 20 people are to be seated at seven tables, three of which have 4 seats and four of which have 2 seats. If the people are randomly seated, find the expected value of the number of married couples that are seated at the same table.

10. Individuals 1 through n, $n > 1$, are to be recruited into a firm in the following manner. Individual 1 starts the firm and recruits individual 2. Individuals 1 and 2 will then compete to recruit individual 3. Once individual 3 is recruited, individuals 1, 2, and 3 will compete to recruit individual 4, and so on. Suppose that when individuals $1, 2, \ldots, i$ compete to recruit individual $i + 1$, each of them is equally likely to be the successful recruiter.
(a) Find the expected number of the individuals $1, \ldots, n$ that did not recruit anyone else.
(b) Derive an expression for the variance of the number of individuals who did not recruit anyone else and evaluate it for $n = 5$.

11. The nine players on a basketball team consist of 2 centers, 3 forwards, and 4 backcourt players. If the players are paired up at random into three groups of size 3 each, find the (a) expected value and the (b) variance of the number of triplets consisting of one of each type of player.

12. A deck of 52 cards is shuffled and a bridge hand of 13 cards is dealt out. Let X and Y denote, respectively, the number of aces and the number of spades in the dealt hand.
(a) Show that X and Y are uncorrelated.
(b) Are they independent?

13. Each coin in a bin has a value attached to it. Each time that a coin with value p is flipped it lands on heads with probability p. When a coin is randomly chosen from the bin, its value is uniformly distributed on (0, 1). Suppose that after the coin is chosen but before it is flipped, you must predict whether it will land heads or tails. You will win 1 if you are correct and will lose 1 otherwise.
(a) What is your expected gain if you are not told the value of the coin?
(b) Suppose now that you are allowed to inspect the coin before it is flipped, with the result of your inspection being that you learn the value of the coin. As a function of p, the value of the coin, what prediction should you make?
(c) Under the conditions of part (b), what is your expected gain?

14. In Self-Test Problem 1 we showed how to use the value of a uniform (0, 1) random variable (commonly called a *random number*) to obtain the value of a random variable whose mean is equal to the expected number of distinct names on a list. However, its use required that one chooses a random position and then determine the number of times that the name in that position appears on the list. Another approach, which can be more efficient when there is a large amount of name replication, is as follows. As before, start by choosing the random variable X as in Problem 3. Now identify the name in position X, and then go through the list starting at the beginning until that name appears. Let I equal 0 if you encounter that name before getting to position X, and let I equal 1 if your first encounter with the name is at position X. Show that $E[mI] = d$.

HINT: Compute $E[I]$ by using conditional expectation.

CHAPTER 8

Limit Theorems

8.1 INTRODUCTION

The most important theoretical results in probability theory are limit theorems. Of these, the most important are those that are classified either under the heading *laws of large numbers* or under the heading *central limit theorems.* Usually, theorems are considered to be laws of large numbers if they are concerned with stating conditions under which the average of a sequence of random variables converges (in some sense) to the expected average. On the other hand, central limit theorems are concerned with determining conditions under which the sum of a large number of random variables has a probability distribution that is approximately normal.

8.2 CHEBYSHEV'S INEQUALITY AND THE WEAK LAW OF LARGE NUMBERS

We start this section by proving a result known as Markov's inequality.

Proposition 2.1 Markov's inequality

If X is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

Proof: For $a > 0$, let

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{otherwise} \end{cases}$$

and note that since $X \geq 0$,

$$I \leq \frac{X}{a}$$

Taking expectations of the above yields that

$$E[I] \leq \frac{E[X]}{a}$$

which, since $E[I] = P\{X \geq a\}$, proves the result.

As a corollary, we obtain Proposition 2.2.

Proposition 2.2 Chebyshev's inequality

If X is a random variable with finite mean μ and variance σ^2, then for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Proof: Since $(X - \mu)^2$ is a nonnegative random variable, we can apply Markov's inequality (with $a = k^2$) to obtain

$$P\{(X - \mu)^2 \geq k^2\} \leq \frac{E[(X - \mu)^2]}{k^2} \tag{2.1}$$

But since $(X - \mu)^2 \geq k^2$ if and only if $|X - \mu| \geq k$, Equation (2.1) is equivalent to

$$P\{|X - \mu| \geq k\} \leq \frac{E[(X - \mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$$

and the proof is complete.

The importance of Markov's and Chebyshev's inequalities is that they enable us to derive bounds on probabilities when only the mean, or both the mean and the variance, of the probability distribution are known. Of course, if the actual distribution were known, then the desired probabilities could be exactly computed and we would not need to resort to bounds.

Example 2a. Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Solution Let X be the number of items that will be produced in a week:

(a) By Markov's inequality

$$P\{X > 75\} \leq \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}$$

(b) By Chebyshev's inequality

$$P\{|X - 50| \geq 10\} \leq \frac{\sigma^2}{10^2} = \frac{1}{4}$$

Hence

$$P\{|X - 50| < 10\} \geq 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

so the probability that this week's production will be between 40 and 60 is at least .75. ∎

As Chebyshev's inequality is valid for all distributions of the random variable X, we cannot expect the bound on the probability to be very close to the actual probability in most cases. For instance, consider Example 2b.

Example 2b. If X is uniformly distributed over the interval (0, 10), then, as $E[X] = 5$, $\text{Var}(X) = \frac{25}{3}$, it follows from Chebyshev's inequality that

$$P\{|X - 5| > 4\} \leq \frac{25}{3(16)} \approx .52$$

whereas the exact result is

$$P\{|X - 5| > 4\} = .20$$

Thus, although Chebyshev's inequality is correct, the upper bound that it provides is not particularly close to the actual probability.

Similarly, if X is a normal random variable with mean μ and variance σ^2, Chebyshev's inequality states that

$$P\{|X - \mu| > 2\sigma\} \leq \tfrac{1}{4}$$

whereas the actual probability is given by

$$P\{|X - \mu| > 2\sigma\} = P\left\{\left|\frac{X - \mu}{\sigma}\right| > 2\right\} = 2[1 - \Phi(2)] \approx .0456$$ ∎

Chebyshev's inequality is often used as a theoretical tool in proving results. This is illustrated first by Proposition 2.3 and then, most importantly, by the weak law of large numbers.

Proposition 2.3

If $\text{Var}(X) = 0$, then

$$P\{X = E[X]\} = 1$$

In other words, the only random variables having variances equal to 0 are those that are constant with probability 1.

Proof: By Chebyshev's inequality we have, for any $n \geq 1$

$$P\left\{|X - \mu| > \frac{1}{n}\right\} = 0$$

Letting $n \to \infty$ and using the continuity property of probability yields

$$0 = \lim_{n\to\infty} P\left\{|X - \mu| > \frac{1}{n}\right\} = P\left\{\lim_{n\to\infty}\left\{|X - \mu| > \frac{1}{n}\right\}\right\}$$
$$= P\{X \neq \mu\}$$

and the result is established.

Theorem 2.1 The weak law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, each having finite mean $E[X_i] = \mu$. Then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \quad \text{as} \quad n \to \infty$$

Proof: We shall prove the result only under the additional assumption that the random variables have a finite variance σ^2. Now, as

$$E\left[\frac{X_1 + \cdots + X_n}{n}\right] = \mu \quad \text{and} \quad \text{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{\sigma^2}{n}$$

it follows from Chebyshev's inequality that

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \leq \frac{\sigma^2}{n\varepsilon^2}$$

and the result is proved.

The weak law of large numbers was originally proved by James Bernoulli for the special case where the X_i are 0—1 (that is, Bernoulli) random variables. His statement and proof of this theorem were presented in his book *Ars Conjectandi,* which was published in 1713, 8 years after his death by his nephew Nicholas Bernoulli. It should be noted that as Chebyshev's inequality was not known in his time, Bernoulli had to resort to a quite ingenious proof to establish the result. The general form of the weak law of large numbers presented in Theorem 2.1 was proved by the Russian mathematician Khintchine.

8.3 THE CENTRAL LIMIT THEOREM

The central limit theorem is one of the most remarkable results in probability theory. Loosely put, it states that the sum of a large number of independent random variables has a distribution that is approximately normal. Hence it not only provides a simple method for computing approximate probabilities for sums of independent random variables, but it also helps explain the remarkable fact that the empirical frequencies of so many natural populations exhibit bell-shaped (that is, normal) curves.

In its simplest form the central limit theorem is as follows.

Theorem 3.1 The central limit theorem

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx \quad \text{as} \quad n \to \infty$$

The key to the proof of the central limit theorem is the following lemma, which we state without proof.

Lemma 3.1

Let $Z_1, Z_2, \ldots$ be a sequence of random variables having distribution functions F_{Z_n} and moment generating functions M_{Z_n}, $n \geq 1$; and let Z be a random variable having distribution function F_Z and moment generating function M_Z. If $M_{Z_n}(t) \to M_Z(t)$ for all t, then $F_{Z_n}(t) \to F_Z(t)$ for all t at which $F_Z(t)$ is continuous.

If we let Z be a unit normal random variable, then, as $M_Z(t) = e^{t^2/2}$, it follows from Lemma 3.1 that if $M_{Z_n}(t) \to e^{t^2/2}$ as $n \to \infty$, then $F_{Z_n}(t) \to \Phi(t)$ as $n \to \infty$.

We are now ready to prove the central limit theorem.

Proof of the Central Limit Theorem: Let us assume at first that $\mu = 0$ and $\sigma^2 = 1$. We shall prove the theorem under the assumption that the

moment generating function of the X_i, $M(t)$, exists and is finite. Now the moment generating function of $X_i/\sqrt{n}$ is given by

$$E\left[\exp\left\{\frac{tX_i}{\sqrt{n}}\right\}\right] = M\left(\frac{t}{\sqrt{n}}\right)$$

and thus the moment generating function of $\sum_{i=1}^{n} X_i/\sqrt{n}$ is given by $\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$. Let

$$L(t) = \log M(t)$$

and note that

$$\begin{aligned} L(0) &= 0 \\ L'(0) &= \frac{M'(0)}{M(0)} \\ &= \mu \\ &= 0 \\ L''(0) &= \frac{M(0)M''(0) - [M'(0)]^2}{[M(0)]^2} \\ &= E[X^2] \\ &= 1 \end{aligned}$$

Now, to prove the theorem, we must show that $[M(t/\sqrt{n})]^n \to e^{t^2/2}$ as $n \to \infty$, or equivalently, that $nL(t/\sqrt{n}) \to t^2/2$ as $n \to \infty$. To show this, note that

$$\begin{aligned} \lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} &= \lim_{n\to\infty} \frac{-L'(t/\sqrt{n})\, n^{-3/2}\, t}{-2n^{-2}} && \text{by L'Hospital's rule} \\ &= \lim_{n\to\infty} \left[\frac{L'(t/\sqrt{n})t}{2n^{-1/2}}\right] \\ &= \lim_{n\to\infty} \left[\frac{-L''(t/\sqrt{n})\, n^{-3/2}\, t^2}{-2n^{-3/2}}\right] && \text{again by L'Hospital's rule} \\ &= \lim_{n\to\infty} \left[L''\left(\frac{t}{\sqrt{n}}\right)\frac{t^2}{2}\right] \\ &= \frac{t^2}{2} \end{aligned}$$

Thus the central limit theorem is proved when $\mu = 0$ and $\sigma^2 = 1$. The result now follows in the general case by considering the standardized random variables $X_i^* = (X_i - \mu)/\sigma$ and applying the result above, since $E[X_i^*] = 0$, $\text{Var}(X_i^*) = 1$.

REMARK. Although Theorem 3.1 only states that for each a,

$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \Phi(a)$$

it can, in fact, be shown that the convergence is uniform in a. [We say that $f_n(a) \to f(a)$ uniformly in a, if for each $\varepsilon > 0$, there exists an N such that $|f_n(a) - f(a)| < \varepsilon$ for all a whenever $n \ge N$.]

The first version of the central limit theorem was proved by DeMoivre around 1733 for the special case where the X_i are Bernoulli random variables with $p = \frac{1}{2}$. This was subsequently extended by Laplace to the case of arbitrary p. (Since a binomial random variable may be regarded as the sum of n independent and identically distributed Bernoulli random variables, this justifies the normal approximation to the binomial that was presented in Section 5.4.1.) Laplace also discovered the more general form of the central limit theorem given in Theorem 3.1. His proof, however, was not completely rigorous and, in fact, cannot easily be made rigorous. A truly rigorous proof of the central limit theorem was first presented by the Russian mathematician Liapounoff in the period 1901–1902.

This important theorem is illustrated by the central limit theorem module on the text diskette. This diskette plots the density function of the sum of n independent and identically distributed random variables that each take on one of the values 0, 1, 2, 3, 4. When using it, one enters the probability mass function and the desired value of n. Figure 8.1 shows the resulting plots for a specified probability mass function when (a) $n = 5$, (b) $n = 10$, (c) $n = 25$, and (d) $n = 100$.

Example 3a. An astronomer is interested in measuring, in light years, the distance from his observatory to a distant star. Although the astronomer has a measuring technique, he knows that, because of changing atmospheric conditions and normal error, each time a measurement is made it will not yield the exact distance but merely an estimate. As a result the astronomer plans to make a series of measurements and then use the average value of these measurements as his estimated value of the actual distance. If the astronomer believes that the values of the measurements are independent and identically distributed random variables having a common mean d (the actual distance) and a common variance of 4 (light years), how many measurements need he make to be reasonably sure that his estimated distance is accurate to within $\pm.5$ light year?

Solution Suppose that the astronomer decides to make n observations. If $X_1, X_2, \ldots, X_n$ are the n measurements, then, from the central limit theorem, it follows that

$$Z_n = \frac{\sum_{i=1}^{n} X_i - nd}{2\sqrt{n}}$$

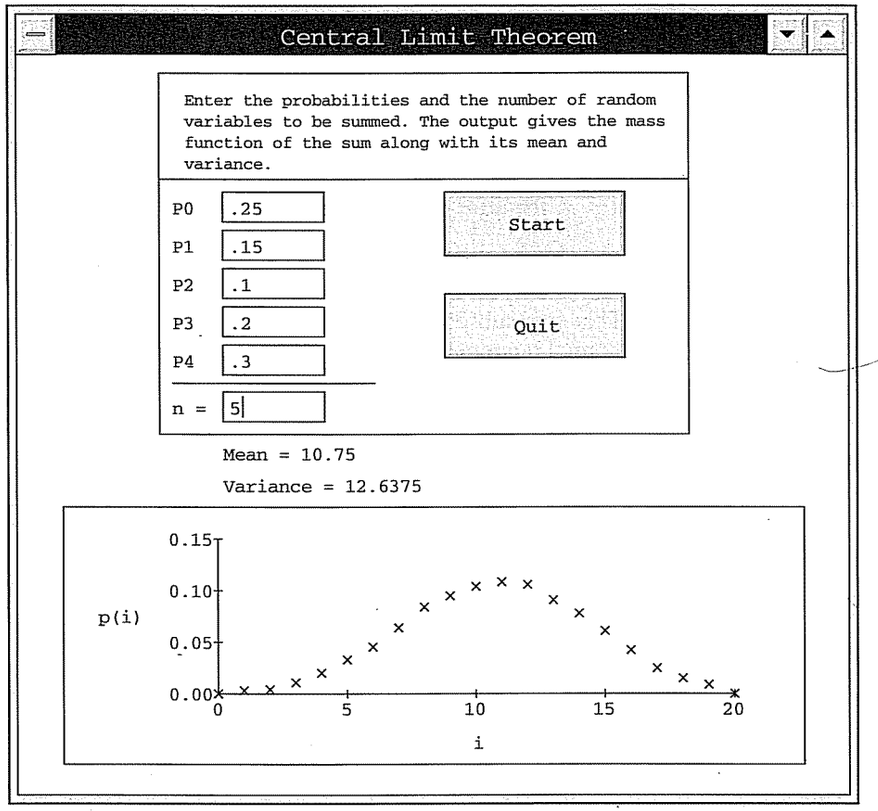


Figure 8.1(a)

has approximately a unit normal distribution. Hence

$$P\left\{-.5 \le \frac{\sum_{i=1}^{n} X_i}{n} - d \le .5\right\} = P\left\{-.5\frac{\sqrt{n}}{2} \le Z_n \le .5\frac{\sqrt{n}}{2}\right\}$$

$$\approx \Phi\left(\frac{\sqrt{n}}{4}\right) - \phi\left(-\frac{\sqrt{n}}{4}\right) = 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$$

Therefore, if the astronomer wanted, for instance, to be 95 percent certain that his estimated value is accurate to within .5 light year, he should make n^* measurements, where n^* is such that

$$2\Phi\left(\frac{\sqrt{n^*}}{4}\right) - 1 = .95 \qquad \text{or} \qquad \Phi\left(\frac{\sqrt{n^*}}{4}\right) = .975$$

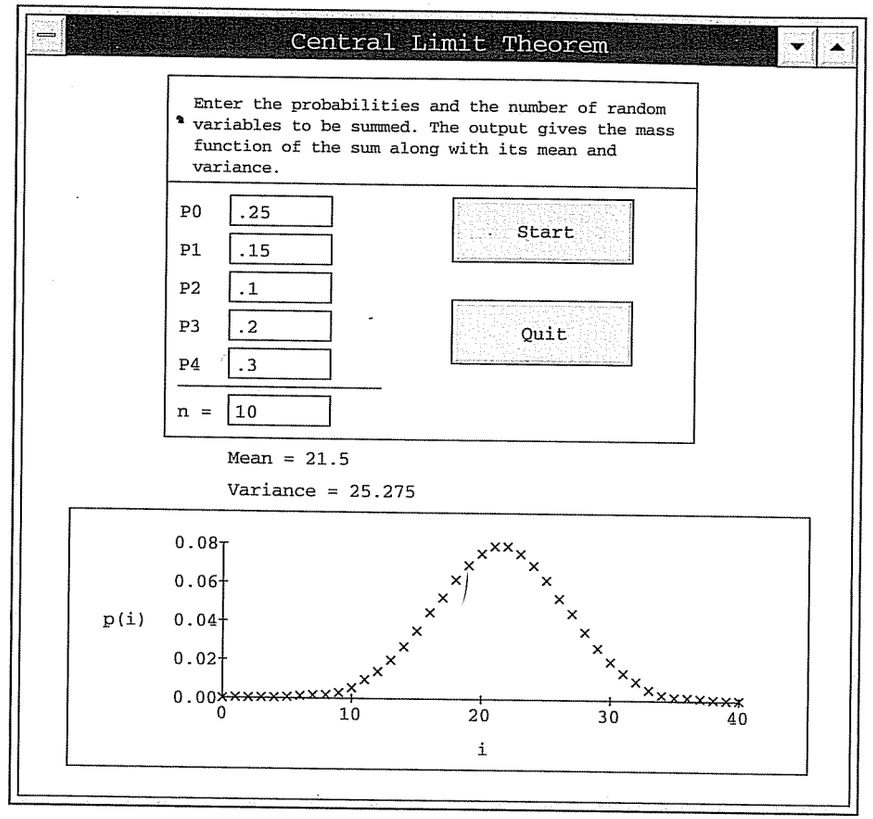


Figure 8.1(b)

and thus from Table 5.1 of Chapter 5,

$$\frac{\sqrt{n^*}}{4} = 1.96 \qquad \text{or} \qquad n^* = (7.84)^2 \approx 61.47$$

As n^* is not integral valued, he should make 62 observations.

It should, however, be noted that the preceding analysis has been done under the assumption that the normal approximation will be a good approximation when $n = 62$. Although this will usually be the case, in general the question of how large n need be before the approximation is "good" depends on the distribution of the X_i. If the astronomer was concerned about this point and wanted to take no chances, he could still solve his problem by using Chebyshev's inequality. Since

$$E\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = d \qquad \text{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) = \frac{4}{n}$$

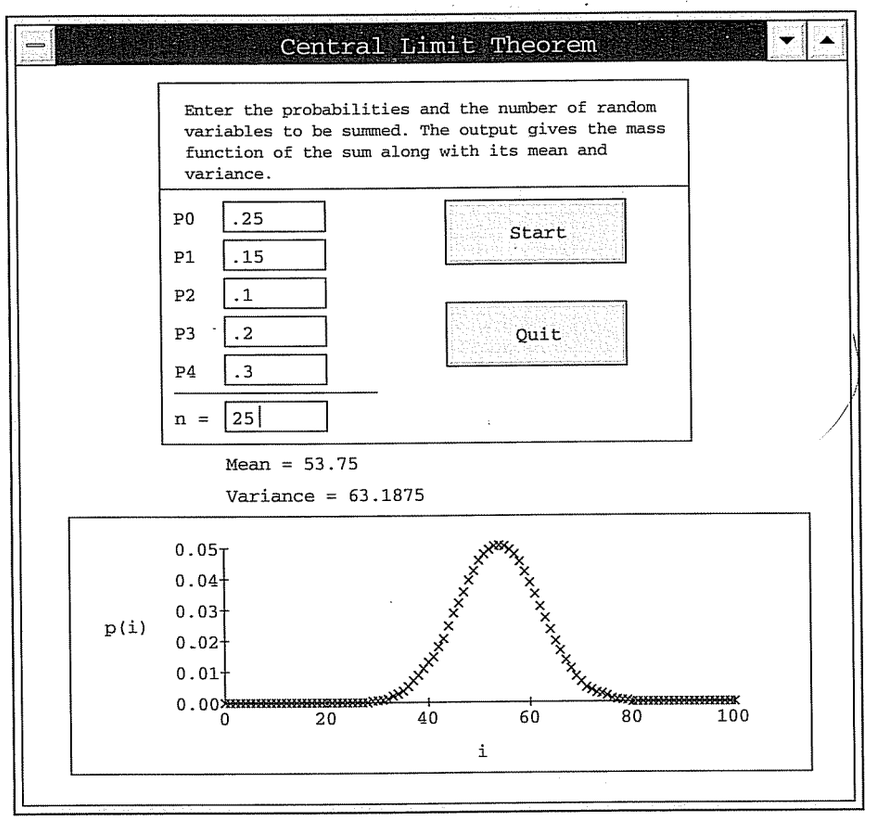


Figure 8.1(c)

Chebyshev's inequality yields that

$$P\left\{\left|\sum_{i=1}^{n}\frac{X_i}{n}-d\right|>.5\right\}\leq\frac{4}{n(.5)^2}=\frac{16}{n}$$

Hence, if he makes $n = 16/.05 = 320$ observations, he can be 95 percent certain that his estimate will be accurate to within .5 light year. ∎

Example 3b. The number of students that enroll in a psychology course is a Poisson random variable with mean 100. The professor in charge of the course has decided that if the number enrolling is 120 or more he will teach the course in two separate sections, whereas if fewer than 120 students enroll he will teach all of the students together in a single section. What is the probability that the professor will have to teach two sections?

Solution The exact solution $e^{-100}\sum_{i=120}^{\infty}(100)^i/i!$ does not readily yield a numerical answer. However, by recalling that a Poisson random variable with mean 100 is the sum of 100 independent Poisson random variables

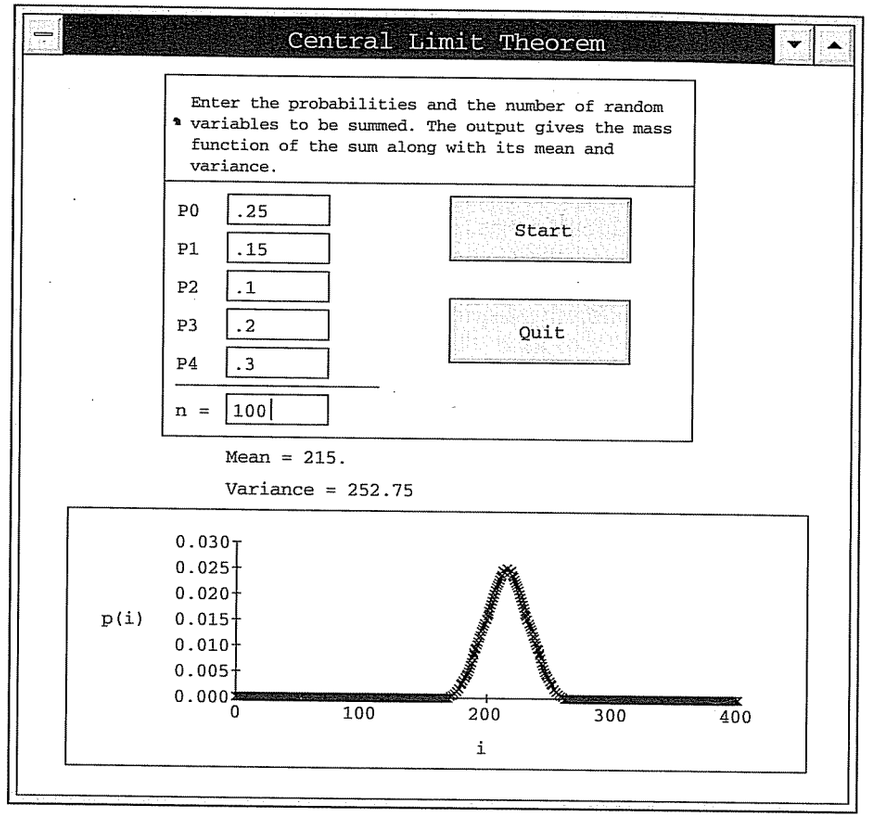


Figure 8.1(d)

each with mean 1, we can make use of the central limit theorem to obtain an approximate solution. If X denotes the number of students that enroll in the course, we have

$$\begin{aligned}P\{X\geq 120\}&=P\left\{\frac{X-100}{\sqrt{100}}\geq\frac{120-100}{\sqrt{100}}\right\}\\&\approx 1-\Phi(2)\\&\approx .0228\end{aligned}$$

where we have used the fact that the variance of a Poisson random variable is equal to its mean. ∎

Example 3c. If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40.

Solution Let X_i denote the value of the ith die, $i = 1, 2, \ldots, 10$. Since $E(X_i) = \frac{7}{2}$, $\mathrm{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = \frac{35}{12}$, the central limit theorem yields

$$P\left\{30 \le \sum_{i=1}^{10} X_i \le 40\right\} = P\left\{\frac{30-35}{\sqrt{\frac{350}{12}}} \le \frac{\sum_{i=1}^{10} X_i - 35}{\sqrt{\frac{350}{12}}} \le \frac{40-35}{\sqrt{\frac{350}{12}}}\right\}$$
$$\approx 2\Phi\left(\sqrt{\tfrac{6}{7}}\right) - 1$$
$$\approx .65$$ ∎

Example 3d. Let X_i, $i = 1, \ldots, 10$ be independent random variables, each uniformly distributed over (0, 1). Calculate an approximation to $P\left\{\sum_{i=1}^{10} X_i > 6\right\}$.

Solution Since $E[X_i] = \frac{1}{2}$, $\text{Var}(X_i) = \frac{1}{12}$, we have by the central limit theorem

$$P\left\{\sum_{1}^{10} X_i > 6\right\} = P\left\{\frac{\sum_{1}^{10} X_i - 5}{\sqrt{10\left(\frac{1}{12}\right)}} > \frac{6-5}{\sqrt{10\left(\frac{1}{12}\right)}}\right\}$$
$$\approx 1 - \Phi(\sqrt{1.2})$$
$$\approx .16$$

Hence only 16 percent of the time will $\sum_{i=1}^{10} X_i$ be greater than 6. ∎

Central limit theorems also exist when the X_i are independent but not necessarily identically distributed random variables. One version, by no means the most general, is as follows.

Theorem 3.2 Central limit theorem for independent random variables

Let $X_1, X_2, \ldots$ be a sequence of independent random variables having respective means and variances $\mu_i = E[X_i]$, $\sigma_i^2 = \text{Var}(X_i)$. If (a) the X_i are uniformly bounded; that is, if for some M, $P\{|X_i| < M\} = 1$ for all i, and (b) $\sum_{i=1}^{\infty} \sigma_i^2 = \infty$, then

$$P\left\{\frac{\sum_{i=1}^{n} (X_i - \mu_i)}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \le a\right\} \to \Phi(a) \quad \text{as} \quad n \to \infty$$

HISTORICAL NOTE

Pierre Simon, Marquis de Laplace

The central limit theorem was originally stated and proved by the French mathematician Pierre Simon, the Marquis de Laplace, who came to this theorem from his observations that errors of measurement (which can usually be regarded as being the sum of a large number of tiny forces) tend to be normally distributed. Laplace, who was also a famous astronomer (and indeed was called "the Newton of France"), was one of the great early contributors to both probability and statistics. Laplace was also a popularizer of the uses of probability in everyday life. He strongly believed in its importance, as is indicated by the following quotations of his taken from his published book *Analytical Theory of Probability*. *"We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it. . . . It is remarkable that this science, which originated in the consideration of games of chance, should become the most important object of human knowledge. . . . The most important questions of life are, for the most part, really only problems of probability."*

The application of the central limit theorem to show that measurement errors are approximately normally distributed is regarded as an important contribution to science. Indeed, in the seventeenth and eighteenth centuries the central limit theorem was often called the "law of frequency of errors." The *law of frequency of errors* was considered a major advance by scientists. Listen to the words of Francis Galton (taken from his book *Natural Inheritance*, published in 1889): "I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the 'Law of Frequency of Error.' The Law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of unreason."

8.4 THE STRONG LAW OF LARGE NUMBERS

The *strong law of large numbers* is probably the best-known result in probability theory. It states that the average of a sequence of independent random variables having a common distribution will, with probability 1, converge to the mean of that distribution.

Theorem 4.1 The strong law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, each having a finite mean $\mu = E[X_i]$. Then, with probability 1,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu \quad \text{as} \quad n \to \infty^{\dagger}$$

As an application of the strong law of large numbers, suppose that a sequence of independent trials of some experiment is performed. Let E be a fixed event of the experiment and denote by $P(E)$ the probability that E occurs on any particular trial. Letting

$$X_i = \begin{cases} 1 & \text{if } E \text{ occurs on the } i\text{th trial} \\ 0 & \text{if } E \text{ does not occur on the } i\text{th trial} \end{cases}$$

we have by the strong law of large numbers that with probability 1,

$$\frac{X_1 + \cdots + X_n}{n} \to E[X] = P(E) \tag{4.1}$$

Since $X_1 + \cdots + X_n$ represents the number of times that the event E occurs in the first n trials, we may interpret Equation (4.1) as stating that, with probability 1, the limiting proportion of time that the event E occurs is just $P(E)$.

Although the theorem can be proven without this assumption, our proof of the strong law of large numbers will assume that the random variables X_i have a finite fourth moment. That is, we will suppose that $E[X_i^4] = K < \infty$.

Proof of the Strong Law of Large Numbers: To begin, assume that μ, the mean of the X_i, is equal to 0. Let $S_n = \sum_{i=1}^{n} X_i$ and consider

$$\begin{aligned} E[S_n^4] = E[&(X_1 + \cdots + X_n)(X_1 + \cdots + X_n) \\ &\times (X_1 + \cdots + X_n)(X_1 + \cdots + X_n)] \end{aligned}$$

Expanding the right side of the above will result in terms of the form

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k, \quad \text{and} \quad X_i X_j X_k X_l$$

† That is, the strong law of large numbers states that

$$P\left\{\lim_{n\to\infty} (X_1 + \cdots + X_n)/n = \mu\right\} = 1$$

where i, j, k, l are all different. As all the X_i have mean 0, it follows by independence that

$$\begin{aligned} E[X_i^3 X_j] &= E[X_i^3]E[X_j] = 0 \\ E[X_i^2 X_j X_k] &= E[X_i^2]E[X_j]E[X_k] = 0 \\ E[X_i X_j X_k X_l] &= 0 \end{aligned}$$

Now, for a given pair i and j there will be $\binom{4}{2} = 6$ terms in the expansion that will equal $X_i^2 X_j^2$. Hence it follows upon expanding the preceding product and taking expectations term by term that

$$\begin{aligned} E[S_n^4] &= nE[X_i^4] + 6\binom{n}{2} E[X_i^2 X_j^2] \\ &= nK + 3n(n-1)E[X_i^2]E[X_j^2] \end{aligned}$$

where we have once again made use of the independence assumption. Now, since

$$0 \le \text{Var}(X_i^2) = E[X_i^4] - (E[X_i^2])^2$$

we see that

$$(E[X_i^2])^2 \le E[X_i^4] = K$$

Therefore, from the preceding we have that

$$E[S_n^4] \le nK + 3n(n-1)K$$

which implies that

$$E\left[\frac{S_n^4}{n^4}\right] \le \frac{K}{n^3} + \frac{3K}{n^2}$$

Therefore, it follows that

$$E\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} E\left[\frac{S_n^4}{n^4}\right] < \infty$$

But the preceding implies that with probability 1, $\sum_{n=1}^{\infty} S_n^4/n^4 < \infty$. (For if there is a positive probability that the sum is infinite, then its expected value is infinite.) But the convergence of a series implies that its nth term goes to 0; so we can conclude that with probability 1,

$$\lim_{n\to\infty} \frac{S_n^4}{n^4} = 0$$

But if $S_n^4/n^4 = (S_n/n)^4$ goes to 0, then so must S_n/n; so we have proven that with probability 1,

$$\frac{S_n}{n} \to 0 \quad \text{as} \quad n \to \infty$$

When μ, the mean of the X_i, is not equal to 0, we can apply the preceding argument to the random variables $X_i - \mu$ to obtain that with probability 1,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{(X_i - \mu)}{n} = 0$$

or, equivalently,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{X_i}{n} = \mu$$

which proves the result.

The strong law is illustrated by two modules on the text diskette that consider independent and identically distributed random variables which take on one of the values 0, 1, 2, 3, 4. The modules simulate the values of n such random variables; the proportions of time that each outcome occurs, as well as the resulting sample mean $\sum_{i=1}^{n} X_i/n$, are then indicated and plotted. When using these modules, which differ only in the type of graph presented, one enters the probabilities and the desired value of n. Figure 8.2 gives the results of a simulation using a specified probability mass function and (a) $n = 100$, (b) $n = 1000$, and (c) $n = 10{,}000$.

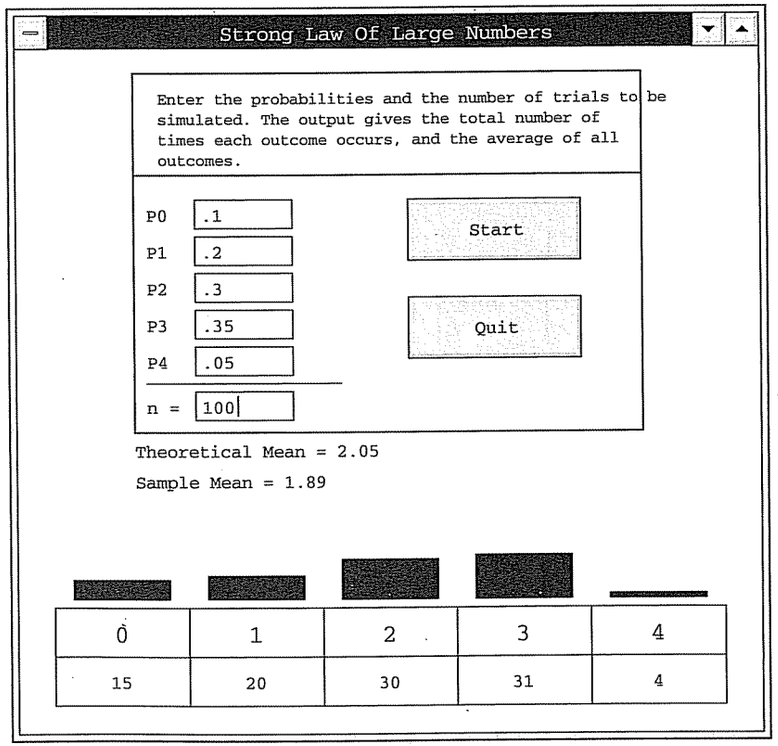


Figure 8.2(a)

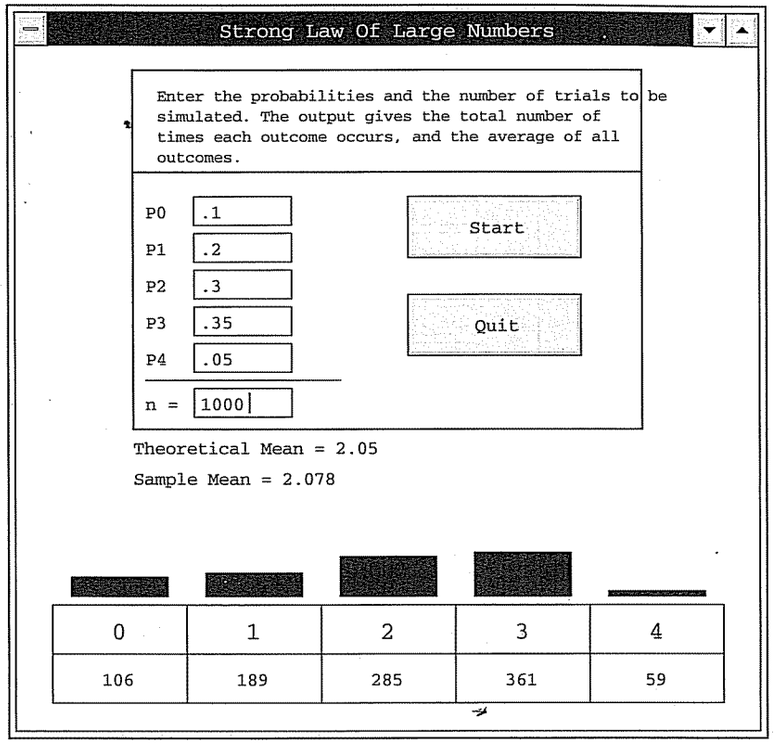


Figure 8.2(b)

Many students are initially confused about the difference between the weak and the strong law of large numbers. The weak law of large numbers states that for any specified large value n^*, $(X_1 + \cdots + X_{n^*})/n^*$ is likely to be near μ. However, it does not say that $(X_1 + \cdots + X_n)/n$ is bound to stay near μ for all values of n larger than n^*. Thus it leaves open the possibility that large values of $|(X_1 + \cdots + X_n)/n - \mu|$ can occur infinitely often (though at infrequent intervals). The strong law shows that this cannot occur. In particular, it implies that with probability 1, for any positive value ε,

$$\left| \sum_{1}^{n} \frac{X_i}{n} - \mu \right|$$

will be greater than ε only a finite number of times.

The strong law of large numbers was originally proved, in the special case of Bernoulli random variables, by the French mathematician Borel. The general form of the strong law presented in Theorem 4.1 was proved by the Russian mathematician A. N. Kolmogorov.

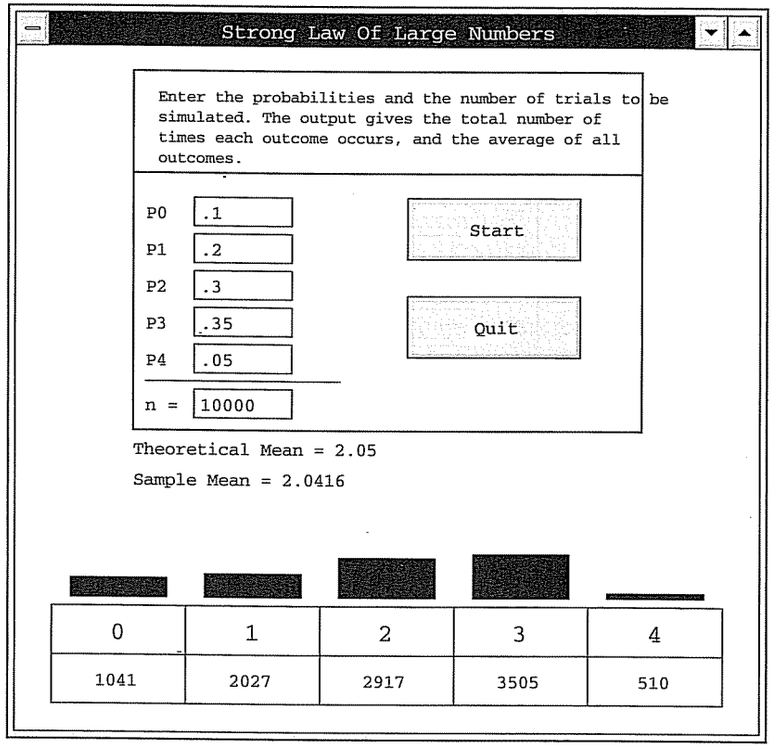


Figure 8.2(c)

8.5 OTHER INEQUALITIES

We are sometimes confronted with situations in which we are interested in obtaining an upper bound for a probability of the form $P\{X - \mu \geq a\}$, where a is some positive value and when only the mean $\mu = E[X]$ and variance $\sigma^2 = \text{Var}(X)$ of the distribution of X are known. Of course, since $X - \mu \geq a > 0$ implies that $|X - \mu| \geq a$, it follows from Chebyshev's inequality that

$$P\{X - \mu \geq a\} \leq P\{|X - \mu| \geq a\} \leq \frac{\sigma^2}{a^2} \qquad \text{when} \quad a > 0$$

However, as the following proposition shows, it turns out that we can do better.

Proposition 5.1 One-sided Chebyshev inequality

If X is a random variable with mean 0 and finite variance σ^2, then for any $a > 0$,

$$P\{X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Proof: Let $b > 0$ and note that

$$X \geq a \quad \text{is equivalent to} \quad X + b \geq a + b$$

Hence

$$\begin{aligned} P\{X \geq a\} &= P\{X + b \geq a + b\} \\ &\leq P\{(X + b)^2 \geq (a + b)^2\} \end{aligned}$$

where the inequality above is obtained by noting that since $a + b > 0$, $X + b \geq a + b$ implies that $(X + b)^2 \geq (a + b)^2$. Upon applying Markov's inequality, the above yields that

$$P\{X \geq a\} \leq \frac{E[(X + b)^2]}{(a + b)^2} = \frac{\sigma^2 + b^2}{(a + b)^2}$$

Letting $b = \sigma^2/a$ [which is easily seen to be the value of b that minimizes $(\sigma^2 + b^2)/(a + b)^2$] gives the desired result.

Example 5a. If the number of items produced in a factory during a week is a random variable with mean 100 and variance 400, compute an upper bound on the probability that this week's production will be at least 120.

Solution It follows from the one-sided Chebyshev inequality that

$$P\{X \geq 120\} = P\{X - 100 \geq 20\} \leq \frac{400}{400 + (20)^2} = \frac{1}{2}$$

Hence the probability that this week's production will be 120 or more is at most $\frac{1}{2}$.

If we attempted to obtain a bound by applying Markov's inequality, then we would have obtained

$$P\{X \geq 120\} \leq \frac{E(X)}{120} = \frac{5}{6}$$

which is a far weaker bound than the preceding one. ∎

Suppose now that X has mean μ and variance σ^2. As both $X - \mu$ and $\mu - X$ have mean 0 and variance σ^2, we obtain from the one-sided Chebyshev inequality that for $a > 0$,

$$P\{X - \mu \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

and

$$P\{\mu - X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Thus we have the following corollary.

Corollary 5.1

If $E[X] = \mu$, $\text{Var}(X) = \sigma^2$, then for $a > 0$,

$$P\{X \geq \mu + a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$$P\{X \leq \mu - a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Example 5b. A set of 200 people, consisting of 100 men and 100 women, is randomly divided into 100 pairs of 2 each. Give an upper bound to the probability that at most 30 of these pairs will consist of a man and a woman.

Solution Number the men, arbitrarily, from 1 to 100 and let for $i = 1, 2, \ldots 100$,

$$X_i = \begin{cases} 1 & \text{if man } i \text{ is paired with a woman} \\ 0 & \text{otherwise} \end{cases}$$

Then X, the number of man–woman pairs, can be expressed as

$$X = \sum_{i=1}^{100} X_i$$

As man i is equally likely to be paired with any of the other 199 people, of which 100 are women, we have

$$E[X_i] = P\{X_i = 1\} = \frac{100}{199}$$

Similarly, for $i \neq j$,

$$\begin{aligned} E[X_i X_j] &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\} P\{X_j = 1 | X_i = 1\} = \frac{100}{199}\frac{99}{197} \end{aligned}$$

where $P\{X_j = 1 | X_i = 1\} = 99/197$ since, given that man i is paired with a woman, man j is equally likely to be paired with any of the remaining 197 people, of which 99 are women. Hence we obtain that

$$\begin{aligned} E[X] &= \sum_{i=1}^{100} E[X_i] \\ &= (100)\frac{100}{199} \\ &\approx 50.25 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \sum_{i=1}^{100} \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j) \\ &= 100\frac{100}{199}\frac{99}{199} + 2\binom{100}{2}\left[\frac{100}{199}\frac{99}{197} - \left(\frac{100}{199}\right)^2\right] \\ &\approx 25.126 \end{aligned}$$

The Chebyshev inequality yields that

$$P\{X \leq 30\} \leq P\{|X - 50.25| \geq 20.25\} \leq \frac{25.126}{(20.25)^2} \approx .061$$

and thus there are fewer than 6 chances in a hundred that fewer than 30 men will be paired with women. However, we can improve on this bound by using the one-sided Chebyshev inequality, which yields that

$$\begin{aligned} P\{X \leq 30\} &= P\{X \leq 50.25 - 20.25\} \\ &\leq \frac{25.126}{25.126 + (20.25)^2} \\ &\approx .058 \end{aligned}$$

When the moment generating function of the random variable X is known, we can obtain even more effective bounds on $P\{X \geq a\}$. Let

$$M(t) = E[e^{tX}]$$

be the moment generating function of the random variable X. Then for $t > 0$,

$$\begin{aligned} P\{X \geq a\} &= P\{e^{tX} \geq e^{ta}\} \\ &\leq E[e^{tX}]e^{-ta} \qquad \text{by Markov's inequality} \end{aligned}$$

Similarly, for $t < 0$,

$$\begin{aligned} P\{X \leq a\} &= P\{e^{tX} \geq e^{ta}\} \\ &\leq E[e^{tX}]e^{-ta} \end{aligned}$$

Thus we have the following inequalities, known as *Chernoff bounds.*

Proposition 5.2 Chernoff bounds

$$P\{X \geq a\} \leq e^{-ta} M(t) \qquad \text{for all } t > 0$$
$$P\{X \leq a\} \leq e^{-ta} M(t) \qquad \text{for all } t < 0$$

Since the Chernoff bounds hold for all t in either the positive or negative quadrant, we obtain the best bound on $P\{X \geq a\}$ by using the t that minimizes $e^{-ta} M(t)$.

Example 5c. ***Chernoff bounds for the standard normal random variable.*** If Z is a standard normal random variable, then its moment generating function is $M(t) = e^{t^2/2}$, so the Chernoff bound on $P\{Z \geq a\}$ is given by

$$P\{Z \geq a\} \leq e^{-ta}e^{t^2/2} \qquad \text{for all} \quad t > 0$$

Now the value of t, $t > 0$, that minimizes $e^{t^2/2 - ta}$ is the value that minimizes $t^2/2 - ta$, which is $t = a$. Thus for $a > 0$ we see that

$$P\{Z \geq a\} \leq e^{-a^2/2}$$

Similarly, we can show that for $a < 0$,

$$P\{Z \leq a\} \leq e^{-a^2/2}$$

■

Example 5d. ***Chernoff bounds for the Poisson random variable.*** If X is a Poisson random variable with parameter λ, then its moment generating function is $M(t) = e^{\lambda(e^t - 1)}$. Hence the Chernoff bound on $P\{X \geq i\}$ is

$$P\{X \geq i\} \leq e^{\lambda(e^t - 1)}\, e^{-it} \qquad t > 0$$

Minimizing the right side of the above is equivalent to minimizing $\lambda(e^t - 1) - it$, and calculus shows that the minimal value occurs when $e^t = i/\lambda$. Provided that $i/\lambda > 1$, this minimizing value of t will be positive. Therefore, assuming that $i > \lambda$ and letting $e^t = i/\lambda$ in the Chernoff bound yields that

$$P\{X \geq i\} \leq e^{\lambda(i/\lambda - 1)} \left(\frac{\lambda}{i}\right)^i$$

or, equivalently,

$$P\{X \geq i\} \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}$$

■

Example 5e. Consider a gambler who on every play is equally likely, independent of the past, to either win or lose 1 unit. That is, if X_i is the gambler's winnings on the ith play, then the X_i are independent and

$$P\{X_i = 1\} = P\{X_i = -1\} = \tfrac{1}{2}$$

Let $S_n = \sum_{i=1}^{n} X_i$ denote the gambler's winnings after n plays. We will use the Chernoff bound on $P\{S_n \geq a\}$. To start, note that the moment generating function of X_i is

$$E[e^{tX}] = \frac{e^t + e^{-t}}{2}$$

Now, using the McLaurin expansions of e^t and e^{-t} we see that

$$e^t + e^{-t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

$$= 2\left\{1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right\}$$

$$= 2\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}$$

$$\leq 2\sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} \qquad \text{since } (2n)! \geq n!\, 2^n$$

$$= 2e^{t^2/2}$$

Therefore,

$$E[e^{tX}] \leq e^{t^2/2}$$

Since the moment generating function of the sum of independent random variables is the product of their moment generating functions, we have that

$$E[e^{tS_n}] = (E[e^{tX}])^n$$
$$\leq e^{nt^2/2}$$

Using the result above along with the Chernoff bound gives that

$$P\{S_n \geq a\} \leq e^{-ta}e^{nt^2/2} \qquad t > 0$$

The value of t that minimizes the right side of the above is the value that minimizes $nt^2/2 - ta$, and this value is $t = a/n$. Supposing that $a > 0$ (so that this minimizing t is positive) and letting $t = a/n$ in the preceding inequality yields that

$$P\{S_n \geq a\} \leq e^{-a^2/2n} \qquad a > 0$$

For instance, this inequality yields that

$$P\{S_{10} \geq 6\} \leq e^{-36/20} \approx .1653$$

whereas the exact probability is

$$P\{S_{10} \geq 6\} = P\{\text{gambler wins at least 8 of the first 10 games}\}$$
$$= \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}} = \frac{56}{1024} \approx .0547$$

■

The next inequality is one having to do with expectations rather than probabilities. Before stating it, we need the following definition.

Definition

A twice-differentiable real-valued function $f(x)$ is said to be *convex* if $f''(x) \geq 0$ for all x; similarly, it is said to be *concave* if $f''(x) \leq 0$.

Some examples of convex functions are $f(x) = x^2$, $f(x) = e^{ax}$, $f(x) = -x^{1/n}$ for $x \geq 0$. If $f(x)$ is convex, then $g(x) = -f(x)$ is concave, and vice versa.

Proposition 5.3 *Jensen's inequality*

If $f(x)$ is a convex function, then

$$E[f(X)] \geq f(E[X])$$

provided that the expectations exist and are finite.

Proof: Expanding $f(x)$ in a Taylor's series expansion about $\mu = E[X]$ yields

$$f(x) = f(\mu) + f'(\mu)(x - \mu) + \frac{f''(\xi)(x - \xi)^2}{2}$$

where ξ is some value between x and μ. Since $f''(\xi) \geq 0$, we obtain

$$f(x) \geq f(\mu) + f'(\mu)(x - \mu)$$

Hence

$$f(X) \geq f(\mu) + f'(\mu)(X - \mu)$$

Taking expectations yields

$$E[f(X)] \geq f(\mu) + f'(\mu)E[X - \mu] = f(\mu)$$

and the inequality is established.

Example 5f. An investor is faced with the following choices: She can either invest all of her money in a risky proposition that would lead to a random return X that has mean m; or she can put the money into a risk-free venture that will lead to a return of m with probability 1. Suppose that her decision will be made on the basis of maximizing the expected value of $u(R)$, where R is her return and u is her utility function. By Jensen's inequality it follows that if u is a concave function, then $E[u(X)] \leq u(m)$, so the risk-free alternative is preferable; whereas if u is convex, then $E[u(X)] \geq u(m)$, so the risk investment alternative would be preferred.

8.6 BOUNDING THE ERROR PROBABILITY WHEN APPROXIMATING A SUM OF INDEPENDENT BERNOULLI RANDOM VARIABLES BY A POISSON RANDOM VARIABLE

In this section we establish bounds on how closely a sum of independent Bernoulli random variables is approximated by a Poisson random variable with the same mean. Suppose that we want to approximate the sum of independent Bernoulli random variables with respective means $p_1, p_2, \ldots, p_n$. Starting with a sequence $Y_1, \ldots, Y_n$ of independent Poisson random variables, with Y_i having mean

p_i, we will construct a sequence of independent Bernoulli random variables $X_1, \ldots, X_n$ with parameters $p_1, \ldots, p_n$ such that

$$P\{X_i \neq Y_i\} \leq p_i^2 \qquad \text{for each } i$$

Letting $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{n} Y_i$, we will use the preceding fact to conclude that

$$P\{X \neq Y\} \leq \sum_{i=1}^{n} p_i^2$$

Finally, we will show that the inequality above implies that for any set of real numbers A,

$$|P\{X \in A\} - P\{Y \in A\}| \leq \sum_{i=1}^{n} p_i^2$$

Since X is the sum of independent Bernoulli random variables and Y is a Poisson random variable, the preceding inequality will yield the desired bound.

To show how the preceding is accomplished, let Y_i, $i = 1, \ldots, n$ be independent Poisson random variables with respective means p_i. Now let $U_1, \ldots, U_n$ be independent random variables that are also independent of the Y_i's and which are such that

$$U_i = \begin{cases} 0 & \text{with probability } (1 - p_i)e^{p_i} \\ 1 & \text{with probability } 1 - (1 - p_i)e^{p_i} \end{cases}$$

The preceding definition implicity makes use of the inequality

$$e^{-p} \geq 1 - p$$

in assuming that $(1 - p_i)e^{p_i} \leq 1$.

Now define the random variables X_i, $i = 1, \ldots, n$ by

$$X_i = \begin{cases} 0 & \text{if } Y_i = U_i = 0 \\ 1 & \text{otherwise} \end{cases}$$

Note that

$$P\{X_i = 0\} = P\{Y_i = 0\}P\{U_i = 0\} = e^{-p_i}(1 - p_i)e^{p_i} = 1 - p_i$$
$$P\{X_i = 1\} = 1 - P\{X_i = 0\} = p_i$$

Now if X_i is equal to 0, then so must Y_i equal 0 (by the definition of X_i). Therefore, we see that

$$\begin{aligned} P\{X_i \neq Y_i\} &= P\{X_i = 1, Y_i \neq 1\} \\ &= P\{Y_i = 0, X_i = 1\} + P\{Y_i > 1\} \\ &= P\{Y_i = 0, U_i = 1\} + P\{Y_i > 1\} \\ &= e^{-p_i}[1 - (1 - p_i)e^{p_i}] + 1 - e^{-p_i} - p_i e^{-p_i} \\ &= p_i - p_i e^{-p_i} \\ &\leq p_i^2 \qquad \text{(since } 1 - e^{-p} \leq p\text{)} \end{aligned}$$

Now let $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{n} Y_i$ and note that X is the sum of independent Bernoulli random variables and Y is Poisson with the expected value $E[Y] = E[X] = \sum_{i=1}^{n} p_i$. Note also that the inequality $X \neq Y$ implies that $X_i \neq Y_i$ for some i, so

$$\begin{aligned} P\{X \neq Y\} &\leq P\{X_i \neq Y_i \text{ for some } i\} \\ &\leq \sum_{i=1}^{n} P\{X_i \neq Y_i\} \qquad \text{(Boole's inequality)} \\ &\leq \sum_{i=1}^{n} p_i^2 \end{aligned}$$

For any event B, let I_B, the indicator variable for the event B, be defined by

$$I_B = \begin{cases} 1 & \text{if } B \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Note that for any set of real numbers A,

$$I_{\{X \in A\}} - I_{\{Y \in A\}} \leq I_{\{X \neq Y\}}$$

The above follows since, as an indicator variable is either 0 or 1, the left-hand side equals 1 only when $I_{\{X \in A\}} = 1$ and $I_{\{Y \in A\}} = 0$. But this would imply that $X \in A$ and $Y \notin A$, which means that $X \neq Y$, so the right side would also equal 1. Upon taking expectations of the preceding inequality, we obtain that

$$P\{X \in A\} - P\{Y \in A\} \leq P\{X \neq Y\}$$

By reversing X and Y, we obtain in the same manner that

$$P\{Y \in A\} - P\{X \in A\} \leq P\{X \neq Y\}$$

and thus we can conclude that

$$|P\{X \in A\} - P\{Y \in A\}| \leq P\{X \neq Y\}$$

Therefore, we have proven that with $\lambda = \sum_{i=1}^{n} p_i$,

$$\left| P\left\{\sum_{i=1}^{n} X_i \in A\right\} - \sum_{i \in A} \frac{e^{-\lambda}\lambda^i}{i!} \right| \leq \sum_{i=1}^{n} p_i^2$$

REMARK. When all the p_i are equal to p, X is a binomial random variable. Thus the above shows that for any set of nonnegative integers A,

$$\left| \sum_{i \in A} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i \in A} \frac{e^{-np}(np)^i}{i!} \right| \leq np^2$$

SUMMARY

Two useful probability bounds are provided by the *Markov* and *Chebyshev* inequalities. The Markov inequality is concerned with nonnegative random variables, and says that for X of that type

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

for every positive value a. The Chebyshev inequality, which is a simple consequence of the Markov inequality, states that if X has mean μ and variance σ^2, then for every positive k,

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

The two most important theoretical results in probability are the *central limit theorem* and the *strong law of large numbers*. Both are concerned with a sequence of independent and identically distributed random variables. The central limit theorem says that if the random variables have a finite mean μ and a finite variance σ^2, then the distribution of the sum of the first n of them is, for large n, approximately that of a normal random variable with mean $n\mu$ and variance $n\sigma^2$. That is, if X_i, $i \geq 1$, is the sequence, then the central limit theorem states that for every real number a,

$$\lim_{n \to \infty} P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx$$

The *strong law of large numbers* requires only that the random variables in the sequence have a finite mean μ. It states that with probability 1, the average of the first n of them will converge to μ as n goes to infinity. This implies that if A is any specified event of an experiment for which independent replications are performed, then the limiting proportion of experiments whose outcomes are in A will, with probability 1, equal $P(A)$. Therefore, if we accept the interpretation that "with probability 1" means "with certainty," we obtain the theoretical justification for the long-run relative frequency interpretation of probabilities.

PROBLEMS

1. Suppose that X is a random variable with mean and variance both equal to 20. What can be said about $P\{0 \leq X \leq 40\}$?

2. From past experience a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(a) Give an upper bound for the probability that a student's test score will exceed 85.

Suppose, in addition, the professor knows that the variance of a student's test score is equal to 25.

(b) What can be said about the probability that a student will score between 65 and 85?

(c) How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem.

3. Use the central limit theorem to solve part (c) of Problem 2.

4. Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to obtain a bound on

$$P\left\{\sum_1^{20} X_i > 15\right\}$$

(b) Use the central limit theorem to approximate $P\left\{\sum_1^{20} X_i > 15\right\}$.

5. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$ what is the probability that the resultant sum differs from the exact sum by more than 3?

6. A die is continually rolled until the total sum of all rolls exceeds 300. What is the probability that at least 80 rolls are necessary?

7. One has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, what is the probability that there is still a working bulb after 525 hours?

8. In Problem 7 suppose that it takes a random time, uniformly distributed over $(0, .5)$, to replace a failed bulb. What is the probability that all bulbs have failed by time 550?

9. If X is a gamma random variable with parameters $(n, 1)$ how large need n be so that

$$P\left\{\left|\frac{X}{n} - 1\right| > .01\right\} < .01?$$

10. Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation .3. How many cars would have to be on the bridge span for the probability of structural damage to exceed .1?

11. Many people believe that the daily change of price of a company's stock on the stock market is a random variable with mean 0 and variance σ^2. That is, if Y_n represents the price of the stock on the nth day, then

$$Y_n = Y_{n-1} + X_n \qquad n \geq 1$$

where $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100. If $\sigma^2 = 1$, what can you say about the probability that the stock's price will exceed 105 after 10 days?

12. We have 100 components that we will put in use in a sequential fashion. That is, component 1 is initially put in use, and upon failure it is replaced by component 2, which is itself replaced upon failure by component 3, and so on. If the lifetime of component i is exponentially distributed with mean $10 + i/10$, $i = 1, \ldots, 100$, estimate the probability that the total life of all components will exceed 1200. Now repeat when the life distribution of component i is uniformly distributed over $(0, 20 + i/5)$, $i = 1, \ldots, 100$.

13. Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

(a) Approximate the probability that the average test score in the class of size 25 exceeds 80.

(b) Repeat part (a) for the class of size 64.

(c) Approximate the probability that the average test score in the larger class exceeds that of the other class by over 2.2 points.

(d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by over 2.2 points.

14. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least .95?

15. An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is \$240 with a standard deviation of \$800. Approximate the probability that the total yearly claim exceeds \$2.7 million.

16. Redo Example 5b under the assumption that the number of man–woman pairs is (approximately) normally distributed. Does this seem like a reasonable supposition?

17. Repeat part (a) of Problem 2 when it is known that the variance of a student's test score is equal to 25.

18. A lake contains 4 distinct types of fish. Suppose that each fish caught is equally likely to be any one of these types. Let Y denote the number of fish that need be caught to obtain at least one of each type.

(a) Give an interval (a, b) such that $P\{a \leq Y \leq b\} \geq .90$.

(b) Using the one-sided Chebyshev inequality, how many fish need we plan on catching so as to be at least 90 percent certain of obtaining at least one of each type?

19. If X is a nonnegative random variable with mean 25, what can be said about:

(a) $E[X^3]$;

(b) $E[\sqrt{X}]$;

(c) $E[\log X]$;

(d) $E[e^{-X}]$?

20. Let X be a nonnegative random variable. Prove that

$$E[X] \leq (E[X^2])^{1/2} \leq (E[X^3])^{1/3} \leq \cdots$$

21. Would the results of Example 5f change if the investor were allowed to divide her money and invest the fraction α, $0 < \alpha < 1$ in the risky proposition and invest the remainder in the risk-free venture? Her return for such a split investment would be $R = \alpha X + (1 - \alpha)m$.

22. Let X be a Poisson random variable with mean 20.

(a) Use the Markov inequality to obtain an upper bound on

$$p = P\{X \geq 26\}$$

(b) Use the one-sided Chebyshev inequality to obtain an upper bound on p.

(c) Use the Chernoff bound to obtain an upper bound on p.

(d) Approximate p by making use of the central limit theorem.

(e) Determine p by running an appropriate program.

THEORETICAL EXERCISES

1. If X has variance σ^2, then σ, the positive square root of the variance, is called the *standard deviation.* If X has mean μ and standard deviation σ, show that

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

2. If X has mean μ and standard deviation σ, the ratio $r \equiv |\mu|/\sigma$ is called the *measurement signal-to-noise ratio* of X. The idea is that X can be expressed as $X = \mu + (X - \mu)$ with μ representing the signal and $X - \mu$ the noise. If we define $|(X - \mu)/\mu| \equiv D$ as the relative deviation of X from its signal (or mean) μ, show that for $\alpha > 0$,

$$P\{D \leq \alpha\} \geq 1 - \frac{1}{r^2\alpha^2}$$

3. Compute the measurement signal-to-noise ratio—that is, $|\mu|/\sigma$ where $\mu = E[X]$, $\sigma^2 = \text{Var}(X)$—of the following random variables:

(a) Poisson with mean λ;

(b) binomial with parameters n and p;

(c) geometric with mean $1/p$;

(d) uniform over (a, b);

(e) exponential with mean $1/\lambda$;

(f) normal with parameters μ, σ^2.

4. Let Z_n, $n \geq 1$ be a sequence of random variables and c a constant such that for each $\varepsilon > 0$, $P\{|Z_n - c| > \varepsilon\} \to 0$ as $n \to \infty$. Show that for any bounded continuous function g,

$$E[g(Z_n)] \to g(c) \qquad \text{as} \quad n \to \infty$$

5. Let $f(x)$ be a continuous function defined for $0 \leq x \leq 1$. Consider the functions

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k}$$

(called *Bernstein polynomials*) and prove that

$$\lim_{n\to\infty} B_n(x) = f(x)$$

HINT: Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with mean x. Show and then use the fact (by making use of the result of Theoretical Exercise 4) that

$$B_n(x) = E\left[f\left(\frac{X_1 + \cdots + X_n}{n}\right)\right]$$

As it can be shown that the convergence of $B_n(x)$ to $f(x)$ is uniform in x, the above provides a probabilistic proof to the famous Weierstrass theorem of analysis that states that any continuous function on a closed interval can be approximated arbitrarily closely by a polynomial.

6. **(a)** Let X be a discrete random variable, whose possible values are $1, 2, \ldots$ If $P\{X = k\}$ is nonincreasing in $k = 1, 2, \ldots$, prove that

$$P\{X = k\} \leq 2\frac{E[X]}{k^2}$$

(b) Let X be a nonnegative continuous random variable having a nonincreasing density function. Show that

$$f(x) \leq \frac{2E[X]}{x^2} \qquad \text{for all} \quad x > 0$$

7. Suppose that a fair die is rolled 100 times. Let X_i be the value obtained on the ith roll. Compute an approximation for

$$P\left\{\prod_{1}^{100} X_i \leq a^{100}\right\} \qquad 1 < a < 6$$

8. Explain why a gamma random variable with parameters (t, λ) has an approximately normal distribution when t is large.

9. Suppose a fair coin is tossed 1000 times. If the first 100 tosses all result in heads, what proportion of heads would you expect on the final 900 tosses? Comment on the statement that "the strong law of large numbers swamps but does not compensate."

10. If X is a Poisson random variable with mean λ, show that for $i < \lambda$,

$$P\{X \leq i\} \leq \frac{e^{-\lambda}(e\lambda)^i}{i^i}$$

11. Let X be a binomial random variable with parameters n and p. Show that for $i > np$:

(a) $\min_{t>0} e^{-ti}E[e^{tX}]$ occurs when t is such that $e^t = \frac{iq}{(n-i)p}$, where $q = 1 - p$.

(b) $P\{X \geq i\} \leq \frac{n^n}{i^i(n-i)^{n-i}} p^i(1-p)^{n-i}$.

12. The Chernoff bound on a standard normal random variable Z gives that $P\{Z > a\} \leq e^{-a^2/2}$, $a > 0$. Show, by considering the density of Z, that the right side of the inequality can be reduced by a factor 2. That is, show that

$$P\{Z > a\} \leq \tfrac{1}{2}e^{-a^2/2} \qquad a > 0$$

13. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, show that $\theta > 0$.

SELF-TEST PROBLEMS AND EXERCISES

1. The number of automobiles sold weekly at a certain dealership is a random variable with expected value 16. Give an upper bound to the probability that
(a) next week's sales exceed 18;
(b) next week's sales exceed 25.

2. Suppose in Problem 1 that the variance of the number of automobiles sold weekly is 9.
(a) Give a lower bound to the probability that next week's sales are between 10 and 22 inclusively.
(b) Give an upper bound to the probability that next week's sales exceed 18.

3. If

$$E[X]=75 \quad E[Y]=75 \quad \text{Var}(X)=10 \quad \text{Var}(Y)=12 \quad \text{Cov}(X, Y)=-3$$

give an upper bound to
(a) $P\{|X - Y| > 15\}$;
(b) $P\{X > Y + 15\}$;
(c) $P\{Y > X + 15\}$.

4. Suppose that the number of units produced daily at factory A is a random variable with mean 20 and standard deviation 3 and the number produced at factory B is a random variable with mean 18 and standard deviation 6. Assuming independence, derive an upper bound for the probability that more units are produced today at factory B than at factory A.

5. The number of days that a certain type of component functions before failing is a random variable with probability density function

$$f(x) = 2x \qquad 0 < x < 1$$

Once the component fails it is immediately replaced by another one of the same type. If we let X_i denote the lifetime of the ith component to be put in use, then $S_n = \sum_{i=1}^{n} X_i$ represents the time of the nth failure. The long-term rate at which failures occur, call it r, is defined by

$$r = \lim_{n\to\infty} \frac{n}{S_n}$$

Assuming that the random variables X_i, $i \geq 1$ are independent, determine r.

6. In Self-Test Problem 5, how many components would one need to have on hand to be approximately 90 percent certain that the stock will last at least 35 days?

7. The servicing of a machine requires two separate steps, with the time needed for the first step being an exponential random variable with mean .2 hour and the time for the second step being an independent exponential random variable with mean .3 hour. If a repairperson has 20 machines to service, approximate the probability that all the work can be completed in 8 hours.

8. On each bet, a gambler loses 1 with probability .7, loses 2 with probability .2, or wins 10 with probability .1. Approximate the probability that the gambler will be losing after his first 100 bets.

9. Determine t so that the probability that the repairperson in Self-Test Problem 7 finishes the 20 jobs within time t is approximately equal to .95.

CHAPTER 9

Additional Topics in Probability

9.1 THE POISSON PROCESS

Before defining a Poisson process, recall that a function f is said to be $o(h)$ if $\lim_{h\to 0} f(h)/h = 0$. That is, f is $o(h)$ if, for small values of h, $f(h)$ is small even in relation to h. Suppose now that "events" are occurring at random time points and let $N(t)$ denote the number of events that occur in the time interval $[0, t]$. The collection of random variables $\{N(t), t \geq 0\}$ is said to be a *Poisson process having rate* λ, $\lambda > 0$ if

(i) $N(0) = 0$.
(ii) The numbers of events that occur in disjoint time intervals are independent.
(iii) The distribution of the number of events that occur in a given interval depends only on the length of that interval and not on its location.
(iv) $P\{N(h) = 1\} = \lambda h + o(h)$.
(v) $P\{N(h) \geq 2\} = o(h)$.

Thus condition (i) states that the process begins at time 0. Condition (ii), the *independent increment* assumption, states, for instance, that the number of events by time t [that is, $N(t)$] is independent of the number of events that occur between t and $t + s$ [that is, $N(t + s) - N(t)$]. Condition (iii), the *stationary increment* assumption, states that the probability distribution of $N(t + s) - N(t)$ is the same for all values of t.

In Chapter 4 we presented an argument, based on the Poisson distribution being a limiting version of the binomial distribution, that the foregoing conditions imply that $N(t)$ has a Poisson distribution with mean λt. We will now obtain this result by a different method.

Lemma 1.1

For a Poisson process with rate λ,

$$P\{N(t) = 0\} = e^{-\lambda t}$$

Proof: Let $P_0(t) = P\{N(t) = 0\}$. We derive a differential equation for $P_0(t)$ in the following manner:

$$\begin{aligned} P_0(t + h) &= P\{N(t + h) = 0\} \\ &= P\{N(t) = 0, N(t + h) - N(t) = 0\} \\ &= P\{N(t) = 0\}P\{N(t + h) - N(t) = 0\} \\ &= P_0(t)[1 - \lambda h + o(h)] \end{aligned}$$

where the final two equations follow from condition (ii) plus the fact that conditions (iv) and (v) imply that $P\{N(h) = 0\} = 1 - \lambda h + o(h)$. Hence

$$\frac{P_0(t + h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}$$

Now, letting $h \to 0$, we obtain

$$P_0'(t) = -\lambda P_0(t)$$

or, equivalently,

$$\frac{P_0'(t)}{P_0(t)} = -\lambda$$

which implies, by integration, that

$$\log P_0(t) = -\lambda t + c$$

or

$$P_0(t) = Ke^{-\lambda t}$$

Since $P_0(0) = P\{N(0) = 0\} = 1$, we arrive at

$$P_0(t) = e^{-\lambda t}$$

∎

For a Poisson process, let us denote by T_1 the time of the first event. Further, for $n > 1$, let T_n denote the elapsed time between the $(n - 1)$st and the nth event. The sequence $\{T_n, n = 1, 2, \ldots\}$ is called the *sequence of interarrival times*. For instance, if $T_1 = 5$ and $T_2 = 10$, then the first event of the Poisson process would have occurred at time 5 and the second at time 15.

We shall now determine the distribution of the T_n. To do so, we first note that the event $\{T_1 > t\}$ takes place if and only if no events of the Poisson process occur in the interval $[0, t]$, and thus

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Hence T_1 has an exponential distribution with mean $1/\lambda$. Now,

$$P\{T_2 > t\} = E[P\{T_2 > t|T_1\}]$$

However,

$$\begin{aligned} P\{T_2 > t|T_1 = s\} &= P\{0 \text{ events in } (s, s+t]|T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} \\ &= e^{-\lambda t} \end{aligned}$$

where the last two equations followed from the assumptions about independent and stationary increments. Therefore, from the above we conclude that T_2 is also an exponential random variable with mean $1/\lambda$, and furthermore, that T_2 is independent of T_1. Repeating the same argument yields Proposition 1.1.

Proposition 1.1

$T_1, T_2, \ldots$ are independent exponential random variables each with mean $1/\lambda$.

Another quantity of interest is S_n, the arrival time of the nth event, also called the *waiting time* until the nth event. It is easily seen that

$$S_n = \sum_{i=1}^{n} T_i \qquad n \geq 1$$

and hence from Proposition 1.1 and the results of Section 5.6.1, it follows that S_n has a gamma distribution with parameters n and λ. That is, the probability density of S_n is given by

$$f_{S_n}(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \qquad x \geq 0$$

We are now ready to prove that $N(t)$ is a Poisson random variable with mean λt.

Theorem 1.1

For a Poisson process with rate λ,

$$P\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Proof: Note that the nth event of the Poisson process will occur before or at time t if and only if the number of events that occur by t is at least n. That is,

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

so

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \\ &= \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}\, dx - \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}\, dx \end{aligned}$$

But the integration by parts formula $\int u\, dv = uv - \int v\, du$ yields, with $u = e^{-\lambda x}$, $dv = \lambda[(\lambda x)^{n-1}/(n-1)!]\, dx$,

$$\int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}\, dx = e^{-\lambda t} \frac{(\lambda t)^n}{n!} + \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}\, dx$$

which completes the proof.

9.2 MARKOV CHAINS

Consider a sequence of random variables $X_0, X_1, \ldots$, and suppose that the set of possible values of these random variables is $\{0, 1, \ldots, M\}$. It will be helpful to interpret X_n as being the state of some system at time n, and, in accordance with this interpretation, we say that the system is in state i at time n if $X_n = i$. The sequence of random variables is said to form a *Markov chain* if each time the system is in state i there is some fixed probability—call it P_{ij}—that it will next be in state j. That is, for all $i_0, \ldots, i_{n-1}, i, j$,

$$P\{X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}$$

The values P_{ij}, $0 \leq i \leq M$, $0 \leq j \leq N$, are called the *transition probabilities* of the Markov chain and they satisfy (why?)

$$P_{ij} \geq 0 \qquad \sum_{j=0}^{M} P_{ij} = 1 \qquad i = 0, 1, \ldots, M$$

It is convenient to arrange the transition probabilities P_{ij} in a square array as follows:

$$\begin{Vmatrix} P_{00} & P_{01} & \cdots & P_{0M} \\ P_{10} & P_{11} & \cdots & P_{1M} \\ \vdots & & & \\ P_{M0} & P_{M1} & \cdots & P_{MM} \end{Vmatrix}$$

Such an array is called a *matrix*.

Knowledge of the transition probability matrix and the distribution of X_0 enables us, in theory, to compute all probabilities of interest. For instance, the joint probability mass function of $X_0, \ldots, X_n$ is given by

$$\begin{aligned} &P\{X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} \\ &\quad = P\{X_n = i_n|X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}\, P\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \\ &\quad = P_{i_{n-1}, i_n} P\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \end{aligned}$$

and continual repetition of this argument yields that the above is equal to

$$P_{i_{n-1},i_n} P_{i_{n-2},i_{n-1}} \cdots P_{i_1,i_2} P_{i_0,i_1} P\{X_0 = i_0\}$$

Example 2a. Suppose that whether or not it rains tomorrow depends on previous weather conditions only through whether or not it is raining today. Suppose further that if it is raining today, then it will rain tomorrow with probability α, and, if it is not raining today, then it will rain tomorrow with probability β.

If we say that the system is in state 0 when it rains and state 1 when it does not, then the system above is a two-state Markov chain having transition probability matrix

$$\begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix}$$

That is, $P_{00} = \alpha = 1 - P_{01}$, $P_{10} = \beta = 1 - P_{11}$. ∎

Example 2b. Consider a gambler who at each play of the game either wins 1 unit with probability p or loses 1 unit with probability $1 - p$. If we suppose that the gambler will quit playing when his fortune hits either 0 or M, then the gambler's sequence of fortunes is a Markov chain having transition probabilities

$$\begin{aligned} P_{i,i+1} &= p = 1 - P_{i,i-1} \qquad i = 1, \ldots, M-1 \\ P_{00} &= P_{MM} = 1 \end{aligned}$$

∎

Example 2c. The physicists P. and T. Ehrenfest considered a conceptual model for the movement of molecules in which M molecules are distributed among 2 urns. At each time point one of the molecules is chosen at random and is removed from its urn and placed in the other one. If we let X_n denote the number of molecules in the first urn immediately after the nth exchange, then $\{X_0, X_1, \ldots\}$ is a Markov chain with transition probabilities

$$\begin{aligned} P_{i,i+1} &= \frac{M-i}{M} \qquad 0 \le i \le M \\ P_{i,i-1} &= \frac{i}{M} \qquad 0 \le i \le M \\ P_{ij} &= 0 \qquad \text{if } |j-i| > 1 \end{aligned}$$

∎

Thus, for a Markov chain, P_{ij} represents the probability that a system in state i will enter state j at the next transition. We can also define the two-stage transition probability, $P_{ij}^{(2)}$, that a system presently in state i will be in state j after two additional transitions. That is,

$$P_{ij}^{(2)} = P\{X_{m+2} = j | X_m = i\}$$

The $P_{ij}^{(2)}$ can be computed from the P_{ij} as follows:

$$\begin{aligned} P_{ij}^{(2)} &= P\{X_2 = j | X_0 = i\} \\ &= \sum_{k=0}^{M} P\{X_2 = j, X_1 = k | X_0 = i\} \\ &= \sum_{k=0}^{M} P\{X_2 = j | X_1 = k, X_0 = i\} P\{X_1 = k | X_0 = i\} \\ &= \sum_{k=0}^{M} P_{kj} P_{ik} \end{aligned}$$

In general, we define the n-stage transition probabilities, denoted as $P_{ij}^{(n)}$, by

$$P_{ij}^{(n)} = P\{X_{n+m} = j | X_m = i\}$$

Proposition 2.1, known as the Chapman–Kolmogorov equations, shows how the $P_{ij}^{(n)}$ can be computed.

Proposition 2.1 The Chapman–Kolmogorov equations

$$P_{ij}^{(n)} = \sum_{k=0}^{M} P_{ik}^{(r)} P_{kj}^{(n-r)} \qquad \text{for all } 0 < r < n$$

Proof

$$\begin{aligned} P_{ij}^{(n)} &= P\{X_n = j | X_0 = i\} \\ &= \sum_k P\{X_n = j, X_r = k | X_0 = i\} \\ &= \sum_k P\{X_n = j | X_r = k, X_0 = i\} P\{X_r = k | X_0 = i\} \\ &= \sum_k P_{kj}^{(n-r)} P_{ik}^{(r)} \end{aligned}$$

Example 2d. ***A random walk.*** An example of a Markov chain having a countably infinite state space is the *random walk*, which tracks a particle as it moves along a one-dimensional axis. Suppose that at each point in time the particle will move either one step to the right or one step to the left with respective probabilities p and $1 - p$. That is, suppose the particle's path follows a Markov chain with transition probabilities

$$P_{i,i+1} = p = 1 - P_{i,i-1} \qquad i = 0, \pm 1, \ldots$$

If the particle is at state i, then the probability it will be at state j after n transitions is the probability that $(n - i + j)/2$ of these steps are to the right and $n - [(n - i + j)/2] = (n + i - j)/2$ are to the left. As each

step will be to the right, independently of the other steps, with probability p, it follows that the above is just the binomial probability

$$P_{ij}^n = \binom{n}{(n-i+j)/2} p^{(n-i+j)/2}(1-p)^{(n+i-j)/2}$$

where $\binom{n}{x}$ is taken to equal 0 when x is not a nonnegative integer less than or equal to n. The above can be rewritten as

$$P_{i,i+2k}^{2n} = \binom{2n}{n+k} p^{n+k}(1-p)^{n-k} \qquad k = 0, \pm 1, \ldots, \pm n$$

$$P_{i,i+2k+1}^{2n+1} = \binom{2n+1}{n+k+1} p^{n+k+1}(1-p)^{n-k}$$

$$k = 0, \pm 1, \ldots, \pm n, -(n+1)$$ ∎

Although the $P_{ij}^{(n)}$ denote conditional probabilities, we can, by conditioning on the initial state, use them to derive expressions for unconditional probabilities. For instance,

$$P\{X_n = j\} = \sum_i P\{X_n = j | X_0 = i\} P\{X_0 = i\}$$
$$= \sum_i P_{ij}^{(n)} P\{X_0 = i\}$$

For a large number of Markov chains it turns out that $P_{ij}^{(n)}$ converges, as $n \to \infty$, to a value Π_j that depends only on j. That is, for large values of n, the probability of being in state j after n transitions is approximately equal to Π_j no matter what the initial state was. It can be shown that a sufficient condition for a Markov chain to possess this property is that for some $n > 0$,

$$P_{ij}^{(n)} > 0 \qquad \text{for all } i, j = 0, 1, \ldots, M \tag{2.1}$$

Markov chains that satisfy Equation (2.1) are said to be *ergodic*. Since Proposition 2.1 yields

$$P_{ij}^{(n+1)} = \sum_{k=0}^{M} P_{ik}^{(n)} P_{kj}$$

it follows, by letting $n \to \infty$, that for ergodic chains

$$\Pi_j = \sum_{k=0}^{M} \Pi_k P_{kj} \tag{2.2}$$

Furthermore, since $1 = \sum_{j=0}^{M} P_{ij}^{(n)}$, we also obtain, by letting $n \to \infty$,

$$\sum_{j=0}^{M} \Pi_j = 1 \tag{2.3}$$

In fact, it can be shown that the Π_j, $0 \le j \le M$, are the unique nonnegative solutions of Equations (2.2) and (2.3). All this is summed up in Theorem 2.1, which we state without proof.

Theorem 2.1

For an ergodic Markov chain

$$\Pi_j = \lim_{n \to \infty} P_{ij}^{(n)}$$

exists, and the Π_j, $0 \le j \le M$, are the unique nonnegative solutions of

$$\Pi_j = \sum_{k=0}^{M} \Pi_k P_{kj}$$
$$\sum_{j=0}^{M} \Pi_j = 1$$

Example 2e. Consider Example 2a, in which we assume that if it rains today, then it will rain tomorrow with probability α; and, if it does not rain today, then it will rain tomorrow with probability β. From Theorem 2.1 it follows that the limiting probabilities of rain and of no rain, Π_0 and Π_1, are given by

$$\Pi_0 = \alpha\Pi_0 + \beta\Pi_1$$
$$\Pi_1 = (1-\alpha)\Pi_0 + (1-\beta)\Pi_1$$
$$\Pi_0 + \Pi_1 = 1$$

which yields

$$\Pi_0 = \frac{\beta}{1+\beta-\alpha} \qquad \Pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$$

For instance, if $\alpha = .6$, $\beta = .3$, then the limiting probability of rain on the nth day is $\Pi_0 = \frac{3}{7}$. ∎

The quantity Π_j is also equal to the long-run proportion of time that the Markov chain is in state j, $j = 0, \ldots, M$. To see intuitively why this might be so, let P_j denote the long-run proportion of time the chain is in state j. (It can be proven, using the strong law of large numbers, that for an ergodic chain such long-run proportions exist and are constants.) Now, since the proportion of time the chain is in state k is P_k and since, when in state k, the chain goes to state j with probability P_{kj}, it follows that the proportion of time the Markov chain is entering state j from state k is equal to $P_k P_{kj}$. Summing over all k shows that P_j, the proportion of time the Markov chain is entering state j, satisfies

$$P_j = \sum_k P_k P_{kj}$$

Since clearly it is also true that

$$\sum_j P_j = 1$$

it thus follows, since by Theorem 2.1 the Π_j, $j = 0, \ldots, M$ are the unique solution of the preceding, that $P_j = \Pi_j$, $j = 0, \ldots, M$.

Example 2f. Suppose in Example 2c that we are interested in the proportion of time there are j molecules in urn 1, $j = 0, \ldots, M$. By Theorem 2.1 these quantities will be the unique solution of

$$\Pi_0 = \Pi_1 \times \frac{1}{M}$$

$$\Pi_j = \Pi_{j-1} \times \frac{M - j + 1}{M} + \Pi_{j+1} \times \frac{j + 1}{M} \qquad j = 1, \ldots, M$$

$$\Pi_M = \Pi_{M-1} \times \frac{1}{M}$$

$$\sum_{j=0}^{M} \Pi_j = 1$$

However, as it is easily checked that

$$\Pi_j = \binom{M}{j}\left(\frac{1}{2}\right)^M \qquad j = 0, \ldots, M$$

satisfy the equations above, it follows that these are the long-run proportions of time that the Markov chain is in each of the states. (See Problem 11 for an explanation of how one might have guessed at the foregoing solution.)

9.3 SURPRISE, UNCERTAINTY, AND ENTROPY

Consider an event E that can occur when an experiment is performed. How surprised would we be to hear that E does, in fact, occur? It seems reasonable to suppose that the amount of surprise engendered by the information that E has occurred should depend on the probability of E. For instance, if the experiment consists of rolling a pair of dice, then we would not be too surprised to hear that E has occurred when E represents the event that the sum of the dice is even (and thus has probability $\frac{1}{2}$), whereas we would certainly be more surprised to hear that E has occurred when E is the event that the sum of the dice is 12 (and thus has probability $\frac{1}{36}$).

In this section we attempt to quantify the concept of surprise. To begin, let us agree to suppose that the surprise one feels upon learning that an event E has occurred depends only on the probability of E; and let us denote by $S(p)$ the surprise evoked by the occurrence of an event having probability p. We determine the functional form of $S(p)$ by first agreeing on a set of reasonable conditions

that $S(p)$ should satisfy, and then proving that these axioms require that $S(p)$ has a specified form. We assume throughout that $S(p)$ is defined for all $0 < p \leq 1$ but is not defined for events having $p = 0$.

Our first condition is just a statement of the intuitive fact that there is no surprise in hearing that an event sure to occur has indeed occurred.

Axiom 1

$$S(1) = 0$$

Our second condition states that the more unlikely an event is to occur, the greater is the surprise evoked by its occurrence.

Axiom 2

$S(p)$ is a strictly decreasing function of p; that is, if $p < q$, then $S(p) > S(q)$.

The third condition is a mathematical statement of the fact that we would intuitively expect a small change in p to correspond to a small change in $S(p)$.

Axiom 3

$S(p)$ is a continuous function of p.

To motivate the final condition, consider two independent events E and F, having respective probabilities $P(E) = p$ and $P(F) = q$. Since $P(EF) = pq$, the surprise evoked by the information that both E and F have occurred is $S(pq)$. Now, suppose that we are first told that E has occurred and then, afterward, that F has also occurred. As $S(p)$ is the surprise evoked by the occurrence of E, it follows that $S(pq) - S(p)$ represents the additional surprise evoked when we are informed that F has also occurred. However, as F is independent of E, the knowledge that E occurred does not change the probability of F, and hence the additional surprise should just be $S(q)$. This reasoning suggests the final condition.

Axiom 4

$$S(pq) = S(p) + S(q) \qquad 0 < p \leq 1, 0 < q \leq 1$$

We are now ready for Theorem 3.1, which yields the structure of $S(p)$.

Theorem 3.1

If $S(\cdot)$ satisfies Axioms 1 through 4, then

$$S(p) = -C \log_2 p$$

where C is an arbitrary positive integer.

Proof: It follows from Axiom 4 that

$$S(p^2) = S(p) + S(p) = 2S(p)$$

and by induction that

$$S(p^m) = m\, S(p) \tag{3.1}$$

Also, since for any integral n, $S(p) = S(p^{1/n} \cdots p^{1/n}) = n\, S(p^{1/n})$, it follows that

$$S(p^{1/n}) = \frac{1}{n} S(p) \tag{3.2}$$

Thus, from Equations (3.1) and (3.2), we obtain

$$S(p^{m/n}) = m\, S(p^{1/n})$$

$$= \frac{m}{n} S(p)$$

which is equivalent to

$$S(p^x) = xS(p) \tag{3.3}$$

whenever x is a positive rational number. But this implies by the continuity of S (Axiom 3) that Equation (3.3) is valid for all nonnegative x. (Reason this out.)

Now, for any p, $0 < p \leq 1$, let $x = -\log_2 p$. Then $p = \left(\frac{1}{2}\right)^x$ and from Equation (3.3),

$$S(p) = S\left(\left(\tfrac{1}{2}\right)^x\right) = xS\left(\tfrac{1}{2}\right) = -C \log_2 p$$

where $C = S\left(\frac{1}{2}\right) > S(1) = 0$ by Axioms 2 and 1. ∎

It is usual to let C equal 1. In this case the surprise is said to be expressed in units of *bits* (short for *binary digits*).

Consider now a random variable X, which must take on one of the values $x_1, \ldots, x_n$ with respective probabilities $p_1, \ldots, p_n$. As $-\log p_i$ represents the surprise evoked if X takes on the value x_i,[†] it follows that the expected amount of surprise we shall receive upon learning the value of X is given by

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i$$

[†] For the remainder of this chapter we write $\log x$ for $\log_2 x$. Also, we use $\ln x$ for $\log_e x$.

The quantity $H(X)$ is known in information theory as the *entropy* of the random variable X. (In case one of the $p_i = 0$, we take $0 \log 0$ to equal 0.) It can be shown (and we leave it as an exercise) that $H(X)$ is maximized when all of the p_i are equal. (Is this intuitive?)

As $H(X)$ represents the average amount of surprise one receives upon learning the value of X, it can also be interpreted as representing the amount of *uncertainty* that exists as to the value of X. In fact, in information theory, $H(X)$ is interpreted as the average amount of *information* received when the value of X is observed. Thus the average surprise evoked by X, the uncertainty of X, or the average amount of information yielded by X, all represent the same concept viewed from three slightly different points of view.

Consider now two random variables X and Y, which take on respective values $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ with joint mass function

$$p(x_i, y_j) = P\{X = x_i, Y = y_j\}$$

It follows that the uncertainty as to the value of the random vector (X, Y), denoted by $H(X, Y)$, is given by

$$H(X, Y) = -\sum_i \sum_j p(x_i, y_j) \log p(x_i, y_j)$$

Suppose now that Y is observed to equal y_j. In this situation the amount of uncertainty remaining in X is given by

$$H_{Y=y_j}(X) = -\sum_i p(x_i|y_j) \log p(x_i|y_j)$$

where

$$p(x_i|y_j) = P\{X = x_i|Y = y_j\}$$

Hence the average amount of uncertainty that will remain in X after Y is observed is given by

$$H_Y(X) = \sum_j H_{Y=y_j}(X) p_Y(y_j)$$

where

$$p_Y(y_j) = P\{Y = y_j\}$$

Proposition 3.1 relates $H(X, Y)$ to $H(Y)$ and $H_Y(X)$. It states that the uncertainty as to the value of X and Y is equal to the uncertainty of Y plus the average uncertainty remaining in X when Y is to be observed.

Proposition 3.1

$$H(X, Y) = H(Y) + H_Y(X)$$

CHAPTER 10

Simulation

10.1 INTRODUCTION

How can we determine the probability of our winning a game of solitaire? (By solitaire we mean any one of the standard solitaire games played with an ordinary deck of 52 playing cards and with some fixed playing strategy.) One possible approach is to start with the reasonable hypothesis that all (52)! possible arrangements of the deck of cards are equally likely to occur and then attempt to determine how many of these lead to a win. Unfortunately, there does not appear to be any systematic method for determining the number of arrangements that lead to a win and, as (52)! is a rather large number and the only way to determine whether or not a particular arrangement leads to a win seems to be by playing the game out, it can be seen that this approach will not work.

In fact, it might appear that the determination of the win probability for solitaire is mathematically intractable. However, all is not lost, for probability falls not only within the realm of mathematics, but also within the realm of applied science; and, as in all applied sciences, experimentation is a valuable technique. For our solitaire example, experimentation takes the form of playing a large number of such games, or, better yet, programming a computer to do so. After playing, say n games, if we let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th game results in a win} \\ 0 & \text{otherwise} \end{cases}$$

then X_i, $i = 1, \ldots, n$ will be independent Bernoulli random variables for which

$$E[X_i] = P\{\text{win at solitaire}\}$$

Hence, by the strong law of large numbers, we know that

$$\sum_{i=1}^{n} \frac{X_i}{n} = \frac{\text{number of games won}}{\text{number of games played}}$$

will, with probability 1, converge to $P\{\text{win at solitaire}\}$. That is, by playing a large number of games we can use the proportion of won games as an estimate of the probability of winning. This method of empirically determining probabilities by means of experimentation is known as *simulation.*

In order to use a computer to initiate a simulation study, we must be able to generate the value of a uniform (0, 1) random variable; such variates are called random numbers. To generate such numbers, most computers have a built-in subroutine, called a random number generator, whose output is a sequence of pseudo random numbers. This is a sequence of numbers that is, for all practical purposes, indistinguishable from a sample from the uniform (0, 1) distribution. Most random number generators start with an initial value X_0, called the seed, and then recursively compute values by specifying positive integers a, c, and m, and then letting

$$X_{n+1} = (aX_n + c) \text{ modulo } m \qquad n \geq 0$$

where the foregoing means that $aX_n + c$ is divided by m and the remainder is taken as the value of X_{n+1}. Thus each X_n is either $0, 1, \ldots, m - 1$ and the quantity X_n/m is taken as an approximation to a uniform (0, 1) random variable. It can be shown that, subject to suitable choices for a, c, and m, the foregoing gives rise to a sequence of numbers that look as if they were generated from independent uniform (0, 1) random variables.

As our starting point in simulation, we shall suppose that we can simulate from the uniform (0, 1) distribution and we shall use the term *random numbers* to mean independent random variables from this distribution.

In the solitaire example we would need to program a computer to play out the game starting with a given ordering of the cards. However, since the initial ordering is supposed to be equally likely to be any of the (52)! possible permutations, it is also necessary to be able to generate a random permutation. Using only random numbers, the following algorithm shows how this can be accomplished. The algorithm begins by randomly choosing one of the elements and then putting it in position n; it then randomly chooses among the remaining elements and puts the choice in position $n - 1$; and so on. It efficiently makes a random choice among the remaining elements by keeping these elements in an ordered list and then randomly choosing a position on that list.

Example 1a. ***Generating a random permutation.*** Suppose we are interested in generating a permutation of the integers $1, 2, \ldots, n$ that is such that all $n!$ possible orderings are equally likely. Starting with any initial permutation we will accomplish this after $n - 1$ steps where at each step we will interchange the positions of two of the numbers of the permutation. Throughout, we will keep track of the permutation by letting $X(i)$, $i = 1, \ldots, n$ denote the number currently in position i. The algorithm operates as follows:

1. Consider any arbitrary permutation and let $X(i)$ denote the element in position i, $i = 1 \ldots, n$. [For instance, we could take $X(i) = i$, $i = 1, \ldots, n$.]

2. Generate a random variable N_n that is equally likely to equal any of the values $1, 2, \ldots, n$.
3. Interchange the values of $X(N_n)$ and $X(n)$. The value of $X(n)$ will now remain fixed. [For instance, suppose that $n = 4$ and initially $X(i) = i$, $i = 1, 2, 3, 4$. If $N_4 = 3$, then the new permutation is $X(1) = 1$, $X(2) = 2$, $X(3) = 4$, $X(4) = 3$, and element 3 will remain in position 4 throughout.]
4. Generate a random variable N_{n-1} that is equally likely to be either $1, 2, \ldots, n - 1$.
5. Interchange the values of $X(N_{n-1})$ and $X(n - 1)$. [If $N_3 = 1$, then the new permutation is $X(1) = 4$, $X(2) = 2$, $X(3) = 1$, $X(4) = 3$.]
6. Generate N_{n-2}, which is equally likely to be either $1, 2, \ldots, n - 2$.
7. Interchange the values of $X(N_{n-2})$ and $X(2)$. [If $N_2 = 1$, then the new permutation is $X(1) = 2$, $X(2) = 4$, $X(3) = 1$, $X(4) = 3$ and this is the final permutation.]
8. Generate N_{n-3}, and so on. The algorithm continues until N_2 is generated and after the next interchange the resulting permutation is the final one.

To implement this algorithm, it is necessary to be able to generate a random variable that is equally likely to be any of the values $1, 2, \ldots, k$. To accomplish this, let U denote a random number—that is, U is uniformly distributed on $(0, 1)$, and note that kU is uniform on $(0, k)$. Hence

$$P\{i - 1 < kU < i\} = \frac{1}{k} \qquad i = 1, \ldots, k$$

so if we take $N_k = [kU] + 1$, where $[x]$ is the integer part of x (that is, it is the largest integer less than or equal to x), then N_k will have the desired distribution.

The algorithm can now be succinctly written as follows:

Step 1. Let $X(1), \ldots, X(n)$ be any permutation of $1, 2, \ldots, n$. [For instance, we can set $X(i) = i$, $i = 1, \ldots, n$.]
Step 2. Let $I = n$.
Step 3. Generate a random number U and set $N = [IU] + 1$.
Step 4. Interchange the values of $X(N)$ and $X(I)$.
Step 5. Reduce the value of I by 1 and if $I > 1$ go to step 3.
Step 6. $X(1), \ldots, X(n)$ is the desired random generated permutation.

The foregoing algorithm for generating a random permutation is extremely useful. For instance, suppose that a statistician is developing an experiment to compare the effects of m different treatments on a set of n subjects. He decides to split the subjects into m different groups of respective sizes $n_1, n_2, \ldots, n_m$ where $\sum_{i=1}^{m} n_i = n$ with the members of the ith group to receive treatment i. To eliminate any bias in the assignment of subjects to treatments (for instance, it would cloud the meaning of the experimental results if it turned out that all the "best" subjects had been put in the same group), it is imperative that the assignment of a subject to a given group be done "at random." How is this to be accomplished?†

A simple and efficient procedure is to arbitrarily number the subjects 1 through n and then generate a random permutation $X(1), \ldots, X(n)$ of $1, 2, \ldots, n$. Now assign subjects $X(1), X(2), \ldots, X(n_1)$ to be in group 1, $X(n_1 + 1), \ldots, X(n_1 + n_2)$ to be in group 2, and in general group j is to consist of subjects numbered $X(n_1 + n_2 + \cdots + n_{j-1} + k)$, $k = 1, \ldots, n_j$.

10.2 GENERAL TECHNIQUES FOR SIMULATING CONTINUOUS RANDOM VARIABLES

In this section we present two general methods for using random numbers to simulate continuous random variables.

10.2.1 The Inverse Transformation Method

A general method for simulating a random variable having a continuous distribution—called the *inverse transformation method*—is based on the following proposition.

Proposition 2.1

Let U be a uniform $(0, 1)$ random variable. For any continuous distribution function F, if we define the random variable Y by

$$Y = F^{-1}(U)$$

then the random variable Y has distribution function F. [$F^{-1}(x)$ is defined to equal that value y for which $F(y) = x$.]

Proof

$$\begin{aligned} F_Y(a) &= P\{Y \le a\} \\ &= P\{F^{-1}(U) \le a\} \end{aligned} \tag{2.1}$$

Now, since $F(x)$ is a monotone function, it follows that $F^{-1}(U) \le a$ if and only if $U \le F(a)$. Hence, from Equation (2.1), we see that

$$\begin{aligned} F_Y(a) &= P\{U \le F(a)\} \\ &= F(a) \end{aligned}$$

† When $m = 2$, another technique for randomly dividing the subjects was presented in Example 2g of Chapter 6. The preceding procedure is faster but requires more space than the one of Example 2g.

It follows from Proposition 2.1 that we can simulate a random variable X having a continuous distribution function F by generating a random number U and then setting $X = F^{-1}(U)$.

Example 2a. ***Simulating an exponential random variable.*** If $F(x) = 1 - e^{-x}$, then $F^{-1}(u)$ is that value of x such that

$$1 - e^{-x} = u$$

or

$$x = -\log(1 - u)$$

Hence, if U is a uniform (0, 1) variable, then

$$F^{-1}(U) = -\log(1 - U)$$

is exponentially distributed with mean 1. Since $1 - U$ is also uniformly distributed on (0, 1), it follows that $-\log U$ is exponential with mean 1. Since cX is exponential with mean c when X is exponential with mean 1, it follows that $-c \log U$ is exponential with mean c. ∎

The results of Example 2a can also be utilized to simulate a gamma random variable.

Example 2b. ***Simulating a gamma (n, λ) random variable.*** To simulate from a gamma distribution with parameters (n, λ), when n is an integer, we use the fact that the sum of n independent exponential random variables each having rate λ has this distribution. Hence, if $U_1, \ldots, U_n$ are independent uniform (0, 1) random variables,

$$X = -\sum_{i=1}^{n} \frac{1}{\lambda} \log U_i = -\frac{1}{\lambda} \log\left(\prod_{i=1}^{n} U_i\right)$$

has the desired distribution. ∎

10.2.2 The Rejection Method

Suppose that we have a method for simulating a random variable having density function $g(x)$. We can use this as the basis for simulating from the continuous distribution having density $f(x)$ by simulating Y from g and then accepting this simulated value with a probability proportional to $f(Y)/g(Y)$.

Specifically, let c be a constant such that

$$\frac{f(y)}{g(y)} \le c \qquad \text{for all } y$$

We then have the following technique for simulating a random variable having density f.

Rejection Method

Step 1. Simulate Y having density g and simulate a random number U.

Step 2. If $U \le f(Y)/cg(Y)$, set $X = Y$. Otherwise return to step 1.

The rejection method is expressed pictorially in Figure 10.1.

We now prove that the rejection method works.

Proposition 2.2

The random variable X generated by the rejection method has density function f.

Proof: Let X be the value obtained and let N denote the number of necessary iterations. Then

$$\begin{aligned} P\{X \le x\} &= P\{Y_N \le x\} \\ &= P\left\{Y \le x \middle| U \le \frac{f(Y)}{cg(Y)}\right\} \\ &= \frac{P\left\{Y \le x, U \le \dfrac{f(Y)}{cg(Y)}\right\}}{K} \end{aligned}$$

where $K = P\{U \le f(Y)/cg(Y)\}$. Now the joint density function of Y and U is, by independence,

$$f(y, u) = g(y) \qquad 0 < u < 1$$

so, using the foregoing, we have

$$\begin{aligned} P\{X \le x\} &= \frac{1}{K} \iint_{\substack{y \le x \\ 0 \le u \le f(y)/cg(y)}} g(y)\, du\, dy \\ &= \frac{1}{K} \int_{-\infty}^{x} \int_{0}^{f(y)/cg(y)} du\, g(y)\, dy \\ &= \frac{1}{cK} \int_{-\infty}^{x} f(y)\, dy \end{aligned} \tag{2.2}$$

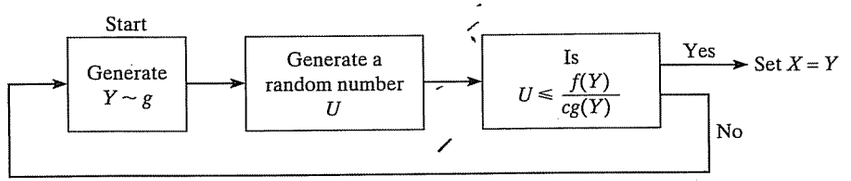


Figure 10.1 Rejection method for simulating a random variable X having density function f.

Letting x approach ∞ and using the fact that f is a density gives

$$1 = \frac{1}{cK}\int_{-\infty}^{\infty} f(y)\,dy = \frac{1}{cK}$$

Hence from Equation (2.2) we obtain that

$$P\{X \le x\} = \int_{-\infty}^{x} f(y)\,dy$$

which completes the proof. ∎

REMARKS. (a) It should be noted that the way in which we "accept the value Y with probability $f(Y)/cg(Y)$" is by generating a random number U and then accepting Y if $U \le f(Y)/cg(Y)$.

(b) Since each iteration will, independently, result in an accepted value with probability $P\{U \le f(Y)/cg(Y)\} = K = 1/c$, it follows that the number of iterations has a geometric distribution with mean c.

Example 2c. ***Simulating a normal random variable.*** To simulate a unit normal random variable Z (that is, one with mean 0 and variance 1), note first that the absolute value of Z has probability density function

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \qquad 0 < x < \infty \tag{2.3}$$

We will start by simulating from the preceding density function by using the rejection method with g being the exponential density function with mean 1—that is

$$g(x) = e^{-x} \qquad 0 < x < \infty$$

Now, note that

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \sqrt{\frac{2}{\pi}} \exp\left\{\frac{-(x^2 - 2x)}{2}\right\} \\
&= \sqrt{\frac{2}{\pi}} \exp\left\{\frac{-(x^2 - 2x + 1)}{2} + \frac{1}{2}\right\} \\
&= \sqrt{\frac{2e}{\pi}} \exp\left\{\frac{-(x-1)^2}{2}\right\} \qquad (2.4) \\
&\le \sqrt{\frac{2e}{\pi}}
\end{aligned}$$

Hence we can take $c = \sqrt{2e/\pi}$; so, from Equation (2.4),

$$\frac{f(x)}{cg(x)} = \exp\left\{\frac{-(x-1)^2}{2}\right\}$$

Therefore, using the rejection method we can simulate the absolute value of a unit normal random variable as follows:

(a) Generate independent random variables Y and U, Y being exponential with rate 1 and U being uniform on (0, 1).
(b) If $U \le \exp\{-(Y-1)^2/2\}$ set $X = Y$. Otherwise, return to (a).

Once we have simulated a random variable X having density function as in Equation (2.3) we can then generate a unit normal random variable Z by letting Z be equally likely to be either X or $-X$.

In step (b), the value Y is accepted if $U \le \exp\{-(Y-1)^2/2\}$, which is equivalent to $-\log U \ge (Y-1)^2/2$. However, in Example 2a it was shown that $-\log U$ is exponential with rate 1, so steps (a) and (b) are equivalent to

(a′) Generate independent exponentials with rate 1, Y_1 and Y_2.
(b′) If $Y_2 \ge (Y_1 - 1)^2/2$, set $X = Y_1$. Otherwise, return to (a).

Suppose now that the foregoing results in Y_1 being accepted—and so we know that Y_2 is larger than $(Y_1 - 1)^2/2$. By how much does the one exceed the other? To answer this recall that Y_2 is exponential with rate 1, and so, given that it exceeds some value, the amount by which Y_2 exceeds $(Y_1 - 1)^2/2$ [that is, its "additional life" beyond the time $(Y_1 - 1)^2/2$] is (by the memoryless property) also exponentially distributed with rate 1. That is, when we accept step (b′), we obtain not only X (the absolute value of a unit normal) but by computing $Y_2 - (Y_1 - 1)^2/2$ we can also generate an exponential random variable (independent of X) having rate 1.

Hence, summing up, we have the following algorithm that generates an exponential with rate 1 and an independent unit normal random variable.

Step 1. Generate Y_1, an exponential random variable with rate 1.
Step 2. Generate Y_2, an exponential random variable with rate 1.
Step 3. If $Y_2 - (Y_1 - 1)^2/2 > 0$ set $Y = Y_2 - (Y_1 - 1)^2/2$ and go to step 4. Otherwise, go to step 1.
Step 4. Generate a random number U and set

$$Z = \begin{cases} Y_1 & \text{if } U \le \frac{1}{2} \\ -Y_1 & \text{if } U > \frac{1}{2} \end{cases}$$

The random variables Z and Y generated by the foregoing are independent with Z being normal with mean 0 and variance 1 and Y being exponential with rate 1. (If we want the normal random variable to have mean μ and variance σ^2, just take $\mu + \sigma Z$.)

REMARKS. (a) Since $c = \sqrt{2e/\pi} \approx 1.32$, the foregoing requires a geometrically distributed number of iterations of step 2 with mean 1.32.

(b) If we want to generate a sequence of unit normal random variables, then we can use the exponential random variable Y obtained in step 3 as the initial

exponential needed in step 1 for the next normal to be generated. Hence, on the average, we can simulate a unit normal by generating 1.64 ($= 2 \times 1.32 - 1$) exponentials and computing 1.32 squares.

Example 2d. ***Simulating normal random variables—the polar method.*** It was shown in Example 7b of Chapter 6 that if X and Y are independent unit normal random variables then their polar coordinates $R = \sqrt{X^2 + Y^2}$, $\Theta = \tan^{-1}(Y/X)$ are independent, with R^2 being exponentially distributed with mean 2 and Θ being uniformly distributed on $(0, 2\pi)$. Hence, if U_1 and U_2 are random numbers then (using the result of Example 2a) we can set

$$R = (-2 \log U_1)^{1/2}$$
$$\Theta = 2\pi U_2$$

which yields that

$$\begin{aligned} X &= R \cos \Theta = (-2 \log U_1)^{1/2} \cos(2\pi U_2) \\ Y &= R \sin \Theta = (-2 \log U_1)^{1/2} \sin(2\pi U_2) \end{aligned} \tag{2.5}$$

are independent unit normals.

The above approach to generating unit normal random variables is called the Box–Muller approach. Its efficiency suffers somewhat from its need to compute the above sine and cosine values. There is, however, a way to get around this potentially time-consuming difficulty. To begin, note that if U is uniform on $(0, 1)$ then $2U$ is uniform on $(0, 2)$ and so $2U - 1$ is uniform on $(-1, 1)$. Thus, if we generate random numbers U_1 and U_2 and set

$$V_1 = 2U_1 - 1$$
$$V_2 = 2U_2 - 1$$

then (V_1, V_2) is uniformly distributed in the square of area 4 centered at $(0, 0)$ (see Figure 10.2).

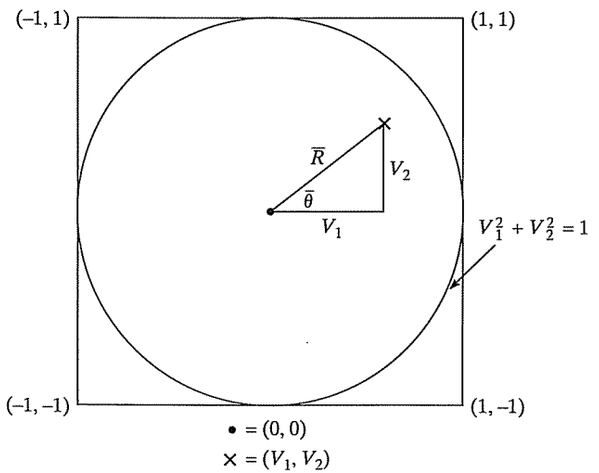


Figure 10.2

Suppose now that we continually generate such pairs (V_1, V_2) until we obtain one that is contained in the disk of radius 1 centered at $(0, 0)$—that is, until (V_1, V_2) is such that $V_1^2 + V_2^2 \leq 1$. It now follows that such a pair (V_1, V_2) is uniformly distributed in the disk. If we let $\overline{R}$, $\overline{\Theta}$ denote the polar coordinates of this pair, then it is easy to verify that $\overline{R}$ and $\overline{\Theta}$ are independent, with $\overline{R}^2$ being uniformly distributed on $(0, 1)$, and $\overline{\Theta}$ uniformly distributed on $(0, 2\pi)$ (see Problem 13).

Since

$$\sin \overline{\Theta} = \frac{V_2}{\overline{R}} = \frac{V_2}{\sqrt{V_1^2 + V_2^2}}$$
$$\cos \overline{\Theta} = \frac{V_1}{\overline{R}} = \frac{V_1}{\sqrt{V_1^2 + V_2^2}}$$

it follows from Equation (2.5) that we can generate independent unit normals X and Y by generating another random number U and setting

$$X = (-2 \log U)^{1/2} V_1/\overline{R}$$
$$Y = (-2 \log U)^{1/2} V_2/\overline{R}$$

In fact, since (conditional on $V_1^2 + V_2^2 \leq 1$) $\overline{R}^2$ is uniform on $(0, 1)$ and is independent of $\overline{\theta}$ we can use it instead of generating a new random number U; thus showing that

$$X = (-2 \log \overline{R}^2)^{1/2} \frac{V_1}{\overline{R}} = \sqrt{\frac{-2 \log S}{S}}\, V_1$$

$$Y = (-2 \log \overline{R}^2)^{1/2} \frac{V_2}{\overline{R}} = \sqrt{\frac{-2 \log S}{S}}\, V_2$$

and independent unit normals, where

$$S = \overline{R}^2 = V_1^2 + V_2^2$$

Summing up, we thus have the following approach to generating a pair of independent unit normals:

Step 1. Generate random numbers U_1 and U_2.
Step 2. Set $V_1 = 2U_1 - 1$, $V_2 = 2U_2 - 1$, $S = V_1^2 + V_2^2$.
Step 3. If $S > 1$, return to step 1.
Step 4. Return the independent unit normals

$$X = \sqrt{\frac{-2 \log S}{S}}\, V_1, \quad Y = \sqrt{\frac{-2 \log S}{S}}\, V_2$$

The above is called the polar method. Since the probability that a random point in the square will fall within the circle is equal to $\pi/4$ (the area of the circle divided by the area of the square), it follows that, on average, the polar method will require $4/\pi \approx 1.273$ iterations of step 1.

Hence it will, on average, require 2.546 random numbers, 1 logarithm, 1 square root, 1 division, and 4.546 multiplications to generate 2 independent unit normals.

Example 2e. ***Simulating a chi-squared random variable.*** The chi-squared distribution with n degrees of freedom is the distribution of $\chi_n^2 = Z_1^2 + \cdots + Z_n^2$, where Z_i, $i = 1, \ldots, n$ are independent unit normals. Now it was shown in Section 3 of Chapter 6 that $Z_1^2 + Z_2^2$ has an exponential distribution with rate $\frac{1}{2}$. Hence, when n is even, say $n = 2k$, χ_{2k}^2 has a gamma distribution with parameters $(k, \frac{1}{2})$. Hence, $-2\log(\Pi_{i=1}^k U_i)$ has a chi-squared distribution with $2k$ degrees of freedom. We can simulate a chi-squared random variable with $2k + 1$ degrees of freedom by first simulating a unit normal random variable Z and then adding Z^2 to the foregoing. That is,

$$\chi_{2k+1}^2 = Z^2 - 2\log\left(\prod_{i=1}^{k} U_i\right)$$

where $Z, U_1, \ldots, U_n$ are independent with Z being a unit normal and the others being uniform (0, 1) random variables.

10.3 SIMULATING FROM DISCRETE DISTRIBUTIONS

All of the general methods for simulating random variables from continuous distributions have analogs in the discrete case. For instance, if we want to simulate a random variable Z having probability mass function

$$P\{X = x_j\} = P_j, \qquad j = 0, 1, \ldots, \qquad \sum_j P_j = 1$$

We can use the following discrete time analog of the inverse transform technique.

To simulate X for which $P\{X = x_j\} = P_j$ let U be uniformly distributed over (0, 1), and set

$$X = \begin{cases} x_1 & \text{if } U < P_1 \\ x_2 & \text{if } P_1 < U < P_1 + P_2 \\ \vdots & \\ x_j & \text{if } \sum_1^{j-1} P_i < U < \sum_i^j P_i \\ \vdots & \end{cases}$$

Since

$$P\{X = x_j\} = P\left\{\sum_1^{j-1} P_i < U < \sum_1^j P_i\right\} = P_j$$

we see that X has the desired distribution.

Example 3a. ***The geometric distribution.*** Suppose that independent trials each of which results in a "success" with probability p, $0 < p < 1$, are continually performed until a success occurs. Letting X denote the number of necessary trials, then

$$P\{X = i\} = (1 - p)^{i-1}p \qquad i \geq 1$$

which is seen by noting that $X = i$ if the first $i - 1$ trials are all failures and the ith is a success. The random variable X is said to be a geometric random variable with parameter p. Since

$$\begin{aligned} \sum_{i=1}^{j-1} P\{X = i\} &= 1 - P\{X > j - 1\} \\ &= 1 - P\{\text{first } j - 1 \text{ are all failures}\} \\ &= 1 - (1 - p)^{j-1} \qquad j \geq 1 \end{aligned}$$

we can simulate such a random variable by generating a random number U and then setting X equal to that value j for which

$$1 - (1 - p)^{j-1} < U < 1 - (1 - p)^j$$

or, equivalently, for which

$$(1 - p)^j < 1 - U < (1 - p)^{j-1}$$

Since $1 - U$ has the same distribution as U, we can thus define X by

$$\begin{aligned} X &= \min\{j : (1 - p)^j < U\} \\ &= \min\{j : j\log(1 - p) < \log U\} \\ &= \min\left\{j : j > \frac{\log U}{\log(1 - p)}\right\} \end{aligned}$$

where the inequality changed sign since $\log(1 - p)$ is negative [since $\log(1 - p) < \log 1 = 0$]. Using the notation $[x]$ for the integer part of x (that is, $[x]$ is the largest integer less than or equal to x), we can write

$$X = 1 + \left[\frac{\log U}{\log(1 - p)}\right]$$

■

As in the continuous case, special simulating techniques have been developed for the more common discrete distributions. We now present two of these.

Example 3b. ***Simulating a binomial random variable.*** A binomial (n, p) random variable can be easily simulated by recalling that it can be expressed as the sum of n independent Bernoulli random variables. That is, if $U_1, \ldots, U_n$ are independent uniform (0, 1) variables, then letting

$$X_i = \begin{cases} 1 & \text{if } U_i < p \\ 0 & \text{otherwise} \end{cases}$$

it follows that $X \equiv \sum_{i=1}^{n} X_i$ is a binomial random variable with parameters n and p.

Example 3c. ***Simulating a Poisson random variable.*** To simulate a Poisson random variable with mean λ, generate independent uniform (0, 1) random variables $U_1, U_2, \ldots$ stopping at

$$N = \min\left\{n: \prod_{i=1}^{n} U_i < e^{-\lambda}\right\}$$

The random variable $X \equiv N - 1$ has the desired distribution. That is, if we continue generating random numbers until their product falls below $e^{-\lambda}$, then the number required, minus 1, is Poisson with mean λ.

That $X \equiv N - 1$ is indeed a Poisson random variable having mean λ can perhaps be most easily seen by noting that

$$X + 1 = \min\left\{n: \prod_{i=1}^{n} U_i < e^{-\lambda}\right\}$$

is equivalent to

$$X = \max\left\{n: \prod_{i=1}^{n} U_i \geq e^{-\lambda}\right\} \qquad \text{where } \prod_{i=1}^{0} U_i \equiv 1$$

or, taking logarithms, to

$$X = \max\left\{n: \sum_{i=1}^{n} \log U_i \geq -\lambda\right\}$$

or

$$X = \max\left\{n: \sum_{i=1}^{n} -\log U_i \leq \lambda\right\}$$

However, $-\log U_i$ is exponential with rate 1 and so X can be thought of as being the maximum number of exponentials having rate 1 that can be summed and still be less than λ. But by recalling that the times between successive events of a Poisson process having rate 1 are independent exponentials with rate 1, it follows that X is equal to the number of events by time λ of a Poisson process having rate 1; and thus X has a Poisson distribution with mean λ.

10.4 VARIANCE REDUCTION TECHNIQUES

Let $X_1, \ldots, X_n$ have a given joint distribution and suppose that we are interested in computing

$$\theta \equiv E[g(X_1, \ldots, X_n)]$$

where g is some specified function. It sometimes turns out that it is extremely difficult to analytically compute the foregoing, and when such is the case we can attempt to use simulation to estimate θ. This is done as follows: generate $X_1^{(1)}, \ldots, X_n^{(1)}$ having the same joint distribution as $X_1, \ldots, X_n$ and set

$$Y_1 = g(X_1^{(1)}, \ldots, X_n^{(1)})$$

Now simulate a second set of random variables (independent of the first set) $X_1^{(2)}, \ldots, X_n^{(2)}$ having the distribution of $X_1, \ldots, X_n$ and set

$$Y_2 = g(X_1^{(2)}, \ldots, X_n^{(2)})$$

Continue this until you have generated k (some predetermined number) sets and so have also computed $Y_1, Y_2, \ldots, Y_k$. Now, $Y_1, \ldots, Y_k$ are independent and identically distributed random variables each having the same distribution of $g(X_1, \ldots, X_n)$. Thus, if we let $\bar{Y}$ denote the average of these k random variables—that is,

$$\bar{Y} = \sum_{i=1}^{k} \frac{Y_i}{k}$$

then

$$E[\bar{Y}] = \theta$$
$$E[(\bar{Y} - \theta)^2] = \text{Var}(\bar{Y})$$

Hence, we can use $\bar{Y}$ as an estimate of θ. Since the expected square of the difference between $\bar{Y}$ and θ is equal to the variance of $\bar{Y}$ we would like this quantity to be as small as possible. [In the preceding situation, $\text{Var}(\bar{Y}) = \text{Var}(Y_i)/k$, which is usually not known in advance but must be estimated from the generated values $Y_1, \ldots, Y_n$]. We now present 3 general techniques for reducing the variance of our estimator.

10.4.1 Use of Antithetic Variables

In the foregoing situation, suppose that we have generated Y_1 and Y_2, which are identically distributed random variables having mean θ. Now

$$\text{Var}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{1}{4}[\text{Var}(Y_1) + \text{Var}(Y_2) + 2\,\text{Cov}(Y_1, Y_2)]$$
$$= \frac{\text{Var}(Y_1)}{2} + \frac{\text{Cov}(Y_1, Y_2)}{2}$$

Hence it would be advantageous (in the sense that the variance would be reduced) if Y_1 and Y_2 rather than being independent were negatively correlated. To see how we could arrange this, let us suppose that the random variables $X_1, \ldots, X_n$ are independent and, in addition, that each is simulated via the inverse transform technique. That is, X_i is simulated from $F_i^{-1}(U_i)$ where U_i is a random number and F_i is the distribution of X_i. Hence, Y_1 can be expressed as

$$Y_1 = g(F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))$$

Now, since $1 - U$ is also uniform over (0, 1) whenever U is a random number (and is negatively correlated with U), it follows that Y_2 defined by

$$Y_2 = g(F_1^{-1}(1 - U_1), \ldots, F_n^{-1}(1 - U_n))$$

will have the same distribution as Y_1. Hence, if Y_1 and Y_2 were negatively correlated, then generating Y_2 by this means would lead to a smaller variance than if it were generated by a new set of random numbers. (In addition, there is a computational savings since rather than having to generate n additional random numbers, we need only subtract each of the previous n from 1). Although we cannot, in general, be certain that Y_1 and Y_2 will be negatively correlated, this often turns out to be the case and indeed it can be proven that it will be so whenever g is a monotonic function.

10.4.2 Variance Reduction by Conditioning

Let us start by recalling the conditional variance formula (see Section 7.4.4)

$$\text{Var}(Y) = E[\text{Var}(Y|Z)] + \text{Var}(E[Y|Z])$$

Now suppose that we are interested in estimating $E[g(X_1, \ldots, X_n)]$ by simulating $\mathbf{X} = (X_1, \ldots, X_n)$ and then computing $Y = g(\mathbf{X})$. Now, if for some random variable Z we can compute $E[Y|Z]$ then, as $\text{Var}(Y|Z) \geq 0$, it follows from the conditional variance formula above that

$$\text{Var}(E[Y|Z]) \leq \text{Var}(Y)$$

implying, since $E[E[Y|Z]] = E[Y]$, that $E[Y|Z]$ is a better estimator of $E[Y]$ than is Y.

Example 4a. ***Estimation of*** π. Let U_1 and U_2 be random numbers and set $V_i = 2U_i - 1, i = 1, 2$. As noted in Example 2d, (V_1, V_2) will be uniformly distributed in the square of area 4 centered at (0, 0). The probability that this point will fall within the inscribed circle of radius 1 centered at (0, 0) (see Figure 10.2) is equal to $\pi/4$ (the ratio of the area of the circle to that of the square). Hence, upon simulating a large number n of such pairs and setting

$$I_j = \begin{cases} 1 & \text{if the } j\text{th pair falls within the circle} \\ 0 & \text{otherwise} \end{cases}$$

it follows that $I_j, j = 1, \ldots, n$ will be independent and identically distributed random variables having $E[I_j] = \pi/4$. Thus, by the strong law of large numbers

$$\frac{I_1 + \cdots + I_n}{n} \longrightarrow \frac{\pi}{4} \quad \text{as} \quad n \to \infty$$

Therefore, it follows that by simulating a large number of pairs (V_1, V_2) and multiplying the proportion of them that fall within the circle by 4, we can accurately approximate π.

The above estimator can, however, be improved upon by using conditional expectation. If we let I be the indicator variable above for the pair (V_1, V_2) then, rather than using the observed value of I, it is better to condition on V_1 and so utilize

$$\begin{aligned} E[I|V_1] &= P\{V_1^2 + V_2^2 \leq 1|V_1\} \\ &= P\{V_2^2 \leq 1 - V_1^2|V_1\} \end{aligned}$$

Now

$$\begin{aligned} P\{V_2^2 \leq 1 - V_1^2|V_1 = v\} &= P\{V_2^2 \leq 1 - v^2\} \\ &= P\{-\sqrt{1 - v^2} \leq V_2 \leq \sqrt{1 - v^2}\} \\ &= \sqrt{1 - v^2} \end{aligned}$$

so

$$E[I|V_1] = E[\sqrt{1 - V_1^2}]$$

Thus, an improvement on using the average value of I to estimate $\pi/4$ is to use the average value of $\sqrt{1 - V_1^2}$. Indeed, since

$$E[\sqrt{1 - V_1^2}] = \int_{-1}^{1} \tfrac{1}{2}\sqrt{1 - v^2}\, dv = \int_0^1 \sqrt{1 - u^2}\, du = E[\sqrt{1 - U^2}]$$

where U is uniform over (0, 1), we can generate n random numbers U and use the average value of $\sqrt{1 - U^2}$ as our estimate of $\pi/4$. (Problem 14 shows that this estimator has the same variance as the average of the n values $\sqrt{1 - V^2}$.)

The above estimator of π can be improved even further by noting that the function $g(u) = \sqrt{1 - u^2}, 0 \leq u \leq 1$ is a monotone decreasing function of u and so the method of antithetic variables will reduce the variance of the estimator of $E[\sqrt{1 - U^2}]$. That is, rather than generating n random numbers and using the average value of $\sqrt{1 - U^2}$ as an estimator of $\pi/4$, an improved estimator would be obtained by generating only $n/2$ random numbers U and then using one-half the average of $\sqrt{1 - U^2} + \sqrt{1 - (1 - U)^2}$ as the estimator of $\pi/4$.

The following table gives the estimates of π resulting from simulations, using $n = 10{,}000$, based on the three estimators above.

Method	Estimate of π
Proportion of the random points that fall in the circle	3.1612
Average value of $\sqrt{1 - U^2}$	3.128448
Average value of $\sqrt{1 - U^2} + \sqrt{1 - (1 - U)^2}$	3.139578

A further simulation using the final approach and $n = 64{,}000$ yielded the estimate 3.143288. ∎

10.4.3 Control Variates

Again suppose that we want to use simulation to estimate $E[g(\mathbf{X})]$ where $\mathbf{X} = (X_1, \ldots, X_n)$. But now suppose that for some function f the expected value of $f(\mathbf{X})$ is known—say $E[f(\mathbf{X})] = \mu$. Then for any constant a we can also use

$$W = g(\mathbf{X}) + a[f(\mathbf{X}) - \mu]$$

as an estimator of $E[g(\mathbf{X})]$. Now

$$\text{Var}(W) = \text{Var}[g(\mathbf{X})] + a^2 \text{Var}[f(\mathbf{X})] + 2a \text{Cov}[g(\mathbf{X}), f(\mathbf{X})] \quad (4.1)$$

Simple calculus shows that the foregoing is minimized when

$$a = \frac{-\text{Cov}[f(\mathbf{X}), g(\mathbf{X})]}{\text{Var}[f(\mathbf{X})]} \quad (4.2)$$

and for this value of a

$$\text{Var}(W) = \text{Var}[g(\mathbf{X})] - \frac{[\text{Cov}[f(\mathbf{X}), g(\mathbf{X})]^2}{\text{Var}[f(\mathbf{X})]} \quad (4.3)$$

Unfortunately, neither $\text{Var}[f(\mathbf{X})]$ nor $\text{Cov}[f(\mathbf{X})], g(\mathbf{X})]$ is usually known, so we cannot usually obtain the foregoing reduction in variance. One approach in practice is to use the simulated data to estimate these quantities. This usually yields almost all of the theoretically possible reduction in variance.

SUMMARY

Let F be a continuous distribution function and U a uniform (0, 1) random variable. The random variable $F^{-1}(U)$ has distribution function F, where $F^{-1}(u)$ is that value x such that $F(x) = u$. Applying this result, we can use the values of uniform (0, 1) random variables, called *random numbers,* to generate the values of other random variables. This is called the *inverse transform* method.

Another technique for generating random variables is based on the *rejection* method. Suppose that we have an efficient procedure for generating a random variable from the density function g, and that we desire to generate a random variable having density function f. The rejection method for accomplishing this starts by determining a constant c such that

$$\max \frac{f(x)}{g(x)} \leq c$$

It then proceeds as follows.

1. Generate Y having density g.
2. Generate a random number U.
3. If $U \leq f(Y)/cg(Y)$, set $X = Y$ and stop.
4. Return to step 1.

The number of passes through step 1 is a geometric random variable with mean c.

Standard normal random variables can be efficiently simulated by using the rejection method (with g being exponential with mean 1) or by using a technique known as the polar algorithm.

To estimate a quantity θ, one often generates the values of a partial sequence of random variables whose expected value is θ. The efficiency of this approach is increased when these random variables have a small variance. Three techniques that can often be used to specify random variables with mean θ and relatively small variances are:

1. The use of antithetic variables
2. The use of conditional expectations
3. The use of control variates

PROBLEMS

1. The following algorithm will generate a random permutation of the elements 1, 2, . . . , n. It is somewhat faster than the one presented in Example 1a but is such that no position is fixed until the algorithm ends. In this algorithm, $P(i)$ can be interpreted as the element in position i.

Step 1. Set $k = 1$.
Step 2. Set $P(1) = 1$.
Step 3. If $k = n$, stop. Otherwise, let $k = k + 1$.
Step 4. Generate a random number U and let

$$P(k) = P([kU] + 1)$$
$$P([kU] + 1) = k$$

Go to step 3.

(a) Explain in words what the algorithm is doing.
(b) Show that at iteration k—that is, when the value of $P(k)$ is initially set—that $P(1), P(2), \ldots, P(k)$ is a random permutation of $1, 2, \ldots, k$.

HINT: Use induction and argue that

$$P_k\{i_1, i_2, \ldots, i_{j-1}, k, i_j, \ldots, i_{k-2}, i\}$$
$$= P_{k-1}\{i_1, i_2, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}\}\frac{1}{k}$$
$$= \frac{1}{k!} \quad \text{by the induction hypothesis}$$

2. Develop a technique for simulating a random variable having density function

$$f(x) = \begin{cases} e^{2x} & -\infty < x < 0 \\ e^{-2x} & 0 < x < \infty \end{cases}$$

3. Give a technique for simulating a random variable having the probability density function

$$f(x) = \begin{cases} \frac{1}{2}(x-2) & 2 \le x \le 3 \\ \frac{1}{2}\left(2 - \frac{x}{3}\right) & 3 < x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

4. Present a method to simulate a random variable having distribution function

$$F(x) = \begin{cases} 0 & x \le -3 \\ \frac{1}{2} + \frac{x}{6} & -3 < x < 0 \\ \frac{1}{2} + \frac{x^2}{32} & 0 < x \le 4 \\ 1 & x > 4 \end{cases}$$

5. Use the inverse transformation method to present an approach for generating a random variable from the Weibull distribution

$$F(t) = 1 - e^{-at^\beta} \qquad t \ge 0$$

6. Give a method for simulating a random variable having failure rate function
(a) $\lambda(t) = c$;
(b) $\lambda(t) = ct$;
(c) $\lambda(t) = ct^2$;
(d) $\lambda(t) = ct^3$.

7. In the following, F is the distribution function

$$F(x) = x^n \qquad 0 < x < \infty$$

(a) Give a method for simulating a random variable having distribution F that uses only a single random number.
(b) Let $U_1, \ldots, U_n$ be independent random numbers. Show that

$$P\{\max(U_1, \ldots, U_n) \le x\} = x^n$$

(c) Use part (b) to give a second method of simulating a random variable having distribution F.

8. Suppose it is relatively easy to simulate from F_i for each $i = 1, \ldots, n$. How can we simulate from

(a) $F(x) = \prod_{i=1}^{n} F_i(x)$;

(b) $F(x) = 1 - \prod_{i=1}^{n} [1 - F_i(x)]$.

9. Suppose we have a method to simulate random variables from the distributions F_1 and F_2. Explain how to simulate from the distribution

$$F(x) = pF_1(x) + (1-p)F_2(x) \qquad 0 < p < 1$$

Give a method for simulating from

$$F(x) = \begin{cases} \frac{1}{3}(1 - e^{-3x}) + \frac{2}{3}x & 0 < x \le 1 \\ \frac{1}{3}(1 - e^{-3x}) + \frac{2}{3} & x > 1 \end{cases}$$

10. In Example 2c we simulated the absolute value of a unit normal by using the rejection procedure on exponential random variables with rate 1. This raises the question of whether we could obtain a more efficient algorithm by using a different exponential density—that is, we could use the density $g(x) = \lambda e^{-\lambda x}$. Show that the mean number of iterations needed in the rejection scheme is minimized when $\lambda = 1$.

11. Use the rejection method with $g(x) = 1, 0 < x < 1$, to determine an algorithm for simulating a random variable having density function

$$f(x) = \begin{cases} 60x^3(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

12. Explain how you could use random numbers to approximate $\int_0^1 k(x)\,dx$ where $k(x)$ is an arbitrary function.

HINT: If U is uniform on (0, 1), what is $E[k(U)]$?

13. Let (X, Y) be uniformly distributed in the circle of radius 1 centered at the origin. Its joint density is thus

$$f(x, y) = \frac{1}{\pi} \qquad 0 \le x^2 + y^2 \le 1$$

Let $R = (X^2 + Y^2)^{1/2}$ and $\theta = \tan^{-1}(Y/X)$ denote its polar coordinates. Show that R and θ are independent with R^2 being uniform on (0, 1) and θ being uniform on $(0, 2\pi)$.

14. In Example 4a we have shown that

$$E[(1 - V^2)^{1/2}] = E[(1 - U^2)^{1/2}] = \frac{\pi}{4}$$

when V is uniform $(-1, 1)$ and U is uniform (0, 1). Show that

$$\text{Var}[(1 - V^2)^{1/2}] = \text{Var}[(1 - U^2)^{1/2}]$$

and find their common value.

15. **(a)** Verify that the minimum of (4.1) occurs when a is as given by (4.2).
(b) Verify that the minimum of (4.1) is given by (4.3).

16. Let X be a random variable on (0, 1) whose density is $f(x)$. Show that we can estimate $\int_0^1 g(x)\,dx$ by simulating X and then taking $g(X)/f(X)$ as our

estimate. This method, called importance sampling, tries to choose f similar in shape to g so that $g(X)/f(X)$ has a small variance.

SELF-TEST PROBLEMS AND EXERCISES

1. The random variable X has probability density function
$$f(x) = Ce^x \qquad 0 < x < 1$$
(a) Find the value of the constant C.
(b) Give a method for simulating such a random variable.
2. Give an approach for simulating a random variable having probability density function
$$f(x) = 30(x^2 - 2x^3 + x^4) \qquad 0 < x < 1$$
3. Give an efficient algorithm to simulate the value of a random variable with probability mass function
$$p_1 = .15 \qquad p_2 = .2 \qquad p_3 = .35 \qquad p_4 = .30$$
4. If X is a normal random variable with mean μ and variance σ^2, define a random variable Y that has the same distribution as X and is negatively correlated with it.
5. Let X and Y be independent standard normal random variables.
(a) Explain how we could use simulation to estimate $E[e^{XY}]$.
(b) Show how to improve the estimation approach in part (a) by using a control variate.
(c) Show how to improve the estimation approach in part (a) by using antithetical variables.

REFERENCE

Ross, S. M. *Simulation.* San Diego, Calif.: Academic Press, Inc., 1997.

APPENDIX A

Answers to Selected Problems

CHAPTER 1

1. 67,600,000; 19,656,000 **2.** 1296 **4.** 24; 4 **5.** 144; 18
6. 2401 **7.** 720; 72; 144; 72 **8.** 120; 1260; 34,650
9. 27,720 **10.** 40,320; 10,080; 1152; 2880; 384 **11.** 720; 72; 144
12. 24,300,000; 17,100,720 **13.** 190 **14.** 2,598,960
16. 42; 94 **17.** 604,800 **18.** 600 **19.** 896; 1000; 910
20. 36; 26 **21.** 35 **22.** 18 **23.** 48 **25.** $52!/(13!)^4$
27. 27,720 **28.** 65,536; 2520 **29.** 12,600; 945 **30.** 564,480
31. 165; 35 **32.** 1287; 14,112 **33.** 220; 572

CHAPTER 2

9. 74 **10.** .4; .1 **11.** 70; 2 **12.** .5; .32; 149/198
13. 20,000; 12,000; 11,000; 68,000; 10,000 **14.** 1.057
15. .0020; .4226; .0475; .0211; .00024 **17.** 9.10946×10^{-6}
18. .048 **19.** 5/18 **20.** .9017 **22.** $(n + 1)/2^n$ **23.** 5/12
25. .4 **26.** .492929 **27.** .58333 **28.** .2477; .2099
30. 1/18; 1/6; 1/2 **31.** 2/9; 1/9 **33.** 70/323 **35.** .8363
36. .0045; .0588 **37.** .0833; .5 **38.** 4 **39.** .48
40. 1/64; 21/64; 36/64; 6/64 **41.** .5177 **44.** .3; .2; .1 **46.** 5
48. 1.0604×10^{-3} **49.** .4329 **50.** 2.6084×10^{-6}
52. .09145; .4268 **53.** 12/35 **54.** .0511 **55.** .2198; .0342

CHAPTER 3

1. 1/3 **2.** 1/6; 1/5; 1/4; 1/3; 1/2; 1 **3.** .339 **5.** 6/91
6. 1/2 **7.** 2/3 **8.** 1/2 **9.** 7/11 **10.** .22 **11.** .4697
12. .9835 **13.** .0792; .264 **14.** .331; .383; .286; 48.62
15. 44.3; 41.18 **16.** .4; 1/26 **17.** .496; 3/14; 9/62
18. .504; .3629 **20.** 35/768 **21.** 4/9; 1/2 **22.** 1/3; 1/2
24. 20/21; 40/41 **26.** 3/128; 29/1536 **27.** .083 **28.** 7/12; 3/5
29. 5/11 **30.** 27/31 **31.** 3/4 **32.** 1/2 **33.** 1/3; 1/5; 1
34. 12/37 **35.** 46/185 **36.** 3/13; 5/13; 5/52; 15/52
37. 43/459 **38.** 34.48 **39.** 4/9 **41.** 1/11 **44.** 2/3
45. 19/268 **46.** 17.5; 38/165; 17/33
47. .65; 56/65; 8/65; 1/65; 14/35; 12/35; 9/35
48. .11; 16/89; 12/27; 3/5; 9/25 **51.** 9 **53.** (c) 2/3
56. 2/3; 1/3; 3/4 **57.** 1/6; 3/20 **61.** 9/13; 1/2
65. 9; 9; 18; 110; 4; 4; 8; 120 all over 128 **66.** 1/9; 1/18
67. 38/64; 13/64; 13/64 **69.** 1/16; 1/32; 5/16; 1/4; 31/32
70. $1/(2 - p)$ for A **71.** $P_1/(P_1 + P_2 - P_1P_2)$ **73.** .3
74. .5550 **76.** .9530 **78.** .5; .6; .8
79. 9/19; 6/19; 4/19; 77/165; 53/165; 35/165 **84.** 97/142; 15/26; 33/102

CHAPTER 4

1. $p(4) = 6/91$; $p(2) = 8/91$; $p(1) = 32/91$; $p(0) = 1/91$; $p(-1) = 16/91$; $p(-2) = 28/91$ **4.** 1/2; 5/18; 5/36; 5/84; 5/252; 1/252; 0; 0; 0; 0
5. $n - 2i$; $i = 0, \ldots, n$ **6.** $p(3) = p(-3) = 1/8$; $p(1) = p(-1) = 3/8$
12. $p(4) = 1/16$; $p(3) = 1/8$; $p(2) = 1/16$; $p(0) = 1/2$; $p(-i) = p(i)$; $p(0) = 1$
13. $p(0) = .28$; $p(500) = .27$, $p(1000) = .315$; $p(1500) = .09$; $p(2000) = .045$
14. $p(0) = 1/2$; $p(1) = 1/6$; $p(2) = 1/12$; $p(3) = 1/20$; $p(4) = 1/5$
17. 1/4; 1/6; 1/12; 1/2 **19.** 1/2; 1/10; 1/5; 1/10; 1/10
20. .5918; no; −.108 **21.** 39.28; 37 **23.** 3.5
24. $p = 11/18$, maximum = 23/72 **26.** 11/2; 17/5
27. $A(p + 1/10)$ **28.** 3/5 **31.** p^* **32.** $110 - 100(.9)^{10}$
33. 3 **35.** −.067; 1.089 **37.** 82.2; 84.5 **39.** 3/8
40. 11/243 **42.** $p \geq 1/2$ **45.** 3 **50.** 1/10; 1/10
51. $e^{-.2}$; $1 - 1.2e^{-.2}$ **53.** $1 - e^{-.6}$; $1 - e^{-219.18}$
56. 365 log (2) **57.** .5768; .6070 **59.** .3935; .3033; .0902
60. .8886 **61.** .4082 **63.** .0821; .2424
65. .3935; .2293; .3935 **66.** .1500; .1012 **68.** 5.8125
69. 32/243; 4864/6561; 160/729; 160/729 **73.** $18(17)^{n-1}/(35)^n$
76. 3/10; 5/6; 75/138 **77.** .3439

CHAPTER 5

2. $3.5e^{-5/2}$ **3.** no; no **4.** 1/2 **5.** $1 - (.01)^{1/5}$
6. 4, 0, ∞ **7.** 3/5; 6/5 **8.** 2 **10.** 2/3; 2/3 **11.** 2/5
13. 2/3; 1/3 **16.** $(.9938)^{10}$ **18.** 22.66 **19.** 14.56
20. .0006; .75; .977 **22.** 9.5; .0019 **23.** .9258; .1762
26. .0606; .0525 **30.** e^{-1}; $e^{-1/2}$ **32.** e^{-1}; 1/3 **36.** 3/5
38. $1/y$

CHAPTER 6

2. (a) 14/39; 10/39; 10/39; 5/39 (b) 84; 70; 70; 70; 40; 40; 40; 15 all divided by 429 **3.** 15/26; 5/26; 5/26; 1/26
4. 25/169; 40/169; 40/169; 64/169 **6.** $p(i, j) = 1/10$
7. $p(i, j) = p^2(1 - p)^{i+j}$ **8.** $c = 1/8$; $E[X] = 0$
9. $(12x^2 + 6x)/7$; 15/56; .8625; 5/7; 8/7 **10.** 1/2; $1 - e^{-a}$
11. .1458 **12.** $39.3e^{-5}$ **13.** 1/6; 1/2 **15.** $\pi/4$
16. $n(1/2)^{n-1}$ **17.** 1/3 **18.** 7/9 **19.** 1/2 **21.** 2/5; 2/5
22. no; 1/3 **23.** 1/2; 2/3; 1/20; 1/18 **25.** $e^{-1}/i!$
30. e^{-2}; $1 - 3e^{-2}$ **32.** .0326 **33.** .3446; .2061
34. .0829; .3766 **35.** 1/3; 2/3; 5/12; 7/12 **36.** 5/13; 8/13
37. 1/6; 5/6; 1/4; 3/4 **42.** $(y + 1)^2xe^{-x(y+1)}$; xe^{-xy}; e^{-x}
43. $1/2 + 3y/(4x) - y^3/(4x^3)$ **47.** $(1 - 2d/L)^3$ **48.** .79297
49. $1 - e^{-5\lambda a}$; $(1 - e^{-\lambda a})^5$ **51.** r/π **52.** r
55. (a) $u/(v + 1)^2$

CHAPTER 7

1. 52.5/12 **2.** 324; 198.8 **5.** 3/2 **6.** 35 **7.** .9; 4.9; 4.2
8. $(1 - (1 - p)^N)/p$ **10.** .6; 0 **11.** $2(n - 1)p(1 - p)$
14. $m/(1 - p)$ **15.** 109/60 **18.** 4 **21.** .9301; 87.5757
22. 14.7 **23.** 147/110 **26.** $n/(n + 1)$; $1/(n + 1)$ **27.** 175/6
29. 14 **30.** 20/19; 360/361 **31.** 21.2; 18.929; 49.214
32. $-n/36$ **33.** 0 **34.** 1/8 **37.** 6; 112/33
38. 100/19; 16,200/6137; 10/19; 3240/6137 **41.** 1/2; 0
43. $1/(n - 1)$ **44.** 6; 7; 5.8192 **45.** 9/5; 6/5; 3/5; 0 **46.** $2y^2$
47. $y^3/4$ **49.** 12 **50.** 8 **52.** $N(1 - e^{-10/N})$ **53.** 12.5
59. −96/145 **61.** 218 **63.** $x[1 + (2p - 1)^2]^n$
65. 1/2; 1/16; 2/81 **66.** 1/2, 1/3 **68.** $1/i$; $[i(i + 1)]^{-1}$; ∞
69. μ; $1 + \sigma^2$; yes; σ^2

CHAPTER 8

1. $\geq 19/20$ **2.** 15/17; $\geq 3/4$; ≥ 10 **3.** ≥ 3 **4.** $\leq 4/3$; .8428
5. .1416 **6.** .9431 **7.** .3085 **8.** .6932 **9.** 66,564
10. 117 **11.** $\geq .6$ **13.** .0162; .0003; .2514; .2514 **14.** $n \geq 23$
17. $\leq .2$ **22.** .769; .357; .1093; .112184

CHAPTER 9

1. 1/9; 5/9 **3.** .0265; .0902; .2642; .4422 **10.** (b) 1/6
14. 2.585; .5417; 3.1267 **15.** 5.5098

APPENDIX B

Solutions to Self-Test Problems and Exercises

CHAPTER 1

1. (a) There are 4! different orderings of the letters C, D, E, F. For each of these orderings, we can obtain an ordering with A and B next to each other by inserting A and B, either in the order A, B or in the order B, A, in any of 5 places. Namely, either before the first letter of the permutation of C, D, E, F, or between the first and second, and so on. Hence, there are $2 \cdot 5 \cdot 4! = 240$ arrangements. Another way of solving this is to imagine that B is glued to the back of A. This yields that there are 5! orderings in which A is immediately before B. As there are also 5! orderings in which B is immediately before A, we again obtain a total of $2 \cdot 5! = 240$ different arrangements.

(b) There are a total of $6! = 720$ possible arrangements, and as there are as many with A before B as with B before A, there are 360 arrangements.

(c) Of the 720 possible arrangements, there are as many that have A before B before C, as have any of the 3! possible orderings of A, B, and C. Hence, there are $720/6 = 120$ possible orderings.

(d) Of the 360 arrangements that have A before B, half will have C before D and half D before C. Hence, there are 180 arrangements having A before B and C before D.

(e) Gluing B to the back of A, and D to the back of C, yields $4! = 24$ different orderings in which B immediately follows A and D immediately follows C. Since the order of A and B and of C and D can be reversed, there are thus $4 \cdot 24 = 96$ different arrangements.

(f) There are 5! orderings in which E is last. Hence, there are $6! - 5! = 600$ orderings in which E is not last.

2. 3!4!3!3! since there are 3! possible orderings of countries and then the countrymen must be ordered.

3. **(a)** $10 \cdot 9 \cdot 8 = 720$

(b) $8 \cdot 7 \cdot 6 + 2 \cdot 3 \cdot 8 \cdot 7 = 672$.
The preceding follows since there are $8 \cdot 7 \cdot 6$ choices not including A or B, and there are $3 \cdot 8 \cdot 7$ choices in which a specified one of A and B, but not the other, serves. The latter following since the serving member of the pair can be assigned to any of the 3 offices, the next position can then be filled by any of the other 8 people, and the final position by any of the remaining 7.

(c) $8 \cdot 7 \cdot 6 + 3 \cdot 2 \cdot 8 = 384$.

(d) $3 \cdot 9 \cdot 8 = 216$.

(e) $9 \cdot 8 \cdot 7 + 9 \cdot 8 = 576$.

4. **(a)** $\binom{10}{7}$ **(b)** $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$

5. $\binom{7}{3,2,2} = 210$

6. There are $\binom{7}{3} = 35$ choices of the three places for the letters. For each choice, there are $(26)^3(10)^4$ different license plates. Hence, altogether there are a total of $35 \cdot (26)^3 \cdot (10)^4$ different plates.

7. Any choice of r of the n items is equivalent to a choice of $n - r$, namely, those items not selected.

8. **(a)** $10 \cdot 9 \cdot 9 \cdots 9 = 10 \cdot 9^{n-1}$

(b) $\binom{n}{i}9^{n-i}$, since there are $\binom{n}{i}$ choices of the i places to put the zeroes, and then each of the other $n - i$ positions can be any of the digits $1, \ldots, 9$.

9. **(a)** $\binom{3n}{3}$ **(b)** $3\binom{n}{3}$ **(c)** $\binom{3}{1}\binom{2}{1}\binom{n}{2}\binom{n}{1} = 3n^2(n-1)$ **(d)** n^3

(e) $\binom{3n}{3} = 3\binom{n}{3} + 3n^2(n-1) + n^3$

10. (number of solutions of $x_1 + \cdots + x_5 = 4$)(number of solutions of $x_1 + \cdots + x_5 = 5$)(number of solutions of $x_1 + \cdots + x_5 = 6$) $= \binom{8}{4}\binom{9}{4}\binom{10}{4}$

11. Since there are $\binom{j-1}{n-1}$ positive vectors whose sum is j, it follows that there are $\sum_{j=n}^{k}\binom{j-1}{n-1}$ such vectors.

CHAPTER 2

1. **(a)** $2 \cdot 3 \cdot 4 = 24$ **(b)** $2 \cdot 3 = 6$ **(c)** $3 \cdot 4 = 12$

(d) $AB = \{(c, \text{pasta}, i), (c, \text{rice}, i), (c, \text{potatoes}, i)\}$

(e) 8 **(f)** $ABC = \{(c, \text{rice}, i)\}$

2. Let A be the event that a suit is purchased, B be the event that a shirt is purchased, and C be the event that a tie is purchased. Then,

$$P(A \cup B \cup C) = .22 + .30 + .28 - .11 - .14 - .10 + .06 = .51$$

(a) $1 - .51 = .49$

(b) The probability that two or more items are purchased is

$$P(AB \cup AC \cup BC) = .11 + .14 + .10 - .06 - .06 - .06 + .06 = .23$$

Hence, the probability that exactly 1 item is purchased is $.51 - .23 = .28$

3. By symmetry the fourteenth card is equally likely to be any of the 52 cards, and thus the probability is 4/52. A more formal argument is to count the number of the 52! outcomes for which the fourteenth card is an ace. This yields,

$$p = \frac{4 \cdot 51 \cdot 50 \cdots 2 \cdot 1}{(52)!} = \frac{4}{52}$$

Letting A be the event that the first ace occurs on the fourteenth card, we have

$$P(A) = \frac{48 \cdot 47 \cdots 36 \cdot 4}{52 \cdot 51 \cdots 40 \cdot 39} = .0312$$

4. Let D denote the event that the minimum temperature is 70 degrees. Then,

$$P(A \cup B) = P(A) + P(B) - P(AB) = .7 - P(AB)$$
$$P(C \cup D) = P(C) + P(D) - P(CD) = .2 + P(D) - P(DC)$$

Subtracting one of the preceding equations from the other yields, upon using the fact that $A \cup B = C \cup D$ and $AB = CD$,

$$0 = .5 - P(D)$$

or, $P(D) = .5$

5. **(a)** $\frac{52 \cdot 48 \cdot 44 \cdot 40}{52 \cdot 51 \cdot 50 \cdot 49} = .6761$ **(b)** $\frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49} = .1055$

6. Let R be the event that both balls are red, and let B be the event that they are both black. Then

$$P(R \cup B) = P(R) + P(B) = \frac{3 \cdot 4}{6 \cdot 10} + \frac{3 \cdot 6}{6 \cdot 10} = 1/2$$

7. **(a)** $\frac{1}{\binom{40}{8}} = 1.3 \times 10^{-8}$ **(b)** $\frac{\binom{8}{7}\binom{32}{1}}{\binom{40}{8}} = 3.3 \times 10^{-6}$

(c) $\dfrac{\binom{8}{6}\binom{32}{2}}{\binom{40}{8}} + 1.3 \times 10^{-8} + 3.3 \times 10^{-6} = 1.8 \times 10^{-4}$

8. (a) $\dfrac{3 \cdot 4 \cdot 4 \cdot 3}{\binom{14}{4}} \doteq .1439$ (b) $\dfrac{\binom{4}{2}\binom{4}{2}}{\binom{14}{4}} = .0360$ (c) $\dfrac{\binom{8}{4}}{\binom{14}{4}} = .0699$

9. Let $S = \bigcup_{i=1}^{n} A_i$, and consider the experiment of randomly choosing an element of S. Then $P(A) = N(A)/N(S)$, and the results follow from Propositions 4.3 and 4.4.

10. Since there are $5! = 120$ outcomes in which the position of horse number 1 is specified, it follows that $N(A) = 360$. Similarly, $N(B) = 120$, and $N(AB) = 2 \cdot 4! = 48$. Hence, from self-test problem 9, we obtain that $N(A \cup B) = 432$.

11. One way to solve this problem is to start with the complementary probability that at least one suit does not appear. Let A_i be the event that no cards from suit i appear, $i = 1, 2, 3, 4$. Then,

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \sum_i P(A_i) - \sum_j \sum_{i:i<j} P(A_iA_j) + \cdots - P(A_1A_2A_3A_4)$$

$$= 4\frac{\binom{39}{5}}{\binom{52}{5}} - \binom{4}{2}\frac{\binom{26}{5}}{\binom{52}{5}} + \binom{4}{3}\frac{\binom{13}{5}}{\binom{52}{5}}$$

$$= 4\frac{\binom{39}{5}}{\binom{52}{5}} - 6\frac{\binom{26}{5}}{\binom{52}{5}} + 4\frac{\binom{13}{5}}{\binom{52}{5}}$$

The desired probability is then 1 minus the preceding. Another way to solve is to let A be the event that all 4 suits are represented, and then use

$$P(A) = P(n, n, n, n, o) + P(n, n, n, o, n) + P(n, n, o, n, n) + P(n, o, n, n, n)$$

where $P(n, n, n, o, n)$, for instance, is the probability that the first card is from a new suit, the second is from a new suit, the third is from a new suit, the fourth is from an old suit (that is, one that has already appeared) and the fifth is from a new suit. This gives

$$P(A) = \frac{52 \cdot 39 \cdot 26 \cdot 13 \cdot 48 + 52 \cdot 39 \cdot 26 \cdot 36 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

$$+ \frac{52 \cdot 39 \cdot 24 \cdot 26 \cdot 13 + 52 \cdot 12 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

$$= \frac{52 \cdot 39 \cdot 26 \cdot 13(48 + 36 + 24 + 12)}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

$$= .2637$$

12. There are $(10)!/2^5$ different divisions of the 10 players into a first roommate pair, a second roommate pair, and so on. Hence, there are $(10)!/(5!2^5)$ divisions into 5 roommate pairs. There are $\binom{6}{2}\binom{4}{2}$ ways of choosing the frontcourt and backcourt players to be in the mixed roommate pairs, and then 2 ways of pairing them up. As there is then 1 way to pair up the remaining two backcourt players, and $4!/(2!2^2) = 3$ ways of making two roommate pairs from the remaining four frontcourt players, we see that the desired probability is

$$P\{\text{2 mixed pairs}\} = \frac{\binom{6}{2}\binom{4}{2}(2)(3)}{(10)!/(5!2^5)} = .5714$$

13.
$$P(A^cB^c) = P((A \cup B)^c)$$
$$= 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(AB)]$$

14. Let $B_1 = A_1$, $B_i = A_i\left(\bigcup_{j=1}^{i-1} A_j\right)^c$, $i > 1$. Then,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$= \sum_{i=1}^{\infty} P(B_i)$$

$$\leq \sum_{i=1}^{\infty} P(A_i)$$

where the final equality uses the fact that the B_i are mutually exclusive, and the inequality follows since $B_i \subset A_i$.

15.
$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right)$$

$$\geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$$

$$= 1$$

CHAPTER 3

1. **(a)** $P(\text{no aces}) = \binom{35}{13}\Big/\binom{39}{13}$ **(b)** $1 - P(\text{no aces}) - \dfrac{4\binom{35}{12}}{\binom{39}{13}}$

(c) $P(i \text{ aces}) = \dfrac{\binom{3}{i}\binom{36}{13-i}}{\binom{39}{13}}$

2. Let L_i denote the event that the life of the battery is greater than $10{,}000 \times i$ miles.

(a) $P(L_2|L_1) = P(L_1L_2)/P(L_1) = P(L_2)/P(L_1) = 1/2$

(b) $P(L_3|L_1) = P(L_1L_3)/P(L_1) = P(L_3)/P(L_1) = 1/8$

3. Put 1 white and 0 black balls in urn one, and the remaining 9 white and 10 black balls in urn two.

4. Let T be the event that the transferred ball is white, and let W be the event that a white ball is drawn from urn B. Then,

$$P(T|W) = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|T^c)P(T^c)} = \frac{(2/7)(2/3)}{(2/7)(2/3) + (1/7)(1/3)} = 4/5$$

5. Let B_i denote the event that ball i is black, and let $R_i = B_i^c$.

$$\begin{aligned} P(B_1|R_2) &= \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)} \\ &= \frac{[r/[(b+r+c)][b/(b+r)]}{[r/(b+r+c)][b/(b+r)] + [(r+c)/(b+r+c)][r/(b+r)]} \\ &= \frac{b}{b+r+c} \end{aligned}$$

6. Let B denote the event that both cards are aces.

(a)
$$\begin{aligned} P\{B|\text{yes to ace of spades}\} &= \frac{P\{B, \text{yes to ace of spades}\}}{P\{\text{yes to ace of spades}\}} \\ &= \frac{\binom{1}{1}\binom{3}{1}}{\binom{52}{2}}\Big/\frac{\binom{1}{1}\binom{51}{1}}{\binom{52}{2}} \\ &= 3/51 \end{aligned}$$

(b) Since the second card is equally likely to be any of the remaining 51, of which 3 are aces, we see that the answer in this situation is also 3/51.

(c) Since we can always interchange which card is considered first and which is considered second, the result should be the same as in part (b). A more formal argument is as follows:

$$\begin{aligned} P\{B|\text{second is ace}\} &= \frac{P\{B, \text{second is ace}\}}{P\{\text{second is ace}\}} \\ &= \frac{P(B)}{P(B) + P\{\text{first is not ace, second is ace}\}} \\ &= \frac{(4/52)(3/51)}{(4/52)(3/51) + (48/52)(4/51)} \\ &= 3/51 \end{aligned}$$

(d)
$$\begin{aligned} P\{B|\text{at least one}\} &= \frac{P(B)}{P\{\text{at least one}\}} \\ &= \frac{(4/52)(3/51)}{1 - (48/52)(47/51)} \\ &= 1/33 \end{aligned}$$

7.
$$\frac{P(H|E)}{P(G|E)} = \frac{P(HE)}{P(GE)} = \frac{P(H)P(E|H)}{P(G)P(E|G)}$$

The hypothesis H is 1.5 times as likely.

8. Let A denote the event that the plant is alive and let W be the event that it was watered.

(a)
$$\begin{aligned} P(A) &= P(A|W)P(W) + P(A|W^c)P(W^c) \\ &= (.85)(.9) + (.2)(.1) = .785 \end{aligned}$$

(b)
$$\begin{aligned} P(W^c|A^c) &= \frac{P(A^c|W^c)P(W^c)}{P(A^c)} \\ &= \frac{(.8)(.1)}{.215} = \frac{16}{43} \end{aligned}$$

9. Since the black rat has a brown sibling we can conclude that both its parents have one black and one brown gene.

$$P(2 \text{ black}|\text{at least one}) = \frac{P(2)}{P(\text{at least one})} = \frac{1/4}{3/4} = \frac{1}{3}$$

(a) Let F be the event that all 5 offspring are black. Let B_2 be the event that the black rat has 2 black genes, and let B_1 be the event that it has 1 black and 1 brown gene.

$$P(B_2|F) = \frac{P(F|B_2)P(B_2)}{P(F|B_2)P(B_2) + P(F|B_1)P(B_1)}$$
$$= \frac{(1)(1/3)}{(1)(1/3) + (1/2)^5(2/3)} = \frac{16}{17}$$

10. Let F be the event that a current flows from A to B, and let C_i be the event that relay i closes. Then

$$P(F) = P(F|C_1)p_1 + P(F|C_1^c)(1 - p_1)$$

Now,

$$P(F|C_1) = P(C_4 \cup C_2C_5)$$
$$= P(C_4) + P(C_2C_5) - P(C_4C_2C_5)$$
$$= p_4 + p_2p_5 - p_4p_2p_5$$

Also,

$$P(F|C_1^c) = P(C_2C_5 \cup C_2C_3C_4)$$
$$= p_2p_5 + p_2p_3p_4 - p_2p_3p_4p_5$$

Hence, for part (a) we obtain

$$P(F) = p_1(p_4 + p_2p_5 - p_4p_2p_5) + (1 - p_1)p_2(p_5 + p_3p_4 - p_3p_4p_5)$$

For part (b), let $q_i = 1 - p_i$. Then,

$$P(C_3|F) = P(F|C_3)P(C_3)/P(F)$$
$$= p_3[1 - P(C_1^cC_2^c \cup C_4^cC_5^c)]/P(F)$$
$$= p_3(1 - q_1q_2 - q_4q_5 + q_1q_2q_4q_5)/P(F)$$

11. Let A be the event that component 1 is working, and let F be the event that the system functions.

(a) $$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(A)}{P(F)} = \frac{1/2}{1 - (1/2)^2} = \frac{2}{3}$$

where $P(F)$ was computed by noting that it is equal to 1 minus the probability that components 1 and 2 are both failed.

(b) $$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(F|A)P(A)}{P(F)} = \frac{(3/4)(1/2)}{(1/2)^3 + 3(1/2)^3} = \frac{3}{4}$$

where $P(F)$ was computed by noting that it is equal to the probability that all 3 components work, plus the three probabilities relating to exactly two of the components working.

12. If we accept that the outcomes of the successive spins are independent then the conditional probability of the next outcome is unchanged by the result that the previous ten spins landed on black.

13. Condition on outcome of initial tosses:

$$P(A \text{ odd}) = P_1(1 - P_2)(1 - P_3) + (1 - P_1)P_2P_3 + P_1P_2P_3P(A \text{ odd})$$
$$+ (1 - P_1)(1 - P_2)(1 - P_3)P(A \text{ odd})$$

so,

$$P(A \text{ odd}) = \frac{P_1(1 - P_2)(1 - P_3) + (1 - P_1)P_2P_3}{P_1 + P_2 + P_3 - P_1P_2 - P_1P_3 - P_2P_3}.$$

14. Let A and B be the events that the first trial is larger and that the second is larger, respectively. Also, let E be the event that the results of the trials are equal. Then

$$1 = P(A) + P(B) + P(E)$$

But, by symmetry, $P(A) = P(B)$, and thus

$$P(B) = \frac{1 - P(E)}{2} = \frac{1 - \sum_{i=1}^n p_i^2}{2}$$

Another way of solving the problem is to note that

$$P(B) = \sum_i \sum_{j>i} P\{\text{first trial results in } i, \text{ second trial results in } j\}$$
$$= \sum_i \sum_{j>i} p_ip_j$$

To see that the two expressions derived for $P(B)$ are equal, note that

$$1 = \sum_{i=1}^n p_i \sum_{j=1}^n p_j$$
$$= \sum_i \sum_j p_ip_j$$
$$= \sum_i p_i^2 + \sum_i \sum_{j \neq i} p_ip_j$$
$$= \sum_i p_i^2 + 2\sum_i \sum_{j>i} p_ip_j$$

15. Let $E = \{A \text{ gets more heads than } B\}$, then

$$P(E) = P(E|A \text{ leads after both flip } n)P(A \text{ leads after both flip } n)$$
$$+ P(E|\text{even after both flip } n)P(\text{even after both flip } n)$$
$$+ P(E|B \text{ leads after both flip } n)P(B \text{ leads after both flip } n)$$
$$= P(A \text{ leads}) + \frac{1}{2}P(\text{even}).$$

Now, by symmetry,

$$P(A \text{ leads}) = P(B \text{ leads})$$
$$= \frac{1 - P(\text{even})}{2}.$$

Hence,

$$P(E) = \frac{1}{2}.$$

16. **(a)** Not true: In rolling 2 dice let $E = \{\text{sum is } 7\}$,

$F = \{1\text{st die does not land on } 4\}$, $G = \{2\text{nd die does not land on } 3\}$.

Then

$$P(E|F \cup G) = \frac{P\{7, \text{not } (4, 3)\}}{P\{\text{not } (4, 3)\}} = \frac{5/36}{35/36} = 5/35 \neq P(E).$$

(b) $P(E(F \cup G)) = P(EF \cup EG)$
$= P(EF) + P(EG)$ since $EFG = \emptyset$
$= P(E)[P(F) + P(G)]$
$= P(E)P(F \cup G)$ since $FG = \emptyset$.

(c) $P(G|EF) = \dfrac{P(EFG)}{P(EF)}$

$= \dfrac{P(E)P(FG)}{P(EF)}$ since E is independent of FG

$= \dfrac{P(E)P(F)P(G)}{P(E)P(F)}$ by independence

$= P(G)$.

17. **(a)** necessarily false, since if they were mutually exclusive then

$$0 = P(AB) \neq P(A)P(B)$$

(b) necessarily false, since if they were independent then

$$P(AB) = P(A)P(B) > 0$$

(c) necessarily false, since if they were mutually exclusive then

$$P(A \cup B) = P(A) + P(B) = 1.2$$

(d) possibly true

18. The probabilities in parts (a), (b), and (c) are .5, $(.8)^3 = .512$, $(.9)^7 \approx .4783$.

19. Let D_i, $i = 1, 2$, denote the event that radio i is defective. Also, let A and B be the events that the radios were produced at factory A and at factory B, respectively. Then,

$$P(D_2|D_1) = \frac{P(D_1D_2)}{P(D_1)}$$
$$= \frac{P(D_1D_2|A)P(A) + P(D_1D_2|B)P(B)}{P(D_1|A)P(A) + P(D_1|B)P(B)}$$
$$= \frac{(.05)^2(1/2) + (.01)^2(1/2)}{(.05)(1/2) + (.01)(1/2)}$$
$$= 13/300$$

CHAPTER 4

1. Since the probabilities sum to 1, we must have that $4P\{X = 3\} + .5 = 1$, implying that $P\{X = 0\} = .375$, $P\{X = 3\} = .125$. Hence $E[X] = 1(.3) + 2(.2) + 3(.125) = 1.075$

2. The relationship implies that $p_i = c^i p_0$, $i = 1, 2$, where $p_i = P\{X = i\}$. As these probabilities sum to 1, we see that

$$p_0(1 + c + c^2) = 1 \Rightarrow p_0 = \frac{1}{1 + c + c^2}$$

Hence,

$$E[X] = p_1 + 2p_2 = \frac{c + 2c^2}{1 + c + c^2}$$

3. Letting X be the number of flips, then the probability mass function of X is

$$p_2 = p^2 + (1 - p)^2, \qquad p_3 = 1 - p_2 = 2p(1 - p)$$

Hence,

$$E[X] = 2p_2 + 3p_3 = 2p_2 + 3(1 - p_2) = 3 - p^2 - (1 - p)^2$$

4. The probability that a randomly chosen family will have i children is n_i/m. Hence,

$$E[X] = \sum_{i=1}^{r} in_i/m$$

Also, since there are in_i children in families having i children, it follows that the probability that a randomly chosen child is from a family with i children is $in_i / \sum_{i=1}^{r} in_i$. Therefore,

$$E[Y] = \frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i}$$

We must, thus, show

$$\frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i} \geq \frac{\sum_{i=1}^{r} in_i}{\sum_{i=1}^{r} n_i}$$

or, equivalently, that

$$\sum_{j=1}^{r} n_j \sum_{i=1}^{r} i^2 n_i \geq \sum_{i=1}^{r} in_i \sum_{j=1}^{r} jn_j$$

or, equivalently, that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} i^2 n_i n_j \geq \sum_{i=1}^{r} \sum_{j=1}^{r} ijn_i n_j$$

But, for a fixed pair i, j, the coefficient of $n_i n_j$ in the left side summation of the preceding is $i^2 + j^2$, whereas its coefficient in the right hand summation is $2ij$. Hence, it suffices to show that

$$i^2 + j^2 \geq 2ij$$

which follows since $(i - j)^2 \geq 0$.

5. Let $p = P\{X = 1\}$. Then, $E[X] = p$, $\text{Var}(X) = p(1 - p)$, and so

$$p = 3p(1 - p)$$

implying that $p = 2/3$. Hence, $P\{X = 0\} = 1/3$.

6. If you wager x on a bet that wins the amount wagered with probability p and loses that amount with probability $1 - p$, then your expected winnings is

$$xp - x(1 - p) = (2p - 1)x$$

which is positive (and increasing in x) if and only $p > 1/2$. Thus, if $p \leq 1/2$ one maximizes one's expected return by wagering 0, and if $p > 1/2$ one maximizes one's expected return by wagering the maximal possible bet. Thus, if the information is that the .6 coin was chosen then you should bet 10, and if the information is that the .3 coin was chosen then you should bet 0. Hence, your expected payoff is

$$\frac{1}{2}(1.2 - 1)10 + \frac{1}{2}0 - C = 1 - C$$

Since your expected payoff is 0 without the information (because in this case the probability of winning is $\frac{1}{2}(.6) + \frac{1}{2}(.3) < 1/2$) it follows that if the information costs less than 1 then it pays to purchase it.

7. **(a)** If you turn over the red paper and observe the value x then your expected return if you switch to the blue paper is

$$2x(1/2) + x/2(1/2) = 5x/4 > x$$

Thus, it would always be better to switch.

(b) Suppose the philanthropist writes the amount x on the red paper and so the amount on the blue paper is either $2x$ or $x/2$. Note that if $x/2 \geq y$ then the amount on the blue paper will be at least y and will thus be accepted. Hence, in this case, the reward is equally likely to be either $2x$ or $x/2$ and so

$$E[R_y(x)] = 5x/4, \qquad \text{if } x/2 \geq y$$

If $x/2 < y \leq 2x$ then the blue paper will be accepted if its value is $2x$ and rejected if it is $x/2$. Therefore,

$$E[R_y(x)] = 2x(1/2) + x(1/2) = 3x/2, \qquad \text{if } x/2 < y \leq 2x$$

Finally, if $2x < y$ then the blue paper will be rejected. Hence, in this case the reward is x, and so

$$R_y(x) = x, \qquad \text{if } 2x < y$$

That is, we have shown that the expected return under the y-policy is, when the amount x is written on the red paper,

$$E[R_y(x)] = \begin{cases} x & \text{if } x < y/2 \\ 3x/2 & \text{if } y/2 \leq x < 2y \\ 5x/4 & \text{if } x \geq 2y \end{cases}$$

8. Suppose that n independent trials each of which results in a success with probability p are performed. Then the number of successes will be less than or equal to i if and only if the number of failures is greater than or equal to $n - i$. But since each trial is a failure with probability $1 - p$, it follows that the number of failures is a binomial random variable with parameters n and $1 - p$. Hence,

$$\begin{aligned} P\{Bin(n, p) \leq i\} &= P\{Bin(n, 1 - p) \geq n - i\} \\ &= 1 - P\{Bin(n, 1 - p) \leq n - i - 1\} \end{aligned}$$

The final equality following since the probability that the number of failures is greater than or equal to $n - i$ is 1 minus the probability that it is less than $n - i$.

9. Since $E[X] = np$, $\text{Var}(X) = np(1 - p)$, we are given that $np = 6$, $np(1 - p) = 2.4$. Thus, $1 - p = .4$, or $p = .6$, $n = 10$. Hence,

$$P\{X = 5\} = \binom{10}{5}(.6)^5(.4)^5$$

10. Let X_i denote the number on the i^{th} ball drawn, $i = 1, \ldots, m$. Then

$$\begin{aligned} P\{X \leq k\} &= P\{X_1 \leq k, X_2 \leq k, \ldots, X_n \leq k\} \\ &= P\{X_1 \leq k\}P\{X_2 \leq k\} \cdots P\{X_n \leq k\} \\ &= \left(\frac{k}{n}\right)^m \end{aligned}$$

Therefore,

$$P\{X = k\} = P\{X \le k\} - P\{X \le k-1\} = \left(\frac{k}{n}\right)^m - \left(\frac{k-1}{n}\right)^m$$

11. **(a)** Given that A wins the first game, it will win the series if from then on it wins 2 games before team B wins 3 games. Thus.

$$P\{A \text{ wins} | A \text{ wins first}\} = \sum_{i=2}^{4} \binom{4}{i} p^i (1-p)^{4-i}$$

12. Condition on whether the team wins this weekend, to obtain the solution:

$$.5 \sum_{i=3}^{4} \binom{4}{i} (.4)^i (.6)^{4-i} + .5 \sum_{i=3}^{4} \binom{4}{i} (.7)^i (.3)^{4-i}$$

13. Assuming that the number of hurricanes can be approximated by a Poisson random variable, we obtain the solution

$$\sum_{i=0}^{3} e^{-5.2} (5.2)^i / i!$$

14.

$$\begin{aligned} E[Y] &= \sum_{i=1}^{\infty} iP\{X = i\}/P\{X > 0\} \\ &= E[X]/P\{X > 0\} \\ &= \frac{\lambda}{1 - e^{-\lambda}} \end{aligned}$$

15. **(a)** $\binom{8}{3}(9/19)^3(10/19)^5(9/19) = \binom{8}{3}(9/19)^4(10/19)^5$

(b) If W is her final winnings and X is the number of bets she makes, then since she would have won 4 bets and lost $X - 4$ bets, it follows that

$$W = 20 - 5(X - 4) = 40 - 5X$$

Hence,

$$E[W] = 40 - 5E[X] = 40 - 5[4/(9/19)] = -20/9$$

16. The probability that a round does not result in an "odd person" is equal to 1/4, the probability that all three coins land on the same side.

(a) $(1/4)^2(3/4) = 3/64$

(b) $(1/4)^4 = 1/256$

17. Let $q = 1 - p$.

$$\begin{aligned} E[1/X] &= \sum_{i=1}^{\infty} \frac{1}{i} q^{i-1} p \\ &= \frac{p}{q} \sum_{i=1}^{\infty} q^i / i \\ &= \frac{p}{q} \sum_{i=1}^{\infty} \int_0^q x^{i-1}\, dx \\ &= \frac{p}{q} \int_0^q \sum_{i=1}^{\infty} x^{i-1}\, dx \\ &= \frac{p}{q} \int_0^q \frac{1}{1-x}\, dx \\ &= \frac{p}{q} \int_p^1 \frac{1}{y}\, dy \\ &= -\frac{p}{q} \log(p) \end{aligned}$$

CHAPTER 5

1. Let X be the number of minutes played.

(a) $P\{X > 15\} = 1 - P\{X \le 15\} = 1 - 5(.025) = .875$

(b) $P\{20 < X < 35\} = 10(.05) + 5(.025) = .625$

(c) $P\{X < 30\} = 10(.025) + 10(.05) = .75$

(d) $P\{X > 36\} = 4(.025) = .1$

2. **(a)** $1 = \int_0^1 cx^n\, dx = c/(n+1) \Rightarrow c = n + 1$

(b) $P\{X > x\} = (n+1) \int_x^1 x^n\, dx = x^{n+1} \Big|_x^1 = 1 - x^{n+1}$

3. First, let us find c by using that

$$1 = \int_0^2 cx^4\, dx = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \frac{5}{32} \int_0^2 x^5\, dx = \frac{5}{32} \frac{64}{6} = 5/3$

(b) $E[X^2] = \frac{5}{32} \int_0^2 x^6\, dx = \frac{5}{32} \frac{128}{7} = 20/7 \Rightarrow \text{Var}(X) = 20/7 - (5/3)^2 = 5/63$

4. Since

$$1 = \int_0^1 (ax + bx^2)\, dx = a/2 + b/3$$

$$.6 = \int_0^1 (ax^2 + bx^3)\, dx = a/3 + b/4$$

we obtain that $a = 3.6$, $b = -2.4$. Hence,

(a) $P\{X < 1/2\} = \int_0^{1/2} (3.6x - 2.4x^2)\,dx = (1.8x^2 - .8x^3)\,\big|_0^{1/2} = .35$

(b) $E[X^2] = \int_0^1 (3.6x^3 - 2.4x^4)\,dx = .42 \Rightarrow \text{Var}(X) = .06$

5. For $i = 1, \ldots, n$

$$\begin{aligned} P\{X = i\} &= P\{\text{Int}(nU) = i - 1\} \\ &= P\{i - 1 \le nU < i\} \\ &= P\left\{\frac{i-1}{n} \le U < \frac{i}{n}\right\} \\ &= 1/n \end{aligned}$$

6. If you bid x, $70 \le x \le 140$, then you will win the bid and make a profit of $x - 100$ with probability $(140 - x)/70$, or lose the bid and make a profit of 0 otherwise. Therefore, your expected profit if you bid x is

$$\frac{1}{70}(x - 100)(140 - x) = \frac{1}{70}(240x - x^2 - 14000)$$

Differentiating and setting the preceding equal to 0 gives that

$$240 - 2x = 0$$

Therefore, you should bid 120 thousand dollars. Your expected profit will be 40/7 thousand dollars.

7. **(a)** $P\{U > .1\} = 9/10$

(b) $P\{U > .2 | U > .1\} = P\{U > .2\}/P\{U > .1\} = 8/9$

(c) $P\{U > .3 | U > .2, U > .1\} = P\{U > .3\}/P\{U > .2\} = 7/8$

(d) $P\{U > .3\} = 7/10$

The answer to part (d) could also have been obtained by multiplying the probabilities in parts (a), (b) and (c).

8. Let X be the test score, and let $Z = (X - 100)/15$. Note that Z is a standard normal random variable.

(a) $P\{X > 125\} = P\{Z > 25/15\} \approx .0478$

(b)
$$\begin{aligned} P\{90 < X < 110\} &= P\{-10/15 < Z < 10/15\} \\ &= P\{Z < 2/3\} - P\{Z < -2/3\} \\ &= P\{Z < 2/3\} - [1 - P\{Z < 2/3\}] \\ &\approx .4950 \end{aligned}$$

9. Let X be the travel time. We want to find x such that

$$P\{X > x\} = .05$$

which is equivalent to

$$P\left\{\frac{X - 40}{7} > \frac{x - 40}{7}\right\} = .05$$

That is, we need to find x such that

$$P\left\{Z > \frac{x - 40}{7}\right\} = .05$$

where Z is a standard normal random variable. But,

$$P\{Z > 1.645\} = .05$$

and thus

$$\frac{x - 40}{7} = 1.645 \quad \text{or} \quad x = 51.515$$

Therefore, you should leave no later than 8.485 minutes after 12 P.M.

10. Let X be the tire life in units of one thousand, and let $Z = (X - 34)/4$. Note that Z is a standard normal random variable.

(a) $P\{X > 40\} = P\{Z > 1.5\} \approx .0668$

(b) $P\{30 < X < 35\} = P\{-1 < Z < .25\} = P\{Z < .25\} - P\{Z > 1\} \approx .44$

(c) $P\{X > 40 | X > 30\} = P\{X > 40\}/P\{X > 30\} = P\{Z > 1.5\}/P\{Z > -1\} \approx .0079$

11. Let X be next year's rainfall and let $Z = (X - 40.2)/8.4$.

(a) $P\{X > 44\} = P\{Z > 3.8/8.4\} \approx P\{Z > .4524\} \approx .3255$

(b) $\binom{7}{3}(.3255)^3(.6745)^4$

12. Let M_i and W_i denote, respectively, the numbers of men and women in the samples that earn, in units of one thousand dollars, at least i per year. Also, let Z be a standard normal random variable.

(a)
$$\begin{aligned} P\{W_{25} \ge 70\} &= P\{W_{25} \ge 69.5\} \\ &= P\left\{\frac{W_{25} - 200(.34)}{\sqrt{200(.34)(.66)}} \ge \frac{69.5 - 200(.34)}{\sqrt{200(.34)(.66)}}\right\} \\ &\approx P\{Z \ge .2239\} \\ &\approx .4114 \end{aligned}$$

8. **(a)** No, advertisements on pages having many ads are less likely to be chosen than are ones on pages with few ads.

(b) $\frac{1}{m}\frac{n(i)}{n}$

(c) $\frac{\sum_{i=1}^{m} n(i)}{nm} = \bar{n}/n$, where $\bar{n} = \sum_{i=1}^{m} n(i)/m$

(d) $(1 - \bar{n}/n)^{k-1}\frac{1}{m}\frac{n(i)}{n}\frac{1}{n(i)} = (1 - \bar{n}/n)^{k-1}/(nm)$

(e) $\sum_{k=1}^{\infty}\frac{1}{nm}(1 - \bar{n}/n)^{k-1} = \frac{1}{\bar{n}m}$.

(f) The number of iterations is geometric with mean $n/\bar{n}$.

9. **(a)** $P\{X = i\} = 1/m,\ i = 1, \ldots, m$.

(b) **Step 2.** Generate a uniform (0, 1) random variable U. If $U < n(X)/n$, go to step 3. Otherwise return to step 1.
Step 3. Generate a uniform (0, 1) random variable U, and select the element on page X in position $[n(X)U] + 1$.

10. Yes, they are independent. This can be easily seen by considering the equivalent question of whether X_N is independent of N. But this is indeed so, since knowing when the first random variable greater than c occurs does not affect the probability distribution of its value, which is the uniform distribution on $(c, 1)$.

11. Letting p_i denote the probability of obtaining i points on a single throw of the dart, then

$$\begin{aligned} p_{30} &= \pi/36 \\ p_{20} &= 4\pi/36 - p_{30} = \pi/12 \\ p_{10} &= 9\pi/36 - p_{20} - p_{30} = 5\pi/36 \\ p_0 &= 1 - p_{10} - p_{20} - p_{30} = 1 - \pi/4 \end{aligned}$$

(a) $\pi/12$
(b) $\pi/9$
(c) $1 - \pi/4$
(d) $\pi(30/36 + 20/12 + 50/36) = 35\pi/9$
(e) $(\pi/4)^2$
(f) $2(\pi/36)(1 - \pi/4) + 2(\pi/12)(5\pi/36)$

12. Let Z be a standard normal random variable.

(a)
$$\begin{aligned} P\left\{\sum_{i=1}^{4} X_i > 0\right\} &= P\left\{\frac{\sum_{i=1}^{4} X_i - 6}{\sqrt{24}} > \frac{-6}{\sqrt{24}}\right\} \\ &\approx P\{Z > -1.2247\} \approx .8897 \end{aligned}$$

(b)
$$\begin{aligned} P\left\{\sum_{i=1}^{4} X_i > 0 \,\middle|\, \sum_{i=1}^{2} X_i = -5\right\} &= P\{X_3 + X_4 > 5\} \\ &= P\left\{\frac{X_3 + X_4 - 3}{\sqrt{12}} > 2/\sqrt{12}\right\} \\ &\approx P\{Z > .5774\} \approx .2818 \end{aligned}$$

(c)
$$\begin{aligned} P\left\{\sum_{i=1}^{4} X_i > 0 \,\middle|\, X_1 = 5\right\} &= P\{X_2 + X_3 + X_4 > -5\} \\ &= P\left\{\frac{X_2 + X_3 + X_4 - 4.5}{\sqrt{18}} > -9.5/\sqrt{18}\right\} \\ &\approx P\{Z > -2.239\} \approx .9874 \end{aligned}$$

13. In the following, C does not depend on n.

$$\begin{aligned} P\{N = n|X = x\} &= f_{X|N}(x|n)P\{N = n\}/f_X(x) \\ &= C\frac{1}{(n-1)!}(\lambda x)^{n-1}(1 - p)^{n-1} \\ &= C(\lambda(1 - p)x)^{n-1}/(n - 1)! \end{aligned}$$

which shows that, conditional on $X = x$, $N - 1$ is a Poisson random variable with mean $\lambda(1 - p)x$. That is,

$$\begin{aligned} P\{N = n|X = x\} &= P\{N - 1 = n - 1|X = x\} \\ &= e^{-\lambda(1-p)x}(\lambda(1 - p)x)^{n-1}/(n - 1)!, n \geq 1. \end{aligned}$$

14. **(a)** The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

As the equations $u = x$, $v = x + y$ imply that $x = u$, $y = v - u$, we obtain that

$$f_{U,V}(u, v) = f_{X,Y}(u, v - u) = 1, \quad 0 < u < 1, 0 < v - u < 1$$

or, equivalently

$$f_{U,V}(u, v) = 1, \quad \max(v - 1, 0) < u < \min(v, 1)$$

(b) For $0 < v < 1$,

$$f_V(v) = \int_0^v du = v$$

For, $1 \le v \le 2$,

$$f_V(v) = \int_{v-1}^1 du = 2 - v$$

CHAPTER 7

1. (a) $d = \sum_{i=1}^m 1/n(i)$

(b) $P\{X = i\} = P\{[mU] = i - 1\} = P\{i - 1 \le mU < i\} = 1/m$, $i = 1, \ldots, m$

(c)
$$E\left[\frac{m}{n(X)}\right] = \sum_{i=1}^m \frac{m}{n(i)} P\{X = i\} = \sum_{i=1}^m \frac{m}{n(i)} \frac{1}{m} = d$$

2. Let I_j equal 1 if the j^{th} ball withdrawn is white and the $(j + 1)^{st}$ is black, and let it equal 0 otherwise. If X is the number of instances in which a white ball is immediately followed by a black one, then we may express X as

$$X = \sum_{j=1}^{n+m-1} I_j$$

and, thus,

$$\begin{aligned} E[X] &= \sum_{j=1}^{n+m-1} E[I_j] \\ &= \sum_{j=1}^{n+m-1} P\{j^{th} \text{ selection is white, } (j+1)^{st} \text{ is black}\} \\ &= \sum_{j=1}^{n+m-1} P\{j^{th} \text{ selection is white}\} P\{j+1)^{st} \text{ is black} | j^{th} \text{ is white}\} \\ &= \sum_{j=1}^{n+m-1} \frac{n}{n+m} \frac{m}{n+m-1} \\ &= \frac{nm}{n+m} \end{aligned}$$

The preceding used the fact that each of the $n + m$ balls is equally likely to be the j^{th} one selected, and given that selection is a white ball each of the other $n + m - 1$ balls is equally likely to be the next ball chosen.

3. Arbitrarily number the couples, and then let I_j equal 1 if married couple number j, $j = 1, \ldots, 10$, is seated at the same table. Then, if X represents the number of married couples that are seated at the same table, we have that

$$X = \sum_{j=1}^{10} I_j$$

and so

$$E[X] = \sum_{j=1}^{10} E[I_j]$$

(a) To compute $E[I_j]$ in this case, consider wife number j. Since each of the $\binom{19}{3}$ groups of size 3 not including her is equally likely to be the remaining members of her table, it follows that the probability that her husband is at her table is

$$\frac{\binom{1}{1}\binom{18}{2}}{\binom{19}{3}} = \frac{3}{19}$$

Hence, $E[I_j] = 3/19$ and so

$$E[X] = 30/19$$

(b) In this case, since the two men at the table of wife j are equally likely to be any of the 10 men, it follows that the probability that one of them is her husband is 2/10, and so

$$E[I_j] = 2/10, \quad \text{and} \quad E[X] = 2$$

4. From Example 2j, we know that the expected number of times that the die need be rolled until all sides have appeared at least once is $6(1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6) = 14.7$. Now, if we let X_i denote the total number of times that side i appears, then since $\sum_{i=1}^6 X_i$ is equal to the total number of rolls, we have that

$$14.7 = E\left[\sum_{i=1}^6 X_i\right] = \sum_{i=1}^6 E[X_i]$$

But, by symmetry, $E[X_i]$ will be the same for all i, and thus it follows from the preceding that $E[X_1] = 14.7/6 = 2.45$.

5. Let I_j equal 1 if we win 1 when the j^{th} red card to show is turned over and let I_j equal 0 otherwise. (For instance, I_1 will equal 1 if the first card turned over is red.) Hence, if X is our total winnings then

$$E[X] = E\left[\sum_{j=1}^n I_j\right] = \sum_{j=1}^n E[I_j]$$

Now, I_j will equal 1 if j red cards appear before j black cards and, by symmetry, the probability of this event is equal to 1/2. Therefore, $E[I_j] = 1/2$, and $E[X] = n/2$.

6. To see that $N \leq n - 1 + I$, note that if all events occur then both sides of the preceding are equal to n, whereas if they do not all occur then the inequality reduces to $N \leq n - 1$, which is clearly true in this case. Taking expectations, yields that

$$E[N] \leq n - 1 + E[I]$$

However, if we let I_i equal 1 if A_i occurs and 0 otherwise, then

$$E[N] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} P(A_i)$$

As $E[I] = P(A_1 \cdots A_n)$, the result follows.

7. **(a)** $X = n - R$

(b) Randomly order the $n - k$ unchosen balls, and let I_j equal 1 if the j^{th} one has a larger number than each of the k selected balls, and let I_j equal 0 otherwise. Then

$$R = \sum_{j=1}^{n-k} I_j$$

(c) Consider the k chosen balls along with the j^{th} unchosen ball. Since each of these $k + 1$ balls is equally likely to have the largest number it follows that $E[I_j] = 1/(k + 1)$. Therefore, by parts (a) and (b)

$$E[X] = n - E[R] = n - \frac{n-k}{k+1} = \frac{k(n+1)}{k+1}$$

8. If $g(x) = x^{1/2}$, then

$$g'(x) = \frac{1}{2}x^{-1/2}, \qquad g''(x) = -\frac{1}{4}x^{-3/2}$$

and so the Taylor series expansion of $\sqrt{x}$ about λ gives

$$\sqrt{X} \approx \sqrt{\lambda} + \frac{1}{2}\lambda^{-1/2}(X - \lambda) - \frac{1}{8}\lambda^{-3/2}(X - \lambda)^2$$

Taking expectations yields

$$\begin{aligned} E[\sqrt{X}] &\approx \sqrt{\lambda} + \frac{1}{2}\lambda^{-1/2}E[X - \lambda] - \frac{1}{8}\lambda^{-3/2}E[(X - \lambda)^2] \\ &= \sqrt{\lambda} - \frac{1}{8}\lambda^{-3/2}\lambda \\ &= \sqrt{\lambda} - \frac{1}{8}\lambda^{-1/2} \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Var}(\sqrt{X}) &= E[X] - (E[\sqrt{X}])^2 \\ &\approx \lambda - \left(\sqrt{\lambda} - \frac{1}{8}\lambda^{-1/2}\right)^2 \\ &= 1/4 - \frac{1}{64\lambda} \\ &\approx 1/4 \end{aligned}$$

9. Number the tables so that tables 1, 2, 3 are the ones with four and tables 4, 5, 6, 7 are the ones with two seats. Also, number the women and let $X_{i,j}$ equal 1 if woman i is seated with her husband at table j. Note that

$$E[X_{i,j}] = \frac{\binom{2}{2}\binom{18}{2}}{\binom{20}{4}} = \frac{3}{95}, \qquad j = 1, 2, 3$$

and

$$E[X_{i,j}] = \frac{1}{\binom{20}{2}} = \frac{1}{190}, \qquad j = 4, 5, 6, 7$$

If X denotes the number of married couples that are seated at the same table, we thus have

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{5}\sum_{j=1}^{7} X_{i,j}\right] \\ &= \sum_{i=1}^{5}\sum_{j=1}^{3} E[X_{i,j}] + \sum_{i=1}^{5}\sum_{j=4}^{7} E[X_{i,j}] \\ &= 15(3/95) + 20(1/190) = 11/19 \end{aligned}$$

10. Let X_i equal 1 if individual i does not recruit anyone and let it equal 0 otherwise. Then

$$\begin{aligned} E[X_i] &= P\{i \text{ does not recruit any of } i + 1, i + 2, \ldots, n\} \\ &= \frac{i-1}{i}\,\frac{i}{i+1}\cdots\frac{n-2}{n-1} \\ &= \frac{i-1}{n-1} \end{aligned}$$

Hence,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \frac{i-1}{n-1} = \frac{n}{2}$$

From the preceding we also obtain that

$$\text{Var}(X_i) = \frac{i-1}{n-1}\left(1 - \frac{i-1}{n-1}\right) = \frac{(i-1)(n-i)}{(n-1)^2}$$

Now, for $i < j$,

$$E[X_iX_j] = \frac{i-1}{i} \cdots \frac{j-2}{j-1}\frac{j-2}{j}\frac{j-1}{j+1} \cdots \frac{n-3}{n-1}$$

$$= \frac{(i-1)(j-2)}{(n-2)(n-1)}$$

and, thus,

$$\text{Cov}(X_i, X_j) = \frac{(i-1)(j-2)}{(n-2)(n-1)} - \frac{i-1}{n-1}\frac{j-1}{n-1}$$

$$= \frac{(i-1)(j-n)}{(n-2)(n-1)^2}$$

Therefore,

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \frac{(i-1)(n-i)}{(n-1)^2} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{(i-1)(j-n)}{(n-2)(n-1)^2}$$

$$= \frac{1}{(n-1)^2}\sum_{i=1}^{n}(i-1)(n-i)$$

$$- \frac{1}{(n-2)(n-1)^2}\sum_{i=1}^{n-1}(i-1)(n-i)(n-i-1)$$

11. Let X_i equal 1 if the i^{th} triple consists of one of each type of player. Then

$$E[X_i] = \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} = \frac{2}{7}$$

Hence, for part (a) we obtain that

$$E\left[\sum_{i=1}^{3} X_i\right] = 6/7$$

It follows from the preceding that

$$\text{Var}(X_i) = (2/7)(1 - 2/7) = 10/49$$

Also, for $i \neq j$,

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\}$$

$$= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\}$$

$$= \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} \frac{\binom{1}{1}\binom{2}{1}\binom{3}{1}}{\binom{6}{3}}$$

$$= 6/70$$

Hence, for part (b) we obtain

$$\text{Var}\left(\sum_{i=1}^{3} X_i\right) = \sum_{i=1}^{3} \text{Var}(X_i) + 2\sum\sum_{j>i} \text{Cov}(X_i, X_j)$$

$$= 30/49 + 2\binom{3}{2}(6/70)$$

$$= \frac{552}{490}$$

12. Let X_i equal 1 if the i^{th} card is an ace and let it be 0 otherwise, and let Y_j equal 1 if the j^{th} card is a spade and let it be 0 otherwise, $i, j = 1, \ldots, 13$. Now,

$$\text{Cov}(X, Y) = \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, Y_j)$$

However, X_i is clearly independent of Y_j because knowing the suit of a particular card clearly gives no information about whether it is an ace and thus cannot affect the probability that another specified card is an ace. More formally, let $A_{i,s}$, $A_{i,h}$, $A_{i,d}$, $A_{i,c}$ be the events, respectively, that card i is a spade, a heart, a diamond, or a club. Then

$$P\{Y_j = 1\} = \frac{1}{4}(P\{Y_j = 1|A_{i,s}\} + P\{Y_j = 1|A_{i,h}\}$$

$$+ P\{Y_j = 1|A_{i,d}\} + P\{Y_j = 1|A_{i,c}\})$$

But, by symmetry, we have that

$$P\{Y_j = 1|A_{i,s}\} = P\{Y_j = 1|A_{i,h}\} = P\{Y_j = 1|A_{i,d}\} = P\{Y_j = 1|A_{i,c}\}$$

Therefore,

$$P\{Y_j = 1\} = P\{Y_j = 1|A_{i,s}\}$$

As the preceding implies that

$$P\{Y_j = 1\} = P\{Y_j = 1 | A_{i,s}^c\}$$

we see that Y_j and X_i are independent. Hence, $\text{Cov}(X_i, Y_j) = 0$, and thus $\text{Cov}(X, Y) = 0$.

The random variables X and Y, although uncorrelated, are not independent. This follows, for instance, from the fact that

$$P\{Y = 13 | X = 4\} = 0 \neq P\{Y = 13\}$$

13. **(a)** Your expected gain without any information is 0.
(b) You should predict heads if $p > 1/2$ and tails otherwise.
(c) Conditioning on V, the value of the coin, gives

$$\begin{aligned} E[\text{Gain}] &= \int_0^1 E[\text{Gain}|V = p]\, dp \\ &= \int_0^{1/2} [1(1-p) - 1(p)]\, dp + \int_{1/2}^1 [1(p) - 1(1-p)]\, dp \\ &= 1/2 \end{aligned}$$

14. Given that the name chosen appears in $n(X)$ different positions on the list it follows, since each of these positions is equally likely to be the one chosen, that

$$E[I|n(X)] = P\{I = 1|n(X)\} = 1/n(X)$$

Hence,

$$E[I] = E[1/n(X)]$$

and thus, $E[mI] = E[m/n(X)] = d$.

CHAPTER 8

1. Let X denote the number of sales made next week, and note that X is integral. From Markov's inequality we obtain the following.

(a) $$P\{X > 18\} = P\{X \geq 19\} \leq \frac{E[X]}{19} = 16/19$$

(b) $$P\{X > 25\} = P\{X \geq 26\} \leq \frac{E[X]}{26} = 16/26$$

2. **(a)**

$$\begin{aligned} P\{10 \leq X \leq 22\} &= P\{|X - 16| \leq 6\} \\ &= P\{|X - \mu| \leq 6\} \\ &= 1 - P\{|X - \mu| > 6\} \\ &\geq 1 - 9/36 = 3/4 \end{aligned}$$

(b) $$P\{X \geq 19\} = P\{X - 16 \geq 3\} \leq \frac{9}{9+9} = 1/2$$

In part (a) we used Chebyshev's inequality and in part (b) its one sided version (see Proposition 5.1).

3. First note that $E[X - Y] = 0$, and

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\,\text{Cov}(X, Y) = 28$$

Using Chebyshev's inequality in part (a) and the one-sided version in parts (b) and (c) gives the following results.

(a) $$P\{|X - Y| > 15\} \leq 28/225$$

(b) $$P\{X - Y > 15\} \leq \frac{28}{28 + 225} = 28/253$$

(c) $$P\{Y - X > 15\} \leq \frac{28}{28 + 225} = 28/253$$

4. If X is the number produced at factory A and Y the number produced at factory B, then

$$E[Y - X] = -2, \qquad \text{Var}(Y - X) = 36 + 9 = 45$$

$$P\{Y - X > 0\} = P\{Y - X \geq 1\} = P\{Y - X + 2 \geq 3\} \leq \frac{45}{45 + 9} = 45/54$$

5. Note first that

$$E[X_i] = \int_0^1 2x^2\, dx = 2/3$$

Now use the strong law of large numbers to obtain

$$\begin{aligned} r &= \lim_{n\to\infty} \frac{n}{S_n} \\ &= \lim_{n\to\infty} \frac{1}{S_n/n} \\ &= \frac{1}{\lim_{n\to\infty} S_n/n} \\ &= 1/(2/3) = 3/2 \end{aligned}$$

6. Since $E[X_i] = 2/3$, and

$$E[X_i^2] = \int_0^1 2x^3\, dx = 1/2$$

we have that $\text{Var}(X_i) = 1/2 - (2/3)^2 = 1/18$. Thus, if there are n components on hand, then

$$P\{S_n \geq 35\} = P\{S_n \geq 34.5\} \qquad \text{(the continuity correction)}$$

$$= P\left\{\frac{S_n - 2n/3}{\sqrt{n/18}} \leq \frac{34.5 - 2n/3}{\sqrt{n/18}}\right\}$$

$$\approx P\left\{Z \geq \frac{34.5 - 2n/3}{\sqrt{n/18}}\right\}$$

where Z is a standard normal random variable. Since

$$P\{Z > -1.284\} = P\{Z < 1.284\} \approx .90$$

we see that n should be chosen so that

$$(34.5 - 2n/3) \approx -1.284\sqrt{n/18}$$

A numerical computation gives the result $n = 55$.

7. If X is the time to service a machine then

$$E[X] = .2 + .3 = .5$$

Also, using that the variance of an exponential random variable is equal to the square of its mean gives

$$\text{Var}(X) = (.2)^2 + (.3)^2 = .13$$

Therefore, with X_i being the time to service job i, $i = 1, \ldots, 20$, and Z being a standard normal random variable,

$$P\{X_1 + \cdots + X_{20} < 8\} = P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{8 - 10}{\sqrt{2.6}}\right\}$$

$$\approx P\{Z < -1.24035\}$$

$$\approx .1074$$

8. Note first that if X is the gambler's winnings on a single bet, then

$$E[X] = -.7 - .4 + 1 = -.1, \quad E[X^2] = .7 + .8 + 10 = 11.5$$

$$\rightarrow \text{Var}(X) = 11.49$$

Therefore, with Z having a standard normal distribution,

$$P\{X_1 + \cdots + X_{100} \leq -.5\} = P\left\{\frac{X_1 + \cdots + X_{100} + 10}{\sqrt{1149}} \leq \frac{-.5 + 10}{\sqrt{1149}}\right\}$$

$$\approx P\{Z \leq .2803\}$$

$$\approx .6104$$

9. Using the notation of Problem 7 we have

$$P\{X_1 + \cdots + X_{20} < t\} = P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{t - 10}{\sqrt{2.6}}\right\}$$

$$\approx P\left\{Z < \frac{t - 10}{\sqrt{2.6}}\right\}$$

Now, $P\{Z < 1.645\} \approx .95$, and so t should be such that

$$\frac{t - 10}{\sqrt{2.6}} \approx 1.645$$

which yields that $t \approx 12.65$.

CHAPTER 9

1. From axiom (iii) it follows that the number of events that occur between times 8 and 10 has the same distribution as the number of events that occur by time 2, and thus is a Poisson random variable with mean 6. Hence, we obtain the following solutions for parts (a) and (b).

(a) $P\{N(10) - N(8) = 0\} = e^{-6}$

(b) $E[N(10) - N(8)] = 6$

(c) It follows from axioms (ii) and (iii) that from any time point onward the process of events occurring is a Poisson process with rate λ. Hence, the expected time of the fifth event after 2 P.M. is $2 + E[S_5] = 2 + 5/3$. That is, the expected time of this event is 3:40 P.M.

2. (a) $P\{N(1/3) = 2 | N(1) = 2\}$

$$= \frac{P\{N(1/3) = 2, N(1) = 2\}}{P\{N(1) = 2\}}$$

$$= \frac{P\{N(1/3) = 2, N(1) - N(1/3) = 0\}}{P\{N(1) = 2\}}$$

$$= \frac{P\{N(1/3) = 2\}P\{N(1) - N(1/3) = 0\}}{P\{N(1) = 2\}} \qquad \text{(by axiom } (ii))$$

$$= \frac{P\{N(1/3) = 2\}P\{N(2/3) = 0\}}{P\{N(1) = 2\}} \qquad \text{(by axiom } (iii))$$

$$= \frac{e^{-\lambda/3}(\lambda/3)^2/2!\, e^{-2\lambda/3}}{e^{-\lambda}\lambda^2/2!}$$

$$= 1/9$$

(b)
$$\begin{aligned}
P\{N(1/2) \geq 1 | N(1) = 2\} &= 1 - P\{N(1/2) = 0 | N(1) = 2\} \\
&= 1 - \frac{P\{N(1/2) = 0, N(1) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2) = 0, N(1) - N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2) = 0\}P\{N(1) - N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2 = 0\}P\{N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{e^{-\lambda/2}e^{-\lambda/2}(\lambda/2)^2/2!}{e^{-\lambda}\lambda^2/2!} \\
&= 1 - 1/4 = 3/4
\end{aligned}$$

3. Fix a point on the road and let X_n equal 0 if the n^{th} vehicle to pass is a car and let it equal 1 if it is a truck, $n \geq 1$. We now suppose that the sequence X_n, $n \geq 1$, is a Markov chain with transition probabilities

$$P_{0,0} = 5/6,\ P_{0,1} = 1/6,\ P_{1,0} = 4/5,\ P_{1,1} = 1/5$$

Hence, the long run proportion of times are the solution of

$$\begin{aligned}
\pi_0 &= \pi_0(5/6) + \pi_1(4/5) \\
\pi_1 &= \pi_0(1/6) + \pi_1(1/5) \\
\pi_0 + \pi_1 &= 1
\end{aligned}$$

Solving the preceding equations give that

$$\pi_0 = 24/29 \qquad \pi_1 = 5/29$$

Thus, $2400/29 \approx 83$ percent of the vehicles on the road are cars.

4. The successive weather classifications constitute a Markov chain. If the states are 0 for rainy, 1 for sunny, and 2 for overcast, then the transition probability matrix is as follows.

$$\mathbf{P} = \begin{matrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{matrix}$$

the long run proportions satisfy

$$\begin{aligned}
\pi_0 &= \pi_1(1/3) + \pi_2(1/3) \\
\pi_1 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\
\pi_2 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\
1 &= \pi_0 + \pi_1 + \pi_2
\end{aligned}$$

The solution of the preceding is

$$\pi_0 = 1/4, \qquad \pi_1 = 3/8, \qquad \pi_2 = 3/8$$

Hence, three-eights of the days are sunny and one-fourth are rainy.

5. (a) A direct computation yields that

$$H(X)/H(Y) \approx 1.06$$

(b) Both random variables take on two of their values with the same probabilities .35 and .05. The difference is that if they do not take on either of those values then X, but not Y, is equally likely to take on any of its three remaining possible values. Hence, from Problem 13 we would expect the result of part (a).

CHAPTER 10

1. (a)
$$1 = C\int_0^1 e^x\,dx \Rightarrow C = 1/(e-1)$$

(b)
$$F(x) = C\int_0^x e^y\,dy = \frac{e^x - 1}{e - 1}, \qquad 0 \leq x \leq 1$$

Hence, if we let $X = F^{-1}(U)$, then

$$U = \frac{e^X - 1}{e - 1}$$

or

$$X = \log(U(e-1) + 1)$$

Thus, we can simulate the random variable X by generating a random number U and then setting $X = \log(U(e-1) + 1)$.

2. Use the acceptance-rejection method with $g(x) = 1$, $0 < x < 1$. Calculus shows that the maximum value of $f(x)/g(x)$ occurs at a value of x, $0 < x < 1$, such that

$$2x - 6x^2 + 4x^3 = 0$$

or, equivalently, when

$$4x^2 - 6x + 2 = (4x - 2)(x - 1) = 0$$

The maximum thus occurs when $x = 1/2$, and

$$C = \max f(x)/g(x) = 30(1/4 - 2/8 + 1/16) = 15/8$$

Hence, the algorithm is as follows:

Step 1: Generate a random number U_1
Step 2: Generate a random number U_2
Step 3: If $U_2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$ set $X = U_1$; else return to Step 1

3. It is most efficient to check the higher probability values first, as in the following algorithm.

Step 1: Generate a random number$U$
Step 2: If $U \leq .35$, set $X = 3$ and stop
Step 3: If $U \leq .65$, set $X = 4$ and stop
Step 4: If $U \leq .85$, set $X = 2$ and stop
Step 5: $X = 1$

4. $2\mu - X$

5. **(a)** Generate $2n$ independent standard normal random variables X_i, Y_i, $i = 1, \ldots, n$, and then use the estimator $\sum_{i=1}^{n} e^{X_iY_i}/n$.

(b) We can use XY as a control variate to obtain an estimator of the type

$$\sum_{i=1}^{n} (e^{X_iY_i} + cX_iY_i)/n$$

Another possibility would be to use $XY + X^2Y^2/2$ as the control variate and so obtain an estimator of the type

$$\sum_{i=1}^{n} (e^{X_iY_i} + c[X_iY_i + X_i^2Y_i^2/2 - 1/2])/n$$

The motivation behind the preceding is based on the fact that the first three terms of the MacLaurin series expansion of e^{xy} are $1 + xy + (x^2y^2)/2$.

(c) The logic of antithetic variables leads to the estimator

$$\sum_{i=1}^{n} (e^{X_iY_i} + e^{-X_iY_i})/2n$$

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Solutions Manual

A First Course in

PROBABILITY

Seventh Edition

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Chapter 1

Problems

1. (a) By the generalized basic principle of counting there are

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 67{,}600{,}000$$

(b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19{,}656{,}000$

2. $6^4 = 1296$

3. An assignment is a sequence $i_1, \ldots, i_{20}$ where i_j is the job to which person j is assigned. Since only one person can be assigned to a job, it follows that the sequence is a permutation of the numbers 1, …, 20 and so there are 20! different possible assignments.

4. There are 4! possible arrangements. By assigning instruments to Jay, Jack, John and Jim, in that order, we see by the generalized basic principle that there are $2 \cdot 1 \cdot 2 \cdot 1 = 4$ possibilities.

5. There were $8 \cdot 2 \cdot 9 = 144$ possible codes. There were $1 \cdot 2 \cdot 9 = 18$ that started with a 4.

6. Each kitten can be identified by a code number i, j, k, l where each of i, j, k, l is any of the numbers from 1 to 7. The number i represents which wife is carrying the kitten, j then represents which of that wife's 7 sacks contain the kitten; k represents which of the 7 cats in sack j of wife i is the mother of the kitten; and l represents the number of the kitten of cat k in sack j of wife i. By the generalized principle there are thus $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ kittens

7. (a) $6! = 720$
(b) $2 \cdot 3! \cdot 3! = 72$
(c) $4!3! = 144$
(d) $6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72$

8. (a) $5! = 120$
(b) $\dfrac{7!}{2!2!} = 1260$
(c) $\dfrac{11!}{4!4!2!} = 34{,}650$
(d) $\dfrac{7!}{2!2!} = 1260$

9. $\dfrac{(12)!}{6!4!} = 27{,}720$

10. (a) $8! = 40{,}320$
(b) $2 \cdot 7! = 10{,}080$
(c) $5!4! = 2{,}880$
(d) $4!2^4 = 384$

11. (a) 6!
(b) 3!2!3!
(c) 3!4!

12. (a) 30^5
(b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

13. $\binom{20}{2}$

14. $\binom{52}{5}$

15. There are $\binom{10}{5}\binom{12}{5}$ possible choices of the 5 men and 5 women. They can then be paired up in 5! ways, since if we arbitrarily order the men then the first man can be paired with any of the 5 women, the next with any of the remaining 4, and so on. Hence, there are $5!\binom{10}{5}\binom{12}{5}$ possible results.

16. (a) $\binom{6}{2}+\binom{7}{2}+\binom{4}{2} = 42$ possibilities.
(b) There are $6 \cdot 7$ choices of a math and a science book, $6 \cdot 4$ choices of a math and an economics book, and $7 \cdot 4$ choices of a science and an economics book. Hence, there are 94 possible choices.

17. The first gift can go to any of the 10 children, the second to any of the remaining 9 children, and so on. Hence, there are $10 \cdot 9 \cdot 8 \cdots 5 \cdot 4 = 604{,}800$ possibilities.

18. $\binom{5}{2}\binom{6}{2}\binom{4}{3} = 600$

19. (a) There are $\binom{8}{3}\binom{4}{3}+\binom{8}{3}\binom{2}{1}\binom{4}{2} = 896$ possible committees.
There are $\binom{8}{3}\binom{4}{3}$ that do not contain either of the 2 men, and there are $\binom{8}{3}\binom{2}{1}\binom{4}{2}$ that contain exactly 1 of them.

(b) There are $\binom{6}{3}\binom{6}{3}+\binom{2}{1}\binom{6}{2}\binom{6}{3} = 1000$ possible committees.

(c) There are $\binom{7}{3}\binom{5}{3}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2} = 910$ possible committees. There are $\binom{7}{3}\binom{5}{3}$ in which neither feuding party serves; $\binom{7}{2}\binom{5}{3}$ in which the feuding women serves; and $\binom{7}{3}\binom{5}{2}$ in which the feuding man serves.

20. $\binom{6}{5}+\binom{2}{1}\binom{6}{4}, \binom{6}{5}+\binom{6}{3}$

21. $\frac{7!}{3!4!} = 35$. Each path is a linear arrangement of 4 *r*'s and 3 *u*'s (*r* for right and *u* for up). For instance the arrangement *r*, *r*, *u*, *u*, *r*, *r*, *u* specifies the path whose first 2 steps are to the right, next 2 steps are up, next 2 are to the right, and final step is up.

22. There are $\frac{4!}{2!2!}$ paths from A to the circled point; and $\frac{3!}{2!1!}$ paths from the circled point to B. Thus, by the basic principle, there are 18 different paths from A to B that go through the circled piont.

23. $3!2^3$

25. $\binom{52}{13, 13, 13, 13}$

27. $\binom{12}{3, 4, 5} = \frac{12!}{3!4!5!}$

28. Assuming teachers are distinct.
(a) 4^8
(b) $\binom{8}{2,2,2,2} = \frac{8!}{(2)^4} = 2520.$

29. (a) (10)!/3!4!2!

(b) $3\binom{3}{2}\frac{7!}{4!2!}$

30. $2 \cdot 9! - 2^2 8!$ since $2 \cdot 9!$ is the number in which the French and English are next to each other and $2^2 8!$ the number in which the French and English are next to each other and the U.S. and Russian are next to each other.

31. (a) number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$.
Hence, answer is $\binom{11}{3} = 165$

(b) here it is the number of positive solutions—hence answer is $\binom{7}{3} = 35$

32. (a) number of nonnegative solutions of $x_1 + \ldots + x_6 = 8$
answer = $\binom{13}{5}$

(b) (number of solutions of $x_1 + \ldots + x_6 = 5$) × (number of solutions of $x_1 + \ldots + x_6 = 3$) = $\binom{10}{5}\binom{8}{5}$

33. (a) $x_1 + x_2 + x_3 + x_4 = 20, x_1 \geq 2, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4$
Let $y_1 = x_1 - 1, y_2 = x_2 - 1, y_3 = x_3 - 2, y_4 = x_4 - 3$

$y_1 + y_2 + y_3 + y_4 = 13, y_i > 0$

Hence, there are $\binom{12}{3} = 220$ possible strategies.

(b) there are $\binom{15}{2}$ investments only in 1, 2, 3

there are $\binom{14}{2}$ investments only in 1, 2, 4

there are $\binom{13}{2}$ investments only in 1, 3, 4

there are $\binom{13}{2}$ investments only in 2, 3, 4

$\binom{15}{2} + \binom{14}{2} + 2\binom{13}{2} + \binom{12}{3} = 552$ possibilities

Theoretical Exercises

2. $\sum_{i=1}^{m} n_i$

3. $n(n-1)\cdots(n-r+1) = n!/(n-r)!$

4. Each arrangement is determined by the choice of the r positions where the black balls are situated.

5. There are $\binom{n}{j}$ different $0-1$ vectors whose sum is j, since any such vector can be characterized by a selection of j of the n indices whose values are then set equal to 1. Hence there are $\sum_{j=k}^{n}\binom{n}{j}$ vectors that meet the criterion.

6. $\binom{n}{k}$

7. $$\binom{n-1}{r}+\binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(n-r)!(r-1)!}$$
$$= \frac{n!}{r!(n-r)!}\left[\frac{n-r}{n}+\frac{r}{n}\right] = \binom{n}{r}$$

8. There are $\binom{n+m}{r}$ gropus of size r. As there are $\binom{n}{i}\binom{m}{r-i}$ groups of size r that consist of i men and $r-i$ women, we see that

$$\binom{n+m}{r} = \sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}.$$

9. $$\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\binom{n}{i}^2$$

10. Parts (a), (b), (c), and (d) are immediate. For part (e), we have the following:

$$k\binom{n}{k} = \frac{k!n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!}$$
$$(n-k+1)\binom{n}{k-1} = \frac{(n-k+1)n!}{(n-k+1)!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$$
$$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(n-k)!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$$

11. The number of subsets of size k that have i as their highest numbered member is equal to $\binom{i-1}{k-1}$, the number of ways of choosing $k-1$ of the numbers $1, \ldots, i-1$. Summing over i yields the number of subsets of size k.

12. Number of possible selections of a committee of size k and a chairperson is $k\binom{n}{k}$ and so $\sum_{k=1}^{n} k\binom{n}{k}$ represents the desired number. On the other hand, the chairperson can be anyone of the n persons and then each of the other $n-1$ can either be on or off the committee. Hence, $n2^{n-1}$ also represents the desired quantity.

(i) $\binom{n}{k}k^2$

(ii) $n2^{n-1}$ since there are n possible choices for the combined chairperson and secretary and then each of the other $n-1$ can either be on or off the committee.

(iii) $n(n-1)2^{n-2}$

(c) From a set of n we want to choose a committee, its chairperson its secretary and its treasurer (possibly the same). The result follows since

(a) there are $n2^{n-1}$ selections in which the chair, secretary and treasurer are the same person.

(b) there are $3n(n-1)2^{n-2}$ selection in which the chair, secretary and treasurer jobs are held by 2 people.

(c) there are $n(n-1)(n-2)2^{n-3}$ selections in which the chair, secretary and treasurer are all different.

(d) there are $\binom{n}{k}k^3$ selections in which the committee is of size k.

13. $(1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-1}$

14. (a) $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

(b) From (a), $\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1} = \binom{n}{i}2^{n-i}$

(c) $\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{n-j} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1}(-1)^{n-j}$

$= \binom{n}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{n-i-k} = 0$

15. (a) The number of vectors that have $x_k = j$ is equal to the number of vectors $x_1 \le x_2 \le \ldots \le x_{k-1}$ satisfying $1 \le x_i \le j$. That is, the number of vectors is equal to $H_{k-1}(j)$, and the result follows.

(b)
$$H_2(1) = H_1(1) = 1$$
$$H_2(2) = H_1(1) + H_1(2) = 3$$
$$H_2(3) = H_1(1) + H_1(2) + H_1(3) = 6$$
$$H_2(4) = H_1(1) + H_1(2) + H_1(3) + H_1(4) = 10$$
$$H_2(5) = H_1(1) + H_1(2) + H_1(3) + H_1(4) + H_1(5) = 15$$
$$H_3(5) = H_2(1) + H_2(2) + H_2(3) + H_2(4) + H_2(5) = 35$$

16. (a) $1 < 2 < 3$, $1 < 3 < 2$, $2 < 1 < 3$, $2 < 3 < 1$, $3 < 1 < 2$, $3 < 2 < 1$,
$1 = 2 < 3$, $1 = 3 < 2$, $2 = 3 < 1$, $1 < 2 = 3$, $2 < 1 = 3$, $3 < 1 = 2$, $1 = 2 = 3$

(b) The number of outcomes in which i players tie for last place is equal to $\binom{n}{i}$, the number of ways to choose these i players, multiplied by the number of outcomes of the remaining $n - i$ players, which is clearly equal to $N(n - i)$.

(c)
$$\sum_{i=1}^{n}\binom{n}{i}N(n-1) = \sum_{i=1}^{n}\binom{n}{n-i}N(n-i)$$
$$= \sum_{j=0}^{n-1}\binom{n}{j}N(j)$$

where the final equality followed by letting $j = n - i$.

(d) $N(3) = 1 + 3N(1) + 3N(2) = 1 + 3 + 9 = 13$
$N(4) = 1 + 4N(1) + 6N(2) + 4N(3) = 75$

17. A choice of r elements from a set of n elements is equivalent to breaking these elements into two subsets, one of size r (equal to the elements selected) and the other of size $n - r$ (equal to the elements not selected).

18. Suppose that r labelled subsets of respective sizes $n_1, n_2, \ldots, n_r$ are to be made up from elements 1, 2, …, n where $n = \sum_{i=1}^{r} n_i$. As $\binom{n-1}{n_1,\ldots,n_i-1,\ldots n_r}$ represents the number of possibilities when person n is put in subset i, the result follows.

19. By induction:

$$(x_1 + x_2 + \ldots + x_r)^n$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} (x_2 + \ldots + x_r)^{n-i_1} \text{ by the Binomial theorem}$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} \sum_{\substack{i_2,\ldots,i_r \\ i_2+\ldots+i_r=n-i_1}} \cdots \sum \binom{n-i_1}{i_2,\ldots,i_r} x_1^{i_2} \ldots x_r^{i_2}$$

$$= \sum_{\substack{i_1,\ldots,i_r \\ i_1+i_2+\ldots+i_r=n}} \sum \cdots \sum \binom{n}{i_1,\ldots,i_r} x_1^{i_1} \ldots x_r^{i_r}$$

where the second equality follows from the induction hypothesis and the last from the identity $\binom{n}{i_1}\binom{n-i_1}{i_2,\ldots,i_n} = \binom{n}{i_1,\ldots,i_r}$.

20. The number of integer solutions of

$$x_1 + \ldots + x_r = n,\ x_i \geq m_i$$

is the same as the number of nonnegative solutions of

$$y_1 + \ldots + y_r = n - \sum_{1}^{r} m_i\ ,\ y_i \geq 0.$$

Proposition 6.2 gives the result $\binom{n - \sum_{1}^{r} m_i + r - 1}{r-1}$.

21. There are $\binom{r}{k}$ choices of the k of the x's to equal 0. Given this choice the other $r - k$ of the x's must be positive and sum to n.
By Proposition 6.1, there are $\binom{n-1}{r-k-1} = \binom{n-1}{n-r+k}$ such solutions.
Hence the result follows.

22. $\binom{n+r-1}{n-1}$ by Proposition 6.2.

23. There are $\binom{j+n-1}{j}$ nonnegative integer solutions of

$$\sum_{i=1}^{n} x_i = j$$

Hence, there are $\sum_{j=0}^{k} \binom{j+n-1}{j}$ such vectors.

Chapter 2

Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$
(b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$

2. $S = \{(n, x_1, \ldots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \ldots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll n, and x_i appears on roll, i, $i = 1, \ldots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.
$E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.
EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

4. $A = \{1,0001,0000001, \ldots\}$ $B = \{01, 00001, 00000001, \ldots\}$
$(A \cup B)^c = \{00000 \ldots, 001, 000001, \ldots\}$

5. (a) $2^5 = 32$
(b)
$W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$
$(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$
$(0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

(c) 8
(d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$

6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$
(b) $A = \{(1, s), (0, s)\}$
(c) $B = \{(0, g), (0, f), (0, s)\}$
(d) $\{(1, s), (0, s), (1, g), (1, f)\}$

7. (a) 6^{15}
(b) $6^{15} - 3^{15}$
(c) 4^{15}

8. (a) .8
(b) .3
(c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

$$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$
Thus, $P(RN) = .1$

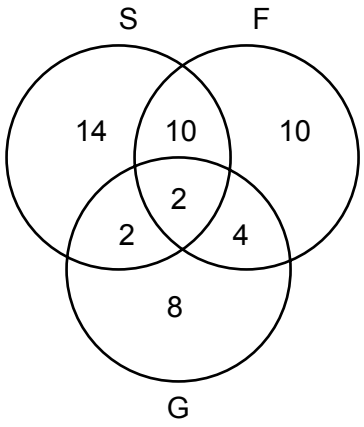
11. Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.

(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^cB) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

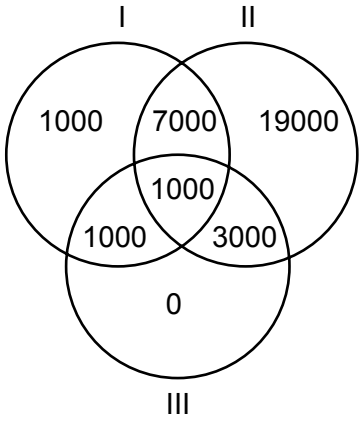
12. (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.



(c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2} \Big/ \binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is 149/198.

13.



(a) 20,000
(b) 12,000
(c) 11,000
(d) 68,000
(e) 10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15. (a) $4\binom{13}{5}\Big/\binom{52}{5}$

(b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}\Big/\binom{52}{5}$

(d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(e) $13\binom{4}{4}\binom{48}{1}\Big/\binom{52}{5}$

16. (a) $\dfrac{6\cdot 5\cdot 4\cdot 3\cdot 2}{6^5}$ (b) $\dfrac{6\binom{5}{2}5\cdot 4\cdot 3}{6^5}$ (c) $\dfrac{\binom{6}{2}4\binom{5}{2}\binom{3}{2}}{6^5}$

(d) $\dfrac{6\cdot 5\cdot 4\binom{5}{3}}{21}$ (e) $\dfrac{6\cdot 5\binom{5}{3}}{6^5}$ (f) $\dfrac{6\cdot 5\binom{5}{4}}{6^5}$

(g) $\dfrac{6}{6^5}$

17. $\dfrac{\prod_{i=1}^{8} i^2}{64\cdot 63\cdots 58}$

18. $\dfrac{2\cdot 4\cdot 16}{52\cdot 51}$

19. 4/36 + 4/36 +1/36 + 1/36 = 5/18

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4\cdot 4\cdot 16}{52\cdot 51} + \frac{4\cdot 4\cdot 16\cdot 3\cdot 15}{52\cdot 51\cdot 50\cdot 49} \\ &= .0983 \end{aligned}$$

where the preceding used that $P(A) = P(B) = 2 \times \dfrac{4\cdot 16}{52\cdot 51}$. Hence, the probability that neither is dealt blackjack is .9017.

21. (a) $p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

22. The ordering will be unchanged if for some k, $0 \le k \le n$, the first k coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1/2^n$

23. The answer is 5/12, which can be seen as follows:

$$\begin{aligned} 1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + 1/6 \end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

25. $$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{6}{36}, \quad \sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

28. $$P\{\text{same}\} = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

$$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1} \Big/ \binom{19}{3}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned} P\{\text{different}\} &= P(RBG) + P\{BRG) + P(\text{RGB}) + \ldots + P(GBR) \\ &= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3} \end{aligned}$$

29. (a) $\dfrac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) Putting all terms over the common denominator $(n + m)^2(n + m - 1)$ shows that we must prove that

$n^2(n + m - 1) + m^2(n + m - 1) \geq n(n - 1)(n + m) + m(m - 1)(n + m)$

which is immediate upon multiplying through and simplifying.

30. (a) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$

(b) $\dfrac{\binom{7}{3}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} - 1/18 = 1/6$

(c) $\dfrac{\binom{7}{3}\binom{8}{4}+\binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$

31. $P(\{\text{complete}\}) = \dfrac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3} = \dfrac{2}{9}$

$P\{\text{same}\} = \dfrac{3}{27} = \dfrac{1}{9}$

32. $\dfrac{g(b+g-1)!}{(b+g)!} = \dfrac{g}{b+g}$

33. $\dfrac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \dfrac{70}{323}$

34. $\binom{32}{13} \Big/ \binom{52}{13}$

35. $1 - \binom{30}{3} \Big/ \binom{54}{3} \approx .8363$

36. (a) $\binom{4}{2}\Big/\binom{52}{2} \approx .0045,$

(b) $13\binom{4}{2}\Big/\binom{52}{2} = 1/17 \approx .0588$

37. (a) $\binom{7}{5}\Big/\binom{10}{5} = 1/12 \approx .0833$

(b) $\binom{7}{4}\binom{3}{1}\Big/\binom{10}{5} + 1/12 = 1/2$

38. $1/2 = \binom{3}{2}\Big/\binom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

39. $\dfrac{5\cdot 4\cdot 3}{5\cdot 5\cdot 5} = \dfrac{12}{25}$

40. $P\{1\} = \dfrac{4}{44} = \dfrac{1}{64}$

$$P\{2\} = \binom{4}{2}\left[4 + \binom{4}{2} + 4\right]\Big/ 4^4 = \frac{84}{256}$$

$$P\{3\} = \binom{4}{3}\binom{3}{1}\frac{4!}{2!}\Big/ 4^4 = \frac{36}{64}$$

$$P\{4\} = \frac{4!}{4^4} = \frac{6}{64}$$

41. $1 - \dfrac{5^4}{6^4}$

42. $1 - \left(\dfrac{35}{36}\right)^n$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

$\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

44. (a) If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

(b) $2 \cdot 2 \cdot 3!/5! = 1/5$

(c) $2 \cdot 3!/5! = 1/10$

45. $1/n$ if discard, $\dfrac{(n-1)^{k-1}}{n^k}$ if do not discard

46. If n in the room,

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \quad \cdot (13-n)}{12 \cdot 12 \cdot \quad \cdot 12}$$

When $n = 5$ this falls below 1/2. (Its value when $n = 5$ is .3819)

47. $12!/(12)^{12}$

48. $\dbinom{12}{4}\dbinom{8}{4}\dfrac{(20)!}{(3!)^4(2!)^4}\Big/(12)^{20}$

49. $\dbinom{6}{3}\dbinom{6}{3}\Big/\dbinom{12}{6}$

50. $\dbinom{13}{5}\dbinom{39}{8}\dbinom{8}{8}\dbinom{31}{5}\Big/\dbinom{52}{13}\dbinom{39}{13}$

51. $\dbinom{n}{m}(n-1)^{n-m}/N^n$

52. (a) $\dfrac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

(b) $\dfrac{\dbinom{10}{1}\dbinom{9}{6}\dfrac{8!}{2!}2^6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^4 A_i) = 4\frac{2 \cdot 7!}{8!} - 6\frac{2^2 \cdot 6!}{8!} + 4\frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.

54. $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(\text{C}) - P(SH) - \ldots - P(SHDC)$

$$= \frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4\binom{13}{13}}{\binom{52}{13}}$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

55. (a) $P(S \cup H \cup D \cup C) = P(S) + \ldots - P(SHDC)$

$$= \frac{4\binom{2}{2}}{\binom{52}{13}} - \frac{6\binom{2}{2}\binom{2}{2}\binom{48}{9}}{\binom{52}{13}} + \frac{4\binom{2}{2}^3\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{2}{2}^4\binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4\binom{50}{11} - 6\binom{48}{9} + 4\binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \ldots \cup 13) = \dfrac{13\binom{48}{9}}{\binom{52}{13}} - \dfrac{\binom{13}{2}\binom{44}{5}}{\binom{52}{13}} + \dfrac{\binom{13}{3}\binom{40}{1}}{\binom{52}{13}}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

Theoretical Exercises

5. $F_i = E_i \bigcap_{j=1}^{i=1} E_j^c$

6. (a) EF^cG^c

   (b) EF^cG

   (c) $E \cup F \cup G$

   (d) $EF \cup EG \cup FG$

   (e) EFG

   (f) $E^cF^cG^c$

   (g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

   (h) $(EFG)^c$

   (i) $EFG^c \cup EF^cG \cup E^cFG$

   (j) S

7. (a) E

   (b) EF

   (c) $EG \cup F$

8. The number of partitions that has $n + 1$ and a fixed set of i of the elements 1, 2, …, n as a subset is T_{n-i}. Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^{n} \binom{n}{i} T_{n-i} = 1 + \sum_{i=0}^{n-1} \binom{n}{i} T_{n-i} = 1 + \sum_{k=1}^{n} \binom{n}{k} T_k \ .$$

11. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

12. $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$= P(E) - P(EF) + P(F) - P(EF)$

13. $E = EF \cup EF^c$

15. $$\frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$$

16. $P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19. $$\frac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}$$

21. Let $y_1, y_2, \ldots, y_k$ denote the successive runs of losses and $x_1, \ldots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \ldots, y_k x_k$ or $x_1 y_1, \ldots x_k, y_k$ where all x_i, y_i are positive, with $x_1 + \ldots + x_k = n$, $y_1 + \ldots + y_k = m$. By Proposition 6.1 there are $2\binom{n-1}{k-1}\binom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2\binom{n-1}{k-1}\binom{m-1}{k-1}\Big/\binom{m+n}{n}.$$

There will be $2k+1$ runs if the outcome is either of the form $x_1, y_1, \ldots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \ldots, y_k, x_k y_{k+1}$ where all are positive and $\sum x_i = n$, $\sum y_i = m$. By Proposition 6.1 there are $\binom{n-1}{k}\binom{m-1}{k-1}$ outcomes of the first type and $\binom{n-1}{k-1}\binom{m-1}{k}$ of the second.

Chapter 3

Problems

1. $P\{6 \mid \text{different}\} = P\{6, \text{different}\}/P\{\text{different}\}$

$$= \frac{P\{1\text{st} = 6, 2\text{nd} \neq 6\} + P\{1\text{st} \neq 6, 2\text{nd} = 6\}}{5/6}$$

$$= \frac{2 \ 1/6 \ \ 5/6}{5.6} = 1/3$$

could also have been solved by using reduced sample space—for given that outcomes differ it is the same as asking for the probability that 6 is chosen when 2 of the numbers 1, 2, 3, 4, 5, 6 are randomly chosen.

2. $P\{6 \mid \text{sum of } 7\} = P\{(6, 1)\}/1/6 = 1/6$

$P\{6 \mid \text{sum of } 8\} = P\{(6, 2)\}/5/36 = 1/5$

$P\{6 \mid \text{sum of } 9\} = P\{(6, 3)\}/4/36 = 1/4$

$P\{6 \mid \text{sum of } 10\} = P\{(6, 4)\}/3/36 = 1/3$

$P\{6 \mid \text{sum of } 11\} = P\{(6, 5)\}/2/36 = 1/2$

$P\{6 \mid \text{sum of } 12\} = 1.$

3. $P\{E \text{ has } 3 \mid N - S \text{ has } 8\} = \dfrac{P\{E \text{ has } 3, N - S \text{ has } 8\}}{P\{N - S \text{ has } 8\}}$

$$= \frac{\binom{13}{8}\binom{39}{18}\binom{5}{3}\binom{21}{10} \Big/ \binom{52}{26}\binom{26}{13}}{\binom{13}{8}\binom{39}{18} \Big/ \binom{52}{26}} = .339$$

4. $P\{\text{at least one } 6 \mid \text{sum of } 12\} = 1.$ Otherwise twice the probability given in Problem 2.

5. $\dfrac{6}{15}\dfrac{5}{14}\dfrac{9}{13}\dfrac{8}{12}$

6. In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is 1/2.

7. $P1\ g \text{ and } 1\ b \mid \text{at least one } b\} = \dfrac{1/2}{3/4} = 2/3$

8. 1/2

9. $P\{A = w \mid 2w\} = \dfrac{P\{A = w, 2w\}}{P\{2w\}}$

$$= \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{2w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{2}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = \frac{7}{11}$$

10. 11/50

11. (a) $P(B \mid A_s) = \dfrac{P(BA_s)}{P(A_s)} = \dfrac{\frac{1}{52}\frac{3}{21} + \frac{3}{52}\frac{1}{51}}{\frac{2}{52}} = \dfrac{1}{17}$

Which could have been seen by noting that, given the ace of spades is chosen, the other card is equally likely to be any of the remaining 51 cards, of which 3 are aces.

(b) $P(B \mid A) = \dfrac{P(B)}{P(A)} = \dfrac{\frac{4}{52}\frac{3}{51}}{1 - \frac{48}{52}\frac{47}{51}} = \dfrac{1}{33}$

12. (a) (.9)(.8)(.7) = .504

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid F_1^c F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1 - .504} = \frac{(.9)(.2)}{.496} = .3629$$

13. $P(E_1) = \dbinom{4}{1}\dbinom{48}{12}\Big/\dbinom{52}{13}$, $\quad P(E_2 \mid E_1) = \dbinom{3}{1}\dbinom{36}{12}\Big/\dbinom{39}{13}$

$P(E_3 \mid E_1E_2) = \dbinom{2}{1}\dbinom{24}{12}\Big/\dbinom{26}{13}$, $\quad P(E_4 \mid E_1E_2E_3) = 1.$

Hence,

$$p = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} \cdot \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} \cdot \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$$

14. $\dfrac{5}{12}\dfrac{7}{14}\dfrac{7}{16}\dfrac{9}{18} - \dfrac{35}{768}$.

15. Let E be the event that a randomly chosen pregnant women has an ectopic pregnancy and S the event that the chosen person is a smoker. Then the problem states that

$$P(E \mid S) = 2P(E \mid S^c),\ P(S) = .32$$

Hence,

$$\begin{aligned} P(S \mid E) &= P(SE)/P(E) \\ &= \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|S^c)P(S^C)} \\ &= \frac{2P(S)}{2P(S) + P(S^c)} \\ &= 32/66 \approx .4548 \end{aligned}$$

16. With S being survival and C being C section of a randomly chosen delivery, we have that

$$\begin{aligned} .98 = P(S) &= P(S \mid C).15 + P(S \mid C^2)\ .85 \\ &= .96(.15) + P(S \mid C^2)\ .85 \end{aligned}$$

Hence

$$P(S \mid C^c) \approx .9835.$$

17. $P(D) = .36$, $P(\mathrm{C}) = .30$, $P(C \mid D) = .22$

(a) $P(DC) = P(D)\ P(C \mid D) = .0792$
(b) $P(D \mid C) = P(DC)/P(C) = .0792/.3 = .264$

18. (a) $$P(\mathrm{Ind} \mid \mathrm{voted}) = \frac{P(\mathrm{voted}|\mathrm{Ind})P(\mathrm{Ind})}{\sum P(\mathrm{voted}|t\mathrm{ype})P(\mathrm{type})}$$
$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

(b) $$P\{\mathrm{Lib} \mid \mathrm{voted}\} = \frac{.62(.30)}{.35(.46) + .62(.3) + .58(.24)} \approx .383$$

(c) $$P\{\mathrm{Con} \mid \mathrm{voted}\} = \frac{.58(.24)}{.35(.46) + .62(.3) + .58(.24)} \approx .286$$

(d) $P\{\mathrm{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

19. Choose a random member of the class. Let A be the event that this person attends the party and let W be the event that this person is a woman.

(a) $P(W \mid A) = \dfrac{P(A|W)P(W)}{P(A|W)P(W) + P(A|M)P(M)}$ where $M = W^c$

$$= \frac{.48(.38)}{.48(.38) + .37(.62)} \approx .443$$

Therefore, 44.3 percent of the attendees were women.

(b) $P(A) = .48(.38) + .37(.62) = .4118$

Therefore, 41.18 percent of the class attended.

20. (a) $P(F \mid C) = \dfrac{P(FC)}{P(C)} = .02/.05 = .40$

(b) $P(C \mid F) = P(FC)/P(F) = .02/.52 = 1/26 \approx .038$

21. (a) $P\{\text{husband under 25}\} = (212 + 36)/500 = .496$

(b) $P\{\text{wife over} \mid \text{husband over}\} = P\{\text{both over}\}/P\{\text{husband over}\}$

$$= (54/500)/(252/500)$$
$$= 3/14 \approx .214$$

(c) $P\{\text{wife over} \mid \text{husband under}\} = 36/248 \approx .145$

22. a. $\dfrac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \dfrac{5}{9}$

b. $\dfrac{1}{3!} = \dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6} = \dfrac{5}{54}$

23. $P(w \mid w \text{ transferred})P\{w \text{ tr.}\} + P(w \mid R \text{ tr.})P\{R \text{ tr.}\} = \dfrac{2}{3}\dfrac{1}{3} + \dfrac{1}{3}\dfrac{2}{3} = \dfrac{4}{9}$.

$$P\{w \text{ transferred} \mid w\} = \frac{P\{w|w \text{ tr.}\}P\{w \text{ tr.}\}}{P\{w\}} = \frac{\frac{2}{3}\frac{1}{3}}{\frac{4}{9}} = 1/2.$$

24. (a) $P\{g-g \mid \text{at least one } g\} = \dfrac{1/4}{3/4} = 1/3.$

(b) Since we have no information about the ball in the urn, the answer is 1/2.

26. Let M be the event that the person is male, and let C be the event that he or she is color blind. Also, let p denote the proportion of the population that is male.

$$P(M \mid C) = \frac{P(C \mid M)P(M)}{P(C \mid M)P(M) + P(C \mid M^c)P(M^c)} = \frac{(.05)p}{(.05)p + (.0025)(1-p)}$$

27. Method (b) is correct as it will enable one to estimate the average number of workers per car. Method (a) gives too much weight to cars carrying a lot of workers. For instance, suppose there are 10 cars, 9 transporting a single worker and the other carrying 9 workers. Then 9 of the 18 workers were in a car carrying 9 workers and so if you randomly choose a worker then with probability 1/2 the worker would have been in a car carrying 9 workers and with probability 1/2 the worker would have been in a car carrying 1 worker.

28. Let A denote the event that the next card is the ace of spades and let B be the event that it is the two of clubs.

(a) $P\{A\} = P\{\text{next card is an ace}\}P\{A \mid \text{next card is an ace}\}$

$$= \frac{3}{32}\frac{1}{4} = \frac{3}{128}$$

(b) Let C be the event that the two of clubs appeared among the first 20 cards.

$$P(B) = P(B \mid C)P(C) + P(B \mid C^c)P(C^c)$$
$$= 0\frac{19}{48} + \frac{1}{32}\frac{29}{48} = \frac{29}{1536}$$

29. Let A be the event that none of the final 3 balls were ever used and let B_i denote the event that i of the first 3 balls chosen had previously been used. Then,

$$P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= \sum_{i=0}^{3} \frac{\binom{6+i}{3}}{\binom{15}{3}} \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}}$$
$$= .083$$

30. Let B and W be the events that the marble is black and white, respectively, and let B be the event that box i is chosen. Then,

$$P(B) = P(B \mid B_1)P(B_1) + P(B \mid B_2)P(B_2) = (1/2)(1/2) = (2/3)(1/2) = 7/12$$
$$P(B_1 \mid W) = \frac{P(W \mid B_1)P(B_1)}{P(W)} = \frac{(1/2)(1/2)}{5/12} = 3/5$$

31. Let C be the event that the tumor is cancerous, and let N be the event that the doctor does not call. Then

$$\begin{aligned}\beta = P(C \mid N) &= \frac{P(NC)}{P(N)} \\ &= \frac{P(N \mid C)P(C)}{P(N \mid C)P(C) + P(N \mid C^c)P(C^c)} \\ &= \frac{\alpha}{\alpha + \frac{1}{2}(1-\alpha)} \\ &= \frac{2\alpha}{1+\alpha} \geq \alpha\end{aligned}$$

with strict inequality unless $\alpha = 1$.

32. Let E be the event the child selected is the eldest, and let F_j be the event that the family has j children. Then,

$$\begin{aligned}P(F_j \mid E) &= \frac{P(EF_j)}{P(E)} \\ &= \frac{P(F_j)P(E \mid F_j)}{\sum_j P(F_j)P(E \mid F_j)} \\ &= \frac{p_j(1/j)}{.1 + .25(1/2) + .35(1/3) + .3(1/4)} = .24\end{aligned}$$

Thus, $P(F_1 \mid E) = .24$, $P(F_4 \mid E) = .18$.

33. Let V be the event that the letter is a vowel. Then

$$P(E \mid V) = \frac{P(V \mid E)P(E)}{P(V \mid E)P(E) + P(V \mid A)P(A)} = \frac{(1/2)(2/5)}{(1/2)(2/5) + (2/5)(3/5)} = 5/11$$

34. $$P(G \mid C) = \frac{P(C \mid G)P(G)}{P(C \mid G)P(G) + P(C \mid G^c)P(G^c)} = 54/62$$

35. $$\begin{aligned}&P\{A = \text{superior} \mid A \text{ fair}, B \text{ poor}\} \\ &= \frac{P\{A \text{ fair}, B \text{ poor} \mid A \text{ superior} \mid A \text{ superior}\}}{P\{A \text{ fair}, B \text{ poor}\}} \\ &= \frac{\frac{10}{30}\frac{15}{30}\frac{1}{2}}{\frac{10}{30}\frac{15}{30}\frac{1}{2} + \frac{10}{30}\frac{5}{30}\frac{1}{2}} = \frac{3}{4}.\end{aligned}$$

36. $P\{C \mid \text{woman}\} = \dfrac{P\{\text{women} \mid C\}P\{C\}}{P\{\text{women} \mid A\}P\{A\} + P\{\text{women} \mid B\}P\{B\} + P\{\text{women} \mid C\}P\{C\}}$

$$= \frac{.7\dfrac{100}{225}}{.5\dfrac{50}{225} + .6\dfrac{75}{225} + .7\dfrac{100}{225}} = \frac{1}{2}$$

37. (a) $P\{\text{fair} \mid h\} = \dfrac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2} + \frac{1}{2}} = \frac{1}{3}$.

(b) $P\{\text{fair} \mid hh\} = \dfrac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}} = \frac{1}{5}$.

(c) 1

38. $P\{\text{tails} \mid w\} = \dfrac{\frac{3}{15}\frac{1}{2}}{\frac{3}{15}\frac{1}{2} + \frac{5}{12}\frac{1}{2}} = \dfrac{36}{36+75} = \dfrac{36}{111}$.

39. $P\{\text{acc.} \mid \text{no acc.}\} = \dfrac{P\{\text{no acc., acc.}}{P\{\text{no acc.}\}}$

$$= \frac{\frac{3}{10}(.4)(.6) + \frac{7}{10}(.2)(.8)}{\frac{3}{10}(.6) + \frac{7}{10}(.8)} = \frac{46}{185}.$$

40. (a) $\frac{7}{12}\frac{8}{13}\frac{9}{14}$

(b) $3\dfrac{7 \cdot 8 \cdot 5}{12 \cdot 13 \cdot 14}$

(c) $\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

(d) $3\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

41. $P\{\text{ace}\} = P\{\text{ace} \mid \text{interchanged selected}\}\,\frac{1}{27}$

$$+P\{\text{ace} \mid \text{interchanged not selected}\}\,\frac{26}{27}$$

$$= 1\frac{1}{27} + \frac{3}{51}\frac{26}{27} = \frac{129}{51 \cdot 27}.$$

42. $$P\{A \mid \text{failure}\} = \frac{(.02)(.5)}{(.02)(.5)+(.03)(.3)+(.05)(.2)} = \frac{10}{29}$$

43. $$P\{2 \text{ headed} \mid \text{heads}\} = \frac{\frac{1}{3}(1)}{\frac{1}{3}(1)+\frac{1}{3}\frac{1}{2}+\frac{1}{3}\frac{3}{4}} = \frac{4}{4+2+3} = \frac{4}{9}.$$

45. $$P\{5\text{th} \mid \text{heads}\} = \frac{P\{\text{heads} \mid 5^{\text{th}}\}P\{5^{\text{th}}\}}{\sum_i P\{h \mid i^{\text{th}}\}P\{i^{th}\}}$$

$$= \frac{\frac{5}{10}\frac{1}{10}}{\sum_{i=1}^{10}\frac{i}{10}\frac{1}{10}} = \frac{1}{11}.$$

46. Let M and F denote, respectively, the events that the policyholder is male and that the policyholder is female. Conditioning on which is the case gives the following.

$$P(A_2 \mid A_1) = \frac{P(A_1A_2)}{P(A_1)}$$

$$= \frac{P(A_1A_2 \mid M)\alpha + P(A_1A_2 \mid F)(1-\alpha)}{P(A_1 \mid M)\alpha + P(A_1 \mid F)(1-\alpha)}$$

$$= \frac{p_m^2\alpha + p_f^2(1-\alpha)}{p_m\alpha + p_f(1-\alpha)}$$

Hence, we need to show that

$$p_m^2\alpha + p_f^2[1-\alpha) > (p_m\alpha + p_f(1-\alpha))^2$$

or equivalently, that

$$p_m^2(\alpha-\alpha^2) + p_f^2[1-\alpha-(1-a)^2] > 2\alpha(1-\alpha)p_fp_m$$

Factoring out $\alpha(1-\alpha)$ gives the equivalent condition

$$p_m^2 + p_f^2 > 2pf_m$$

or

$$(p_m - p_f)^2 > 0$$

which follows because $p_m \neq p_f$. Intuitively, the inequality follows because given the information that the policyholder had a claim in year 1 makes it more likely that it was a type policyholder having a larger claim probability. That is, the policyholder is more likely to me male if $p_m > p_f$ (or more likely to be female if the inequality is reversed) than without this information, thus raising the probability of a claim in the following year.

47. $$P\{\text{all white}\} = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right]$$

$$P\{3 \mid \text{all white}\} = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{all white}\}}$$

48. (a) $P\{\text{silver in other} \mid \text{silver found}\}$

$$= \frac{P\{S \text{ in other}, S \text{ found}\}}{P\{S \text{ found}\}}.$$

To compute these probabilities, condition on the cabinet selected.

$$= \frac{1/2}{P\{S \text{ found}|A\}\,1/2 + P\{S \text{ found}|B\}\,1/2}$$

$$= \frac{1}{1+1/2} = \frac{2}{3}.$$

49. Let C be the event that the patient has cancer, and let E be the event that the test indicates an elevated PSA level. Then, with $p = P(C)$,

$$P(C \mid E) = \frac{P(E|C)P(C)}{P(E|C)P(C) + P(E|C^c)P(C^c)}$$

Similarly,

$$P(C \mid E^c) = \frac{P(E^c|C)P(C)}{P(E^c|C)P(C) + P(E^c|C^c)P(C^c)}$$

$$= \frac{.732p}{.732p + .865(1-p)}$$

50. Choose a person at random

$$P\{\text{they have accident}\} = P\{\text{acc.} \mid \text{good}\}P\{g\} + P\{\text{acc.} \mid \text{ave.}\}P\{\text{ave.}\} + P\{\text{acc.} \mid \text{bad } P(\text{b})\}$$
$$= (.05)(.2) + (.15)(.5) + (.30)(.3) = .175$$

$$P\{A \text{ is good} \mid \text{ no accident}\} = \frac{.95(2)}{.825}$$
$$P\{A \text{ is average} \mid \text{no accident}\} = \frac{(.85)(.5)}{.825}$$

51. Let R be the event that she receives a job offer.

(a) $P(R) = P(R \mid \text{strong})P(\text{strong}) + P(R \mid \text{moderate})P(\text{moderate}) + P(R \mid \text{weak})P(\text{weak})$
$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65$

(b) $$P(\text{strong} \mid R) = \frac{P(R \mid \text{strong})P(\text{strong})}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}$$

Similarly,

$$P(\text{moderate} \mid R) = \frac{8}{65}, P(\text{weak} \mid R) = \frac{1}{65}$$

(c) $$P(\text{strong} \mid R^c) = \frac{P(R^c \mid \text{strong})P(\text{strong})}{P(R^c)} = \frac{(.2)(.7)}{.35} = \frac{14}{35}$$

Similarly,

$$P(\text{moderate} \mid R^c) = \frac{12}{35}, P(\text{weak} \mid R^c) = \frac{9}{35}$$

52. Let M, T, W, Th, F be the events that the mail is received on that day. Also, let A be the event that she is accepted and R that she is rejected.

(a) $P(M) = P(M \mid A)P(A) + P(M \mid R)P(R) = (.15)(.6) + (.05)(.4) = .11$

(b) $P(T \mid M^c) = \dfrac{P(T)}{P(M^c)}$

$$= \frac{P(T \mid A)P(A) + P(T \mid R)P(R)}{1 - P(M)}$$

$$= \frac{(.2)(.6) + (.1)(.4)}{.89} \frac{16}{89}$$

(c) $P(A \mid M^c T^c W^c) = \dfrac{P(M^c T^c W^c \mid A)P(A)}{P(M^c T^c W^c)}$

$$= \frac{(1 - .15 - .20 - .25)(.6)}{(.4)(.6) + (.75)(.4)} = \frac{12}{27}$$

(d) $P(A \mid Th) = \dfrac{P(Th \mid A)P(A)}{P(Th)}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.15)(.4)} = \frac{3}{5}$$

(e) $P(A \mid \text{no mail}) = \dfrac{P(\text{no mail} \mid A)P(A)}{P(\text{no mail})}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.4)(.4)} = \frac{9}{25}$$

53. Let W and F be the events that component 1 works and that the system functions.

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

55. $P\{\text{Boy}, F\} = \dfrac{4}{16 + x}$ $P\{\text{Boy}) = \dfrac{10}{16 + x}$ $P\{F\} = \dfrac{10}{16 + x}$

so independence $\Rightarrow 4 = \dfrac{10 \cdot 10}{16 + x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

56. $P\{\text{new}\} = \sum_i P\{\text{new} \mid \text{type } i\} p_i = \sum_i (1 - p_i)^{n-1} p_i$

57. (a) $2p(1-p)$

(b) $\binom{3}{2}p^2(1-p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$
$= P\{\text{up first, up 1 after 3}\}/[3p^2(1-p)]$
$= p2p(1-p)/[3p^2(1-p)] = 2/3.$

58. (a) All we know when the procedure ends is that the two most flips were either H, T, or T, H. Thus,

$$P(\text{heads}) = P(H, T \mid H, T \text{ or } T, H) = \frac{P(H,T)}{P(H,T)+P(T,H)} = \frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$$

(b) No, with this new procedure the result will be heads (tails) whenever the first flip is tails (heads). Hence, it will be heads with probability $1-p$.

59. (a) 1/16

(b) 1/16

(c) The only way in which the pattern H, H, H, H can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so T, H, H, H will appear first). Hence, the probability that T, H, H, H occurs first is 15/16.

60. From the information of the problem we can conclude that both of Smith's parents have one blue and one brown eyed gene. Note that at birth, Smith was equally likely to receive either a blue gene or a brown gene from each parent. Let X denote the number of blue genes that Smith received.

(a) $P\{\text{Smith blue gene}\} = P\{X = 1 \mid X \le 1\} = \dfrac{1/2}{1-1/4} = 2/3$

(b) Condition on whether Smith has a blue-eyed gene.
$P\{\text{child blue}\} = P\{\text{blue} \mid \text{blue gene}\}(2/3) + P\{\text{blue} \mid \text{no blue}\}(1/3)$
$= (1/2)(2/3) = 1/3$

(c) First compute

$$P\{\text{Smith blue} \mid \text{child brown}\} = \frac{P\{\text{child brown} \mid \text{Smith blue}\}2/3}{2/3} = 1/2$$

Now condition on whether Smith has a blue gene given that first child has brown eyes.
$P\{\text{second child brown}\} = P\{\text{brown} \mid \text{Smith blue}\}1/2 + P\{\text{brown} \mid \text{Smith no blue}\}1/2$
$= 1/4 + 1/2 = 3/4$

61. Because the non-albino child has an albino sibling we know that both its parents are carriers. Hence, the probability that the non-albino child is not a carrier is

$$P(A, A \mid A, a \text{ or } a, A \text{ or } A, A) = \frac{1}{3}$$

Where the first gene member in each gene pair is from the mother and the second from the father. Hence, with probability 2/3 the non-albino child is a carrier.

(a) Condition on whether the non-albino child is a carrier. With C denoting this event, and O_i the event that the i^{th} offspring is albino, we have:

$$\begin{aligned} P(O_1) &= P(O_1 \mid C)P(C) + P(O_1 \mid C^c)P(C^c) \\ &= (1/4)(2/3) + 0(1/3) = 1/6 \end{aligned}$$

(b) $$\begin{aligned} P(O_2 \mid O_1^c) &= \frac{P(O_1^c O_2)}{P(O_1^c)} \\ &= \frac{P(O_1^c O_2 \mid C)P(C) + P(O_1^c O_2 \mid C^c)P(C^c)}{5/6} \\ &= \frac{(3/4)(1/4)(2/3) + 0(1/3)}{5/6} = \frac{3}{20} \end{aligned}$$

62. (a) $$P\{\text{both hit} \mid \text{at least one hit}\} = \frac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}}$$

$$= p_1 p_2/(1 - q_1 q_2)$$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1 q_2)$
$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

63. Consider the final round of the duel. Let $q_x = 1 - p_x$

(a) $$\begin{aligned} P\{A \text{ not hit}\} &= P\{A \text{ not hit} \mid \text{at least one is hit}\} \\ &= P\{A \text{ not hit, } B \text{ hit}\}/P\{\text{at least one is hit}\} \\ &= q_B p_A/(1 - q_A q_B) \end{aligned}$$

(b) $$\begin{aligned} P\{\text{both hit}\} &= P\{\text{both hit} \mid \text{at least one is hit}\} \\ &= P\{\text{both hit}\}/P\{\text{at least one hit}\} \\ &= p_A p_B/(1 - q_A q_B) \end{aligned}$$

(c) $(q_A q_B)^{n-1}(1 - q_A q_B)$

(d) $$\begin{aligned} P\{n \text{ rounds} \mid A \text{ unhit}\} &= P\{n \text{ rounds, } A \text{ unhit}\}/P\{A \text{ unhit}\} \\ &= \frac{(q_A q_B)^{n-1} p_A q_B}{q_B p_A/(1 - q_A q_B)} \\ &= (q_A q_B)^{n-1}(1 - q_A q_B) \end{aligned}$$

(e) $P(n \text{ rounds} \mid \text{both hit}) = P\{n \text{ rounds both hit}\}/P\{\text{both hit}\}$

$$= \frac{(q_A q_B)^{n-1} p_A p_B}{p_B p_A/(1-q_A q_B)}$$

$$= (q_A q_B)^{n-1}(1 - q_A q_B)$$

Note that (c), (d), and (e) all have the same answer.

64. If use (a) will win with probability p. If use strategy (b) then

$$\begin{aligned} P\{\text{win}\} &= P\{\text{win} \mid \text{both correct}\}p^2 + P\{\text{win} \mid \text{exactly 1 correct}\}2p(1-p) \\ &\quad + P\{\text{win} \mid \text{neither correct}\}(1-p)^2 \\ &= p^2 + p(1-p) + 0 = p \end{aligned}$$

Thus, both strategies give the same probability of winning.

65. (a) $P\{\text{correct} \mid \text{agree}\} = P\{\text{correct, agree}\}/P\{\text{agree}\}$
$= p^2/[p^2 + (1-p)^2]$
$= 36/52 = 9/13 \quad$ when $p = .6$

(b) 1/2

66. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

67. $P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4)$
$- P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4)$
$+ P(E_1E_3E_4) + P(E_2E_3E_4) - P(E_1E_2E_3E_4)$

$= P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4$
$- P_1P_2P_3P_5 - P_2P_3P_4P_5 - 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5.$

(a) $P_1P_2(1 - P_3)(1 - P_4) + P_1(1 - P_2)P_3(1 - P_4) + P_1(1 - P_2(1 - P_3)P_4$
$+ P_2P_3(1 - P_1)(1 - P_4) + (1 - P_1)P_2(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_3P_4$
$+ P_1P_2P_3(1 - P_4) + P_1P_2(1 - P_3)P_4 + P_1(1 - P_2)P_3P_4 + (1 - P_1)P_2P_3P_4 + P_1P_2P_3P_4.$

(c) $\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

68. Let C_i denote the event that relay i is closed, and let F be the event that current flows from A to B.

$$\begin{aligned} P(C_1C_2 \mid F) &= \frac{P(C_1C_2F)}{P(F)} \\ &= \frac{P(F \mid C_1C_2)P(C_1C_2)}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)} \\ &= \frac{p_5p_1p_2}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)} \end{aligned}$$

69. 1. (a) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(b) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(c) $\frac{18}{128}$

(d) $\frac{110}{128}$

2. (a) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(b) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(c) $\frac{1}{16}$

(d) $\frac{15}{16}$

70. (a) $P\{\text{carrier} \mid \text{3 without}\}$

$$= \frac{1/8\,1/2}{1/8\,1/2 + 1\,1/2} = 1/9.$$

(b) 1/18

71. $P\{\text{Braves win}\} = P\{B \mid B \text{ wins 3 of 3}\}\ 1/8 + P\{B \mid B \text{ wins 2 of 3}\}\ 3/8$
$+ P\{B \mid B \text{ wins 1 of 3}\}\ 3/8 + P\{B \mid B \text{ wins 0 of 3}\}\ 1/8$

$$= \frac{1}{8} + \frac{3}{8}\left[\frac{1}{4}\frac{1}{2} + \frac{3}{4}\right] + \frac{3}{8}\quad \frac{3}{4}\frac{1}{2} = \frac{38}{64}$$

where $P\{B \mid B \text{ wins } i \text{ of 3}\}$ is obtained by conditioning on the outcome of the other series. For instance

$$\begin{aligned} P\{B \mid B \text{ win 2 of 3}\} &= P\{B \mid D \text{ or } G \text{ win 3 of 3}, B \text{ win 2 of 3}\}\ 1/4 \\ &= P\{B \mid D \text{ or } G \text{ win 2 of 3}, B \text{ win 2 of 3}\}\ 3/4 \\ &= \frac{1}{2}\frac{1}{4} + \frac{3}{4}. \end{aligned}$$

By symmetry $P\{D \text{ win}\} = P\{G \text{ win}\}$ and as the probabilities must sum to 1 we have.

$$P\{D \text{ win}\} = P\{G \text{ win}\} = \frac{13}{64}.$$

72. Let *f* denote for and *a* against a certain place of legislature. The situations in which a given steering committees vote is decisive are as follows:

given member	other members of S.C.	other council members
for	both for	3 or 4 against
for	one for, one against	at least 2 for
against	one for, one against	at least 2 for
against	both for	3 of 4 against

$$P\{\text{decisive}\} = p^3 4p(1-p)^3 + p^2 p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) + (1-p)2p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) + (1-p)p^2 4p(1-p)^3.$$

73. (a) 1/16, (b) 1/32, (c) 10/32, (d) 1/4, (e) 31/32.

74. Let P_A be the probability that A wins when A rolls first, and let P_B be the probability that B wins when B rolls first. Using that the sum of the dice is 9 with probability 1/9, we obtain upon conditioning on whether A rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B)$$

Similarly,

$$P_B = \frac{5}{36} + \frac{31}{36}(1 - P_A)$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)

75. (a) The probability that a family has 2 sons is 1/4; the probability that a family has exactly 1 son is 1/2. Therefore, on average, every four families will have one family with 2 sons and two families with 1 son. Therefore, three out of every four sons will be eldest sons.

Another argument is to choose a child at random. Letting E be the event that the child is an eldest son, letting S be the event that it is a son, and letting A be the event that the child's family has at least one son,

$$\begin{aligned} P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E \mid A)\frac{3}{4} + P(E \mid A^c)\frac{1}{4} \right] \\ &= 2\left[\frac{1}{2}\frac{3}{4} + 0\frac{1}{4} \right] = 3/4 \end{aligned}$$

(b) Using the preceding notation

$$\begin{aligned} P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E \mid A)\frac{7}{8} + P(E \mid A^c)\frac{1}{8} \right] \\ &= 2\left[\frac{1}{3}\frac{7}{8} \right] = 7/12 \end{aligned}$$

76. Condition on outcome of initial trial

$$\begin{aligned} P(E \text{ before } F) &= P(E \text{ b } F \mid E)P(E) + P(E \text{ b } F \mid F)P(F) \\ &\quad + P(E \text{ b } F \mid \text{neither } E \text{ or } F)[1 - P(E) - P(F)] \\ &= P(E) + P(E \text{ b } F)(1 - P(E) - P(F)]. \end{aligned}$$

Hence,

$$P(E \text{ b } F) = \frac{P(E)}{P(E) + P(F)}.$$

77. (a) This is equal to the conditional probability that the first trial results in outcome 1 (F_1) given that it results in either 1 or 2, giving the result 1/2. More formally, with L_3 being the event that outcome 3 is the last to occur

$$P(F_1 \mid L_3) = \frac{P(L_3 \mid F_1)P(F_1)}{P(L_3)} = \frac{(1/2)(1/3)}{1/3} = 1/2$$

(b) With S_1 being the event that the second trial results in outcome 1, we have

$$P(F_1S_1 \mid L_3) = \frac{P(L_3 \mid F_1S_1)P(F_1S_1)}{P(L_3)} = \frac{(1/2)(1/9)}{1/3} = 1/6$$

78. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$\begin{aligned} P(A &= P(A \mid a, a)p^2 + P(A \mid a, b)p(1-p) + P(A \mid b, a)(1-p)p + P(A \mid b, b)(1-p)^2 \\ &= p^2 + P(A)2p(1-p) \end{aligned}$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A \mid a, b) = P(A \mid b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1 - 2p(1-p)}$$

79. Each roll that is either a 7 or an even number will be a 7 with probability

$$p = \frac{P(7)}{P(7) + P(\text{even})} = \frac{1/6}{1/6 + 1/2} = 1/4$$

Hence, from Example 4*i* we see that the desired probability is

$$\sum_{i=2}^{7} \binom{7}{i} (1/4)^i (3/4)^{7-i} = 1 - (3/4)^7 - 7(3/4)^6(1/4)$$

80. (a) $P(A_i) = (1/2)^i$, if $i < n$
$= (1/2)^{n-1}$, if $i = n$

(b) $$\frac{\sum_{i=1}^{n} i(1/2)^i + n(1/2)^{n-1}}{2^n - 1} = \frac{1}{2^{n-1}}$$

(c) Condition on whether they initially play each other. This gives

$$P_n = \frac{1}{2^n - 1} + \frac{2^n - 2}{2^n - 1}\left(\frac{1}{2}\right)^2 P_{n-1}$$

where $\left(\frac{1}{2}\right)^2$ is the probability they both win given they do not play each other.

(d) There will be $2^n - 1$ losers, and thus that number of games.

(e) Since the 2 players in game *i* are equally likely to be any of the $\binom{2^n}{2}$ pairs it follows that $P(B_i) = 1 \Big/ \binom{2^n}{2}$.

(f) Since the events B_i are mutually exclusive

$$P(\cup B_i) = \sum P(B_i) = (2^n - 1) \Big/ \binom{2^n}{2} = (1/2)^{n-1}$$

81. $$\frac{1 - (9/11)^{15}}{1 - (9/11)^{30}}$$

82. (a) $P(A) = P_1^2 + \left(1 - P_1^2\right)\left[\left(1 - P_2^2\right)P(A)\right]$ or $P(A) = \dfrac{P_1^2}{P_1^2 + P_2^2 - P_1^2 P_2^2}$

(c) similar to (a) with P_i^3 replacing P_i^2.

(b) and (d) Let $P_{ij}(\overline{P}_{ij})$ denote the probability that A wins when A needs i more and B needs j more and $A(B)$ is to flip. Then

$$P_{ij} = P_1 P_{i-1,j} + (1-P_1)\overline{P}_{ij}$$
$$\overline{P}_{ij} = P_2 \overline{P}_{i,j-1} + (1-P_2)P_{ij}.$$

These equations can be recursively solved starting with

$$P_{01} = 1, P_{1,0} = 0.$$

83. (a) Condition on the coin flip

$$P\{\text{throw } n \text{ is red}\} = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}$$

(b) $$P\{r \mid rr\} = \frac{P\{rrr\}}{P\{rr\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{1}{2}\left(\frac{2^2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)^2} = \frac{3}{5}$$

(c) $$P\{A \mid rr\} = \frac{P\{rr|A\}P(A)}{P\{rr\}} = \frac{\left(\frac{2^2}{3}\right)\frac{1}{2}}{\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)^2\frac{1}{2}} = 4/5$$

84. (b) $$P(A \text{ wins}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}$$
$$P(B \text{ wins}) = \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}$$
$$P(C \text{ wins}) = \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}$$

85. Part (a) remains the same. The possibilities for part (b) become more numerous.

86. Using the hint

$$P\{A \subset B\} = \sum_{i=0}^{n} (2^i/2^n)\binom{n}{i} \Big/ 2^n = \sum_{i=0}^{n} \binom{n}{i} 2^i/4^n = (3/4)^n$$

where the final equality uses

$$\sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i} = (2+1)^n$$

(b) $P(AB = \phi) = P(A \subset B^c) = (3/4)^n$, by part (a), since B^c is also equally likely to be any of the subsets.

87. $P\{i^{\text{th}} \mid \text{all heads}\} = \dfrac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$.

88. No—they are conditionally independent given the coin selected.

89. (a) $P(J_3 \text{ votes guilty} \mid J_1 \text{ and } J_2 \text{ vote guilty}\}$

$= P\{J_1, J_2, J_3 \text{ all vote guilty}\}/P\{J_1 \text{ and } J_2 \text{ vote guilty}\}$

$$= \frac{\frac{7}{10}(.7)^3 + \frac{3}{10}(.2)^3}{\frac{7}{10}(.7)^2 + \frac{3}{10}(.2)^2} = \frac{97}{142}.$$

(b) $P(J_3 \text{ guilty} \mid \text{one of } J_1, J_2 \text{ votes guilty}\}$

$$= \frac{\frac{7}{10}(.7)2(.7)(.3) + \frac{3}{10}(2.)2(.2)(.8)}{\frac{7}{10}2(.7)(.3) + \frac{3}{10}2(.2)(.8)} = \frac{15}{26}.$$

(c) $P\{J_3 \text{ guilty} \mid J_1, J_2 \text{ vote innocent}\}$

$$= \frac{\frac{7}{10}(.7)(.3)^2 + \frac{3}{10}(.2)(.8)^2}{\frac{7}{10}(.3)^2 + \frac{3}{10}(.8)^2} = \frac{33}{102}.$$

E_i are conditionally independent given the guilt or innocence of the defendant.

90. Let N_i denote the event that none of the trials result in outcome i, $i = 1, 2$. Then

$$\begin{aligned} P(N_1 \cup N_2) &= P(N_1) + P(N_2) - P(N_1N_2) \\ &= (1 - p_1)^n + (1 - p_2)^n - (1 - p_1 - p_2)^n \end{aligned}$$

Hence, the probability that both outcomes occur at least once is
$1 - (1 - p_1)^n - (1 - p_2)^n + (p_0)^n$.

Theoretical Exercises

1. $$P(AB \mid A) = \frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)} = P(AB \mid A \cup B)$$

2. If $A \subset B$

$$P(A \mid B) = \frac{P(A)}{P(B)}, \; P(A \mid B^c) = 0, \qquad P(B \mid A) = 1, \qquad P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}$$

3. Let F be the event that a first born is chosen. Also, let S_i be the event that the family chosen in method a is of size i.

$$P_a(F) = \sum_i P(F \mid S_i) P(S_i) = \sum_i \frac{1}{i} \frac{n_i}{m}$$

$$P_b(F) = \frac{m}{\sum_i i n_i}$$

Thus, we must show that

$$\sum_i i n_i \sum_i n_i / i \geq m^2$$

or, equivalently,

$$\sum_i i n_i \sum_j n_j / j \geq \sum_i n_i \sum_j n_j$$

or,

$$\sum \sum_{i \neq j} \frac{i}{j} n_i n_j \geq \sum \sum_{i \neq j} n_i n_j$$

Considering the coefficients of the term $n_i n_j$, shows that it is sufficient to establish that

$$\frac{i}{j} + \frac{j}{i} \geq 2$$

or equivalently

$$i^2 + j^2 \geq 2ij$$

which follows since $(i - j)^2 \geq 0$.

4. Let N_i denote the event that the ball is not found in a search of box i, and let B_j denote the event that it is in box j.

$$P(B_j \mid N_i) = \frac{P(N_i \mid B_j)P(B_j)}{P(N_i \mid B_i)P(B_i) + P(N_i \mid B_i^c)P(B_i^c)}$$
$$= \frac{P_j}{(1-\alpha_i)P_i + 1 - P_i} \text{ if } j \neq i$$
$$= \frac{(1-\alpha_i)P_i}{(1-\alpha_i)P_i + 1 - P_i} \text{ if } j = i$$

5. None are true.

6. $$P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n E_i^c\right) = 1 - \prod_1^n [1 - P(E_i)]$$

7. (a) They will all be white if the last ball withdrawn from the urn (when all balls are withdrawn) is white. As it is equally likely to by any of the $n + m$ balls the result follows.

(b) $P(RBG) = \frac{g}{r+b+g} P(RBG \mid G \text{ last}) = \frac{g}{r+b+g}\frac{b}{r+b}$.

Hence, the answer is $\frac{bg}{(r+b)(r+b+g)} + \frac{b}{r+b+g}\frac{g}{r+g}$.

8. (a) $P(A) = P(A \mid C)P(C) + P(A \mid C^c)P(C^c) > P(B \mid C)P(C) + P(B \mid C^c)P(C^c) = P(B)$

(b) For the events given in the hint

$$P(A \mid C) = \frac{P(C \mid A)P(A)}{3/36} = \frac{(1/6)(1/6)}{3/36} = 1/3$$

Because 1/6 = $P(A$ is a weighted average of $P(A \mid C)$ and $P(A \mid C^c)$, it follows from the result $P(A \mid C) > P(A)$ that $P(A \mid C^c) < P(A)$. Similarly,

$$1/3 = P(B \mid C) > P(B) > P(B \mid C^c)$$

However, $P(AB \mid C) = 0 < P(AB \mid C^c)$.

9. $P(A) = P(B) = P(C) = 1/2$, $P(AB) = P(AC) = P(BC) = 1/4$. But, $P(ABC) = 1/4$.

10. $P(A_{i,j}) = 1/365$. For $i \neq j \neq k$, $P(A_{i,j}A_{j,k}) = 365/(365)^3 = 1/(365)^2$. Also, for $i \neq j \neq k \neq r$, $P(A_{i,j}A_{k,r}) = 1/(365)^2$.

11. $1 - (1-p)^n \geq 1/2$, or, $n \geq -\frac{\log(2)}{\log(1-p)}$

12. $a_i \prod_{j=1}^{i-1}(1-a_j)$ is the probability that the first head appears on the i^{th} flip and $\prod_{i=1}^{\infty}(1-a_i)$ is the probability that all flips land on tails.

13. Condition on the initial flip. If it lands on heads then A will win with probability $P_{n-1,m}$ whereas if it lands tails then B will win with probability $P_{m,n}$ (and so A will win with probability $1 - P_{m,n}$).

14. Let N go to infinity in Example 4*j*.

15. $P\{r \text{ successes before } m \text{ failures}\}$
$= P\{r^{\text{th}} \text{ success occurs before trial } m + r\}$
$= \sum_{n=r}^{m+r-1} \binom{n-1}{r-1} p^r (1-p)^{n-r}$.

16. If the first trial is a success, then the remaining $n - 1$ must result in an odd number of successes, whereas if it is a failure, then the remaining $n - 1$ must result in an even number of successes.

17. $P_1 = 1/3$
$P_2 = (1/3)(4/5) + (2/3)(1/5) = 2/5$
$P_3 = (1/3)(4/5)(6/7) + (2/3)(4/5)(1/7) + (1/3)(1/5)(1/7) = 3/7$
$P_4 = 4/9$

(b) $P_n = \dfrac{n}{2n+1}$

(c) Condition on the result of trial n to obtain

$$P_n = (1 - P_{n-1})\frac{1}{2n+1} + P_{n-1}\frac{2n}{2n+1}$$

(d) Must show that

$$\frac{n}{2n+1} = \left[1 - \frac{n-1}{2n-1}\right]\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

or equivalently, that

$$\frac{n}{2n+1} = \frac{n}{2n-1}\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

But the right hand side is equal to

$$\frac{n + 2n(n-1)}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

18. Condition on when the first tail occurs.

19. $P_{n,i} = p^{P}_{n-1,i+1} + (1-p)P_{n-1,i-1}$

20. $\alpha_{n+1} = \alpha_n p + (1-\alpha_n)(1-p^1)$
$P_n = \alpha_n p + (1-\alpha_n)p^1$

21. (b) $P_{n,1} = P\{A \text{ receives first 2 votes}\} = \dfrac{n(n-1)}{(n+1)n} = \dfrac{n-1}{n+1}$

$P_{n,2} = P\{A \text{ receives first 2 and at least 1 of the next 2}\}$

$$= \frac{n}{n+2}\frac{n-1}{n+1}\left\{1 - \frac{2\cdot 1}{n(n-1)}\right\} = \frac{n-2}{n+2}$$

(c) $P_{n,m} = \dfrac{n-m}{n+m}$, $n \geq m$.

(d) $P_{n,m} = P\{A \text{ always ahead}\}$

$= P\{A \text{ always} \mid A \text{ receives last vote}\} \ \dfrac{n}{n+m}$

$\quad + P\{A \text{ always} \mid B \text{ receives last vote}\} \ \dfrac{m}{n+m}$

$$= \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1}$$

(e) The conjecture of (c) is true when $n + m = 1$ ($n = 1$, $m = 0$).
Assume it when $n + m = k$. Now suppose that $n + m = k + 1$. By (d) and the induction hypothesis we have that

$$P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

which completes the proof.

22. $P_n = P_{n-1}p + (1-P_{n-1})(1-p)$

$= (2p-1)P_{n-1} + (1-p)$

$= (2p-1)\left[\dfrac{1}{2} + \dfrac{1}{2}(2p-1)^{n-1}\right] + 1 - p$ by the induction hypothesis

$$= \frac{2p-1}{2} + \frac{1}{2}(2p-1)^n + 1 - p$$

$$= \frac{1}{2} + \frac{1}{2}(2p-1)^n .$$

23. $P_{1,1} = 1/2$. Assume that $P_{a,b} = 1/2$ when $k \geq a + b$ and now suppose $a+ b = k + 1$. Now

$$P_{a,b} = P\{\text{last is white} \mid \text{first } a \text{ are white}\} \frac{1}{\binom{a+b}{a}}$$

$$+ P\{\text{last is white} \mid \text{first } b \text{ are black}\} \frac{1}{\binom{b+a}{b}}$$

$$+ P\{\text{last is white} \mid \text{neither first } a \text{ are white nor first } b \text{ are black}\}$$

$$\left[1 - \frac{1}{\binom{a+b}{a}} - \frac{1}{\binom{b+a}{b}}\right] = \frac{a!b!}{(a+b)!} + \frac{1}{2}\left[1 - \frac{a!b!}{(a+b)!} - \frac{a!b!}{(a+b)!}\right] = \frac{1}{2}$$

where the induction hypothesis was used to obtain the final conditional probability above.

24. The probability that a given contestant does not beat all the members of some given subset of k other contestants is, by independence, $1 - (1/2)^k$. Therefore $P(B_i)$, the probability that none of the other $n - k$ contestants beats all the members of a given subset of k contestants, is $[1 - (1/2)^k]^{n-k}$. Hence, Boole's inequality we have that

$$P(\cup B_i) \leq \binom{n}{k}[1 - (1/2)^k]^{n-k}$$

Hence, if $\binom{n}{k}[1 - (1/2)^k]^{n-k} < 1$ then there is a positive probability that none of the $\binom{n}{k}$ events B_i occur, which means that there is a positive probability that for every set of k contestants there is a contestant who beats each member of this set.

25. $P(E \mid F) = \mathrm{P}(EF)/P(F)$

$$P(E \mid FG)P(G \mid F) = \frac{P(EFG)}{P(FG)}\frac{P(FG)}{P(F)} = \frac{P(EFG)}{P(F)}$$

$$P(E \mid FG^c)P(G^c \mid F) = \frac{P(EFG^c)}{P(F)}.$$

The result now follows since

$$P(EF) = P(EFG) + P(EFG^c)$$

27. $E_1, E_2, \ldots, E_n$ are conditionally independent given F if for all subsets $i_1, \ldots, i_r$ of $1, 2, \ldots, n$

$$P\left(E_{i_1} \ldots E_{i_r} \mid F\right) = \prod_{j=1}^{r} P\left(E_{i_j} \mid F\right).$$

28. Not true. Let $F = E_1$.

29. $P\{\text{next } m \text{ heads} \mid \text{first } n \text{ heads}\}$
$= P\{\text{first } n + m \text{ are heads}\}/P(\text{first } n \text{ heads}\}$
$$= \int_0^1 p^{n+m}dp \Big/ \int_0^1 p^n dp = \frac{n+1}{n+m+1}.$$

Chapter 4

Problems

1. $P\{X=4\} = \dfrac{\binom{4}{2}}{\binom{14}{2}} = \dfrac{6}{91}$ $\quad$ $\mathrm{P}\{X=0\} = \dfrac{\binom{2}{2}}{\binom{14}{2}} = \dfrac{1}{91}$

$P\{X=2\} = \dfrac{\binom{4}{2}\binom{2}{1}}{\binom{14}{2}} = \dfrac{8}{91}$ $\quad$ $P\{X=-1\} = \dfrac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \dfrac{16}{91}$

$P\{X=1\} = \dfrac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \dfrac{32}{91}$ $\quad$ $P\{X=-2\} = \dfrac{\binom{8}{2}}{\binom{14}{2}} = \dfrac{28}{91}$

2. $p(1) = 1/36$ $\quad$ $p(5) = 2/36$ $\quad$ $p(9) = 1/36$ $\quad$ $p(15) = 2/36$ $\quad$ $p(24) = 2/36$
$p(2) = 2/36$ $\quad$ $p(6) = 4/36$ $\quad$ $p(10) = 2/36$ $\quad$ $p(16) = 1/36$ $\quad$ $p(25) = 1/36$
$p(3) = 2/36$ $\quad$ $p(7) = 0$ $\quad$ $p(11) = 0$ $\quad$ $p(18) = 2/36$ $\quad$ $p(30) = 2/36$
$p(4) = 3/36$ $\quad$ $p(8) = 2/36$ $\quad$ $p(12) = 4/36$ $\quad$ $p(20) = 2/36$ $\quad$ $p(36) = 1/36$

4. $P\{X=1\} = 1/2$, $P\{X=2\} = \dfrac{5}{10}\dfrac{5}{9} = \dfrac{5}{18}$, $P\{X=3\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{5}{8} = \dfrac{5}{36}$,
$P\{X=4\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{3}{8}\dfrac{5}{7} = \dfrac{10}{168}$, $P\{X=5\} = \dfrac{5\cdot4\cdot3\cdot2}{10\cdot9\cdot8\cdot7}\dfrac{5}{6} = \dfrac{5}{252}$,
$P\{X=6\} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6} = \dfrac{1}{252}$

5. $n-2i$, $i = 0, 1, \ldots, n$

6. $P(X=3\} = 1/8$, $P\{X=1\} = 3/8$, $P\{X=-1\} = 3/8$, $P\{X=-3\} = 1/8$

8. (a) $p(6) = 1 - (5/6)^2 = 11/36$, $p(5) = 2\ 1/6\ \ 4/6 + (1/6)^2 = 9/36$
$p(4) = 2\ 1/6\ \ 3/6 + (1/6)^2 = 7/36$, $p(3) = 2\ 1/6\ \ 2/6 + (1/6)^2 = 5/36$
$p(2) = 2\ 1/6\ \ 1/6 + (1/6)^2 = 3/36$, $p(1) = 1/36$

(d) $p(5) = 1/36$, $p(4) = 2/36$, $p(3) = 3/36$, $p(2) = 4/36$, $p(1) = 5/36$
$p(0) = 6/36$, $p(-j) = p(j)$, $j > 0$

11. (a) $P\{\text{divisible by 3}\} = \dfrac{333}{1000}$ $P\{\text{divisible by 105}\} = \dfrac{9}{1000}$

$P\{\text{divisible by 7}\} = \dfrac{142}{1000}$

$P\{\text{divisible by 15}\} = \dfrac{66}{1000}$

In limiting cases, probabilities converge to 1/3, 1/7, 1/15, 1/10

(b) $P\{\mu(N) \neq 0\} = P\{N \text{ is not divisible by } p_i^2, i \geq 1\}$

$= \prod_i P\{N \text{ is not divisible by } p_i^2\}$

$= \prod_i (1 - 1/p_i^2) = 6/\pi^2$

13. $p(0) = P\{\text{no sale on first and no sale on second}\}$
$= (.7)(.4) = .28$

$p(500) = P\{1 \text{ sale and it is for standard}\}$
$= P\{1 \text{ sale}\}/2$
$= [P\{\text{sale, no sale}\} + P\{\text{no sale, sale}\}]/2$
$= [(.3)(.4) + (.7)(.6)]/2 = .27$

$p(1000) = P\{2 \text{ standard sales}\} + P\{1 \text{ sale for deluxe}\}$
$= (.3)(.6)(1/4) + P\{1 \text{ sale}\}/2$
$= .045 + .27 = .315$

$p(1500) = P\{2 \text{ sales, one deluxe and one standard}\}$
$= (.3)(.6)(1/2) = .09$

$p(2000) = P\{2 \text{ sales, both deluxe}\} = (.3)(.6)(1/4) = .045$

14. $P\{X = 0\} = P\{1 \text{ loses to } 2\} = 1/2$

$P\{X = 1\} = P\{\text{of 1, 2, 3: 3 has largest, then 1, then 2}\}$
$= (1/3)(1/2) = 1/6$

$P\{X = 2\} = P\{\text{of 1, 2, 3, 4: 4 has largest and 1 has next largest}\}$
$= (1/4)(1/3) = 1/12$

$P\{X = 3\} = P\{\text{of 1, 2, 3, 4, 5: 5 has largest then 1}\}$
$= (1/5)(1/4) = 1/20$

$P\{X = 4\} = P\{1 \text{ has largest}\} = 1/5$

15. $P\{X = 1\} = 11/66$

$$P\{X = 2\} = \sum_{j=2}^{11} \left(\frac{12-j}{66}\right)\left(\frac{11}{54+j}\right)$$

$$P\{X = 3\} = \sum_{\substack{k\neq 1\\ k\neq j}} \sum_{j=2} \left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+j+k}\right)$$

$$P\{X = 4\} = 1 - \sum_{i=1}^{3} P\{X = 1\}$$

16. $P\{Y_1 = i\} = \dfrac{12-i}{66}$

$$P\{Y_2 = i\} = \sum_{j\neq i} \left(\frac{12-j}{66}\right)\left(\frac{12-i}{54+j}\right)$$

$$P\{Y_3 = i\} = \sum_{\substack{k\neq j\\ k\neq i}} \sum_{j\neq i} \left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+k+j}\right)$$

All sums go from 1 to 11, except for prohibited values.

20. (a) $P\{x > 0\} = P\{\text{win first bet}\} + P\{\text{lose, win, win}\}$
$= 18/38 + (20/38)(18/38)^2 \approx .5918$

(b) No, because if the gambler wins then he or she wins \$1.
However, a loss would either be \$1 or \$3.

(c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148$, $i = 40, 33, 25, 50$

$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$
$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

22. Let N denote the number of games played.

(a) $E(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$

The final equality could also have been obtained by using that $N = 2 +]$ where I is 0 if two games are played and 1 if three are played. Differentiation yields that

$$\frac{d}{dp}E[N] = 2 - 4p$$

and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$.

(b) $E[N] = 3[p^3 + (1-p)^3 + 4[3p^2(1-p)p + 3p(1-p)^2(1-p)]$
$+ 5[6p^2(1-p)^2 = 6p^4 - 12p^3 + 3p^2 + 3p + 3$

Differentiation yields

$$\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

23. (a) Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is

$$E[\text{money}] = \frac{1}{2}500 + \frac{1}{2}2000 = 1250$$

(b) Do not immediately buy but use your money to buy after one week. Then

$$E[\text{ounces of commodity}] = \frac{1}{2}1000 + \frac{1}{2}250 = 625$$

24. (a) $p - (1-p)\frac{3}{4} = \frac{7}{4}p - 3/4$, (b) $-\frac{3}{4}p + (1-p)2 = -\frac{11}{4}p + 2$

$\frac{7}{4}p - 3/4 = -\frac{11}{4}p + 2 \Rightarrow p = 11/18$, maximum value = 23.72

(c) $q - \frac{3}{4}(1-q)$ (d) $-\frac{3}{4}q + 2(1-q)$, minimax value = 23/72
attained when $q = 11/18$

25. (a) $\frac{1}{10}(1 + 2 + ... + 10) = \frac{11}{2}$

(b) after 2 questions, there are 3 remaining possibilities with probability 3/5 and 2 with probability 2/5. Hence.

$$E[\text{Number}] = \frac{2}{5}(3) + \frac{3}{5}\left[2 + \frac{1}{3} + 2\frac{2}{3}\right] = \frac{17}{5}.$$

The above assumes that when 3 remain, you choose 1 of the 3 and ask if that is the one.

27. $C - Ap = \frac{A}{10} \Rightarrow C = A\left(p + \frac{1}{10}\right)$

28. $3 \cdot \frac{4}{20} = 3/5$

29. If check 1, then (if desired) 2: Expected Cost = $C_1 + (1-p)C_2 + pR_1 + (1-p)R_2$;
if check 2, then 1: Expected Cost = $C_2 + pC_1 + pR_1 + (1-p)R_2$ so 1, 2, best if
$C_1 + (1-p)C_2 \le C_2 + pC_1$, or $C_1 \le \dfrac{p}{1-p}C_2$

30. $E[X] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \infty$

(a) probably not

(b) yes, if you could play an arbitrarily large number of games

31. $E[\text{score}] = p^*[1-(1-P)^2 + (1-p^*)(1-p^2)$

$$\frac{d}{dp} = 2(1-p)p^* - 2p(1-p^*)$$
$$= 0 \Rightarrow p = p^*$$

32. If T is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1-(.9)^{10}] = 11 - 10(.9)^{10}$$

35. If X is the amount that you win, then

$$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$$
$$E[X] = (1.1)4/9 - 5/9 = -.6/9 \approx = -.067$$
$$\text{Var}(X) = (1.1)^2(4/9) + 5/9 - (.6/9)^2 \approx 1.089$$

36. Using the representation

$$N = 2 + I$$

where I is 0 if the first two games are won by the same team and 1 otherwise, we have that

$$\text{Var}(N) = \text{Var}(I) = E[I]^2 - E^2[I]$$

Now, $E[I]^2 = E[I\} = P\{I = 1\} = 2p\{1-p\}$ and so
$\text{Var}(N) = 2p(1-p)[1-2p(1-p)] = 8p^3 - 4p^4 - 6p^2 + 2p$

Differentiation yields

$$\frac{d}{dp}\text{Var}(N) = 24p^2 - 16p^3 - 12p + 2$$

and it is easy to verify that this is equal to 0 when $p = 1/2$.

37. $E[X^2] = [(40)^3 + (33)^3 + (25)^3 + (50)^3]/148 \approx 1625.4$

$\text{Var}(X = E[X^2] - (E[X])^2 \approx 82.2$

$E[Y^2] = = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/4 = 1453.5, \qquad \text{Varr}(Y) = 84.5$

38. (a) $E[(2 + X)^2] = \text{Var}(2 + X) + (E[2 + X])^2 = \text{Var}(X) + 9 = 14$

(b) $\text{Var}(4 + 3X) = 9\ \text{Var}(X) = 45$

39. $\binom{4}{2}(1/2)^4 = 3/8$ 40. $\binom{5}{4}(1/3)^4(2/3)^1 + (1/3)^5 = 11/243$

41. $\sum_{i=7}^{10}\binom{10}{i}(1/2)^{10}$

42. $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 \geq \binom{3}{2}p^2(1-p) + p^3$

$\Leftrightarrow 6p^3 - 15p^2 + 12p - 3 \geq 0$

$\Leftrightarrow 6(p - 1/2)(p - 1)^2 \geq 0$

$\Leftrightarrow p \geq 1/2$

43. $\binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8) + (.2)^5$

44. $\alpha\sum_{i=k}^{n}\binom{n}{i}p_1^i(1-p_1)^{n-i} + (1-\alpha)\sum_{i=k}^{n}\binom{n}{i}p_2^i(1-p_2)^{n-i}$

45. with 3: $P\{\text{pass}\} = \frac{1}{3}\left[\binom{3}{2}(.8)^2(.2) + (.8)^3\right] + \frac{2}{3}\left[\binom{3}{2}(.4)^2(.6) + (.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i} + \frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$

47. (a) and (b): (i) $\sum_{i=5}^{9}\binom{9}{i}p^i(1-p)^{9-i}$, (ii) $\sum_{i=5}^{8}\binom{8}{i}p^i(1-p)^{8-i}$,

(iii) $\sum_{i=4}^{7}\binom{7}{i}p^i(1-p)^{7-i}$ where $p = .7$ in (a) and $p = .3$ in (b).

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

49. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3 + \frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $$\frac{\frac{1}{2}\binom{9}{6}.4^7.6^3 + \frac{1}{2}.7^7.3^3}{.55}$$

50. (a) $P\{H, T, T \mid 6 \text{ heads}\}$
$= P(H, T, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}$
$= P\{H, T, T\}P\{6 \text{ heads} \mid H, T, T\}/P\{6 \text{ heads}\}$
$= pq^2\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4$
$=1/10$

(b) $P\{T, H, T \mid 6 \text{ heads}\}$
$= P(T, H, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}$
$= P\{T, H, T\}P\{6 \text{ heads} \mid T, H, T\}/P\{6 \text{ heads}\}$
$= q^2p\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4$
$=1/10$

51. (a) $e^{-.2}$ (b) $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$
Since each letter has a small probability of being a typo, the number of errors should approximately have a Poisson distribution.

52. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$

(b) $4.5e^{-3.5}$

Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

54. (a) $e^{-2.2}$ (b) $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$

55. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2}$

56. The number of people in a random collection of size n that have the same birthday as yourself is approximately Poisson distributed with mean $n/365$. Hence, the probability that at least one person has the same birthday as you is approximately $1 - e^{-n/365}$. Now, $e^{-x} = 1/2$ when $x = \log(2)$. Thus, $1 - e^{-n/365} \geq 1/2$ when $n/365 \geq \log(2)$. That is, there must be at least $365 \log(2)$ people.

57. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3}$

(b) $P\{X \geq 3 \mid X \geq 1\} = \frac{P\{X \geq 3\}}{P\{X \geq 1\}} = \frac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}}$

59. (a) $1 - e^{-1/2}$

(b) $\frac{1}{2}e^{-1/2}$

(c) $1 - e^{-1/2} = \frac{1}{2}e^{-1/2}$

60. $P\{\text{beneficial} \mid 2\} = \frac{P\{2 \mid \text{beneficial}\}3/4}{P\{2 \mid \text{beneficial}\}3/4 + P\{2 \mid \text{not beneficial}\}1/4}$

$$= \frac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4} + e^{-5}\frac{5^2}{2}\frac{1}{4}}$$

61. $1 - e^{-1.4} - 1.4e^{-1.4}$

62. If A_i is the event that couple number i are seated next to each other, then these events are, when n is large, roughly independent. As $P(A_i = 2/(2n - 1)$ it follows that, for n large, the number of wives that sit next to their husbands is approximately Poisson with mean $2n/(2n - 1) \approx 1$. Hence, the desired probability is $e^{-1} = .368$ which is not particularly close to the exact solution of .2656 provided in Example 5n of Chapter 2, thus indicating that $n = 10$ is not large enough for the approximation to be a good one.

63. (a) $e^{-2.5}$

(b) $1 - e^{-2.5} - 2.5e^{-2.5} - \frac{(2.5)^2}{2}e^{-2.5} - \frac{(2.5)^3}{3!}e^{-2.5}$

64. (a) $1 - \sum_{i=0}^{7} e^{-4} 4^i / i! \equiv p$

(b) $1 - (1-p)^{12} - 12p(1-p)^{11}$

(c) $(1-p)^{i-1}p$

65. (a) $1 - e^{-1/2}$

(b) $P\{X \geq 2 \,|\, X \geq 1\} = \dfrac{1 - e^{-1/2} - \frac{1}{2}e^{-1/2}}{1 - e^{-1/2}}$

(c) $1 - e^{-1/2}$

(d) $1 - \exp\{-500 - i)/1000\}$

66. Assume $n > 1$.

(a) $\dfrac{2}{2n-1}$

(b) $\dfrac{2}{2n-2}$

(c) $\exp\{-2n/(2n-1)\} \approx \mathrm{e}^{-1}$

67. Assume $n > 1$.

(a) $\dfrac{2}{n}$

(b) Conditioning on whether the man of couple j sits next to the woman of couple i gives the result: $\dfrac{1}{n-1}\dfrac{1}{n-1} + \dfrac{n-2}{n-1}\dfrac{2}{n-1} = \dfrac{2n-3}{(n-1)^2}$

(c) e^{-2}

68. $\exp(-10e^{-5}\}$

69. With P_j equal to the probability that 4 consecutive heads occur within j flips of a fair coin, $P_1 = P_2 = P + 3 = 0$, and

$P_4 = 1/16$
$P_5 = (1/2)P_4 + 1/16 = 3/32$
$P_6 = (1/2)P_5 + (1/4)P_4 + 1/16 = 1/8$
$P_7 = (1/2)P_6 + (1/4)P_5 + (1/8)P_4 + 1/16 = 5/32$
$P_8 = (1/2)P_7 + (1/4)P_6 + (1/8)P_5 + (1/16)P_4 + 1/16 = 6/32$
$P_9 = (1/2)P_8 + (1/4)P_7 + (1/8)P_6 + (1/16)P_5 + 1/16 = 111/512$
$P_{10} = (1/2)P_9 + (1/4)P_8 + (1/8)P_7 + (1/16)P_6 + 1/16 = 251/1024 = .2451$

The Poisson approximation gives

$$P_{10} \approx 1 - \exp\{-6/32 - 1/16\} = 1 - e^{-.25} = .2212$$

70. $e^{-\lambda t} + (1 - e^{-\lambda t})p$

71. (a) $\left(\frac{26}{38}\right)^5$

(b) $\left(\frac{26}{38}\right)^3 \frac{12}{38}$

72. $P\{\text{wins in } i \text{ games}\} = \binom{i-1}{3}(.6)^4(.4)^{i-4}$

73. Let N be the number of games played. Then

$P\{N=4\} = 2(1/2)^4 = 1/8, \qquad P\{N=5\} = 2\binom{4}{1}(1/2)(1/2)^4 = 1/4$

$P\{N=6\} = 2\binom{5}{2}(1/2)^2(1/2)^4 = 5/16, \qquad P\{N=7\} = 5/16$

$E[N] = 4/8 + 5/4 + 30/16 + 35.16 = 93/16 = 5.8125$

74. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

76. $\binom{N_1+N_2-k}{N_1}(1/2)^{N_1+N_2-k}(1/2) + \binom{N_1+N_2-k}{N_2}(1/2)^{N_1+N_2-k}(1/2)$

77. $2\binom{2N-k}{N}(1/2)^{2N-k}$

$2\binom{2N-k-1}{N-1}(1/2)^{2N-k-1}(1/2)$

79. (a) $P\{X=0\} = \dfrac{\binom{94}{10}}{\binom{100}{10}}$

(b) $P\{X>2\} = 1 - \dfrac{\binom{94}{10}+\binom{94}{9}\binom{6}{1}+\binom{94}{8}\binom{6}{2}}{\binom{100}{10}}$

80. $P\{\text{rejected} \mid 1 \text{ defective}\} = 3/10$

$P\{\text{rejected} \mid 4 \text{ defective}\} = 1 - \binom{6}{3}\Big/\binom{10}{3} = 5/6$

$P\{4 \text{ defective} \mid \text{rejected}\} = \dfrac{\frac{5}{6}\frac{3}{10}}{\frac{5}{6}\frac{3}{10}+\frac{3}{10}\frac{7}{10}} = 75/138$

81. $P\{\text{rejected}\} = 1 - (.9)^4$

Theoretical Exercises

1. Let E_i = {no type i in first n selections}

$$P\{T > n\} = P\left(\bigcup_{i=1}^{N} E_i\right)$$
$$= \sum_i (1-P_i)^n - \sum\sum_{I<J} (1-P_i-P_j)^n + \sum\sum\sum_{i<j<k} (1-p_{_J}-p_k)^n$$
$$\ldots + (-1)^N \sum_i P_i^n$$

$$P\{T = n\} = P\{\mathrm{T} > n-1\} - P\{T > n\}$$

3. $1 - \lim_{h\to 0} F(a-h)$

4. Not true. Suppose $P\{X = b\} = \varepsilon > 0$ and $b_n = b + 1/n$. Then $\lim_{b_n \to b} P(X < b_n\} = P\{X \le b\} \neq P\{X < b\}$.

5. When $\alpha > 0$

$$P\{\alpha X + \beta \le x\} = P\left\{x \le \frac{x-\beta}{\alpha}\right\} = F\left(\frac{x-\beta}{\alpha}\right)$$

When $\alpha < 0$

$$P\{\alpha X + \beta \le x\} = P\left\{X \ge \frac{x-\beta}{\alpha}\right\} = 1 - \lim_{h\to 0^+} F\left(\frac{x-\beta}{\alpha} - 1\right).$$

6. $$\sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} P\{N = K\}$$
$$= \sum_{k=1}^{\infty} kP\{N = k\} = E[N].$$

7. $$\sum_{i=0}^{\infty} i\, P\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty} P\{N = k\} \sum_{i=0}^{k-1} i$$
$$= \sum_{k=1}^{\infty} P\{N = k\}(k-1)k/2$$
$$= \left(\sum_{k=1}^{\infty} k^2 P\{N = k\} - \sum_{k=1}^{\infty} kP\{N = k\}\right) \Big/ 2$$

8. $E[c^X] = cp + c^{-1}(1-p)$

Hence, $1 = E[c^X]$ if

$cp + c^{-1}(1-p) = 1$

or, equivalently

$pc^2 - c + 1 - p = 0$

or

$(pc - 1 + p)(c - 1) = 0$

Thus, $c = (1-p)/p$.

9. $E[Y] = E[X/\sigma - \mu/\sigma] = \frac{1}{\sigma}E[X] - \mu/\sigma = \mu/\sigma - \mu/\sigma = 0$

$\text{Var}(Y) = (1/\sigma)^2 \text{Var}(X) = \sigma^2/\sigma^2 = 1.$

10. $$\begin{aligned} E[1/(X+1)] &= \sum_{i=0}^{n} \frac{1}{i+1}\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i} \\ &= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!}p^i(1-p)^{n-i} \\ &= \frac{1}{(n+1)p}\sum_{i=0}^{n}\binom{n+1}{i+1}p^{i+1}(1-p)^{n-i} \\ &= \frac{1}{(n+1)p}\sum_{j=1}^{n+1}\binom{n+1}{j}p^j(1-p)^{n+1-j} \\ &= \frac{1}{(n+1)p}\left[1-\binom{n+1}{0}p^0(1-p)^{n+1-0}\right] \\ &= \frac{1}{(n+1)p}[1-(1-p)^{n+1}] \end{aligned}$$

11. For any given arrangement of k successes and $n-k$ failures:

$$P\{\text{arrangement} \mid \text{total of } k \text{ successes}\}$$

$$= \frac{P\{\text{arrangement}\}}{P\{k \text{ successes}\}} = \frac{p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

12. Condition on the number of functioning components and then use the results of Example 4c of Chapter 1:

$$\text{Prob} = \sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}\left[\binom{i+1}{n-i}\Big/\binom{n}{i}\right]$$

where $\binom{i+1}{n-i} = 0$ if $n - i > i + 1$. We are using the results of Exercise 11.

13. Easiest to first take log and then determine the p that maximizes $\log P\{X = k\}$.

$$\log P\{X = k\} = \log\binom{n}{k} + k \log p + (n - k) \log (1 - p)$$

$$\frac{\partial}{\partial p} \log P\{x = k\} = \frac{k}{p} - \frac{n-k}{1-p}$$
$$= 0 \Rightarrow p = k/n \text{ maximizes}$$

14. (a) $1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \frac{\alpha p}{1-p}$

(b) Condition on the number of children: For $k > 0$

$$P\{k \text{ boys}\} = \sum_{n=1}^{\infty} P\{k \mid n \text{ children}\} \alpha p^n$$
$$= \sum_{n=k}^{\infty} \binom{n}{k} (1/2)^n \alpha p^n$$

$$P\{0 \text{ boys}\} = 1 - \frac{\alpha p}{1-p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n$$

17. (a) If X is binomial (n, p) then, from exercise 15,

$$P\{X \text{ is even}\} = [1 + (1 - 2p)^n]/2$$
$$= [1 + (1 - 2\lambda/n)^n]/2 \text{ when } \lambda = np$$
$$\to (1 + e^{-2\lambda})/2 \text{ as } n \text{ approaches infinity}$$

(b) $P\{X \text{ is even}\} = e^{-\lambda} \sum_n \lambda^{2n}/(2n)! = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2$

18. $\log P\{X = k\} = -\lambda + k \log \lambda - \log (k!)$

$$\frac{\partial}{\partial \lambda} \log P\{X = k\} = -1 + \frac{k}{\lambda}$$
$$= 0 \Rightarrow \lambda = k$$

19. $E[X^n] = \sum_{i=0}^{\infty} i^n e^{-\lambda}\lambda^i / i!$

$= \sum_{i=1}^{\infty} i^n e^{-\lambda}\lambda^i / i!$

$= \sum_{i=1}^{\infty} i^{n-1} e^{-\lambda}\lambda^i /(i-1)!$

$= \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j+1} / j!$

$= \lambda\sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^j / j!$

$= \lambda E[(X+1)^{n-1}]$

Hence $[X^3] = \lambda E(X+1)^2]$

$= \lambda\sum_{i=0}^{\infty} (i+1)^2 e^{-\lambda}\lambda^i / i!$

$= \lambda\left[\sum_{i=0}^{\infty} i^2 e^{-\lambda}\lambda^i / i! + 2\sum_{i=0}^{\infty} i e^{-\lambda}\lambda^i / i! + \sum_{i=0}^{\infty} e^{-\lambda}\lambda^i / i!\right]$

$= \lambda[E[X^2] + 2E[X] + 1)$

$= \lambda(\text{Var}(X) = E^2[X] + 2E[X] + 1)$

$= \lambda(\lambda + \lambda^2 + 2\lambda + 1) = \lambda(\lambda^2 + 3\lambda + 1)$

20. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a Poisson random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S > 0$,

$$P\{X = 1\} = P\{S = 1 \mid S > 0\} = \frac{P\{S = 1\}}{P\{S > 0\}} \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

21. (i) 1/365
(ii) 1/365
(iii) 1 The events, though independent in pairs, are not independent.

22. (i) Say that trial i is a success if the i^{th} pair selected have the same number. When n is large trials 1, …, k are roughly independent.
(ii) Since, $P\{\text{trial } i \text{ is a success}\} = 1/(2n - 1)$ it follows that, when n is large, M_k is approximately Poisson distributed with mean $k/(2n - 1)$. Hence,

$$P\{M_k = 0\} \approx \exp[-k/(2n - 1)]$$

(iii) and (iv) $P\{T > \alpha n\} = P\{M_{\alpha n} = 0\} \approx \exp[-\alpha n/(2n - 1)] \to e^{-\alpha/2}$

23. (a) $P(E_i) = 1 - \sum_{j=0}^{2} \binom{365}{j} (1/365)^j (364/365)^{365-j}$

(b) $\exp(-365P(E_1))\}$

24. (a) There will be a string of k consecutive heads within the first n trials either if there is one within the first $n-1$ trials, or if the first such string occurs at trial n; the latter case is equivalent to the conditions of 2.

(b) Because cases 1 and 2 are mutually exclusive

$$P_n = P_{n-1} + (1 - P_{n-k-1})(1-P)p^k$$

25. $$\begin{aligned} P(m \text{ counted}) &= \sum_n P(m \mid n \text{ events}) e^{-\lambda}\lambda^n / n! \\ &= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda}\lambda^n / n! \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{n=m}^{\infty} \frac{[\lambda(1-p)]^{n-m}}{(n-m)!} e^{-\lambda(1-p)} \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \end{aligned}$$

Intuitively, the Poisson λ random variable arises as the approximate number of successes in n (large) independent trials each having a small success probability α (and $\lambda \approx n\alpha$). Now if each successful trial is counted with probability p, than the number counted is Binomial with parameters n (large) and αp (small) which is approximately Poisson with parameter $\alpha pn = \lambda p$.

27. $$\begin{aligned} P\{X = n+k \mid X > n\} &= \frac{P\{X = n+k\}}{P\{X > n\}} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^n} \\ &= p(1-p)^{k-1} \end{aligned}$$

If the first n trials are fall failures, then it is as if we are beginning anew at that time.

28. The events $\{X > n\}$ and $\{Y < r\}$ are both equivalent to the event that there are fewer than r successes in the first n trials; hence, they are the same event.

29. $$\begin{aligned} \frac{P\{X = k+1\}}{P\{X = k\}} &= \frac{\binom{Np}{k+1}\binom{N-np}{n-k-1}}{\binom{Np}{k}\binom{N-Np}{n-k}} \\ &= \frac{(Np-k)(n-k)}{(k+1)(N-Np-n+k+1)} \end{aligned}$$

30. $P\{Y=j\} = \binom{j-1}{n-1} \Big/ \binom{N}{n}, \; n \le j \le N$

$$E[Y] = \sum_{j=n}^{N} \binom{j-1}{n-1} \Big/ \binom{N}{n}$$

$$= \frac{n}{\binom{N}{n}} \sum_{j=n}^{N} \binom{j}{n}$$

$$= \frac{n}{\binom{N}{n}} \sum_{i=n+1}^{N+1} \binom{i-1}{n+1-1}$$

$$= \frac{n}{\binom{N}{n}} \binom{N+1}{n+1}$$

$$= \frac{n(N+1)}{n+1}$$

31. Let Y denote the largest of the remaining m chips. By exercise 28

$$P\{Y=j\} = \binom{j-1}{m-1} \Big/ \binom{m+n}{m}, \; m \le j \le n+m$$

Now, $X = n + m - Y$ and so

$$P\{X=i\} = P\{Y=m+n-i\} = \binom{m+n-i-1}{m-1} \Big/ \binom{m+n}{m}, \; i \le n$$

32. $P\{X=k\} = \frac{k-1}{n} \prod_{i=0}^{k-2} \frac{n-i}{n}, \; k > 1$

34. $$E[X] = \sum_{k=0}^{n} \frac{k\binom{n}{k}}{2^n - 1} = \frac{n2^{n-1}}{2^n - 1}$$

$$E[X^2] = \sum_{k=0}^{n} \frac{k^2\binom{n}{k}}{2^n - 1} = \frac{2^{n-2} n(n+1)}{2^n - 1}$$

$$\text{Var}(X) = E[X^2] - \{E[X]\}^2 = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

$$\sim \frac{n2^{2n-2}}{2^{2n}} = \frac{n}{4}$$

$$E[Y] = \frac{n+1}{2},\ E[Y^2] = \sum_{i=1}^{n} i^2/n \sim \int_1^{n+1} \frac{x^2\,dx}{n} \sim \frac{n^2}{3}$$

$$\text{Var}(Y) \sim \frac{n^2}{3} - \left(\frac{n+1}{2}\right)^2 \sim \frac{n^2}{12}$$

35. (a) $P\{X > i\} = \frac{1}{2}\frac{2}{3}\cdots\frac{i}{i+1} = \frac{1}{i+1}$

(b) $P(X < \infty) = \lim_{i\to\infty} P\{X \le i\}$

$= \lim_{i}(1 - 1/(i+1)) = 1$

(c) $E[X] = \sum_i iP\{X = i\}$

$= \sum_i i(P\{X > i-1\} - P\{X > i\}$

$= \sum_i i\left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \sum_i \frac{1}{i+1}$

$= \infty$

Chapter 5

Problems

1. (a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \frac{3}{4}\int_{-1}^{x}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right), -1 < x < 1$

2. $\int xe^{-x/2}dx = -2xe^{-x/2} - 4e^{-x/2}$. Hence,

$$c\int_{0}^{\infty} xe^{-x/2}dx = 1 \Rightarrow c = 1/4$$

$$P\{X > 5\} = \frac{1}{4}\int_{5}^{\infty} xe^{-x/2}dx = \frac{1}{4}[10e^{-5/2} + 4e^{-5/2}]$$

$$= \frac{14}{4}e^{-5/2}$$

3. No. $f(5/2) < 0$

4. (a) $\int_{20}^{\infty}\frac{10}{x^2}dx = \frac{-10}{x}\Big|_{20}^{\infty} = 1/2$.

(b) $F(y) = \int_{10}^{y}\frac{10}{x^2}dx = 1 - \frac{10}{y}$, $y > 10$. $F(y) = 0$ for $y < 10$.

(c) $\sum_{i=3}^{6}\binom{6}{i}\left(\frac{2}{3}\right)^i\left(\frac{1}{3}\right)^{6-i}$ since $\overline{F}(15) = \frac{10}{15}$. Assuming independence of the events that the devices exceed 15 hours.

5. Must choose c so that

$$.01 = \int_{c}^{1} 5(1-x)^4 dx = (1-c)^5$$

so $c = 1 - (.01)^{1/.5}$.

6. (a) $E[X] = \frac{1}{4}\int_0^\infty x^2 e^{-x/2}dx = 2\int_0^\infty y^2 e^{-y}dx = 2\Gamma(3) = 4$

(b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$

(c) $E[X] = \int_5^\infty \frac{5}{x}dx = \infty$

7. $\int_0^1 (a + bx^2)dx = 1$ or $a + \frac{b}{3} = 1$

$\int_0^1 x(a + bx^2)dx = \frac{3}{5}$ or $\frac{a}{2} + \frac{b}{4} = 3/5$. Hence,

$a = \frac{3}{5}$, $b = \frac{6}{5}$

8. $E[X] = \int_0^\infty x^2 e^{-x}dx = \Gamma(3) = 2$

9. If s units are stocked and the demand is X, then the profit, $P(s)$, is given by

$$\begin{aligned} P(s) &= bX - (s - X)P && \text{if } X \le s \\ &= sb && \text{if } X > s \end{aligned}$$

Hence

$$\begin{aligned} E[P(s)] &= \int_0^s (bx - (s-x)\ell) f(x)dx + \int_s^\infty sbf(x)dx \\ &= (b+\ell)\int_0^s xf(x)dx - s\ell\int_0^s f(x)dx + sb\left[1 - \int_0^s f(x)dx\right] \\ &= sb + (b+\ell)\int_0^s (x-s)f(x)dx \end{aligned}$$

Differentiation yields

$$\begin{aligned} \frac{d}{ds}E[P(s)] &= b + (b+\ell)\frac{d}{ds}\left[\int_0^s xf(x)dx - s\int_0^s f(x)dx\right] \\ &= b + (b+\ell)\left[sf(s) - sf(s) - \int_0^s f(s)dx\right] \\ &= b - (b+\ell)\int_0^s f(x)dx \end{aligned}$$

Equating to zero shows that the maximal expected profit is obtained when s is chosen so that

$$F(s) = \frac{b}{b+\ell}$$

where $F(s) = \int_0^s f(x)dx$ is the cumulative distribution of demand.

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.
$= 2/3$ since X is uniform (0, 60).

(b) same answer as in (a).

11. X is uniform on $(0, L)$.

$$\begin{aligned}
&P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < 1/4\right\} \\
&= 1 - P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > 1/4\right\} \\
&= 1 - P\left\{\frac{X}{L-X} > 1/4, \frac{L-X}{X} > 1/4\right\} \\
&= 1 - P\{X > L/5, X < 4L/5\} \\
&= 1 - P\left\{\frac{L}{5} < X < 4L/5\right\} \\
&= 1 - \frac{3}{5} = \frac{2}{5}.
\end{aligned}$$

13. $P\{X > 10\} = \frac{2}{3}$, $P\{X > 25 \mid X > 15\} = \frac{P\{X > 25}{P\{X > 15\}} = \frac{5/30}{15/30} = 1/3$
where X is uniform (0, 30).

14. $E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}$
$P\{X^n \le x\} = P\{X \le x^{1/n}\} = x^{1/n}$
$E[X^n] = \int_0^1 x \frac{1}{n} x^{\left(\frac{1}{n}-1\right)} dx = \frac{1}{n}\int_0^1 x^{1/n} dx = \frac{1}{n+1}$

15. (a) $\Phi(.8333) = .7977$
(b) $2\Phi(1) - 1 = .6827$
(c) $1 - \Phi(.3333) = .3695$
(d) $\Phi(1.6667) = .9522$
(e) $1 - \Phi(1) = .1587$

16. $P\{X > 50\} = P\left\{\dfrac{X-40}{4} > \dfrac{10}{4}\right\} = 1 - \Phi(2.5) = 1 - .9938$

Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$

17. $E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$

18. $.2 = P\left\{\dfrac{X-5}{\sigma} > \dfrac{9-5}{\sigma}\right\} = P\{Z > 4/\sigma\}$ where Z is a standard normal. But from the normal table $P\{Z < .84) \approx .80$ and so

$$.84 \approx 4/\sigma \text{ or } \sigma \approx 4.76$$

That is, the variance is approximately $(4.76)^2 = 22.66$.

19. Letting $Z = (X - 12)/2$ then Z is a standard normal. Now, $.10 = P\{Z > (c - 12)/2\}$. But from Table 5.1, $P\{Z < 1.28\} = .90$ and so

$$(c - 12)/2 = 1.28 \text{ or } c = 14.56$$

20. Let X denote the number in favor. Then X is binomial with mean 65 and standard deviation $\sqrt{65(.35)} \approx 4.77$. Also let Z be a standard normal random variable.

(a) $P\{X \geq 50\} = P\{X \geq 49.5\} = P\{X - 65\}/4.77 \geq -15.5/4.77$
$\approx P\{Z \geq -3.25\} \approx .9994$

(b) $P\{59.5 \leq X \leq 70.5\} \approx P\{-5.5/4.77 \leq Z \leq 5.5/4.77\}$
$= 2P\{Z \leq 1.15\} - 1 \approx .75$

(c) $P\{X \leq 74.5\} \approx P\{Z \leq 9.5/4.77\} \approx .977$

22. (a) $P\{.9000 - .005 < X < .9000 + .005\}$
$= P\left\{-\dfrac{.005}{.003} < Z < \dfrac{.005}{.003}\right\}$
$= P\{-1.67 < Z < 1.67\}$
$= 2\Phi(1.67) - 1 = .9050.$

Hence 9.5 percent will be defective (that is each will be defective with probability $1 - .9050 = .0950$).

(b) $P\left\{-\dfrac{.005}{\sigma} < Z < \dfrac{.005}{\sigma}\right\} = 2\Phi\left(\dfrac{.005}{\sigma}\right) - 1 = .99$ when

$$\Phi\left(\frac{.005}{\sigma}\right) = .995 \Rightarrow \frac{.005}{\sigma} = 2.575 \Rightarrow \sigma = .0019\,.$$

23. (a) $P\{149.5 < X < 200.5\} = P\left\{\dfrac{149.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}} < Z < \dfrac{200.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}}\right\}$

$= \Phi\left(\dfrac{200.5 - 166.7}{\sqrt{5000/36}}\right) - \Phi\left(\dfrac{149.5 - 166.7}{\sqrt{5000/36}}\right)$

$\approx \Phi(2.87) + \Phi(1.46) - 1 = .9258.$

(b) $P\{X < 149.5\} = P\left\{Z < \dfrac{149.5 - 800(1/5)}{\sqrt{800\frac{1}{5}\frac{4}{5}}}\right\}$

$= P\{Z < -.93\}$

$= 1 - \Phi(.93) = .1762.$

24. With C denoting the life of a chip, and ϕ the standard normal distribution function we have

$$P\{C < 1.8 \times 10^6\} = \phi\left(\frac{1.8\times10^6 - 1.4\times10^6}{3\times10^5}\right)$$

$$= \phi(1.33)$$

$$= .9082$$

Thus, if N is the number of the chips whose life is less than 1.8×10^6 then N is a binomial random variable with parameters (100, .9082). Hence,

$$P\{N > 19.5\} \approx 1 - \phi\left(\frac{19.5 - 90.82}{90.82(.0918)}\right) = 1 - \phi(-24.7) \approx 1$$

25. Let X denote the number of unacceptable items among the next 150 produced. Since X is a binomial random variable with mean 150(.05) = 7.5 and variance 150(.05)(.95) = 7.125, we obtain that, for a standard normal random variable Z.

$P\{X \le 10\} = P\{X \le 10.5\}$

$= P\left\{\dfrac{X - 7.5}{\sqrt{7.125}} \le \dfrac{10.5 - 7.5}{\sqrt{7.125}}\right\}$

$\approx P\{Z \le 1.1239\}$

$= .8695$

The exact result can be obtained by using the text diskette, and (to four decimal places) is equal to .8678.

27. $P\{X > 5{,}799.5\} = P\left\{Z > \dfrac{799.5}{\sqrt{2{,}500}}\right\}$

$= P\{Z > 15.99\}$ = negligible.

28. Let X equal the number of lefthanders. Assuming that X is approximately distributed as a binomial random variable with parameters $n = 200$, $p = .12$, then, with Z being a standard normal random variable,

$$\begin{aligned} P\{X > 19.5\} &= P\left\{\frac{X - 200(.12)}{\sqrt{200(.12)(.88)}} > \frac{19.5 - 200(.12)}{\sqrt{200(.12)(.88)}}\right\} \\ &\approx P\{Z > -.9792\} \\ &\approx .8363 \end{aligned}$$

29. Let s be the initial price of the stock. Then, if X is the number of the 1000 time periods in which the stock increases, then its price at the end is

$$su^X d^{1000-X} = sd^{1000}\left(\frac{u}{d}\right)^X$$

Hence, in order for the price to be at least $1.3s$, we would need that

$$d^{1000}\left(\frac{u}{d}\right)^X > 1.3$$

or

$$X > \frac{\log(1.3) - 1000\log(d)}{\log(u/d)} = 469.2$$

That is, the stock would have to rise in at least 470 time periods. Because X is binomial with parameters 1000, .52, we have

$$\begin{aligned} P\{X > 469.5\} &= P\left\{\frac{X - 1000(.52)}{\sqrt{1000(.52)(.48)}} > \frac{469.5 - 1000(.52)}{\sqrt{1000(.52)(.48)}}\right\} \\ &\approx P\{Z > -3.196\} \\ &\approx .9993 \end{aligned}$$

30. $$\begin{aligned} P\{\text{in black}\} &= \frac{P\{5 \mid \text{black}\}\alpha}{P\{5 \mid \text{black}\}\alpha + P\{5 \mid \text{white}\}(1-\alpha)} \\ &= \frac{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8}\alpha}{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8}\alpha + (1-\alpha)\frac{1}{3\sqrt{2\pi}} e^{-(5-6)^2/18}} \\ &= \frac{\frac{\alpha}{2} e^{-1/8}}{\frac{\alpha}{2} e^{-1/8} + \frac{(1-\alpha)}{3} e^{-1/8}} \end{aligned}$$

α is the value that makes preceding equal 1/2

31. (a) $E[|X-a|] = \int_a^A (x-a)\frac{dx}{A} + \int_0^a (a-x)\frac{dx}{A} = \frac{A}{2} - \left(a - \frac{a^2}{A}\right)$

$$\frac{d}{da}(\quad) = \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2$$

(b) $E[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

$$= a(1-e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}$$

Differentiation yields that the minimum is attained at $\bar{a}$ where

$$e^{-\lambda \bar{a}} = 1/2 \text{ or } \bar{a} = \log 2/\lambda$$

(c) Minimizing a = median of F

32. (a) e^{-1}
(b) $e^{-1/2}$

33. e^{-1}

34. (a) $P\{X > 20\} = e^{-1}$

(b) $P\{X > 30 \mid X > 10 = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{1/4}{3/4} = 1/3$

35. (a) $\exp\left[-\int_{40}^{50} \lambda(t)dt\right] = e^{-.35}$

(b) $e^{-1.21}$

36. (a) $1 - F(2) = \exp\left[-\int_0^2 t^3 dt\right] = e^{-4}$

(b) $\exp[-(.4)^4/4] - \exp[-(1.4)^4/4]$

(c) $\exp\left[-\int_1^2 t^3 dt\right] = e^{-15/4}$

37. (a) $P\{|X| > 1/2\} = P\{X > 1/2\} + P\{X < -1/2\} = 1/2$

(b) $P\{|X| \le a\} = P\{-a \le X \le a\} = a$, $0 < a < 1$. Therefore,
$f_{|X|}(a) = 1$, $0 < a < 1$
That is, $|X|$ is uniform on (0, 1).

38. For both roots to be real the discriminant $(4Y)^2 - 44(Y + 2)$ must be ≥ 0. That is, we need that $Y^2 \geq Y + 2$. Now in the interval $0 < Y < 5$.

$$Y^2 \geq Y + 2 \Leftrightarrow Y \geq 2 \text{ and so}$$
$$P\{Y^2 \geq Y + 2\} = P\{Y \geq 2\} = 3/5.$$

39. $$F_Y(y) = P\{\log X \leq y\}$$
$$= P\{X \leq e^y\} = F_X(e^y)$$

$$f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y}$$

40. $$F_Y(y) = P\{e^X \leq y\}$$
$$= F_X(\log y)$$

$$f_Y(y) = f_X(\log y)\frac{1}{y} = \frac{1}{y},\ 1 < y < e$$

Theoretical Exercises

1. The integration by parts formula $\int udv = uv - \int vdu$ with $dv = -2bxe^{-bx^2}$, $u = -x/2b$ yields that

$$\begin{aligned}\int_0^\infty x^2 e^{-bx^2} dx &= \frac{-xe^{-bx^2}}{2b}\Big|_0^\infty + \frac{1}{2b}\int_0^\infty e^{-bx^2} dx \\ &= \frac{1}{(2b)^{3/2}}\int_0^\infty e^{-y^2/2} dy \text{ by } y = x\sqrt{2b} \\ &= \frac{\sqrt{2\pi}}{2}\frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}}\end{aligned}$$

where the above uses that $\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y^2/2} dy = 1/2$. Hence, $a = \frac{4b^{3/2}}{\sqrt{\pi}}$

2. $$\begin{aligned}\int_0^\infty P\{Y < -y\} dy &= \int_0^\infty \int_{-\infty}^{-y} f_Y(x) dx\, dy \\ &= \int_{-\infty}^0 \int_0^{-x} f_Y(x) dy\, dx = -\int_{-\infty}^0 x f_Y(x)\, dx\end{aligned}$$

Similarly,

$$\int_0^\infty P\{Y > y\} dy = \int_0^\infty x f_Y(x)\, dx$$

Subtracting these equalities gives the result.

4. $$\begin{aligned}E[aX + b] = \int (ax + b) f(x)\, dx &= a\int x f(x) dx + b\int f(x) dx \\ &= aE[X] + b\end{aligned}$$

5. $$\begin{aligned}E[X^n] &= \int_0^\infty P\{X^n > t\} dt \\ &= \int_0^\infty P\{X^n > x^n\} n x^{n-1} dx \text{ by } t = x^n,\ dt = nx^{n-1} dx \\ &= \int_0^\infty P\{X > x\} n x^{n-1} dx\end{aligned}$$

6. Let X be uniform on (0, 1) and define E_a to be the event that X is unequal to a. Since $\cap_a E_a$ is the empty set, it must have probability 0.

7. $SD(aX + b) = \sqrt{\text{Var}(aX + b)} = \sqrt{a^2\sigma^2} = |a|\sigma$

8. Since $0 \le X \le c$, it follows that $X^2 \le cX$. Hence,

$$\begin{aligned}\text{Var}(X) &= E[X^2] - (E[X])^2 \\ &\le E[cX] - (E[X])^2 \\ &= cE[X] - (E[X])^2 \\ &= E[X](c - E[X]) \\ &= c^2[\alpha(1 - \alpha)] \quad \text{where } \alpha = E[X]/c \\ &\le c^2/4\end{aligned}$$

where the last inequality first uses the hypothesis that $P\{0 \le X \le c\} = 1$ to calculate that $0 \le \alpha \le 1$ and then uses calculus to show that $\underset{0 \le \alpha \le 1}{\text{maximum}}\, \alpha(1 - \alpha) = 1/4$.

9. The final step of parts (a) and (b) use that $-Z$ is also a standard normal random variable.

(a) $P\{Z > x\} = P\{-Z < -x\} = P\{Z < -x\}$

(b) $$\begin{aligned}P\{|Z| > x\} &= P\{Z > x\} + P\{Z < -x\} = P\{Z > x\} + P\{-Z > x\} \\ &= 2P\{Z > x\}\end{aligned}$$

(c) $$\begin{aligned}P\{|Z| < x\} &= 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\} \text{ by (b)} \\ &= 1 - 2(1 - P\{Z < x\})\end{aligned}$$

10. With $c = 1/\left(\sqrt{2\pi}\sigma\right)$ we have

$$f(x) = ce^{-(x-\mu)^2/2\sigma^2}$$

$$f'(x) = -ce^{-(x-\mu)^2/2\sigma^2}(x - \mu)/\sigma^2$$

$$f''(x) = c\sigma^{-4}e^{-(x-\mu)^2/2\sigma^2}(x - \mu)^2 - c\sigma^{-2}e^{-(x-\mu)^2/2\sigma^2}$$

Therefore,

$$f''(\mu + \sigma) = f''(\mu - \sigma) = c\sigma^{-2}e^{-1/2} - c\sigma^{-2}e^{-1/2} = 0$$

11. $$E[X^2] = \int_0^\infty P\{X > x\}2x^{2-1}dx = 2\int_0^\infty xe^{-\lambda x}dx = \frac{2}{\lambda}E[X] = 2/\lambda^2$$

12. (a) $\dfrac{b + a}{2}$

(b) μ

(c) $1 - e^{-\lambda m} = 1/2$ or $m = \dfrac{1}{\lambda}\log 2$

13. (a) all values in (a, b)

(b) μ

(c) 0

14. $P\{cX < x\} = P\{X < x/c\} = 1 - e^{-\lambda x/c}$

15. $\lambda(t) = \dfrac{f(t)}{\bar{F}(t)} = \dfrac{1/a}{(a-t)/a} = \dfrac{1}{a-t}$, $0 < t < a$

16. If X has distribution function F and density f, then for $a > 0$

$$F_{aX}(t) = P\{aX \le t\} = F(t/a)$$

and

$$f_{ax} = \frac{1}{a} f(t/a)$$

Thus,

$$\lambda_{aX}(t) = \frac{\frac{1}{a} f(t/a)}{1 - F(t/a)} = \frac{1}{a}\lambda_X(t/a)\,.$$

18. $$E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} dx = \lambda^{-k} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^k dx$$
$$= \lambda^{-k}\,\Gamma(\mathrm{k}+1) = \mathrm{k}!/\lambda^k$$

19. $$E[X^k] = \frac{1}{\Gamma(t)} \int_0^\infty x^k \lambda e^{-\lambda x} (\lambda x)^{t-1} dx$$
$$= \frac{\lambda^{-k}}{\Gamma(t)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{t+k-1} dx$$
$$= \frac{\lambda^{-k}}{\Gamma(t)} \Gamma(t+k)$$

Therefore,

$E[X] = t/\lambda$,
$E[X^2] = = \lambda^{-2}\Gamma(t+2)/\Gamma(t) = (t+1)t/\lambda^2$

and thus

$$\mathrm{Var}(X) = (t+1)t/\lambda^2 - t^2/\lambda^2 = t/\lambda^2$$

20. $\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} dx$

$= \sqrt{2}\int_0^\infty e^{-y^2/2} dy$ by $x = y^2/2, dx = ydy = \sqrt{2x}\ dy$

$= 2\sqrt{\pi}\int_0^\infty (2\pi)^{-1/2} e^{-y^2/2} dy$

$= 2\sqrt{\pi} P\{Z > 0\}$ where Z is a standard normal

$= \sqrt{\pi}$

21. $1/\lambda(s) = \int_{x \geq s} \lambda e^{-\lambda x} (\lambda x)^{t-1} dx / \lambda e^{-\lambda s} (\lambda s)^{t-1}$

$= \int_{x \geq s} e^{-\lambda(x-s)} (x/s)^{t-1} dx$

$= \int_{y \geq 0} e^{-\lambda y} (1 + y/s)^{t-1} dy$ by letting $y = x - s$

As the above, equal to the inverse of the hazard rate function, is clearly decreasing in s when $t \geq 1$ and increasing when $t \leq 1$ the result follows.

22. $\lambda(s) = c(s - v)^{\beta-1}$, $s > v$ which is clearly increasing when $\beta \geq 1$ and decreasing otherwise.

23. $F(\alpha) = 1 - e^{-1}$

24. Suppose X is Weibull with parameters v, α, β. Then

$$P\left\{\left(\frac{X - v}{\alpha}\right)^\beta \leq x\right\} = P\left\{\frac{X - v}{\alpha} \leq x^{1/\beta}\right\}$$

$= P\{X \leq v + \alpha x^{1/\beta}\}$

$= 1 - \exp\{-x\}$.

25. We use Equation (6.3).

$$E[X] = B(a + 1, b)/B(A, b) = \frac{\Gamma(a+1)}{\Gamma(a+b+1)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{a}{a+b}$$

$$E[X^2] = B(a + 2, b)/B(a, b) = \frac{\Gamma(a+2)}{\Gamma(a+b+2)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)}$$

Thus,

$$\text{Var}(X) = \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$

26. $(X - a)/(b - a)$

28. $P\{F(X \le x\} = P\{X \le F^{-1}(x)\}$
$= F(F^{-1}(x))$
$= x$

29. $F_Y(x) = P\{aX + b \le x\}$
$= P\left\{X \le \dfrac{x-b}{a}\right\}$ when $a > 0$
$= F_X((x-b)/a)$ when $a > 0$.

$f_Y(x) = \dfrac{1}{a} f_X((x-b)/a)$ if $a > 0$.

When $a < 0$, $F_Y(x) = P\left\{X \ge \dfrac{x-b}{a}\right\} = 1 - F_X\left(\dfrac{x-b}{a}\right)$ and so

$f_Y(x) = -\dfrac{1}{a} f_X\left(\dfrac{x-b}{a}\right)$.

30. $F_Y(x) = P\{e^X \le x\}$
$= P\{X \le \log x\}$
$F_X(\log x)$

$f_Y(x) = f_X(\log x)/x$

$= \dfrac{1}{x\sqrt{2\pi}\sigma} e^{-(\log x - \mu)^2 / 2\sigma^2}$

Chapter 6

Problems

2. (a) $p(0, 0) = \dfrac{8 \cdot 7}{13 \cdot 12} = 14/39$,

$p(0, 1) = p(1, 0) = \dfrac{8 \cdot 5}{13 \cdot 12} = 10/39$

$p(1, 1) = \dfrac{5 \cdot 4}{13 \cdot 12} = 5/39$

(b) $p(0, 0, 0) = \dfrac{8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11} = 28/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = \dfrac{8 \cdot 7 \cdot 5}{13 \cdot 12 \cdot 11} = 70/429$

$p(0, 1, 1) = p(1, 0, 1) = p(1, 1, 0) = \dfrac{8 \cdot 5 \cdot 4}{13 \cdot 12 \cdot 11} = 40/429$

$p(1, 1, 1) = \dfrac{5 \cdot 4 \cdot 3}{13 \cdot 12 \cdot 11} = 5/143$

3. (a) $p(0, 0) = (10/13)(9/12) = 15/26$

$p(0, 1) = p(1, 0) = (10/13)(3/12) = 5/26$

$p(1, 1) = (3/13)(2/12) = 1/26$

(b) $p(0, 0, 0) = (10/13)(9/12)(8/11) = 60/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = (10/13)(9/12)(3/11) = 45/286$

$p(i, j, k) = (3/13)(2/12)(10/11) = 5/143$ if $i + j + k = 2$

$p(1, 1, 1) = (3/13)(2/12)(1/11) = 1/286$

4. (a) $p(0, 0) = (8/13)^2$, $p(0, 1) = p(1, 0) = (5/13)(8/13)$, $p(1, 1) = (5/13)^2$

(b) $p(0, 0, 0) = (8/13)^3$

$p(i, j, k) = (8/13)^2(5/13)$ if $i + j + k = 1$

$p(i, j, k) = (8/13)(5/13)^2$ if $i + j + k = 2$

5. $p(0, 0) = (12/13)^3(11/12)^3$

$p(0, 1) = p(1, 0) = (12/13)^3[1 - (11/12)^3]$

$p(1, 1) = (2/13)[(1/13) + (12.13)(1/13)] + (11/13)(2/13)(1/13)$

8. $f_Y(y) = c\int_{-y}^{y}(y^2 - x^2)e^{-y}dx$

$= \frac{4}{3}cy^3e^{-y}$, $-0 < y < \infty$

$\int_0^\infty f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \frac{y^3e^{-y}}{6}$, $0 < y < \infty$

$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$

$= \frac{1}{4}e^{-|x|}(1 + |x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

9. (b) $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^{2}\int_{0}^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\int_0^{1/2}(2x^2 + x)dx}$$

10. (a) $f_X(x) = e^{-x}$, $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

$P\{X < Y\} = 1/2$

(b) $P\{X < a\} = 1 - e^{-a}$

11. $\frac{5!}{2!1!2!}(.45)^2(.15)(.40)^2$

12. $e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5}$

14. Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{|Y-X|<a\} = P\{Y<X<Y+a\} + P\{X<Y<X+a\}$$

$$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dxdy + \int_{L-a}^{L}\int_y^{L} dxdy\right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2-\frac{a}{L}\right),\ 0<a<L$$

15. (a) $1 = \int\int f(x,y)dydx = \int_{(x,y)\in R}\int c\ dydx = cA(R)$

where $A(R)$ is the area of the region R.

(b) $f(x, y) = 1/4, -1 \le x, y \le 1$

$= f(x)f(y)$

where $f(v) = 1/2, -1 \le v \le 1$.

(c) $P\{X^2 + Y^2 \le 1\} = \frac{1}{4}\int\int_c dydx$ = (area of circle)/4 = $\pi/4$.

16. (a) $A = \cup A_i$,

(b) yes

(c) $P(A) = \sum P(A_i) = n(1/2)^{n-1}$

17. $\frac{1}{3}$ since each of the 3 points is equally likely to be the middle one.

18. $P\{Y-X>L/3\} = \int_{y-x>L/3}\int \frac{4}{L^2}dydx$

$$\frac{L}{2} < y < L$$

$$0 < x < \frac{L}{2}$$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dydx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = 7/9$$

19. $\int_0^1 \int_0^x \frac{1}{x}\, dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x} dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x} dy = 1$, $0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y \ln(y) dy = -1 - \int_0^1 (y\ln(y) - y) dy$$

yielding the result

$$E[Y] = -\int_0^1 y\ln(y) dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}, f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y) dy = 2(1-x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y) dx = 2y, \; 0 < y < 1$$

21. (a) We must show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dxdy = 1$. Now,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dxdy &= \int_0^1 \int_0^{1-y} 24xy \; dxdy \\ &= \int_0^1 12y(1-y)^2 dy \\ &= \int_0^1 12(y - 2y^2 + y^3) dy \\ &= 12(1/2 - 2/3 + 1/4) = 1 \end{aligned}$$

(b) $$\begin{aligned} E[X] &= \int_0^1 x f_X(x) dx \\ &= \int_0^1 x \int_0^{1-x} 24xy \; dydx \\ &= \int_0^1 12x^2(1-x)^2 dx = 2/5 \end{aligned}$$

(c) 2/5

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2, \ 0 < x < 1.$

(c) $P\{X+Y<1\} = \int_0^1 \int_0^{1-x} (x+y)dydx$

$= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3$

23. (a) yes

$f_X(x) = 12x(1-x)\int_0^1 ydy = 6x(1-x), \ 0 < x < 1$

$f_Y(y) = 12y\int_0^1 x(1-x)dx = 2y, \ 0 < y < 1$

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $\text{Var}(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20$

(e) $\text{Var}(Y) = \int_0^1 2y^3 dy - 4/9 = 1/18$

24. $P\{N=n\} = p_0^{n-1}(1-p_0)$

(b) $P\{X=j\} = p_j/(1-p_0)$

(c) $P\{N=n, X=j\} = p_0^{n-1}p_j$

25. $\dfrac{e^{-1}}{i!}$ by the Poisson approximation to the binomial.

26. (a) $F_{A,B,C}(a,b,c) = abc \quad 0 < a, b, c < 1$

(b) The roots will be real if $B^2 \geq 4AC$. Now

$$P\{AC \leq x\} = \int\int_{\substack{c \leq x/a \\ 0 \leq a \leq 1 \\ 0 \leq c \leq 1}} dadc = \int_0^x \int_0^1 dcda + \int_x^1 \int_0^{x/a} dcda$$

$= x - x \log x.$

Hence, $F_{AC}(x) = x - x \log x$ and so

$f_{AC}(x) = -\log x, \ 0 < x < 1$

$$P\{B^2/4 \ge AC\} = -\int_0^1 \int_0^{b^2/4} \log x\,dx\,db$$

$$= \int_0^1 \left[\frac{b^2}{4} - \frac{b^2}{4}\log(b^2/4) \right] db$$

$$= \frac{\log 2}{6} + \frac{5}{36}$$

where the above uses the identity

$$\int x^2 \log x\,dx = \frac{x^3 \log x}{3} - \frac{x^3}{9}.$$

27. (a) $P\{X + Y \le a\} = \int_0^a \int_0^{a-x} e^{-y}\,dy\,dx = a - 1 + e^{-a},\ a < 1$

$$= \int_0^1 \int_0^{a-x} e^{-y}\,dy\,dx = 1 - e^{-a}(e-1),\ a > 1$$

(b) $P\{Y > X/a\} = \int_0^1 \int_{x/a}^{\infty} e^{-y}\,dy\,dx = a(1 - e^{-1/a})$

28. $P\{X_1/X_2 < a\} = \int_0^{\infty} \int_0^{ay} \lambda_1 e^{-\lambda x} 1^x \lambda_2 e^{-\lambda_2 y}\,dx\,dy$

$$= \int_0^{\infty} \left(1 - e^{-\lambda_1 a y}\right) \lambda_2 e^{-\lambda_2 y}\,dy$$

$$= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1 a}{a\lambda_1 + \lambda_2}$$

$$P\{X_1/X_2 < 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

29. $P\{I^2R \le w\} = \int_{\substack{x^2 \\ 0 \\ 0}} \int_{\substack{y \le w \\ x \le 1 \\ y \le 1}} 6x(1-x)2y\,dy\,dx$

$$= \int_0^{\sqrt{w}} \int_0^1 12x(1-x)y\,dy\,dx + \int_{\sqrt{w}}^1 \int_0^{w/x^2} 12x(1-x)y\,dy\,dx$$

$$= 3w - 2w^{3/2} = 6w(1 + (\log w)/2 - \sqrt{w}\,)$$

$$= 4w^{3/2} - 3w(1 + \log w),\ 0 < w < 1$$

30. (a) e^{-2}

(b) $1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$
The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable.

31. (a) $1 - e^{-2.2} - 2.2e^{-2.2} - e^{-2.2}(2.2)^2/2!$

(b) $1 - \sum_{i=0}^{4} e^{-4.4}(4.4)^i / i!$, (c) $1 - \sum_{i=0}^{5} e^{-6.6}(6.6)^i / i!$

The reasoning is the same as in Problem 26.

32. (a) If $W = X_1 + X_2$ is the sales over the next two weeks, then W is normal with mean 4,400 and standard deviation $\sqrt{2(230)^2} = 325.27$. Hence, with Z being a standard normal, we have

$$\begin{aligned} P\{W > 5000\} &= P\left\{Z > \frac{5000 - 4400}{325.27}\right\} \\ &= P\{Z > 1.8446\} = .0326 \end{aligned}$$

(b) $P\{X > 2000\} = P\{Z > (2000 - 2200)/230\}$
$= P\{Z > -.87\} = P\{Z < .87\} = .8078$

Hence, the probability that weekly sales exceeds 2000 in at least 2 of the next 3 weeks $p^3 + 3p^2(1 - p)$ where $p = .8078$.

We have assumed that the weekly sales are independent.

33. Let X denote Jill's score and let Y be Jack's score. Also, let Z denote a standard normal random variable.

(a) $$\begin{aligned} P\{Y > X\} &= P\{Y - X > 0\} \\ &\approx P\{Y - X > .5\} \\ &= P\left\{\frac{Y - X - (160 - 170)}{\sqrt{(20)^2 + (15)^2}} > \frac{.5 - (160 - 170)}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .42\} \approx .3372 \end{aligned}$$

(b) $$\begin{aligned} P\{X + Y > 350\} &= P\{X + Y > 350.5\} \\ &= P\left\{\frac{X + Y - 330}{\sqrt{(20)^2 + (15)^2}} > \frac{20.5}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .82\} \approx .2061 \end{aligned}$$

34. Let X and Y denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E[X] = 50.4,\ \mathrm{Var}(X) = 37.6992,\ \ E[Y] = 47.2,\ \mathrm{Var}(Y) = 36.0608$$

it follows from the normal approximation to the binomial that is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that Y is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. Let Z be a standard normal random variable.

(a) $P\{X + Y \geq 110\} = P\{X + Y \geq 109.5\}$

$$= P\left\{\frac{X + Y - 97.6}{\sqrt{73.76}} \geq \frac{109.5 - 97.6}{\sqrt{73.76}}\right\}$$

$$\approx P\{Z > 1.3856\} \approx .0829$$

(b) $P\{Y \geq X\} = P\{Y - X \geq -.5\}$

$$= P\left\{\frac{Y - X - (-3.2)}{\sqrt{73.76}} \geq \frac{-.5 - (-3.2)}{\sqrt{73.76}}\right\}$$

$$\approx P\{Z \geq .3144\} \approx .3766$$

35. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 4/12 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) $P\{X_1 = 1 \mid X_2 = 0\} = 5/12 = 1 - P\{X_1 = 0 \mid X_2 = 0\}$

36. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 5/13 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) same as in (a)

37. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = 2/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = 3/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$

38. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = p(1, 1)/[1 - (12/13)^3] = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = p(1, 0)/(12/13)^3 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$
where $p(1, 1)$ and $p(1, 0)$ are given in the solution to Problem 5.

39. (a) $P\{X = j, Y = i\} = \frac{1}{5}\frac{1}{j}, j = 1, \ldots, j, i = 1, \ldots, j$

(b) $P\{X = j \mid Y = i\} = \frac{1}{5j}\Big/\sum_{k=i}^{5} 1/5\ k = \frac{1}{j}\Big/\sum_{k=i}^{5} 1/k,\ 5 \geq j \geq i.$

(c) No.

40. For $j = i$: $P\{Y = i \mid X = i\} = \dfrac{P\{Y = i, X = i\}}{P\{X = i\}} = \dfrac{1}{36P\{X = i\}}$

For $j < i$: $P\{Y = j \mid X = i\} = \dfrac{2}{36P\{X = i\}}$

Hence

$$1 = \sum_{j=1}^{i} P\{Y = j \mid X = i\} = \frac{2(i-1)}{36P\{X = i\}} + \frac{1}{36P\{X = i\}}$$

and so, $P\{X = i\} = \dfrac{2i-1}{36}$ and

$$P\{Y = j \mid X = i\} = \begin{cases} \dfrac{1}{2i-i} & j = i \\ \dfrac{2}{2i-1} & j < i \end{cases}$$

42. (a) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)}$, $0 < x$

(b) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy}$, $0 < y$

$$P\{XY < a\} = \int_0^{\infty}\int_0^{a/x} xe^{-x(y+1)}dydx$$
$$= \int_0^{\infty}(1-e^{-a})e^{-x}dx = 1 - e^{-a}$$

$f_{XY}(a) = e^{-a}$, $0 < a$

43. $$f_{Y|X}(y \mid x) = \frac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x}(x^2 - y^2)e^{-x}dx}$$
$$= \frac{3}{4x^3}(x^2 - y^2), \quad -x < y < x$$

$$F_{Y|X}(y \mid x) = \frac{3}{4x^3}\int_{-x}^{y}(x^2 - y^2)dy$$
$$= \frac{3}{4x^3}(x^2y - y^3/3 + 2x^3/3), \quad -x < y < x$$

44. $$f(\lambda \mid n) = \frac{P\{N = n \mid \lambda\} g(\lambda)}{P\{N = n\}}$$
$$= C_1 e^{-\lambda} \lambda^n \alpha e^{-\alpha\lambda} (\alpha\lambda)^{s-1}$$
$$= C_2 e^{-(\alpha+1)\lambda} \lambda^{n+s-1}$$

where C_1 and C_2 do not depend on λ. But from the preceding we can conclude that the conditional density is the gamma density with parameters $\alpha + 1$ and $n + s$. The conditional expected number of accidents that the insured will have next year is just the expectation of this distribution, and is thus equal to $(n + s)/(\alpha + 1)$.

45. $$P\{X_1 > X_2 + X_3\} + P\{X_2 > X_1 + X_3\} + P\{X_3 > X_1 + X_2\}$$
$$= 3P\{X_1 > X_2 + X_3\}$$
$$= 3 \iiint_{\substack{x_1 > x_2 > x_3 \\ 0 \le x_i \le 1 \\ i = 1,2,3}} dx_1 dx_2 dx_3 \qquad \text{(take } a = 0, b = 1\text{)}$$
$$= 3 \int_0^1 \int_0^{1-x_3} \int_{x_2 + x_3}^1 dx_1 dx_2 dx_3 = 3 \int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3) dx_2 dx_3$$
$$= 3 \int_0^1 \frac{(1 - x_3)^2}{2} dx_3 = 1/2 .$$

46. $$f_{X_{(3)}}(x) = \frac{5!}{2!2!} \left[\int_0^x x e^{-x} dx \right]^2 x e^{-x} \left[\int_x^\infty x e^{-x} dx \right]^2$$
$$= 30(x + 1)^2 e^{-2x} x e^{-x} [1 - e^{-x}(x + 1)]^2$$

47. $$\left(\frac{L - 2d}{L} \right)^3$$

48. $$\int_{1/4}^{3/4} f_{X_{(3)}}(x) dx = \frac{5!}{2!2!} \int_{1/4}^{3/4} x^2 (1 - x)^2 dx$$

49. (a) $P\{\min X_i \le a\} = 1 - P\{\min X_i > a\} = 1 - \prod P\{X_i > a\} = 1 - e^{-5\lambda a}$

(b) $P\{\max X_i \le a\} = \prod P\{X_i \le a\} = (1 - e^{-\lambda a})^5$

50. $$f_{X_{(1)}, X_{(4)}}(x, y) = \frac{4!}{2!} 2x \left(\int_X^Y 2z dz \right)^2 2y , \quad x < y$$
$$= 48xy(y^2 - x^2).$$

$$P(X_{(4)} - X_{(1)} \le a\} = \int_0^{1-a}\int_0^{a+x} 48xy(y^2 - x^2)dydx + \int_{1-a}^{1}\int_0^{1} 48xy(y^2 - x^2)dydx$$

51. $f_{R_1}(r,\theta) = \frac{r}{\pi} = 2r\frac{1}{2\pi}$, $0 \le r \le 1$, $0 \le \theta < 2\pi$.

Hence, R and θ are independent with θ being uniformly distributed on $(0, 2\pi)$ and R having density $f_R(r) = 2r$, $0 < r < 1$.

52. $f_{R,\theta}(r,\theta) = r$, $0 < r \sin\theta < 1$, $0 < r\cos\theta < 1$, $0 < \theta < \pi/2$, $0 < r < \sqrt{2}$

53. $$J = \begin{vmatrix} \frac{1}{2}x^{-1/2}\cos u\sqrt{2} & \frac{1}{2}z^{-1/2}\sin u\sqrt{2} \\ -\sqrt{2z}\sin u & \sqrt{2z}\cos u \end{vmatrix} = \cos^2 u + \sin^2 u = 1$$

$f_{u,z}(u, z)$ - $\frac{1}{2\pi}e^{-z}$. But $x^2 + y^2 = 2z$ so

$$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

54. (a) If $u = xy$, $v = xy$, then $J = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\frac{x}{y}$ and

$y = \sqrt{u/v}$, $x = \sqrt{vu}$. Hence,

(b) $f_{u,v}(u, v) = \frac{1}{2v}f_{X,Y}\left(\sqrt{vy}, \sqrt{u/v}\right) = \frac{1}{2vu^2}$, $u \ge 1$, $\frac{1}{u} < v < u$

$$f_u(u) = \int_{1/u}^{u} \frac{1}{2vu^2}dv = \frac{1}{u^2}\log u, \; u \ge 1.$$

For $v > 1$

$$f_V(v) = \int_v^{\infty} \frac{1}{2vu^2}du = \frac{1}{2v^2}, \; v > 1$$

For $v < 1$

$$f_V(v) = \int_{1/2}^{\infty} \frac{1}{2vu^2}du = \frac{1}{2}, \; 0 < v < 1.$$

55. (a) $u = x + y,\ v = x/y \Rightarrow y = \dfrac{u}{v+1},\ x = \dfrac{uv}{v+1}$

$$J = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\left(\frac{x}{y^2} + \frac{1}{y}\right) = \frac{-1}{y^2}(x+y) = \frac{-(v+1)^2}{u}$$

$$f_{u,v}(u, v) = \frac{u}{(v+1)^2},\ 0 < uv < 1 + v,\ 0 < u < 1 + v$$

57. $y_1 = x_1 + x_2,\ y_2 = e^{x_1}.\ J = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = -e^{x_1} = -y_2$

$x_1 = \log y_2,\ x_2 = y_1 - \log y_2$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_2}\lambda e^{-\lambda \log y_2}\lambda e^{-\lambda(y_1 - \log y_2)}$$

$$= \frac{1}{y_2}\lambda^2 e^{-\lambda y_1},\ 1 \le y_2,\ y_1 \ge \log y_2$$

58. $u = x + y,\ v = x + z,\ w = y + z \Rightarrow z = \dfrac{v + w - u}{2},\ x = \dfrac{v - w + u}{2},\ y = \dfrac{w - v + u}{2}$

$$J = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

$$f(u, v, w) = \frac{1}{2}\exp\left\{-\frac{1}{2}(u + v + w)\right\},\ u + v > w,\ u + w > v,\ v + w + u$$

59. $P(Y_j = i_j, j = 1, \ldots, k+1\} = P\{Y_j = i_j, j = 1, \ldots, k\}\ P(Y_{k+1} = i_{k+1} \mid Y_j = i_j, j = 1, \ldots, k\}$

$$= \frac{k!(n-k)!}{n!} P\{n + 1 - \sum_{i=1}^{k} Y_i = i_{k+1} \Big| Y_j = i_j, j = 1,\ldots,k\}$$

$$= \begin{cases} k!(n-k)!/n!, \text{ if } \sum_{j=1}^{k+1} i_j = n+1 \\ 0, \text{ otherwise} \end{cases}$$

Thus, the joint mass function is symmetric, which proves the result.

60. The joint mass function is

$$P\{X_i = x_i, i = 1, \ldots, n\} = 1/\binom{n}{k},\ x_i \in \{0, 1\},\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = k$$

As this is symmetric in $x_1, \ldots, x_n$ the result follows.

Theoretical Exercises

1. $$\begin{aligned} P\{X \le a_2,\ Y \le b_2\} &= P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\} \\ &\quad + P\{X \le a_1,\ b_1 < Y \le b_2\} \\ &\quad + P\{a_1 < X \le a_2,\ Y \le b_1\} \\ &\quad + P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

The above following as the left hand event is the union of the 4 mutually exclusive right hand events. Also,

$$\begin{aligned} P\{X \le a_1, Y \le b_2\} &= P\{X \le a_1,\ b_1 < Y \le b_2\} \\ &\quad + P\{X \le a_1,\ Y \le b_1\} \end{aligned}$$

and similarly,

$$\begin{aligned} P\{X \le a_2, Y \le b_1\} &= P\{a_1 \le X \le a_2, < Y \le b_1\} \\ &\quad + P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

Hence, from the above

$$\begin{aligned} F(a_2, b_2) &= P\{a_1 < X \le a_2, b_1 < Y \le b_2\} + F(a_1, b_2) - F(a_1, b_1) \\ &\quad + F(a_2, b_1) - F(a_1, b_1) + F(a_1, b_1). \end{aligned}$$

2. Let X_i denote the number of type i events, $i = 1, \ldots, n$.

$$\begin{aligned} P\{X_1 = r_1, \ldots, X_n = r_n\} &= P\left\{X_1 = r_1,\ldots,X_n = r_n \middle| \sum_1^n r_i \text{ events}\right\} \\ &\quad \times e^{-\lambda}\lambda^{\sum_1^n r_i} \Big/ \left(\sum_1^n r_i\right)! \\ &= \frac{\left(\sum_1^n r_i\right)!}{r_1!\ldots r_n!} P_1^{r_1}\ldots p_n^{r_n} \frac{e^{-\lambda}\lambda^{\sum_1^n r_i}}{\left(\sum_1^n r_i\right)!} \\ &= \prod_{i=1}^n e^{-\lambda P_i}(\lambda_{p_i})^{r_i} \Big/ r_i! \end{aligned}$$

3. Throw a needle on a table, ruled with equidistant parallel lines a distance D apart, a large number of times. Let L, $L < D$, denote the length of the needle. Now estimate π by $\dfrac{2L}{fD}$ where f is the fraction of times the needle intersects one of the lines.

5. (a) For $a > 0$

$$F_Z(a) = P\{X \le aY\}$$
$$= \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y) dx dy$$
$$= \int_0^\infty F_X(ay) f_Y(y) dy$$
$$f_Z(a) = \int_0^\infty f_X(ay) y f_Y(y) dy$$

(b)
$$F_Z(a) = P\{XY < a\}$$
$$= \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y) dx dy$$
$$= \int_0^\infty F_X(a/y) f_Y(y) dy$$
$$f_Z(a) = \int_0^\infty f_X(a/y) \frac{1}{y} f_Y(y) dy$$

If X is exponential with rate λ and Y is exponential with rate μ then (a) and (b) reduce to

(a) $F_Z(a) = \int_0^\lambda \lambda e^{-\lambda a y} y \mu e^{-\mu y} dy$

(b) $F_Z(a) = \int_0^\infty \lambda e^{-\lambda a/y} \frac{1}{y} \mu e^{-\mu y} dy$

6. Interpret X_i as the number of trials needed after the $(i-1)^{st}$ success until the i^{th} success occurs, $i = 1, \ldots, n$, when each trial is independent and results in a success with probability p. Then each X_i is an identically distributed geometric random variable and $\sum_{i=1}^n X_i$, representing the number of trials needed to amass n successes, is a negative binomial random variable.

7. (a) $P\{cX \le a\} = P\{X \le a/c\}$ and differentiation yields

$$f_{cX}(a) = \frac{1}{c} f_X(a/c) = \frac{\lambda}{c} e^{-\lambda a/c} (\lambda a/c)^{t-1} \Gamma(t).$$

Hence, cX is gamma with parameters $(t, \lambda/c)$.

(b) A chi-squared random variable with $2n$ degrees of freedom can be regarded as being the sum of n independent chi-square random variables each with 2 degrees of freedom (which by Example is equivalent to an exponential random variable with parameter λ). Hence by Proposition X^2_{2n} is a gamma random variable with parameters $(n, 1/2)$ and the result now follows from part (a).

8. (a) $P\{W \le t\} = 1 - P\{W > t\} = 1 - P\{X > t, Y > t\} = 1 - [1 - F_X(t)]\,[1 - F_Y(t)]$

(b) $f_W(t) = f_X(t)[1 - F_Y(t)] + f_Y(t)\,[1 - F_X(t)]$

Dividing by $[1 - F_X(t)][1 - F_Y(t)]$ now yields

$$\lambda_W(t) = f_X(t)/[1 - F_X(t)] + f_Y(t)/[1 - F_Y(t)] = \lambda_X(t) + \lambda_Y(t)$$

9. $$\begin{aligned} P\{\min(X_1, \ldots, X_n) > t\} &= P\{X_1 > t, \ldots, X_n > t\} \\ &= e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t} \end{aligned}$$

thus showing that the minimum is exponential with rate $n\lambda$.

10. If we let X_i denote the time between the i^{th} and $(i + 1)^{\text{st}}$ failure, $i = 0, \ldots, n - 2$, then it follows from Exercise 9 that the X_i are independent exponentials with rate 2λ. Hence, $\sum_{i=0}^{n-2} X_i$ the amount of time the light can operate is gamma distributed with parameters $(n - 1, 2\lambda)$.

11. $$\begin{aligned} I &= \int\int\int\int\int_{x_1 < x_2 > x_3 < x_4 > x_5} f(x_1) \ldots f(x_5)dx_1 \ldots dx_5 \\ &= \int\int\int\int\int_{\substack{u_1 < u_2 > u_3 < u_4 > u_5 \\ 0 < u_i < 1}} du_1 \ldots du_5 \qquad \text{by } u_i = F(x_i),\ i = 1, \ldots, 5 \\ &= \int\int\int\int u_2 du_2 \ldots du_5 \\ &= \int\int\int (1 - u_3^2)/2 \ \ du_3 \ldots \\ &= \int\int [u_4 - u_4^3/3]/2 du_4 du_5 \\ &= \int_0^1 [u^2 - u^4/3]/2 du \ = 2/15 \end{aligned}$$

12. Assume that the joint density factors as shown, and let

$$C_i = \int_{-\infty}^{\infty} g_i(x)dx,\ i = 1, \ldots, n$$

Since the n-fold integral of the joint density function is equal to 1, we obtain that

$$1 = \prod_{i=1}^{n} C_i$$

Integrating the joint density over all x_i except x_j gives that

$$f_{X_j}(x_j) = g_j(x_j) \prod_{i \ne j} C_i = g_j(x_j)/C_j$$

If follows from the preceding that

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} f_{X_j}(x_j)$$

which shows that the random variables are independent.

13. No. Let $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & -- \end{cases}$. Then

$$f_{X|X_1,\ldots,X_{n+m}}(x \,|\, x_1,\ldots,x_{n+m}) = \frac{P\{x_1,\ldots,x_{n+m} \,|\, X = x\}}{P\{x_1,\ldots,x_{n+m}\}} f_X(x)$$
$$= cx^{\sum x_i}(1-x)^{n+m-\sum x_i}$$

and so given $\sum_{1}^{n+m} X_i = n$ the conditional density is still beta with parameters $n + 1$, $m + 1$.

14. $P\{X = i \,|\, X + Y = n\} = P\{X = i,\ Y = n - i\}/P\{X + Y = n\}$

$$= \frac{p(1-p)^{i-1}p(1-p)^{n-i-1}}{\binom{n-1}{1}p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

15. $P\{X = k \,|\, X + Y = m\} = \dfrac{P\{X = k, X + Y = m\}}{P\{X + Y = m\}}$

$$= \frac{P\{X = k, Y = m - k\}}{P\{X + Y = m\}}$$
$$= \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}}$$
$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

16. $P(X = n,\ Y = m) = \sum_i P(X = n, Y = m \,|\, X_2 = i)P(X_2 = i)$

$$= e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)!}\frac{\lambda_3^{m-i}}{(m-i)!}\frac{\lambda_2^i}{i!}$$

17. (a) $P\{X_1 > X_2 \mid X_1 > X_3\} = \dfrac{P\{X_1 = \max(X_1, X_2, X_3)\}}{P\{X_1 > X_3\}} = \dfrac{1/3}{1/2} = 2/3$

(b) $P\{X_1 > X_2 \mid X_1 < X_3\} = \dfrac{P\{X_3 > X_1 > X_2\}}{P\{X_1 < X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(c) $P\{X_1 > X_2 \mid X_2 > X_3\} = \dfrac{P\{X_1 > X_2 > X_3\}}{P\{X_2 > X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(d) $P\{X_1 > X_2 \mid X_2 < X_3\} = \dfrac{P\{X_2 = \min(X_1, X_2, X_3)\}}{P\{X_2 < X_3\}} = \dfrac{1/3}{1/2} = 2/3$

18. $P\{U > s \mid U > a\} = P\{U > s\}/P\{U > a\}$
$= \dfrac{1-s}{1-a}, \; a < s < 1$

$P\{U < s \mid U < a\} = P\{U < s\}/P\{U < a\}$
$= s/a, \; 0 < s < \text{a}$

Hence, $U \mid U > a$ is uniform on $(a, 1)$, whereas $U \mid U < a$ is uniform over $(0, a)$.

19. $f_{W|N}(w \mid n) = \dfrac{P\{N = n \mid W = w\} f_W(w)}{P\{N = n\}}$

$= Ce^{-w} \dfrac{w^n}{n!} \beta e^{-\beta w} (\beta w)^{t-1}$

$= C_1 e^{-(\beta+1)w} w^{n+t-1}$

where C and C_1 do not depend on w. Hence, given $N = n$, W is gamma with parameters $(n + t, \beta + 1)$.

20. $f_{W|X_i, i=1,\ldots,n}(w|x_1,\ldots,x_n) = \dfrac{f(x_1,\ldots,x_n|w) f_w(w)}{f(x_1,\ldots,x_n)}$

$= C \prod_{\text{i}=1}^{\text{n}} we^{-wx_i} e^{-\beta w} (\beta w)^{t-1}$

$= Ke^{-w\left(\beta + \sum_1^n x_i\right)} w^{n+t-1}$

21. Let X_{ij} denote the element in row i, column j.

$$P\{X_{ij} \text{ is s saddle point}\}$$

$$= P\left\{\min_{k=1,\ldots,m} X_{ik} > \max_{k\neq i} X_{kj}, X_{ij} = \min_{k} X_{ik}\right\}$$

$$= P\left\{\min_{k} X_{ik} > \max_{k\neq i} X_{kj}\right\} P\left\{X_{ij} = \min_{k} X_{ik}\right\}$$

where the last equality follows as the events that every element in the i^{th} row is greater than all elements in the j^{th} column excluding X_{ij} is clearly independent of the event that X_{ij} is the smallest element in row i. Now each size ordering of the $n + m - 1$ elements under consideration is equally likely and so the probability that the m smallest are the ones in row i is $1\Big/\binom{n+m-1}{m}$. Hence

$$P\{X_{ij} \text{ is a saddlepoint}\} = \frac{1}{\binom{n+m-1}{m}}\frac{1}{m} = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

and so

$$\begin{aligned} P\{\text{there is a saddlepoint}\} &= P\left(\bigcup_{i,j}\{X_{ij} \text{ is a saddlepoint}\}\right) \\ &= \sum_{\text{i,j}} P\{X_{ij} \text{ is a saddlepoint}\} \\ &= \frac{m!n!}{(n+m-1)!} \end{aligned}$$

22. For $0 < x < 1$

$$P([X] = n, X - [X] < x) = P(n < X < n + x) = e^{-n\lambda} - e^{-(n+x)\lambda} = e^{-n\lambda}(1 - e^{-x\lambda})$$

Because the joint distribution factors, they are independent. $[X] + 1$ has a geometric distribution with parameter $p = 1 - e^{-\lambda}$ and $x - [X]$ is distributed as an exponential with rate λ conditioned to be less than 1.

23. Let $Y = \max(X_1, \ldots, X_n)$, $Z = \min(X_1, \ldots, X_n)$

$$P\{Y \le x\} = P\{X_i \le x, i = 1, \ldots, n\} = \prod_{1}^{n} P\{X_i \le x\} = F^n(x)$$

$$P\{Z > x\} = P\{X_i > x, i = 1, \ldots, n\} = \prod_{1}^{n} P\{X_i > x\} = [1 - F(x)]^n .$$

24. (a) Let $d = D/L$. Then the desired probability is

$$n! \int_0^{1-(n-1)d} \int_{x_1+d}^{1-(n-2)d} \cdots \int_{x_{n-3}+d}^{1-2d} \int_{x_{n-2}+d}^{1-d} \int_{x_{n-1}+d}^{1} dx_n dx_{n-1} \ldots dx_2 dx_1$$
$$= [1 - (n-1)\mathrm{d}]^n.$$

(b) 0

25. $$F_{x_{(j)}}(x) = \sum_{i=j}^{n} \binom{n}{i} F^i(x)[1-F(x)]^{n-i}$$

$$f_{X_{(j)}}(x) = \sum_{i=j}^{n} \binom{n}{i} iF^{i-1}(x)f(x)[1-F(x)]^{n-i}$$
$$- \sum_{i=j}^{n} \binom{n}{i} F^i(x)(n-i)[1-F(x)]^{n-i-1} f(x)$$

$$= \sum_{i=j}^{n} \frac{n!}{(n-i)!(i-1)!} F^{i-1}(x)f(x)[1-F(x)]^{n-i}$$
$$- \sum_{k=j+1}^{n} \frac{n!}{(n-k)!(k-1)!} F^{k-1}(x)f(x)[1-F(x)]^{n-k} \text{ by } k = i+1$$
$$= \frac{n!}{(n-j)!(j-1)!} F^{j-1}(x)f(x)[1-F(x)]^{n-j}$$

26. $$f_{X(n+1)}(x) = \frac{(2n+1)!}{n!n!} x^n (1-x)^n$$

27. In order for $X_{(i)} = x_i$, $X_{(j)} = x_j$, $i < j$, we must have

(i) $i - 1$ of the X's less than x_i

(ii) 1 of the X's equal to x_i

(iii) $j - i - 1$ of the X's between x_i and x_j

(iv) 1 of the X's equal to x_j

(v) $n - j$ of the X's greater than x_j

Hence,

$$f_{x_{(i)},X_{(j)}}(x_i, x_j)$$
$$= \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} F^{i-1}(x_i)f(x_i)[F(x_j)-F(x_i)]^{j-i-1} f(x_j) \times [1 - F(x_j)^{n-j}$$

29. Let $X_1, \ldots, X_n$ be n independent uniform random variables over $(0, a)$. We will show by induction on n that

$$P\{X_{(k)} - X_{(k-1)} > t\} = \begin{cases} \left(\frac{a-t}{a}\right)^n & \text{if } t < a \\ 0 & \text{if } t > a \end{cases}$$

It is immediate when $n = 1$ so assume for $n - 1$. In the n case, consider

$$P\{X_{(k)} - X_{(k-1)}) > t \,|\, X_{(n)} = s\}.$$

Now given $X_{(n)} = s$, $X_{(1)}, \ldots, X_{(n-1)}$ are distributed as the order statistics of a set of $n - 1$ uniform $(0, s)$ random variables. Hence, by the induction hypothesis

$$P\{X_{(k)} - X_{(k-l)} > t \,|\, X_{(n)} = s\} = \begin{cases} \left(\frac{s-t}{s}\right)^{n-1} & \text{if } t < s \\ 0 & \text{if } t > s \end{cases}$$

and thus, for $t < a$,

$$P\{X_{(k)} - X_{(k-l)} > t = \int_t^a \left(\frac{s-t}{s}\right)^{n-1} \frac{ns^{n-1}}{a^n} ds = \left(\frac{a-t}{a}\right)^n$$

which completes the induction. (The above used that $f_{X_{(n)}}(s) = n\left(\frac{s}{a}\right)^{n-1} \frac{1}{a} = \frac{ns^{n-1}}{a^n}$).

30. (a) $P\{X > X_{(n)}\} = P\{X \text{ is largest of } n + 1\} = 1/(n + 1)$

(b) $P\{X > X_{(1)}\} = P\{X \text{ is not smallest of } n + 1\} = 1 - 1/(n + 1) = n/(n + 1)$

(c) This is the probability that X is either the $(i + 1)^{st}$ or $(i + 2)^{nd}$ or $\ldots j^{th}$ smallest of the $n + 1$ random variables, which is clearly equal to $(j - 1)/(n + 1)$.

33. The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 1/y \\ 0 & -x/y^2 \end{vmatrix} = -x/y^2$$

Hence, $|J|^{-1} = y^2/|x|$. Therefore, as the solution of the equations $u = x$, $v = x/y$ is $x = u$, $y = u/v$, we see that

$$f_{u,v}(u, v) = \frac{|u|}{v^2} f_{X,Y}(u, u/v) = \frac{|u|}{v^2} \frac{1}{2\pi} e^{-(u^2 + u^2/v^2)/2}$$

Hence,

$$
\begin{aligned}
f_{V(u)} &= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2(1+1/v^2)/2}du \\
&= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2/2\sigma^2}du \text{ , where } \sigma^2 = v^2/(1+v^2) \\
&= \frac{1}{\pi v^2}\int_{0}^{\infty}ue^{-u^2/2\sigma^2}du \\
&= \frac{1}{\pi v^2}\sigma^2\int_{0}^{\infty}e^{-y}dy \\
&= \frac{1}{\pi(1+v^2)}
\end{aligned}
$$

Chapter 7

Problems

1. Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let Y denote the value that shows up on the die. Then, with $p(i, j) = P\{X = i, Y = j\}$

$$E[\text{return}] = \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j)$$
$$= \frac{1}{12}(42 + 10.5) = 52.5/12$$

2. (a) $6 \cdot 6 \cdot 9 = 324$

(b) $X = (6 - S)(6 - W)(9 - R)$

(c) $E[X] = 6(6)(6)P\{S = 0, W = 0, R = 3\} + 6(3)(9)P\{S = 0, W = 3, R = 0\}$
$+ 3(6)(9)P\{S = 3, W = 0, R = 0\} + 6(5)(7)P\{S = 0, W = 1, R = 2\}$
$+ 5(6)(7)P\{S = 1, W = 0, R = 2\} + 6(4)(8)P\{S = 0, W = 2, R = 1\}$
$+ 4(6)(8)P\{S = 2, W = 0, R = 1\} + 5(4)(9)P\{S = 1, W = 2, R = 0\}$
$+ 4(5)(9)P\{S = 2, W = 1, R = 0\} + 5(5)(8)P\{S = 1, W = 1, R = 1\}$

$$= \frac{1}{\binom{21}{3}}\left[216\binom{9}{3} + 324\binom{6}{3} + 420 \cdot 6\binom{9}{2} + 384\binom{6}{2}9 + 360\binom{6}{2}6 + 200(6)(6)(9)\right]$$
$$\approx 198.8$$

3. $E[|X - Y|^a] = \int_0^1 \int_0^1 |x - y|^a dy dx$. Now

$$\int_0^1 |x - y|^a dy = \int_0^x (x - y)^a dy + \int_x^1 (y - x)^a dy$$
$$= \int_0^x u^a du + \int_0^{1-x} u^a du$$
$$= [x^{a+1} + (1 - x)^{a+1}]/(a + 1)$$

Hence,

$$E[|X - Y|^a] = \frac{1}{a+1}\int_0^1 [x^{a+1} + (1 - x)^{a+1}]dx$$
$$= \frac{2}{(a + 1)(a + 2)}$$

4. $E[|X-Y|] = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}|i-j|$. Now,

$$\sum_{j=1}^{m}|i-j| = \sum_{j=1}^{i}(i-j) + \sum_{j=i+1}^{m}(j-i)$$
$$= [i(i-1) + (m-i)(m-i+1)]/2$$

Hence, using the identity $\sum_{j=1}^{m} j^2 = m(m+1)(2m+1)/6$, we obtain that

$$E[|X-Y|] = \left[\frac{1}{m^2}\frac{m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2}\right] = \frac{(m+1)(m-1)}{3m}$$

5. The joint density of the point (X, Y) at which the accident occurs is

$$f(x, y) = \frac{1}{9}, \quad -3/2 < x, y < 3/2$$
$$= f(x)\, f(y)$$

where

$$f(a) = 1/3, \quad -3/2 < a < 3/2.$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$ Therefore,

$$E[\,|X| + |Y|\,] = 2\int_{-3/2}^{3/2}\frac{1}{3}x\, dx = \frac{4}{3}\int_{0}^{3/2} x dx = 3/2\,.$$

6. $E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10(7/2) = 35.$

8. E[number of occupied tables] $= E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$

Now,

$E[X_i] = P\{i^{\text{th}}$ arrival is not friends with any of first $i-1\}$
$= (1-p)^{i-1}$

and so

$$E[\text{number of occupied tables}] = \sum_{i=1}^{N}(1-p)^{i-1}$$

7. Let X_i equal 1 if both choose item i and let it be 0 otherwise; let Y_i equal 1 if neither A nor B chooses item i and let it be 0 otherwise. Also, let W_i equal 1 if exactly one of A and B choose item i and let it be 0 otherwise. Let

$$X = \sum_{i=1}^{10} X_i\,, \quad Y = \sum_{i=1}^{10} Y_i\,, \quad W = \sum_{i=1}^{10} W_i$$

(a) $$E[X] = \sum_{i=1}^{10} E[X_i] = 10(3/10)^2 = .9$$

(b) $$E[Y] = \sum_{i=1}^{10} E[Y_i] = 10(7/10)^2 = 4.9$$

(c) Since $X + Y + W = 10$, we obtain from parts (a) and (b) that

$$E[W] = 10 - .9 - 4.9 = 4.2$$

Of course, we could have obtained $E[W]$ from

$$E[W] = \sum_{i=1}^{10} E[W_i] = 10(2)(3/10)(7/10) = 4.2$$

9. Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$$E[X_j] = P\{\text{ball } i \text{ is not in urn } j,\ i \geq j\} = \prod_{i=j}^{n}(1-1/i)$$

Hence,

(a) $$E[\text{number of empty urns}] = \sum_{j=1}^{n}\sum_{i=j}^{n}(1-1/i)$$

(b) $$P\{\text{none are empty}\} = P\{\text{ball } j \text{ is in urn } j, \text{ for all } j\} = \prod_{j=1}^{n} 1/j$$

10. Let X_i equal 1 if trial i is a success and 0 otherwise.

(a) .6. This occurs when $P\{X_1 = X_2 = X_3\} = 1$. It is the largest possible since $1.8 = \sum P\{X_i = 1\} = 3P\{X_i = 1\}$. Hence, $P\{X_i = 1\} = .6$ and so

$$P\{X = 3\} = P\{X_1 = X_2 = X_3 = 1\} \leq P\{X_1 = 1\} = .6.$$

(b) 0. Letting

$$X_1 = \begin{matrix}1 \text{ if } U \leq .6 \\ 0 \text{ otherwise}\end{matrix}\,, \qquad X_2 = \begin{matrix}1 \text{ if } U \leq .4 \\ 0 \text{ otherwise}\end{matrix}\,, \qquad X_3 = \begin{matrix}1 \text{ if } U \leq .3 \\ 0 \text{ otherwise}\end{matrix}$$

Hence, it is not possible for all X_i to equal 1.

11. Let X_i equal 1 if a changeover occurs on the i^{th} flip and 0 otherwise. Then

$$\begin{aligned} E[X_i] &= P\{i-1 \text{ is } H, i \text{ is } T\} + P\{i-1 \text{ is } T, i \text{ is } H\} \\ &= 2(1-p)p, \quad i \geq 2. \end{aligned}$$

$$E[\text{number of changeovers}] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$$

12. (a) Let X_i equal 1 if the person in position i is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \frac{1}{2}\frac{n}{2n-1}, & \text{if } i = 1, 2n \\ \frac{1}{2}\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{2n} E[X_i] \\ &= \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) \\ &= \frac{3n^2 - n}{4n-2} \end{aligned}$$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result

$$n\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2}$$

where the right side equality assumes that $n > 1$.

13. Let X_i be the indicator for the event that person i is given a card whose number matches his age. Because only one of the cards matches the age of the person i

$$E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E[X_i] = 1$$

14. The number of stages is a negative binomial random variable with parameters m and $1 - p$. Hence, its expected value is $m/(1-p)$.

15. Let $X_{i,j}$, $i \neq j$ equal 1 if i and j form a matched pair, and let it be 0 otherwise.

Then

$$E[X_{i,j}] = P\{i, j \text{ is a matched pair}\} = \frac{1}{n(n-1)}$$

Hence, the expected number of matched pairs is

$$E\left[\sum_{i<j} X_{i,j}\right] = \sum_{i<j} E[X_{i,j}] = \binom{n}{2}\frac{1}{n(n-1)} = \frac{1}{2}$$

16. $$E[X] = \int_{y>x} y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

17. Let I_i equal 1 if guess i is correct and 0 otherwise.

(a) Since any guess will be correct with probability $1/n$ it follows that

$$E[N] = \sum_{i=1}^{n} E[I_i] = n/n = 1$$

(b) The best strategy in this case is to always guess a card which has not yet appeared. For this strategy, the i^{th} guess will be correct with probability $1/(n - i + 1)$ and so

$$E[N] = \sum_{i=1}^{n} 1/(n-i+1)$$

(c) Suppose you will guess in the order 1, 2, …, n. That is, you will continually guess card 1 until it appears, and then card 2 until it appears, and so on. Let J_i denote the indicator variable for the event that you will eventually be correct when guessing card i; and note that this event will occur if among cards 1 thru i, card 1 is first , card 2 is second, …, and card i is the last among these i cards. Since all $i!$ orderings among these cards are equally likely it follows that

$$E[J_i] = 1/i! \text{ and thus } E[N] = E\left[\sum_{i=1}^{n} J_i\right] = \sum_{i=1}^{n} 1/i!$$

18. $$E[\text{number of matches}] = E\left[\sum_{1}^{52} I_i\right], \quad I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{---} \end{cases}$$

$$= 52\frac{1}{13} = 4 \quad \text{since } E[I_i] = 1/13$$

19. (a) E[time of first type 1 catch] $- 1 = \dfrac{1}{p_1} - 1$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1 & \text{a type } j \text{ is caught before a type 1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E\left[\sum_{j\neq 1} X_j\right] &= \sum_{j\neq 1} E[X_j] \\ &= \sum_{j\neq 1} P\{\text{type } j \text{ before type 1}\} \\ &= \sum_{j\neq 1} P_j/(P_j + P_1), \end{aligned}$$

where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give.

$$P\{\text{type } j \text{ before type 1}\} = P\{j \,|\, j \text{ or } 1\} = \frac{P_j}{P_j + P_1}$$

20. Similar to (b) of 19. Let

$$X_j = \begin{cases} 1 & \text{ball } j \text{ removed before ball 1} \\ 0 & \text{- - -} \end{cases}$$

$$\begin{aligned} E\left[\sum_{j\neq 1} X_j\right] = \sum_{j\neq 1} E[X_j] &= \sum_{j\neq 1} P\{\text{ball } j \text{ before ball 1}\} \\ &= \sum_{j\neq 1} P\{j \,|\, j \text{ or } 1\} \\ &= \sum_{j\neq 1} W(j)/W(1) + W(j) \end{aligned}$$

21. (a) $365\binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$

(b) Let $X_j = \begin{cases} 1 & \text{if day } j \text{ is someones birthday} \\ 0 & \text{---} \end{cases}$

$$E\left[\sum_1^{365} X_j\right] = \sum_1^{365} E[X_j] = 365\left[1 - \left(\frac{364}{365}\right)^{100}\right]$$

22. From Example 3g, $1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6$

23. $$E\left[\sum_1^5 X_i + \sum_1^8 Y_i\right] = \sum_1^5 E[X_i] + \sum_1^8 E(Y_i)$$
$$= 5\frac{2}{11}\frac{3}{20} + 8\frac{3}{120} = \frac{147}{110}$$

24. Number the small pills, and let X_i equal 1 if small pill i is still in the bottle after the last large pill has been chosen and let it be 0 otherwise, $i = 1, \ldots, n$. Also, let Y_i, $i = 1, \ldots, m$ equal 1 if the i^{th} small pill created is still in the bottle after the last large pill has been chosen and its smaller half returned.

Note that $X = \sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i$. Now,

$E[X_i] = P\{\text{small pill } i \text{ is chosen after all } m \text{ large pills}\}$
$= 1/(m + 1)$

$E[Y_i] = P\{i^{\text{th}} \text{ created small pill is chosen after } m - i \text{ existing large pills}\}$
$= 1/(m - i + 1)$

Thus,

(a) $E[X] = n/(m + 1) + \sum_{i=1}^{m} 1/(m - i + 1)$

(b) $Y = n + 2m - X$ and thus

$E[Y] = n + 2m - E[X]$

25. $P\{N \geq n\}\ P\{X_1 \geq X_2 \geq \ldots \geq X_n\} = \frac{1}{n!}$

$$E[N] = \sum_{n=1}^{\infty} P\{N \geq n\} = \sum_{n=1}^{\infty} \frac{1}{n!} = e$$

26. (a) $E[\max] = \int_0^1 P\{\max > t\}dt$

$$= \int_0^1 (1 - P\{\max \le t)\}dt$$

$$= \int_0^1 (1 - t^n / dt = \frac{n}{n+1}$$

(b) $\mathrm{E}[\min] = \int_0^1 p\{\min > t\}4t$

$$= \int_0^1 (1-t)^n dt = \frac{1}{n+1}$$

27. Let X denote the number of items in a randomly chosen box. Then, with X_i equal to 1 if item i is in the randomly chosen box

$$E[X] = E\left[\sum_{i=1}^{101} X_i\right] = \sum_{i=1}^{101} E[X_i] = \frac{101}{10} > 10$$

Hence, X can exceed 10, showing that at least one of the boxes must contain more than 10 items.

28. We must show that for any ordering of the 47 components there is a block of 12 consecutive components that contain at least 3 failures. So consider any ordering, and randomly choose a component in such a manner that each of the 47 components is equally likely to be chosen. Now, consider that component along with the next 11 when moving in a clockwise manner and let X denote the number of failures in that group of 12. To determine $E[X]$, arbitrarily number the 8 failed components and let, for $i = 1, \ldots, 8$,

$$X_i = \begin{cases} 1, & \text{if failed component } i \text{ is among the group of 12 components} \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$X = \sum_{i=1}^{8} X_i$$

and so

$$E[X] = \sum_{i=1}^{8} E[X_i]$$

Because X_i will equal 1 if the randomly selected component is either failed component number i or any of its 11 neighboring components in the counterclockwise direction, it follows that $E[X_i] = 12/47$. Hence,

$$E[X] = 8(12/47) = 96/47$$

Because $E[X] > 2$ it follows that there is at least one possible set of 12 consecutive components that contain at least 3 failures.

29. Let X_{ii} be the number of coupons one needs to collect to obtain a type i. Then

$$\begin{aligned}
E[X_{i)}] &= 8, \quad i = 1,2 \\
E\{X_i] &= 8/3, \quad i = 3,4 \\
E[\min(X_1, X_2)] &= 4 \\
E[\min(X_i, X_j)] &= 2, \quad i = 1,2, \quad j = 3,4 \\
E[\min(X_3, X_4)] &= 4/3 \\
E[\min(X_1, X_2, X_j)] &= 8/5, \quad j = 3,4 \\
E[\min(X_i, X_{3,} X_4)] &= 8/7, \quad i = 1,2 \\
E[\min(X_1, X_2, X_3, X_4] &= 1
\end{aligned}$$

(a) $E[\max X_i] = 2 \cdot 8 + 2 \cdot 8/3 - (4 + 4 \cdot 2 + 4/3) + (2 \cdot 8/5 + 2 \cdot 8/7) - 1 = \dfrac{437}{35}$

(b) $E[\max(X_1, X_2)] = 8 + 8 - 4 = 12$

(c) $E[\max(X_3, X_4)] = 8/3 + 8/3 - 4/3 = 4$

(d) Let $Y_1 = \max(X_1, X_2)$, $Y_2 = \max(X_3, X_4)$. Then

$$E[\max(Y_1, Y_2)] = E[Y_1] + E[Y_2] - E[\min(Y_1, Y_2)]$$

giving that

$$E[\min(Y_1, Y_2)] = 12 + 4 - \frac{437}{35} = \frac{123}{35}$$

30. $E[(X - Y)]^2 = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(-Y) = 2\sigma^2$

31. $\text{Var}\left(\sum_{i=1}^{10} X_i\right) = 10\ \text{Var}(X_1)$. Now

$$\begin{aligned}
\text{Var}(X_1) &= E[X_1^2] - (7/2)^2 \\
&= [1 + 4 + 9 + 16 + 25 + 36]/6 - 49/4 \\
&= 35/12
\end{aligned}$$

and so $\text{Var}\left(\sum_{i=1}^{10} X_i\right) = 350/12.$

32. Use the notation in Problem 9,

$$X = \sum_{j=1}^{n} X_j$$

where X_j is 1 if box j is empty and 0 otherwise. Now, with

$E[X_j] = P\{X_j = 1\} = \prod_{i=j}^{n}(1-1/i)$, we have that

$\text{Var}(X_j) = E[X_j](1 - E[X_j])$.

Also, for $j < k$

$$E[X_jX_k] = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i)$$

Hence, for $j < k$,

$$\text{Cov}(X_j, X_k) = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i) - \prod_{i=j}^{n}(1-1/i)\prod_{i=k}^{n}(1-1/i)$$

$$\text{Var}(X) = \sum_{j=1}^{n} E[X_j](1 - E[X_j]) + 2\text{Cov}(X_j, X_k)$$

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

34. Let $X_j = \begin{cases} 1 & \text{if couple } j \text{ are seated next to each other} \\ 0 & \text{otherwise} \end{cases}$

(a) $E\left[\sum_{1}^{10} X_j\right] = 10\frac{2}{19} = \frac{20}{19}$; $P\{X_j = 1\} = \frac{2}{19}$ since there are 2 people seated next to wife j and so the probability that one of them is her husband is $\frac{2}{19}$.

(b) For $i \neq j$, $E[X_iX_j] = P\{X_i = 1, X_j = 1\}$

$= P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\}$

$= \frac{2}{19}\frac{2}{18}$ since given $X_i = 1$ we can regard couple i as a single entity.

$$\text{Var}\left(\sum_{j=1}^{10} X_j\right) = 10\frac{2}{19}\left(1-\frac{2}{19}\right) + 10 \cdot 9\left[\frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2\right]$$

35. (a) Let X_1 denote the number of nonspades preceding the first ace and X_2 the number of nonspades between the first 2 aces. It is easy to see that

$$P\{X_1 = i,\ X_2 = j\} = P\{X_1 = j,\ X_2 = i\}$$

and so X_1 and X_2 have the same distribution. Now $E[X_1] = \frac{48}{5}$ by the results of Example 3j and so $E[2 + X_1 + X_2] = \frac{106}{5}$.

(b) Same method as used in (a) yields the answer $5\left(\frac{39}{14} + 1\right) = \frac{265}{14}$.

(c) Starting from the end of the deck the expected position of the first (from the end) heart is, from Example 3j, $\frac{53}{14}$. Hence, to obtain all 13 hearts we would expect to turn over $52 - \frac{53}{14} + 1 = \frac{13}{14}(53)$.

36. Let $X_i = \begin{cases} 1 & \text{roll } i \text{ lands on 1} \\ 0 & \text{otherwise} \end{cases}$, $Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on 2} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Cov}(X_i, Y_j) &= E[X_i Y_j] - E[X_i]E[Y_j] \\ &= \begin{cases} -\frac{1}{36} & i = j \text{ (since } X_i Y_j = 0 \text{ when } i = j \\ \frac{1}{36} - \frac{1}{36} = 0 & i \neq j \end{cases} \end{aligned}$$

$$\begin{aligned} \text{Cov}\sum_i X_i, \sum_j Y_j &= \sum_i \sum_j \text{Cov}(X_i, Y_j) \\ &= -\frac{n}{36} \end{aligned}$$

37. Let W_i, $i = 1, 2$, denote the i^{th} outcome.

$$\begin{aligned} \text{Cov}(X, Y) &= \text{Cov}(W_1 + W_2,\ W_1 - W_2) \\ &= \text{Cov}(W_1, W_1) - \text{Cov}(W_2, W_2) \\ &= \text{Var}(W_1) - \text{Var}(W_2) = 0 \end{aligned}$$

38. $$E[XY] = \int_0^\infty \int_0^x y 2e^{-2x} dy dx$$

$$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$$

$$E[X] = \int_0^\infty x f_x(x)dx,\ f_x(x) = \int_0^x \frac{2e^{-2x}}{x}dy\ = 2e^{-2x}$$
$$= \frac{1}{2}$$

$$E[Y] = \int_0^\infty y f_Y(y)dy,\ f_Y(y) = \int_0^\infty \frac{2e^{-2x}}{x}dx$$
$$= \int_0^\infty \int_y^\infty y\frac{2e^{-2x}}{x}dxdy$$
$$= \int_0^\infty \int_0^x y\frac{2e^{-2x}}{x}dydx$$
$$= \int_0^\infty xe^{-2x}dx = \frac{1}{4}\int ye^{-2}dy = \frac{\Gamma(2)}{4} = \frac{1}{4}$$

$$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

39. $\text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2$
$\text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$
$= \text{Cov}(X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2}) = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2$
$\text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_{n+2}, X_{n+2}) = \sigma^2$
$\text{Cov}(Y_n, Y_{n+j}) = 0$ when $j \geq 3$

40. $f_Y(y) = e^{-y}\int \frac{1}{y}e^{-x/y}dx = e^{-y}$. In addition, the conditional distribution of X given that $Y = y$ is exponential with mean y. Hence,

$E[Y] = 1,\ E[X] = E[E[X \mid Y]] = E[Y] = 1$

Since, $E[XY] = E[E[XY \mid Y]] = E[YE[X \mid Y]] = E[Y^2] = 2$ (since Y is exponential with mean 1, it follows that $E[Y^2] = 2$). Hence, $\text{Cov}(X, Y) = 2 - 1 = 1$.

41. The number of carp is a hypergeometric random variable.

$$E[X] = \frac{60}{10} = 6$$

$$\text{Var}(X) = \frac{20(80)}{99}\frac{3}{10}\frac{7}{10} = \frac{336}{99}$$ from Example 5c.

42. (a) Let $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = P\{X_i = 1\} = \frac{10}{19}$$

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1 \mid X_2 = 1\}$$
$$= \frac{10}{19}\frac{9}{17}, \; i \neq \text{j}$$

$$E\left[\sum_1^{10} X_i\right] = \frac{100}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{10}{19}\left(1 - \frac{10}{19}\right) + 10 \cdot 9\left[\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right] = \frac{900}{(19)^2}\frac{18}{17}$$

(b) $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = \frac{1}{19}, \; E[X_iX_j] = P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\} = \frac{1}{19}\frac{1}{17}, \; i \neq j$$

$$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{1}{19}\frac{15}{19} + 10 \cdot 9\left[\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right] = \frac{180}{(19)^2}\frac{18}{17}$$

43. $E[R] = n(n + m + 1)/2$

$$\text{Var}(R) = \frac{nm}{n+m-1}\left[\frac{\sum_{i=1}^{n+m} i^2}{n+m} - \left(\frac{n+m+1}{2}\right)^2\right]$$

The above follows from Example 3d since when $F = G$, all orderings are equally likely and the problem reduces to randomly sampling n of the $n + m$ values 1, 2, …, $n + m$.

44. From Example 8l $\frac{n}{n+m}+\frac{nm}{n+m}$. Using the representation of Example 2l the variance can be computed by using

$$E[I_1 I_{l+j}] = \begin{cases} 0 & , \quad j = 1 \\ \frac{n}{n+m}\frac{m}{n+m-1}\frac{n-1}{n+m-2} & , \quad n-1 \le j < 1 \end{cases}$$

$$E[I_i I_{i+j}] = \begin{cases} 0 & , \quad j = 1 \\ \frac{mn(m-1)(n-1)}{(n+m)(n+m-1)(n+m-2)(n+m-3)} & , \quad n-1 \le j < 1 \end{cases}$$

45. (a) $\frac{\text{Cov}(X_1+X_2, X_2+X_3)}{\sqrt{\text{Var}(X_1+X_2)}\sqrt{\text{Var}(X_2+X_3)}} = \frac{1}{2}$

(b) 0

46. $E[I_1 I_2] = \sum_{i=2}^{12} E[I_1 I_2 \mid \text{bank rolls } i]\text{P}\{\text{bank rolls } i\}$

$= \sum_i (P\{\text{roll is greater than } i\})^2 P\{\text{bank rolls } i\}$

$= E[I_1^2]$

$\ge (E[I_1])^2$

$= E[I_1]\, E[I_2]$

47. (a) It is binomial with parameters $n - 1$ and p.

(b) Let $x_{i,j}$ equal 1 if there is an edge between vertices i and j, and let it be 0 otherwise. Then, $D_i = \sum_{k \ne i} X_{i,k}$, and so, for $i \ne j$

$$\begin{aligned} \text{Cov}(D_i, D_j) &= \text{Cov}\left(\sum_{k \ne i} X_{i,k}, \sum_{r \ne j} X_{r,j}\right) \\ &= \sum_{k \ne i}\sum_{r \ne j} Cov(X_{i,k}, X_{r,j}) \\ &= \text{Cov}(X_{i,j}, X_{i,j}) \\ &= \text{Var}(X_{i,j}) \\ &= p(1-p) \end{aligned}$$

where the third equality uses the fact that except when $k = j$ and $r = i$, $X_{i,k}$ and $X_{r,j}$ are independent and thus have covariance equal to 0. Hence, from part (a) and the preceding we obtain that for $i \ne j$,

$$\rho(D_i, D_j) = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

48. (a) $E[X] = 6$

(b) $E[X \mid Y = 1] = 1 + 6 = 7$

(c) $1\frac{1}{5} + 2\frac{4}{5}\frac{1}{5} + 3\left(\frac{4}{5}\right)^2\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4(5+6)$

49. Let C_i be the event that coin i is being flipped (where coin 1 is the one having head probability .4), and let T be the event that 2 of the first 3 flips land on heads. Then

$$P(C_1 \mid T) = \frac{P(T|C_1)P(C_1)}{P(T|C_1)P(C_1) + P(T|C_2)P(C_2)}$$
$$= \frac{3(.4)^2(.6)}{3(.4)^2(.6) + 3(.7)^2(.3)} = .395$$

Now, with N_j equal to the number of heads in the final j flips, we have

$$E[N_{10} \mid T] = 2 + E[N_7 \mid T]$$

Conditioning on which coin is being used, gives

$$E[N_7 \mid T] = E[N_7 \mid TC_1]P(C_1T) + E[N_7TC_2]P(C_2 \mid T) = 2.8(.395) + 4.9(.605) = 4.0705$$

Thus, $E[N_{10} \mid T] = 6.0705$.

50. $$f_{X|Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\, dx} = \frac{1}{y}e^{-x/y}, \quad 0 < x < \infty$$

Hence, given $Y = y$, X is exponential with mean y, and so

$$E[X^2 \mid Y = y] = 2y^2$$

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\, dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3 \frac{1}{y} dx = y^3/4$$

52. The average weight, call it $E[W]$, of a randomly chosen person is equal to average weight of all the members of the population. Conditioning on the subgroup of that person gives

$$E[W] = \sum_{i=1}^{r} E\{W \mid \text{member of subgroup } i\}p_i = \sum_{i=1}^{r} w_i p_i$$

53. Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$\begin{aligned} E[X] &= E[X \mid I = 1](.5) + E[X \mid I = 2](.3) + E[X \mid I = 3](.2) \\ &= (2 + E[X])(.5) + (4 + E[X])(.3) + .2 \end{aligned}$$

Therefore,

$$E[X] = 12$$

54. Let R_i denote the return from the policy that stops the first time a value at least as large as i appears. Also, let X be the first sum, and let $p_i = P\{X = i\}$. Conditioning on X yields

$$\begin{aligned} E[R_5] &= \sum_{i=2}^{12} E[R_5 \mid X = i\} p_i \\ &= E[R_5](p_2 + p_3 + p_4) + \sum_{i=5}^{12} i p_i - 7p_7 \\ &= \frac{6}{36} E[R_5] + 5(4/36) + 6(5/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36) \\ &= \frac{6}{36} E[R_5] + 190/36 \end{aligned}$$

Hence, $E[R_5] = 19/3 \approx 6.33$. In the same fashion, we obtain that

$$E[R_6] = \frac{10}{36} E[R_6] + \frac{1}{36}[30 + 40 + 36 + 30 + 22 + 12]$$

implying that

$$E[R_6] = 170/26 \approx 6.54$$

Also,

$$E[R_8] = \frac{15}{36} E[R_8] + \frac{1}{36}(140)$$

or,

$$E[R_8] = 140/21 \approx 6.67$$

In addition,

$$E[R_9] = \frac{20}{26} E[R_9] + \frac{1}{36}(100)$$

or

$$E[R_9] = 100/16 = 6.25$$

And

$$E[R_{10}] = \frac{24}{36}E[R_{10}] + \frac{1}{36}(64)$$

or

$$E[R_{10}] = 64/12 \approx 5.33$$

The maximum expected return is $E[R_8]$.

55. Let N denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$

$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck i with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left(1 - \frac{.6}{n}\right)^{10} e^{-6}6^n / n!$$

56. Let $I_i = \begin{cases} 1 & \text{elevator stops at floor } i \\ 0 & \text{otherwise} \end{cases}$. Let X be the number that enter on the ground floor.

$$E\left[\sum_{i=1}^{N} I_i \middle| X = k\right] = \sum_{i=1}^{N} E[I_i \mid X = k] = N\left[1 - \left(\frac{N-1}{N}\right)^k\right]$$

$$E\left[\sum_{i=1}^{N} I_i\right] = N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k e^{-10}\frac{(10)^k}{k!}$$

$$= N - Ne^{-10/N} = N(1 - e^{-10/N})$$

57. $$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 12.5$$

58. Let X denote the number of flips required. Condition on the outcome of the first flip to obtain.

$$E[X] = E[X \mid \text{heads}]p + E[x \mid \text{tails}](1 - p)$$
$$= [1 + 1/(1 - p)]p + [1 + 1/p](1 - p)$$
$$= 1 + p/(1 - p) + (1 - p)/p$$

59. (a) $E[\text{total prize shared}] = P\{\text{someone wins}\} = 1 - (1-p)^{n+1}$

(b) Let X_i be the prize to player i. By part (a)

$$E\left[\sum_{i=1}^{n+1} X_i\right] = 1-(1-p)^{n+1}$$

But, by symmetry all $E[X_i]$ are equal and so

$$E[X] = [1-(1-p)^{n+1}]/(n+1)$$

(c) $E[X] = p\,E[1/(1+B)]$ where B, which is binomial with parameters n and p, represents the number of other winners.

60. (a) Since the sum of their number of correct predictions is n (one for each coin) it follows that one of them will have more than $n/2$ correct predictions. Now if N is the number of correct predictions of a specified member of the syndicate, then the probability mass function of the number of correct predictions of the member of the syndicate having more than $n/2$ correct predictions is

$$\begin{aligned} P\{i \text{ correct}\} &= P\{N=i\} + P(N=n-i\} \quad i > n/2 \\ &= 2P\{N=i\} \\ &= P\{N=i \,|\, N > n/2\} \end{aligned}$$

(b) X is binomial with parameters m, 1/2.

(c) Since all of the $X+1$ players (including one from the syndicate) that have more than $n/2$ correct predictions have the same expected return we see that

$$(X+1)\cdot \text{Payoff to syndicate} = m+2$$

implying that

$$E[\text{Payoff to syndicate}] = (m+2)\,E[(X+1)^{-1}]$$

(d) This follows from part (b) above and (c) of Problem 56.

61. (a) $$P(M \le x) = \sum_{n=1}^{\infty} P(M \le x \,|\, N=n)P(N=n) = \sum_{n=1}^{\infty} F^n(x)p(1-p)^{n-1} = \frac{pF(x)}{1-(1-p)F(x)}$$

(b) $P(M \le x \,|\, N=1) = F(x)$

(c) $P(M \le x \,|\, N>1) = F(x)P(M \le x)$

(d) $$\begin{aligned} P(M \le x) &= P(M \le x \,|\, N=1)P(N=1) + P(M \le x \,|\, N>1)P(N>1) \\ &= F(x)p + F(x)P(M \le x)(1-p) \end{aligned}$$

again giving the result

$$P(M \le x) = \frac{pF(x)}{1-(1-p)F(x)}$$

62. The result is true when $n = 0$, so assume that

$$P\{N(x) \geq n\} = x^n/(n-1)!$$

Now,

$$\begin{aligned} P\{N(x) \geq n+1\} &= \int_0^1 P\{N(x) \geq n+1 \mid U_1 = y\}dy \\ &= \int_0^x P\{N(x-y) \geq n\}dy \\ &= \int_0^x P\{N(u) \geq n\}du \\ &= \int_0^x u^{n-1}/(n-1)!\ du \text{ by the induction hypothesis} \\ &= x^n/n! \end{aligned}$$

which completes the proof.

(b) $E[N(x)] = \sum_{n=0}^{\infty} P\{N(x) > n = \sum_{n=0}^{\infty} P\{N(x) \geq n+1\} = \sum_{n=0}^{\infty} x^n/n! = e^x$

63. (a) Number the red balls and the blue balls and let X_i equal 1 if the i^{th} red ball is selected and let it by 0 otherwise. Similarly, let Y_j equal 1 if the j^{th} blue ball is selected and let it be 0 otherwise.

$$\text{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$

Now,

$E[X_i] = E[Y_j] = 12/30$

$E[X_iY_j] = P\{\text{red ball } i \text{ and blue ball } j \text{ are selected}\} = \binom{28}{10} \Big/ \binom{30}{12}$

Thus,

$$\text{Cov}(X, Y) = 80\left[\binom{28}{10} \Big/ \binom{30}{12} - (12/30)^2\right] = -96/145$$

(b) $E[XY \mid X] = XE[Y \mid X] = X(12 - X)8/20$

where the above follows since given X, there are 12-X additional balls to be selected from among 8 blue and 12 non-blue balls. Now, since X is a hypergeometric random variable it follows that

$E[X] = 12(10/30) = 4$ and $E[X^2] = 12(18)(1/3)(2/3)/29 + 4^2 = 512/29$

As $E[Y] = 8(12/30) = 16/5$, we obtain

$$E[XY] = \frac{2}{5}(48 - 512/29) = 352/29,$$

and

$$\text{Cov}(X, Y) = 352/29 - 4(16/5) = -96/145$$

64. (a) $E[X] = E[X \mid \text{type 1}]p + E[X \mid \text{type 2}](1 - p) = p\mu_1 + (1 - p)\mu_2$

(b) Let I be the type.

$E[X \mid I] = \mu_I$, $\text{Var}(X \mid I) = \sigma_I^2$

$$\text{Var}(X) = E[\sigma_I^2] + \text{Var}(\mu_I)$$
$$= p\sigma_1^2 + (1-p)\sigma_2^2 + p\mu_1^2 + (1-p)\mu_2^2 - [p\mu_1 + (1-p)\mu_2]^2$$

65. Let X be the number of storms, and let $G(B)$ be the events that it is a good (bad) year. Then

$$E[X] = E[X \mid G]P(G) + E[X \mid B]P(B) = 3(.4) + 5(.6) = 4.2$$

If Y is Poisson with mean λ, then $E[Y^2] = \lambda + \lambda^2$. Therefore,

$$E[X^2] = E[X^2 \mid G]P(G) + E[X^2 \mid B]P(B) = 12(.4) + 30(.6) = 22.8$$

Consequently,

$$\text{Var}(X) = 22.8 - (4.2)^2 = 5.16$$

66. $$E[X^2] = \frac{1}{3}\{E[X^2 \mid Y = 1] + E[X^2 \mid Y = 2] + E[X^2 \mid Y = 3]\}$$
$$= \frac{1}{3}\{9 + E[(5 + X)^2] + E[(7 + X)^2]\}$$
$$= \frac{1}{3}\{83 + 24E[X] + 2E[X^2]\}$$
$$= \frac{1}{3}\{443 + 2E[X^2]\} \quad \text{since } E[X] = 15$$

Hence,

$$\text{Var}(X) = 443 - (15)^2 = 218.$$

67. Let F_n denote the fortune after n gambles.

$$\begin{aligned} E[F_n] = E[E[F_n \mid F_{n-1}]] &= E[2(2p-1)F_{n-1}p + F_{n-1} - (2p-1)F_{n-1}] \\ &= (1 + (2p-1)^2)E[F_{n-1}] \\ &= [1 + (2p-1)^2]^2 E[F_{n-2}] \\ &\vdots \\ &= [1 + (2p-1)^2]^n E[F_0] \end{aligned}$$

68. (a) $.6e^{-2} + .4e^{-3}$

(b) $.6e^{-2}\dfrac{2^3}{3!} + .4e^{-3}\dfrac{3^3}{3!}$

(c) $P\{3 \mid 0\} = \dfrac{P\{3,0\}}{P\{0\}} = \dfrac{.6e^{-2}e^{-2}\dfrac{2^3}{3!} + .4e^{-3}e^{-3}\dfrac{3^3}{3!}}{.6e^{-2} + .4e^{-3}}$

69. (a) $\displaystyle\int_0^\infty e^{-x}e^{-x}dx = \frac{1}{2}$

(b) $\displaystyle\int_0^\infty e^{-x}\frac{x^3}{3!}e^{-x}dx = \frac{1}{96}\int_0^\infty e^{-y}y^3dy = \frac{\Gamma(4)}{96} = \frac{1}{16}$

(c) $\dfrac{\displaystyle\int_0^\infty e^{-x}e^{-x}\frac{x^3}{3!}e^{-x}dx}{\displaystyle\int_0^\infty e^{-x}e^{-x}dx} = \frac{2}{3^4} = \frac{2}{81}$

70. (a) $\displaystyle\int_0^1 pdp = 1/2$

(b) $\displaystyle\int_0^1 p^2dp = 1/3$

71. $$\begin{aligned} P\{X=i\} = \int_0^1 P\{X=i \mid p\}dp &= \int_0^1 \binom{n}{i}p^i(1-p)^{n-i}dp \\ &= \binom{n}{i}\frac{i!(n-i)!}{(n+1)!} = 1/(n+1) \end{aligned}$$

72. (a) $P\{N \geq i\} = \int_0^1 P\{N \geq i | p\} dp = \int_0^1 (1-p)^{i-1} dp = 1/i$

(b) $P\{N = i\} = P\{N \geq i\} - P\{N \geq i+1\} = \dfrac{1}{i(i+1)}$

(c) $E[N] = \sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} 1/i = \infty$.

73. (a) $E[R] = E[E[R \mid S]] = E[S] = \mu$

(b) $\text{Var}(R \mid S) = 1$, $E[R \mid S] = S$
$\text{Var}(R) = 1 + \text{Var}(S) = 1 + \sigma^2$

(c) $f_R(r) = \int f_S(s) F_{R|S}(r \mid s) ds$

$= C \int e^{-(s-\mu)^2/2\sigma^2} e^{-(r-s)^2/2} ds$

$= K \int \exp\left\{ -\left(S - \dfrac{\mu + r\sigma^2}{1+\sigma^2} \right) \Big/ 2\left(\dfrac{\sigma^2}{1+\sigma^2} \right) \right\} ds \exp\{-(ar^2 + br)\}$

Hence, R is normal.

(d) $E[RS] = E[E[RS \mid S]] = E[SE[R \mid S]] = E[S^2] = \mu^2 + \sigma^2$

$\text{Cov}(R, S) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

75. X is Poisson with mean $\lambda = 2$ and Y is Binomial with parameters 10, 3/4. Hence

(a) $P\{X+Y=2\} = P\{X=0\}P\{Y=2\} + P\{X=1\}P\{Y=1\} + P\{X=2\}P\{Y=0\}$

$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$

(b) $P\{XY=0\} = P\{X=0\} + P\{Y=0\} - P\{X=Y=0\}$
$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \dfrac{3}{4} = 15$

77. The joint moment generating function, $E[e^{tX+sY}]$ can be obtained either by using

$$E[e^{tX+sY}] = \int\int e^{tX+sY} f(x,y)dy\ dx$$

or by noting that Y is exponential with rate 1 and, given Y, X is normal with mean Y and variance 1. Hence, using this we obtain

$$E[e^{tX+sY} \mid Y] = e^{sY}E[E^{tX} \mid Y] = e^{sY}e^{Yt+t^2/2}$$

and so

$$\begin{aligned} E[e^{tX+sY}] &= e^{t^2/2}E[e^{(s+t)Y}] \\ &= e^{t^2/2}(1-s-t)^{-1},\ s+t<1 \end{aligned}$$

Setting first s and then t equal to 0 gives

$$\begin{aligned} E[e^{tX}] &= e^{t^2/2}(1-t)^{-1},\ t<1 \\ E[e^{sY}] &= (1-s)^{-1},\ s<1 \end{aligned}$$

78. Conditioning on the amount of the initial check gives

$$\begin{aligned} E[\text{Return}] &= E[\text{Return} \mid A]/2 + E[\text{Return} \mid \text{B}]/2 \\ &= \{AF(A) + B[1-F(A)]\}/2 + \{BF(B) + A[1-F(B)]\}/2 \\ &= \{A + B + [B-A][F(B)-F(A)]\}/2 \\ &> (A+B)/2 \end{aligned}$$

where the inequality follows since $[B-A]$ and $[F(B)-F(A)$ both have the same sign.

(b) If $x < A$ then the strategy will accept the first value seen: if $x > B$ then it will reject the first one seen; and if x lies between A and B then it will always yield return B. Hence,

$$E[\text{Return of } x\text{-strategy}] = \begin{cases} B & \text{if } A < x < B \\ (A+B)/2 & \text{otherwise} \end{cases}$$

(c) This follows from (b) since there is a positive probability that X will lie between A and B.

79. Let X_i denote sales in week i. Then

$$\begin{aligned} E[X_1 + X_2] &= 80 \\ \text{Var}(X_1 + X_2) &= \text{Var}(X_1) + \text{Var}(X_2) + 2\ \text{Cov}(X_1, X_2) \\ &= 72 + 2[.6(6)(6)] = 93.6 \end{aligned}$$

(a) With Z being a standard normal

$$\begin{aligned} P(X_1 + X_2 > 90) &= P\left(Z > \frac{90-80}{\sqrt{93.6}}\right) \\ &= P(Z > 1.034) \approx .150 \end{aligned}$$

(b) Because the mean of the normal $X_1 + X_2$ is less than 90 the probability that it exceeds 90 is increased as the variance of $X_1 + X_2$ increases. Thus, this probability is smaller when the correlation is .2.

(c) In this case,

$$P(X_1 + X_2 > 90) = P\left(Z > \frac{90 - 80}{\sqrt{72 + 2[.2(6)(6)]}} \right)$$
$$= P(Z > 1.076) \approx .141$$

Theoretical Exercises

1. Let $\mu = E[X]$. Then for any a

$$\begin{aligned} E[(X-a)^2] &= E[(X-\mu+\mu-a)^2] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\ &= E[(X-\mu)^2 + (\mu-a)^2 \end{aligned}$$

2. $$\begin{aligned} E[\,|X-a|\,] &= \int_{x<a} (a-x)f(x)dx + \int_{x>a} (x-a)f(x)dx \\ &= aF(a) - \int_{x<a} xf(x)dx + \int_{x>a} xf(x)dx - a[1-F(a)] \end{aligned}$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

3. $$E[g(X, Y)] = \int_0^\infty P\{g(X,Y) > a\}da$$

$$= \int_0^\infty \iint_{\substack{x,y:\\ g(x,y)>a}} f(x,y)dydxda = \iint \int_0^{g(x,y)} daf(x,y)dydx$$

$$= \iint g(x,y)dydx$$

4. $$g(X) = g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\,\frac{(X-\mu)^2}{2} + \ldots$$

$$\approx g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\,\frac{(X-\mu)^2}{2}$$

Now take expectations of both sides.

5. If we let X_k equal 1 if A_k occurs and 0 otherwise then

$$X = \sum_{k=1}^{n} X_k$$

Hence,

$$E[X] = \sum_{k=1}^{n} E[X_k] = \sum_{k=1}^{n} P(A_k)$$

But

$$E[X] = \sum_{k=1}^{n} P\{X \geq k\} = \sum_{k=1}^{n} P(C_k)\,.$$

6. $X = \int_0^\infty X(t)dt$ and taking expectations gives

$$E[X] = \int_0^\infty E[X(t)]\,dt = \int_0^\infty P\{X > t\}dt$$

7. (a) Use Exercise 6 to obtain that

$$E[X] = \int_0^\infty P\{X > t\}dt \geq \int_0^\infty P\{Y > t\}dt = E[Y]$$

(b) It is easy to verify that

$$X^+ \geq_{\text{st}} Y^+ \text{ and } Y^- \geq_{\text{st}} X^-$$

Now use part (a).

8. Suppose $X \geq_{\text{st}} Y$ and f is increasing. Then

$$\begin{aligned} P\{f(X) > a\} &= P\{X > f^{-1}(a)\} \\ &\geq P\{Y > f^{-1}(a)\} \text{ since } x \geq_{st} Y \\ &= P\{f(Y) > a\} \end{aligned}$$

Therefore, $f(X) \geq_{st} f(Y)$ and so, from Exercise 7,
$E[f(X)] \geq E[f(Y)]$.

On the other hand, if $E[f(X)] \geq E[f(Y)]$ for all increasing functions f, then by letting f be the increasing function

$$f(x) = \begin{matrix} 1 & \text{if } x > t \\ 0 & \text{otherwise} \end{matrix}$$

then

$$P\{X > t\} = E[f(X)] \geq E[f(Y)] = P\{Y > t\}$$

and so $X >_{\text{st}} Y$.

9. Let

$$I_j = \begin{cases} 1 & \text{if a run of size } k \text{ begins at the } j^{\text{th}} \text{ flip} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\text{Number of runs of size } k = \sum_{j=1}^{n-k+1} I_j$$

$$E[\text{Number of runs of size } k = E\left[\sum_{j=1}^{n-k+1} I_j\right]$$

$$= P(I_1 = 1) + \sum_{j=2}^{n-k} P(I_j = 1) + P(I_{n-k+1} = 1)$$

$$= p^k(1-p) + (n-k-1)p^k(1-p)^2 + p^k(1-p)$$

10. $$1 = E\left[\sum_1^n X_i \Big/ \sum_1^n X_i\right] = \sum_1^n E\left[X_i \Big/ \sum_1^n X_i\right] = nE\left[X_1 \Big/ \sum_1^n X_i\right]$$

Hence,

$$E\left[\sum_1^k X_i \Big/ \sum_1^n X_i\right] = k/n$$

11. Let

$$I_j = \begin{cases} 1 & \text{outcome } j \text{ never occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = \sum_1^r I_j$ and $E[X] = \int_{j=1}^r (1-p_j)^n$

12. Let

$$I_j = \begin{cases} 1 & \text{success on trial } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n P_j \quad \text{independence not needed}$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n p_j(1-p_j) \quad \text{independence needed}$$

13. Let

$$I_j = \begin{cases} 1 & \text{record at } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n P\{X_j \text{ is largest of } X_1,\ldots,X_j\} = \sum_1^n 1/j$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n \text{Var}(I_j) = \sum_1^n \frac{1}{j}\left(1-\frac{1}{j}\right)$$

15. $\mu = \sum_{i=1}^{n} p_i$ by letting Number = $\sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & i \text{ is success} \\ 0 & \text{---} \end{cases}$

Var(Number) = $\sum_{i=1}^{n} p_i(1-p_i)$

maximization of variance occur when $p_i \equiv \mu/n$

minimization of variance when $p_i = 1$, $i = 1, \ldots, [\mu]$, $p_{[\mu]+1} = \mu - [\mu]$

To prove the maximization result, suppose that 2 of the p_i are unequal—say $p_i \neq p_j$. Consider a new p-vector with all other p_k, $k \neq i, j$, as before and with $\overline{p}_i = \overline{p}_j = \dfrac{p_i + p_j}{2}$. Then in the variance formula, we must show

$$2\left(\frac{p_i + p_j}{2}\right)\left(1 - \frac{p_i + p_j}{2}\right) \geq p_i(1-p_i) + p_j(1-p_j)$$

or equivalently,

$$p_i^2 + p_j^2 - 2p_ip_j = (p_i - p_j)^2 \geq 0.$$

The maximization is similar.

16. Suppose that each element is, independently, equally likely to be colored red or blue. If we let X_i equal 1 if all the elements of A_i are similarly colored, and let it be 0 otherwise, then $\sum_{i=1}^{r} X_i$ is the number of subsets whose elements all have the same color. Because

$$E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = \sum_{i=1}^{r} 2(1/2)^{|A_i|}$$

it follows that for at least one coloring the number of monocolored subsets is less than or equal to $\sum_{i=1}^{r} (1/2)^{|A_i|-1}$

17. $\mathrm{Var}(\lambda X_1 + (1-\lambda)X_2) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

$\dfrac{d}{d\lambda}(\quad) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda = \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

As $\mathrm{Var}(\lambda X_1 + (1-\lambda)X_2) = E\left[(\lambda X_1 + (1-\lambda)X_2 - \mu)^2\right]$ we want this value to be small.

18. (a. Binomial with parameters m and $P_i + P_j$.

(b) Using (a) we have that $\text{Var}(N_i + N_j) = m(P_i + P_j)(1 - P_i - P_j)$ and thus

$$m(P_i + P_j)(1 - P_i - P_j) = mP_i(1 - P_i) + mP_j(1 - P_j) + 2\,\text{Cov}(N_i, N_j)$$

Simplifying the above shows that

$$\text{Cov}(N_i, N_j) = -mP_iP_j.$$

19. $$\begin{aligned}\text{Cov}(X + Y, X - Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y)\\ &= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y)\\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

20. (a) $\text{Cov}(X, Y \mid Z)$

$$\begin{aligned}&= E[XY - E[X \mid Z]Y - XE[Y \mid Z] + E[X \mid Z]E[Y \mid Z]\,[Z]\\ &= E[XY \mid Z] - E[X \mid Z]\,E[Y \mid Z] - E[X \mid Z]E[Y \mid Z] + E[X \mid Z]E[Y \mid Z]\\ &= E[XY \mid Z] - E[X \mid Z]E[Y \mid Z]\end{aligned}$$

where the next to last equality uses the fact that given Z, $E[X \mid Z]$ and $E[Y \mid Z]$ can be treated as constants.

(b) From (a)

$$E[\text{Cov}(X, Y \mid Z)] = E[XY] - E[E[X \mid Z]E[Y \mid Z]]$$

On the other hand,

$$\text{Cov}(E[X \mid Z], E[Y \mid Z] = E[E[X \mid Z]E[Y \mid Z]] - E[X]E[Y]$$

and so

$$\begin{aligned}E[\text{Cov}(X, Y \mid Z)] + \text{Cov}(E[X \mid Z], E[Y \mid Z]) &= E[XY] - E[X]E[Y]\\ &= \text{Cov}(X, Y)\end{aligned}$$

(c) Noting that $\text{Cov}(X, X \mid Z) = \text{Var}(X \mid Z)$ we obtain upon setting $Y = Z$ that

$$\text{Var}(X) = E[\text{Var}(X \mid Z)] + \text{Var}(E[X \mid Z])$$

21. (a) Using the fact that f integrates to 1 we see that

$c(n, i) \equiv \int_0^1 x^{i-1}(1-x)^{n-i}\,dx = (i-1)!(n-i)!/n!$. From this we see that

$$E[X_{(i)}] = c(n+1, i+1)/c(n, i) = i/(n+1)$$

$$E[X_{(i)}^2] = c(n+2, i+2)/c(n, i) = \frac{i(i+1)}{(n+2)(n+1)}$$

and thus

$$\text{Var}(X_{(i)}) = \frac{i(n+1-i)}{(n+1)^2(n+2)}$$

(b) The maximum of $i(n + 1 - i)$ is obtained when $i = (n + 1)/2$ and the minimum when i is either 1 or n.

22. $\text{Cov}(X, Y) = b\ \text{Var}(X)$, $\text{Var}(Y) = b^2\ \text{Var}(X)$

$$\rho(X,Y) = \frac{b\ \text{Var}(X)}{\sqrt{b^2}\text{Var}(X)} = \frac{b}{|b|}$$

26. Follows since, given X, $g(X)$ is a constant and so

$$E[g(X)Y \mid X] = g(X)E[Y \mid X]$$

27. $E[XY] = E[E[XY \mid X]]$
$= E[XE[Y \mid X]]$

Hence, if $E[Y \mid X] = E[Y]$, then $E[XY] = E[X]E[Y]$. The example in Section 3 of random variables uncorrelated but not independent provides a counterexample to the converse.

28. The result follows from the identity

$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$ which is obtained by noting that, given X, X may be treated as a constant.

29. $$x = E[X_1 + \ldots + X_n \mid X_1 + \ldots + X_n = x] = E\left[X_1 \middle| \sum X_i = x\right] + \ldots + E\left[X_n \middle| \sum X_i = x\right]$$
$$= nE\left[X_1 \middle| \sum X_i = x\right]$$

Hence, $E[X_1 \mid X_1 + \ldots + X_n = x] = x/n$

30. $E[N_iN_j \mid N_i] = N_iE[N_j \mid N_i] = N_i(n - N_i)\dfrac{p_j}{1 - p_i}$ since each of the $n - N_i$ trials no resulting in outcome i will independently result in j with probability $p_j/(1 - p_i)$. Hence,

$$E[N_iN_j] = \frac{p_j}{1-p_i}\left(nE[N_i] - E\left[N_i^2\right]\right) = \frac{p_j}{1-p_i}\left[n^2p_i - n^2p_i^2 - np_i(1-p_i)\right]$$
$$= n(n-1)p_ip_j$$

and

$$\text{Cov}(N_i, N_j) = n(n-1)p_ip_j - n^2p_ip_j = -np_ip_j$$

31. By induction: true when $t = 0$, so assume for $t - 1$. Let $N(t)$ denote the number after stage t.

$$E[N(t) \mid N(t-1)] = N(t-1) - E[\text{number selected}]$$
$$= N(t-1) - N(t-1)\frac{r}{b+w+r}$$
$$E[N(t) \mid N(t-1)] = N(t-1)\frac{b+w}{b+w+r}$$
$$E[N(t)] = \left(\frac{b+w}{b+w+r}\right)^t w$$

32. $E[X_1X_2 \mid Y = y] = E[X_1 \mid Y = y]E[X_2 \mid Y = y] = y^2$

Therefore, $E[X_1X_2 \mid Y] = Y^2$. As $E[X_i \mid Y] = Y$, this gives that

$$E[X_1X_2] = E[E[X_1X_2 \mid Y]] = Ei[\mathrm{Y}^2], \quad E[X_i] = E[E[X_i \mid Y]] = E[Y]$$

Consequently,

$$\mathrm{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] = \mathrm{Var}(Y)$$

34. (a) $E[T_r \mid T_{r-1}] = T_{r-1} + 1 + (1-p)E[T_r]$

(b) Taking expectations of both sides of (a) gives

$$E[T_r] = E[T_{r-1}] + 1 + (1-p)E[T_r]$$

or

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$

(c) Using the result of part (b) gives

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$
$$= \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}E[T_{r-2}]\right)$$
$$= 1/p + (1/p)^2 + (1/p)^2E[T_{r-2}]$$
$$= 1/p + (1/p)^2 + (1/p)^3 + (1/p)^3E[T_{r-3}]$$
$$= \sum_{i=1}^{r}(1/p)^i + (1/p)^r E[T_0]$$
$$= \sum_{i=1}^{r}(1/p)^i \quad \text{since } E[T_0] = 0.$$

35. $$\begin{aligned} P(Y > X) &= \sum_j P(Y > X \mid X = j) p_j \\ &= \sum_j P(Y > j \mid X = j) p_j \\ &= \sum_j P(Y > j) p_j \\ &= \sum_j (1-p)^j p_j \end{aligned}$$

36. Condition on the first ball selected to obtain

$$M_{a,b} = \frac{a}{a+b} M_{a-1,b} + \frac{b}{a+b} M_{a,b-1}, \; a, b > 0$$

$$M_{a,0} = a, \qquad M_{0,b} = b, \qquad M_{a,b} = M_{b,a}$$

$$M_{2,1} = \frac{4}{3}, \qquad M_{3,1} = \frac{7}{4}, \qquad M_{3,2} = 3/2$$

37. Let X_n denote the number of white balls after the n^{th} drawing

$$E[X_{n+1} \mid X_n] = X_n \frac{X_n}{a+b} + (X_n + 1)\left(1 - \frac{X_n}{a+b}\right) = \left(1 - \frac{1}{a+b}\right) X_n + 1$$

Taking expectations now yields (a).

To prove (b), use (a) and the boundary condition $M_0 = a$

(c) $P\{(n+1)\text{st is white}\} = E[P\{(n+1)\text{st is white} \mid X_n\}]$

$$= E\left[\frac{X_n}{a+b}\right] = \frac{M_n}{a+b}$$

40. For (a) and (c), see theoretical Exercise 18 of Chapter 6. For (c)

$$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$$

$$= E\left[X\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)\right)\right]$$

$$= \mu_x\mu_y - \rho\frac{\sigma_y}{\sigma_x}\mu_x^2 + \rho\frac{\sigma_y}{\sigma_x}\left(\mu_x^2 + \sigma_x^2\right)$$

and so

$$\text{Corr}(X, Y) = \frac{\rho\sigma_y\sigma_x}{\sigma_y\sigma_x} = \rho$$

41. (a) No

(b) Yes, since $f_Y(x \mid I = 1) = f_X(x) = f_X(-x) = f_Y(x \mid I = 0)$

(c) $f_Y(x) = \frac{1}{2} f_X(x) + \frac{1}{2} f_X(-x) = f_X(x)$

(d) $E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]] = 0$

(e) No, since X and Y are not jointly normal.

42. If $E[Y \mid X]$ is linear in X, then it is the best linear predictor of Y with respect to X.

43. Must show that $E[Y^2] = E[XY]$. Now

$$\begin{aligned} E[XY] &= E[XE[X \mid Z]] \\ &= E[E[XE[X \mid Z] \mid Z]] \\ &= E[E^2[X \mid Z]] = E[Y^2] \end{aligned}$$

44. Write $X_n = \sum_{i=1}^{X_{n-1}} Z_i$ where Z_i is the number of offspring of the ith individual of the $(n-1)$st generation. Hence,

$$E[X_n] = E[E[X_n \mid X_{n-1}]] = E[\mu X_{n-1}] = \mu E[X_{n-1}]$$

so,

$$E[X_n] = \mu E[X_{n-1}] = \mu^2 E[X_{n-2}] \ \ldots = \mu^n E[X_0] = \mu^n$$

(c) Use the above representation to obtain

$$E[X_n \mid X_{n-1}] = \mu X_{n-1}, \ \mathrm{Var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$$

Hence, using the conditional Variance Formula,

$$\mathrm{Var}(X_n) = \mu^2 \ \mathrm{Var}(X_{n-1}) + \sigma^2 \mu^{n-1}$$

(d) $\pi = P\{\text{dies out}\}$

$= \sum_j P\{\text{dies out} \mid X_i = j\} p_j$

$= \sum_j \pi^j p_j$, since each of the j members of the first generation can be thought of as starting their own (independent) branching process.

46. It is easy to see that the n^{th} derivative of $\sum_{j=0}^{\infty}(t^2/2)^j/j!$ will, when evaluated at $t = 0$, equal 0 whenever n is odd (because all of its terms will be constants multiplied by some power of t). When $n = 2j$ the n^{th} derivative will equal $\frac{d^n}{dt^n}\{t^n\}/(j!2^j)$ plus constants multiplied by powers of t. When evaluated at 0, this gives that

$$E[Z^{2j}] - (2j)!/(j!2^j)$$

47. Write $X = \sigma Z + \mu$ where Z is a standard normal random variable. Then, using the binomial theorem,

$$E[X^n] = \sum_{i=0}^{n}\binom{n}{i}\sigma^i E[Z^i]\mu^{n-i}$$

Now make use of theoretical exercise 46.

48. $\phi_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb}E[e^{taX}] = e^{tb}\phi_X(ta)$

49. Let $Y = \log(X)$. Since Y is normal with mean μ and variance σ^2 it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t+\sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that

$$E[X] = M(1) = e^{\mu+\sigma^2/2}$$

and

$$E[X^2] = M(2) = e^{2\mu+2\sigma^2}$$

Therefore,

$$\text{Var}(X) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

50. $\psi(t) = \log \phi(t)$

$\psi'(t) = \phi'(t)/\phi(t)$

$$\psi''(t) = \frac{\phi(t)\phi''(t) - (\phi'(t))^2}{\phi^2(t)}$$

$$\psi''(t)\Big|_{t=0} = E[X^2] - (E[X])^2 = \text{Var}(X).$$

51. Gamma (n, λ)

52. Let $\phi(s, t) = E[e^{sX+tY}]$

$$\frac{\partial^2}{\partial s \partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[XYe^{sX+tY}]\Big|_{\substack{s=0\\t=0}} = E[XY]$$

$$\frac{\partial}{\partial s}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[X], \quad \frac{\partial}{\partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[Y]$$

53. Follows from the formula for the joint moment generating function.

54. By symmetry, $E[Z^3] = E[Z] = 0$ and so $\text{Cov}(Z, Z^3) = 0$.

55. (a) This follows because the conditional distribution of $Y + Z$ given that $Y = y$ is normal with mean y and variance 1, which is the same as the conditional distribution of X given that $Y = y$.

(b) Because $Y + Z$ and Y are both linear combinations of the independent normal random variables Y and Z, it follows that $Y + Z$, Y has a bivariate normal distribution.

(c) $\mu_x = E[X] = E[Y + Z] = \mu$

$\sigma_x^2 = \text{Var}(X) = \text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) = \sigma^2 + 1$

$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(Y + Z, Y)}{\sigma\sqrt{\sigma^2 + 1}} = \frac{\sigma}{\sqrt{\sigma^2 + 1}}$$

(d) and (e) The conditional distribution of Y given $X = x$ is normal with mean

$$E[Y \mid X = x] = \mu + \rho\frac{\sigma}{\sigma_x}(x - \mu_x) = \mu + \frac{\sigma^2}{1 + \sigma^2}(x - \mu)$$

and variance

$$\text{Var}(Y \mid X = x) = \sigma^2\left(1 - \frac{\sigma^2}{\sigma^2 + 1}\right) = \frac{\sigma^2}{\sigma^2 + 1}$$

Chapter 8

Problems

1. $P\{0 \le X \le 40\} = 1 - P\{|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let Z be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \approx P\{|Z| > \sqrt{n}\} \le .1 \text{ when } n = 3$$

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b) $$\begin{aligned} P\left\{\sum_{i=1}^{20} X_i > 15\right\} &= P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \\ &\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \\ &= P\{Z > -1.006\} \\ &\approx .8428 \end{aligned}$$

5. Letting X_i denote the i^{th} roundoff error it follows that $E\left[\sum_{i=1}^{50} X_i\right] = 0$,

   $\text{Var}\left(\sum_{i=1}^{50} X_i\right) = 50\ \text{Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform (0, 1) and so $\text{Var}(X) = \text{Var}(.5 + X) = 1/12$. Hence,

$$\begin{aligned} P\left\{\left|\sum X_i\right| > 3\right\} &\approx P\{|N(0, 1)| > 3(12/50)^{1/2}\} \text{ by the central limit theorem} \\ &= 2P\{N(0, 1) > 1.47 = .1416 \end{aligned}$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$\begin{aligned} P\left\{\sum_{i=1}^{79} X_i \le 300\right\} &= P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\} \\ &\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429 \end{aligned}$$

7. $$P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525-500}{\sqrt{(100\times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance $2500 + 99/48 = 2502$ it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$
It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

9. Use the fact that a gamma $(n, 1)$ random variable is the sum of n independent exponentials with rate 1 and thus has mean and variance equal to n, to obtain:

$$\begin{aligned} P\left\{\left|\frac{X-n}{n}\right| > .01\right\} &= P\left\{|X-n|/\sqrt{n} > .01\sqrt{n}\right\} \\ &\approx P\left\{|N(0,1)| > .01\sqrt{n}\right\} \\ &= 2P\left\{N(0,1) > .01\sqrt{n}\right\} \end{aligned}$$

Now $P\{N(0, 1) > 2.58\} = .005$ and so $n = (258)^2$.

10. If W_n is the total weight of n cars and A is the amount of weight that the bridge can withstand then $W_n - A$ is normal with mean $3n - 400$ and variance $.09n + 1600$. Hence, the probability of structural damage is

$$P\{W_n - A \ge 0\} \approx P\left\{Z \ge (400-3n)/\sqrt{.09n+1600}\right\}$$

Since $P\{Z \ge 1.28\} = .1$ the probability of damage will exceed .1 when n is such that

$$400 - 3n \le 1.28\sqrt{.09n+1600}$$

The above will be satisfied whenever $n \ge 117$.

12. Let L_i denote the life of component i.

$$E\left[\sum_{i=1}^{100} L_i\right] = 1000 + \frac{1}{10}50(101) = 1505$$

$$\text{Var}\left(\sum_{i=1}^{100} L_i\right) = \sum_{i=1}^{100}\left(10 + \frac{i}{10}\right)^2 = (100)^2 + (100)(101) + \frac{1}{100}\sum_{i=1}^{100} i^2$$

Now apply the central limit theorem to approximate.

13. (a) $P\{\overline{X} > 80\} = P\left\{\dfrac{\overline{X} - 74}{14/5} > 15/7\right\} \approx P\text{PZ} > 2.14\} \approx .0162$

(b) $P\{\overline{Y} > 80\} = P\left\{\dfrac{\overline{Y} - 74}{14/8} > 24/7\right\} \approx P\{Z > 3.43\} \approx .0003$

(c) Using that $SD(\overline{Y} - \overline{X}) = \sqrt{196/64 + 196/25} \approx 3.30$ we have

$$\begin{aligned} P\{\overline{Y} - \overline{X} > 2.2\} &= P\{\overline{Y} - \overline{X}\}/3.30 > 2.2/3.30\} \\ &\approx P\{Z > .67\} \approx .2514 \end{aligned}$$

(d) same as in (c)

14. Suppose n components are in stock. The probability they will last for at least 2000 hours is

$$p = P\left\{\sum_{i=1}^{n} X_i \geq 2000\right\} \approx P\left\{Z \geq \frac{2000 - 100n}{30\sqrt{n}}\right\}$$

where Z is a standard normal random variable. Since

$.95 = P\{Z \geq -1.64\}$ it follows that $p \geq .95$ if

$$\frac{2000 - 100n}{30\sqrt{n}} \leq -1.64$$

or, equivalently,

$$(2000 - 100n)/\sqrt{n} \leq -49.2$$

and this will be the case if $n \geq 23$.

15. $$P\left\{\sum_{i=1}^{10,000} X_i > 2{,}700{,}000\right\} \approx P\{Z \geq (2{,}700{,}000 - 2{,}400{,}000)/(800 \cdot 100)\} = P\{Z \geq 3.75\} \approx 0$$

18. Let Y_i denote the additional number of fish that need to be caught to obtain a new type when there are at present i distinct types. Then Y_i is geometric with parameter $\dfrac{4-i}{4}$.

$$E[Y] = E\left[\sum_{i=0}^{3} Y_i\right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}$$

$$\text{Var}[Y] = \text{Var}\left(\sum_{i=0}^{3} Y_i\right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}$$

Hence,

$$P\left\{\left|Y-\frac{25}{3}\right|>\frac{25}{3}\sqrt{\frac{1300}{9}}\right\}\le\frac{1}{10}$$

and so we can take $a=\dfrac{25-\sqrt{1300}}{3}$, $b=\dfrac{25+\sqrt{1300}}{3}$.

Also,

$$P\left\{Y-\frac{25}{3}>a\right\}\le\frac{130}{130+9a^2}=\frac{1}{10}\ \text{ when } a=\frac{\sqrt{1170}}{3}.$$

Hence $P\left\{Y>\dfrac{25+\sqrt{1170}}{3}\right\}\le .1.$

20. $g(x)=x^{n(n-1)}$ is convex. Hence, by Jensen's Inequality

$E[Y^{n/(n-1)}]\ge E[Y])^{n/(n-1)}$ Now set $Y=X^{n-1}$ and so
$E[X^n]\ge(E[X^{n-1}])^{n/(n-1)}$ or $(E[X^n])^{1/n}\ge(E[X^{n-1}])^{1/(n-1)}$

21. No

22. (a) $20/26\approx .769$

(b) $20/(20+36)=5/14\approx .357$

(d) $p\approx P\{Z\ge(25.5-20)/\sqrt{20}\}\approx P\{Z\ge 1.23\}\approx .1093$

(e) $p=.112184$

Theoretical Exercises

1. This follows immediately from Chebyshev's inequality.

2. $P\{D > \alpha\} = P\{|X - \mu| > \alpha\mu\} \le \dfrac{\varsigma^2}{\alpha^2\mu^2} = \dfrac{1}{\alpha^2 r^2}$

3. (a) $\dfrac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$

   (b) $\dfrac{np}{\sqrt{np(1-p)}} = \sqrt{np/(1-p)}$

   (c) answer = 1

   (d) $\dfrac{1/2}{\sqrt{1/12}} = \sqrt{3}$

   (e) answer = 1

   (d) answer = $|\mu|/\sigma$

4. For $\varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(c)| < \varepsilon$ whenever $|x - c| \le \delta$. Also, let B be such that $|g(x)| < B$. Then,

$$\begin{aligned} E[g(Z_n)] &= \int_{|x-c|\le\delta} g(x)dF_n(x) + \int_{|x-c|>\delta} g(x)dF_n(x) \\ &\le (\varepsilon + g(c))P\{|Z_n - c| \le \delta\} + BP\{|Z_n - c| > \delta\} \end{aligned}$$

   In addition, the same equality yields that

$$E[g(Z_n)] \ge (g(c) - \varepsilon)P\{|Z_n - c| \le \delta\} - BP\{|Z_n - c| > \delta\}$$

   Upon letting $n \to \infty$, we obtain that

$$\begin{aligned} &\text{lim sup } E[g(Z_n)] \le g(c) + \varepsilon \\ &\text{lim inf } E[g(Z_n)] \ge g(c) - \varepsilon \end{aligned}$$

   The result now follows since ε is arbitrary.

5. Use the notation of the hint. The weak law of large numbers yields that

$$\lim_{n\to\infty} P\{|(X_1 + \ldots + X_n)/n - c| > \varepsilon\} = 0$$

Since $X_1 + \ldots + X_n$ is binomial with parameters n, x, we have

$$E\left[f\left(\frac{X_1 + \ldots + X_n}{n}\right)\right] = \sum_{k=1}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

The result now follows from Exercise 4.

6. $$\begin{aligned} E[X] &= \sum_{i=1}^{k} i\ P\{X = i\} + \sum_{i=k+1}^{\infty} i\ P\{X = i\} \\ &\geq \sum_{i=1}^{k} i\ P\{X = k\} \\ &= P\{X = k\}^{k(k+1)/2} \\ &\geq \frac{k^2}{2} P\{X = k\} \end{aligned}$$

7. Take logs and apply the central limit theorem

8. It is the distribution of the sum of t independent exponentials each having rate λ.

9. 1/2

10. Use the Chernoff bound: $e^{-ti}M(t) = e^{\lambda(e^t - 1) - ti}$ will obtain its minimal value when t is chosen to satisfy

$$\lambda e^t = i, \text{ and this value of } t \text{ is negative provided } i < \lambda.$$

Hence, the Chernoff bound gives

$$P\{X \leq i\} \leq e^{i-\lambda}(\lambda/i)^i$$

11. $e^{-ti}M(t) = (pe^t + q)^n e^{-ti}$ and differentiation shows that the value of t that minimizes it is such that

$$npe^t = i(pe^t + q) \text{ or } e^t = \frac{iq}{(n-i)p}$$

Using this value of t, the Chernoff bound gives that

$$\begin{aligned} P\{X \geq i\} &\leq \left(\frac{iq}{n-i} + q\right)^n (n-i)^i p^i / (iq)^i \\ &= \frac{(nq)^n (n-i)^i p^i}{i^i q^i (n-i)^n} \end{aligned}$$

12. $1 = E[e^{\theta X}] \geq e^{\theta E[X]}$ by Jensen's inequality.

Hence, $\theta E[X] \leq 0$ and thus $\theta > 0$.

Chapter 9

Problems and Theoretical Exercises

1. (a) $P(2 \text{ arrivals in } (0, s) \mid 2 \text{ arrivals in } (0, 1))$

$= P\{2 \text{ in } (0, s), 0 \text{ in } (s, 1)\}/e^{-\lambda}\lambda^2/2)$
$= [e^{-\lambda s}(\lambda s)^2/2][e^{-(1-s)\lambda}]/(e^{-\lambda}\lambda^2/2) = s^2 = 1/9 \text{ when } s = 1/3$

(b) $1 - P\{\text{both in last 40 minutes}) = 1 - (2/3)^2 = 5/9$

2. $e^{-3s/60}$

3. $e^{-3s/60} + (s/20)e^{-3s/60}$

8. The equations for the limiting probabilities are:

$$\prod_c = .7\prod_c + .4\prod_s + .2\prod_g$$
$$\prod_s = .2\prod_x + .3\prod_s + .4\prod_g$$
$$\prod_g = .1\prod_c + .3\prod_s + .4\prod_g$$
$$\prod_c + \prod_s + \prod_g = 1$$

and the solution is: $\prod_c = 30/59$, $\prod_s = 16/59$, $\prod_g = 13/59$. Hence, Buffy is cheerful 3000/59 percent of the time.

9. The Markov chain requires 4 states:

0 = RR = Rain today and rain yesterday

1 = RD = Dry today, rain yesterday

2 = DR = Rain today, dry yesterday

3 = DD = Dry today and dry yesterday

with transition probability matrix

$$\underline{P} = \begin{vmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .3 & .7 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \end{vmatrix}$$

The equations for the limiting probabilities are:

$$\Pi_0 = .8\Pi_0 + .4\Pi_2$$
$$\Pi_1 = .2\Pi_0 + .6\Pi_2$$
$$\Pi_2 = .3\Pi_1 + .2\Pi_3$$
$$\Pi_3 = .7\Pi_1 + .8\Pi_3$$
$$\Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = 1$$

which gives

$$\Pi_0 = 4/15,\ \Pi_1 = \Pi_2 = 2/15,\ \Pi_3 = 7/15.$$

Since it rains today when the state is either 0 or 2 the probability is 2/5.

10. Let the state be the number of pairs of shoes at the door he leaves from in the morning. Suppose the present state is i, where $i > 0$. Now after his return it is equally likely that one door will have i and the other $5 - i$ pairs as it is that one will have $i - 1$ ant the other $6 - i$. Hence, since he is equally likely to choose either door when he leaves tomorrow it follows that

$$P_{i,i} = P_{i,5-i} = P_{i,i-1} = P_{i,6-i} = 1/4$$

provided all the states i, $5 - i$, $i - 1$, $6 - i$ are distinct. If they are not then the probabilities are added. From this it is easy to see that the transition matrix P_{ij}, $i, j = 0, 1, \ldots, 5$ is as follows:

$$\underline{P} = \begin{matrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \end{matrix}$$

Since this chain is doubly stochastic (the column sums as well as the row sums all equal to one) it follows that $\Pi_i = 1/6$, $i = 0, \ldots, 5$, and thus he runs barefooted one-sixth of the time.

11. (b) 1/2

(c) Intuitively, they should be independent.

(d) From (b) and (c) the (limiting) number of molecules in urn 1 should have a binomial distribution with parameters (M, 1/2).

Chapter 10

1. (a) After stage k the algorithm has generated a random permutation of 1, 2, …, k. It then puts element $k + 1$ in position $k + 1$; randomly chooses one of the positions 1, …, $k + 1$ and interchanges the element in that position with element $k + 1$.

   (b) The first equality in the hint follows since the permutation given will be the permutation after insertion of element k if the previous permutation is $i_1, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}$ and the random choice of one of the k positions of this permutation results in the choice of position j.

2. Integrating the density function yields that that distribution function is

$$F(x) = \begin{cases} e^{2x}/2, & x > 0 \\ 1 - e^{-2s}/2, & x > 0 \end{cases}$$

which yields that the inverse function is given by

$$F^{-1}(u) = \begin{cases} \log(2u)/2 & \text{if } u < 12 \\ -\log(2[1-u])/2 & \text{if } u > 1/2 \end{cases}$$

Hence, we can simulate X from F by simulating a random number U and setting $X = F^{-1}(U)$.

3. The distribution function is given by

$$F(x) = \begin{cases} x^2/4 - x + 1, & 2 \le x \le 3, \\ x - x^2/12 - 2, & 3 \le x \le 6 \end{cases}$$

Hence, for $u \le 1/4$, $F^{-1}(u)$ is the solution of

$$x^2/4 - x + 1 = u$$

that falls in the region $2 \le x \le 3$. Similarly, for $u \ge 1/4$, $F^{-1}(u)$ is the solution of

$$x - x^2/12 - 2 = u$$

that falls in the region $3 \le x \le 6$. We can now generate X from F by generating a random number U and setting $X = F^{-1}(U)$.

4. Generate a random number U and then set $X = F^{-1}(U)$. If $U \le 1/2$ then $X = 6U - 3$, whereas if $U \ge 1/2$ then X is obtained by solving the quadratic $1/2 + X^2/32 = U$ in the region $0 \le X \le 4$.

5. The inverse equation $F^{-1}(U) = X$ is equivalent to

or

$$1 - e^{-\alpha X^{\beta}} = U$$
$$X = \{-\log(1 - U)/\alpha\}^{1/\beta}$$

Since $1 - U$ has the same distribution as U we can generate from F by generating a random number U and setting $X = \{-\log(U)/\alpha\}^{1/\beta}$.

6. If $\lambda(t) = ct^n$ then the distribution function is given by

$$1 - F(t) = \exp\{-kt^{n+1}\},\ t \geq 0 \text{ where } k = c/(n + 1)$$

Hence, using the inverse transform method we can generate a random number U and then set X such that

$$\exp\{-kX^{n+1}\} = 1 - U$$

or

$$X = \{-\log(1 - U)/k\}^{1/(n+1)}$$

Again U can be used for $1 - U$.

7. (a) The inverse transform method shows that $U^{1/n}$ works.

(b) $P\{\text{Max}U_i \leq v\} = P\{U_1 \leq x, \ldots, U_n \leq x\}$
$= \prod P\{U_i \leq x\}$ by independence
$= x^n$

(c) Simulate n random numbers and use the maximum value obtained.

8. (a) If X_i has distribution F_i , $i = 1, \ldots, n$, then, assuming independence, F is the distribution of $\text{Max}X_i$. Hence, we can simulate from F by simulating X_i, $i = 1, \ldots, n$ and setting $X = \text{Max}X_i$.

(b) Use the method of (a) replacing Max by Min throughout.

9. (a) Simulate X_i from F_i, $i = 1, 2$. Now generate a random number U and set X equal to X_1 if $U < p$ and equal to X_2 if $U > p$.

(b) Note that

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-3x}, \quad x > 0,\ F_2(x) = x, \quad 0 < x < 1$$

Hence, using (a) let U_1, U_2, U_3 be random numbers and set

$$X = \begin{cases} -\log(U_1)/3 & \text{if } U_3 < 1/3 \\ U_2 & \text{if } U_3 > 1/3 \end{cases}$$

where the above uses that $-\log(U_1)/3$ is exponential with rate 3.

10. With $g(x) = \lambda e^{-\lambda x}$

$$\frac{f(x)}{g(x)} = \frac{2e^{-x^2/2}}{\lambda(2\pi)^{1/2}e^{-\lambda x}} = \frac{2}{\lambda(2\pi)^{1/2}}\exp\{-[(x-\lambda)^2 - \lambda^2]/2\}$$
$$\frac{2e^{\lambda^2/2}}{\lambda(2\pi)^{1/2}}\exp\{-(x-\lambda)^2/2\}$$

Hence, $c = 2e^{\lambda^2/2}/[\lambda(2\pi)^{1/2}]$ and simple calculus shows that this is minimized when $\lambda = 1$.

11. Calculus yields that the maximum value of $f(x)/g(x) = 60x^3(1-x)^2$ is attained when $x = 3/5$ and is thus equal to 1296/625. Hence, generate random numbers U_1 and U_2 and set $X = U_1$ if $U_2 \le 3125U_1^3(1-U_1)^2/108$. If not, repeat.

12. Generate random numbers $U_1, \ldots, U_n$, and approximate the integral by $[k(U_1) + \ldots + k(U_n)]/n$. This works by the law of large numbers since $E[k(U)] = \int_0^1 k(x)dx$.

16. $E[g(X)/f(X)] = \int [g(x)/f(x)]f(x)dx = \int g(x)dx$

Corrections to Ross, A FIRST COURSE IN PROBABILITY, seventh ed.

- p. 79, line 2: $P(\sum_{j=1}^{n} R_j) \rightarrow P(\bigcup_{j=1}^{n} R_j)$
- p. 95, centered eq. on line 12: $P_{n,m-1,m} \rightarrow P_{n,m-1}$
- p. 288, l. 2: change "when $j > r$." to "when $j < 0$."
- p. 309, first line of Example 8b: change "let Y_i denote the selection " to "let Y_i denote the selection "
- p. 373, line -8: on the centered equation following "the preceding equation yields" add a right paren at the very end. That is,
  $(1-(1-p_i)^n \rightarrow (1-(1-p_i)^n))$
- p. 416, line 1: change "Let $X_1, \ldots, X_n$ be independent" to "Let $X_1, \ldots$ be independent"
- p. 509, lines 6 and 7: 73. should be 83. and 74. should be 84.
- p. 509, Solution to Problem 68 of Chapter 4: change
  $(1-e^{-5})^{80}$ to $(1-e^{-5})^{10}$