TAREA II MÉTODOS DEL ANÁLISIS NO-LINEAL 2009/1

ENTREGA: LUNES 14 DE DICIEMBRE, 2009

Problem 1.

Consider the Liouville-Gelfand problem

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\lambda > 0$.

a) Prove that there exists λ^* such that (1) has a minimal classical solution for all $0 \le \lambda < \lambda^*$ and has no solution if $\lambda > \lambda^*$.

You can use the sub and supersolution method. Show that for $\lambda > 0$ small (1) has a classical solution. Then, if for $\bar{\lambda} > 0$ (1) has a classical solution show that for all $0 \le \lambda \le \bar{\lambda}$ there is a classical solution. Prove also that for $\lambda > 0$ large there is no solution.

b) Given a bounded function $u : \Omega \to \mathbb{R}$ and a fixed $\lambda > 0$ consider the eigenvalue problem

$$\begin{cases} \Delta \phi + \lambda e^u \phi + \mu\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

Verify that the eigenvalues μ for which there is a nontrivial solution form a sequence $\mu_1 < \mu_2 \le \ldots \to \infty$. The eigenvalue μ_1 is simple and the associated eigenfunction is of constant sign. We call μ_1 the principal eigenvalue.

For $0 \le \lambda < \lambda^*$ let u_λ be the minimal solution of (1). Show that if $0 \le \lambda < \lambda^*$ and $u = u_\lambda$ then $\mu_1 > 0$. Prove also the converse: if u is a classical solution of (1) for some $\lambda > 0$ and the linearization at u has a nonnegative principal eigenvalue, then u is the minimal solution of (1).

For the last part suppose that u is a classical solution for some $\lambda > 0$. Then $u \ge u_\lambda$. Multiply the equation satisfied by $u - u_\lambda$ by $u - u_\lambda$ and integrate. Then use $u - u_\lambda$ as a test function in the Rayleigh quotient that gives the principal eigenvalue of the linearization at u_λ. Recall that e^u is convex.

c) Assume now Ω is the unit ball in $\mathbb{R}^2$. Then all solutions of (1) are radially symmetric (by the result of Gidas, Ni, Nirenberg, 1979). Find the value λ^* and show that for all $0 < \lambda < \lambda^*$ (1) has exactly 2 solutions. Describe the behavior of the solutions as $\lambda \to 0$.

To do this, consider the family of functions

$$w_\mu(r) = \log \frac{8\mu}{(\mu + r^2)^2}$$

Date: 6 de Octubre, 2006.

where $\mu > 0$ is a parameter. Compute Δw_μ. You may use that for any given $\alpha \in \mathbb{R}$ the initial value probelm

$$u'' + \frac{1}{r}u' + \lambda e^u = 0 \quad r \in (0,1)$$
$$u(0^+) = \alpha, \quad u'(0^+) = 0$$

has unique solution.

d) Assume now $\Omega \subset \mathbb{R}^2$ is a bounded domain, with smooth boundary. Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega e^u, \quad u \in H_0^1(\Omega).$$

Verify that $J : H_0^1(\Omega) \to \mathbb{R}$ is C^1, that its critical points are weak solutions of (1) and that J satisfies the Palais-Smale condition. Using L^p and Schauder estimates verify that weak solutions are classical.

For this it is convenient that you find and state the Trudinger-Moser inequality in 2 dimensional domains.

e) Let $0 < \lambda < \lambda^*$. Prove that the minimal solution u_λ is a local minimum of the functional J. Using this show that there exists at least another solution of (1).

f) Assume now that $\Omega \subseteq \mathbb{R}^N$ with $N \geq 3$ and is starshaped. Prove in this case that for $\lambda > 0$ small the only classcial solution of (1) is the minimal one.

Suppose that v is another classical solution, so $v \geq u_\lambda$. Let $w = v - u_\lambda$. Multiply the equation satisfied by w by $x \cdot \nabla w$.

Problem 2. (The Hilbert transform) Let $f \in C_0^\infty(\mathbb{R})$ and consider the following harmonic extension of f to the upper half plane

$$u(x,t) = P_t * f(x), \quad x \in \mathbb{R}, t > 0$$

where

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2 + x^2} \quad x \in \mathbb{R}, t > 0.$$

Let v be the harmonic conjugate of u such that $v(x,t) \to 0$ as $|(x,t)| \to \infty$. One can show that v is given by $v(x,t) = Q_t * f(x)$ where

$$Q_t(x) = \frac{1}{\pi}\frac{x}{t^2 + x^2} \quad x \in \mathbb{R}, t > 0.$$

The Hilbert transform of $f \in C_0^\infty(\mathbb{R})$ is then defined as

$$Hf(x) = \lim_{t \downarrow 0} Q_t * f(x).$$

a) Verify that for $f \in C_0^\infty(\mathbb{R})$, one has

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon \downarrow 0}\int_{|y-x|>\varepsilon} \frac{f(y)}{x-y}\,dy \equiv \frac{1}{\pi} p.v.\frac{1}{x} * f(x).$$

Prove also that if $f \in C_0^\infty(\mathbb{R})$ then

$$\widehat{Hf}(\xi) = -i\, sign(\xi)\, \hat{f}(\xi).$$

Deduce that H is bounded from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

b) Let $K \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$ be such that it defines a tempered distribution in $\mathbb{R}^N$. Assume that

$$\hat{K} \in L^\infty$$

and that K satisfies the Hörmander condition

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq C \quad \forall y \in \mathbb{R}^N.$$

Show that there exists C such that

$$\big|\{x \in \mathbb{R}^N / \, |K * f(x)| \geq \lambda\,\}\big| \leq \frac{C}{\lambda}\|f\|_{L^1} \quad \forall \lambda > 0$$

for all $f \in C_0^\infty(\mathbb{R}^N)$. Deduce that for all $1 < p < \infty$ there is C_p such that

$$\|K * f\|_{L^p} \leq C_p \|f\|_{L^p}.$$

c) Show that if $K \in C^1(\mathbb{R}^N \setminus \{0\})$ and

$$|\nabla K(x)| \leq \frac{C}{|x|^{N+1}}$$

then K satisfied the Hörmander conditions.

d) Deduce that for $1 < p < \infty$ there is $C_p > 0$ such that

$$\|Hf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.$$