## Tarea I Métodos del Análisis No-Lineal 2009/1

(1) Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$. Consider the problem with *infinite boundary condition*

$$-\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \Omega,$$

$$u(x) \to +\infty \quad \text{when dist}\,(x, \partial\Omega) \to 0.$$

Assume that $p > 1$. Prove that this problem has a classical solution. To do so, prove first that the approximate problem

$$-\Delta u_n + u_n^p = 0, \quad u_n > 0 \quad \text{in } \Omega,$$

$$u_n(x) = n \quad \text{on } \partial\Omega.$$

has a classical solution for all $n > 0$. Moreover,

$$u_n(x) < u_{n+1}(x) \quad \text{for all } n.$$

Let

$$u_*(x) \equiv \lim_{n \to +\infty} u_n(x)$$

Prove that this limit indeed exists and that it defines a classical solution of (1). To do so, use the super-subsolution method to prove that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \text{dist}\,(x, \partial\Omega)^{-\frac{2}{p-1}} \leq u_*(x) \leq C_2 \text{dist}\,(x, \partial\Omega)^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega.$$

(2) (Strauss, 1977). Consider the problem

$$\Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

$$u(x) \to 0 \quad \text{as } |x| \to +\infty,$$

where $p > 1$, $N \geq 2$, and $p < \frac{N+2}{N-2}$ if $N \geq 3$. Prove that there exists a solution $u$, radially symmetric ($u(x) = v(|x|)$) for this problem.

To do so, prove that if

$$H^1_{rad}(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \;/\; u \text{ is radially symmetric } \}$$

then the embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is compact (Strauss Lemma). Prove first that there is a $C > 0$ such that for any $u \in H^1_{rad}(\mathbb{R}^N)$ we have

$$|u(x)| \;\leq\; \frac{C}{|x|^{\frac{N-1}{2}}} \|u\|_{H^1(\mathbb{R}^N)} \quad \text{for all } x \in \mathbb{R}^N$$

(3) Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$\Delta u + \lambda u + u^{\frac{N+2}{N-2}} = 0, \quad u > 0 \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Consider the number

$$S_\lambda(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{\left(\int_\Omega |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}.$$

Denote by $S_N > 0$ the best constant in the Sobolev-Gagliardo-Nirenberg inequality, that is

$$S_N := \inf_{u \in C_0^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}.$$

(a) Prove that if $\lambda < \lambda_1$ and that if $S_\lambda(\Omega) < S_N$, then the infimum $S_\lambda(\Omega)$ is attained. To do so, consider a minimizing sequence $u_n$ with $\int_\Omega |u_n|^{\frac{2N}{N-2}} = 1$. Prove that this sequence is bounded in $H_0^1(\Omega)$ and that, passing to a subsequence, can be assumed to be convergent to a weak limit $u$. Show that this convergence is actually strong in $H_0^1(\Omega)$, establishing first the asymptotic relations

$$\int_\Omega |u_n|^{\frac{2N}{N-2}} = \int_\Omega |u|^{\frac{2N}{N-2}} + \int_\Omega |u_n - u|^{\frac{2N}{N-2}} + o(1),$$

$$\int_\Omega |\nabla u_n|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u_n - \nabla u|^2 + o(1),$$

with $o(1) \to 0$ as $n \to \infty$.

(b) Prove that if $N \geq 4$ y $0 < \lambda < \lambda_1$, then (1)-(2) has at least one classical solution (Brezis-Nirenberg, 1983). To do so, accept the fact (Aubin, Talenti, 1979) that the infimum $S_N$ is attained at the functions

$$u(x) = \frac{1}{(\mu^2 + |x - x_0|^2)^{\frac{N-2}{2}}},$$

for any $\mu > 0$, $x_0 \in \mathbb{R}^N$. Use these objects to build up test functions that show $S_\lambda(\Omega) < S_N$.