CHAPTER 3

Sub and super solutions

3.1 Simple monotone iterations

In this Chapter we describe an often used procedure for solving second order elliptic boundary value problems. It is an iterative method involving monotone iterations. We begin with a very simple problem:

$$\text{(1)} \qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $f(x,u)$ is a smooth function from $\overline{\Omega} \times \mathbb{R}$ into $\mathbb{R}$ and $\varphi \in C^{2+\mu}(\partial\Omega)$, $0 < \mu < 1$.

The main assumption is the existence of a **subsolution** $\underline{U} \in C^2(\overline{\Omega})$ and a **supersolution** $\overline{U} \in C^2(\overline{\Omega})$, i.e.,

$$\text{(2)} \qquad \begin{cases} \Delta \underline{U} + f(x,\underline{U}) \geq 0 & \text{in } \Omega, \\ \underline{U} \leq \varphi & \text{on } \partial\Omega, \end{cases}$$

$$\text{(3)} \qquad \begin{cases} \Delta \overline{U} + f(x,\overline{U}) \leq 0 & \text{in } \Omega, \\ \overline{U} \geq \varphi & \text{on } \partial\Omega, \end{cases}$$

such that

$$\text{(4)} \qquad \underline{U} \leq \overline{U}.$$

The main result is the following:

THEOREM 1. *Under the assumptions (2), (3) and (4), problem (1) has a least solution $\underline{u}$ and a greatest solution $\overline{u}$ in the "interval" $[\underline{U}, \overline{U}]$. (Of course $\underline{u}$ and $\overline{u}$ may coincide.)*

Proof. Without loss of generality we may suppose that $\varphi = 0$ on $\partial\Omega$: it suffices to subtract φ (extended inside Ω) from $u, \underline{U}$ and $\overline{U}$.

Fix a constant $k \geq 0$ such that

$$k + f_u(x,u) \geq 0 \quad \forall u \in [\underline{U}(x), \overline{U}(x)], \quad \forall x \in \overline{\Omega}.$$

Rewrite problem (1) in the form

$$-\Delta u + ku = f(x,u) + ku \equiv g(x,u).$$

Note that the function $u \mapsto g(x,u)$ is nondecreasing in the interval $[\underline{U}(x), \overline{U}(x)]$.

Starting the iteration with $\underline{U}$ we will construct the least solution $\underline{u}$ of (1) in the interval $[\underline{U}, \overline{U}]$. If we start the same iteration with $\overline{U}$ instead of $\underline{U}$ we will obtain the greatest solution $\overline{u}$. Let u_1 be the solution of the (linear) problem.

$$\begin{cases} -\Delta u_1 + k u_1 = g(x, \underline{U}) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By (2) we have

$$g(x, \underline{U}) \geq -\Delta \underline{U} + k \underline{U}$$

and thus

$$\begin{aligned} -\Delta(u_1 - \underline{U}) + k(u_1 - \underline{U}) \geq 0 & \quad \text{in } \Omega \\ u_1 - \underline{U} = -\underline{U} \geq 0 & \quad \text{on } \partial\Omega. \end{aligned}$$

It follows from the maximum principle (see Appendix XXX) that $u_1 \geq \underline{U}$.

On the other hand we have

$$g(x, \underline{U}) \leq g(x, \overline{U})$$

and therefore

$$\begin{aligned} -\Delta u_1 + k u_1 \leq -\Delta \overline{U} + k \overline{U} & \quad \text{in } \Omega \\ u_1 - \overline{U} = -\overline{U} \leq 0 & \quad \text{on } \partial\Omega. \end{aligned}$$

Using again the maximum principle we see that $u_1 \leq \overline{U}$.

Next, construct iteratively u_{j+1} as the solution of

$$\begin{cases} -\Delta u_{j+1} + k u_{j+1} = g(x, u_j) & \text{in } \Omega, \\ u_{j+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

One finds inductively, as above, that

$$u_j \leq u_{j+1} \leq \overline{U} \qquad \text{in } \Omega.$$

The right-hand side of (5) is uniformly bounded and thus we have the estimate (see Appendix XXX)

$$|u_{j+1}|_{C^1} \leq C \quad \text{independent of } j.$$

Consequently

$$|u_{j+1}|_{C^{2+\mu}} \leq C \text{ independent of } j.$$

Hence a subsequence of the u_j converges uniformly. Since the sequence (u_j) is monotone, the whole sequence converges uniformly to some limit $\underline{u} \in C^{2+\mu}(\overline{\Omega})$. It follows then that $|u_j - \underline{u}|_{C^2} \to 0$ and so $\underline{u}$ is the least solution of (1) in the interval $[\underline{U}, \overline{U}]$.

Indeed, suppose $\hat{u}$ is any solution of (1) in the interval $[\underline{U}, \overline{U}]$. The sequence (u_j) constructed above satisfies $u_j \leq \hat{u}$ for all j (since we could use $\hat{u}$ as a supersolution instead of $\overline{U}$). Passing to the limit as $j \to \infty$ we see that $\underline{u} \leq \hat{u}$.

Remark 1. The proof of Theorem 1 requires only that $f(x,u)$ be locally Lipschitz with respect to u.

Remark 2. Note that if $\underline{U}$ is a subsolution and $\overline{U}$ is a supersolution, they do not necessarily satisfy the inequality $\underline{U} \le \overline{U}$—it may even happen that $\overline{U} \le \underline{U}$ (construct such an example). In Theorem 1 it is essential to assume that $\underline{U} \le \overline{U}$—otherwise the conclusion may fail (construct such an example).

The least and greatest solutions $\underline{u}$ and $\overline{u}$ obtained above have some "stability" properties as described in the following result:

THEOREM 2. *Let $\underline{U} \le \overline{U}$ be sub and supersolutions as above. Assume $\underline{U}$ is not a solution of (1). Then the first eigenvalue (with Dirichlet boundary condition) of the linearized problem at $\underline{u}$ satisfies*

$$\lambda_1(-\Delta - f_u(x,\underline{u})) \ge 0. \tag{6}$$

Similarly if $\overline{U}$ is not a solution, then

$$\lambda_1(-\Delta - f_u(x,\overline{u})) \ge 0. \tag{7}$$

Moreover, if f is concave (resp. convex) then (6) (resp. (7)) holds with strict inequality.

Proof. We will only treat (6). Let $\varphi_1 > 0$ be an eigenfunction associated with λ_1. Suppose (6) does not hold. We claim that for $0 < \varepsilon$ sufficiently small $v = \underline{u} - \varepsilon\varphi_1$ is a supersolution and $v \ge \underline{U}$ in Ω. But then there would be a solution of (1) between v and $\underline{U}$, contradicting the fact that $\underline{u}$ is the least solution between $\underline{U}$ and $\overline{U}$.

To see that v is a supersolution first check that

$$\Delta v + f(x,v) = \Delta\underline{u} + \lambda_1\varepsilon\varphi_1 + f(x,\underline{u}) + o(\varepsilon)\varphi_1 = (\lambda_1\varepsilon + o(\varepsilon))\,\varphi_1 < 0$$

for ε small.

On the other hand since $\underline{U}$ is not a solution, it follows from the maximum principle and the Hopf Lemma (see Appendix XXX) that $\underline{u} > \underline{U}$ in Ω and the outward normal derivative $(\underline{u} - \underline{U})_\nu < 0$ at any point on $\partial\Omega$ where $\underline{U} = 0$. So for ε small $\underline{u} - \varepsilon\varphi_1 > \underline{U}$ in Ω.

If f is concave in u and if $\lambda_1 = 0$, then

$$\Delta v + f(x,v) \le \Delta\underline{u} - \varepsilon\Delta\varphi_1 + f(x,\underline{u}) + \varepsilon f_u(x,\underline{u})\varphi_1 = 0.$$

As before $v \ge \underline{U}$ and we obtain a contradiction. □

Remark 3. Property (6) (or (7)) suggests that the solution u may be a local minimizer of the corresponding energy

$$E(u) = \frac{1}{2}\int |\nabla u| - \int F(x,u)$$

where $F(x,u) = \int_0^u f(x,s)ds$. This assertion would hold if one knew that $\lambda_1(\dots) > 0$, which need not be the case (see Remark 4). In fact, in Part IV we will prove that there is a local minimum of E between $\underline{U}$ and $\overline{U}$.

Remark 4. In Theorem 2 suppose neither $\underline{U}$ nor $\overline{U}$ is a solution of (1). One may wonder whether (6) (or (7)) holds with a strict inequality—or whether there is a solution u of (1) between $\underline{U}$ and $\overline{U}$ for which $\lambda_1(-\Delta - f_u(x,u)) > 0$. This would imply that u is strongly stable in the following sense: solutions of the evolution equation

$$(8) \qquad \begin{cases} v_t = \Delta v + f(x,v) & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

with an initial value at $t = 0$ close to u converge to u as $t \to \infty$. In general such a solution u, with $\lambda_1(-\Delta - f_u(x,u)) > 0$ need **not** exist.

Here is a simple example. Let μ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on Ω and let $f(x,u) = \mu_1 u - u^3$. Clearly $\underline{U} = -K$ and $\overline{U} = K$ with K large are sub and supersolutions (and not solutions). We claim that $u = 0$ is the only solution of (1). For if u is a solution then

$$\Delta u + \mu_1 u - u^3 = 0.$$

Multiplying by u and integrating we find

$$\mu_1 \int u^2 = \int |\nabla u|^2 + \int u^4 \geq \mu_1 \int u^2 + \int u^4.$$

So $u = 0$.

On the other hand $\lambda_1(-\Delta - f_u(0)) = 0$.

In case $\underline{U} < \overline{U}$ are not solutions of (1), H. Matano [1],[2] and M. Crandall, P. Fife and L. A. Peletier [1] have proved that there is some solution u strictly between $\underline{U}$ and $\overline{U}$ which is stable in the sense that solutions of (8) with initial values close to u remain close to u for all time (see also Appendix XXX).

Sup of a family of subsolutions is a subsolution

We consider a more general notion of subsolution u,

$$\Delta u + g(x,u) \geq 0,$$

with g continuous on $\Omega \times \mathbb{R}$ for simplicity. Namely we assume that $u \in L^\infty_{\text{loc}}$ and that

$$(9) \qquad \int_\Omega u\Delta\zeta + g(x,u)\zeta \geq 0 \qquad \forall \zeta \geq 0, \zeta \in C_0^\infty(\Omega).$$

The main result is

THEOREM 3. *Consider a general family $\mathcal{F} = (u_\alpha)$ of continuous subsolutions which is bounded in L^∞_{loc}, i.e., for every compact set $K \subset \Omega$, there is a constant C_K such that*

$$u_\alpha \leq C_K \qquad \text{in } K.$$

Then

$$u(x) = \operatorname*{Sup}_\alpha u_\alpha(x)$$

is a subsolution.

The proof makes use of several lemmas.

LEMMA 1. *A subsolution as defined above belongs to* H^1_{loc}.

Proof: The function

$$f(x) = g(x,u)$$

is in L^∞_{loc}. Let j_ε be a family of mollifiers (see e.g. XXX XXX) and consider the convolutions

$$u_\varepsilon = j_\varepsilon \star u, \quad f_\varepsilon = j_\varepsilon \star f,$$

in a compact subset K of Ω. Then for small ε,

$$\Delta u_\varepsilon + f_\varepsilon \geq 0 \qquad \text{on } K. \tag{10}$$

Let K' be a compact set in the interior of K and let $1 \geq \zeta \geq 0$, be a function in $C_0^\infty(K), \zeta \equiv 1$ on K'.

On K, $|u_\varepsilon| \leq$ some constant A independent of ε—because $u \in L^\infty_{\text{loc}}$. Multiplying (10) by $\zeta^2(u_\varepsilon + A)$, and integrating, we find

$$\int \zeta^2 |\nabla u_\varepsilon|^2 \leq 2 \int \zeta |\nabla \zeta| (u_\varepsilon + A) |\nabla u_\varepsilon| + C$$

with C independent of ε. It follows easily that

$$\int \zeta^2 |\nabla u_\varepsilon|^2 \leq \text{ Constant independent of } \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the desired conclusion.

In view of Lemma 1 one has: if u is a subsolution as above then

$$\int -\nabla u \cdot \nabla \zeta + g(x,u)\zeta \geq 0 \; \forall \zeta \geq 0, \zeta \text{ in } H^1 \text{ with compact support.} \tag{11}$$

LEMMA 2. *If* u_1, u_2 *are subsolutions in* L^∞_{loc} *then*

$$u(x) = Max\,(u_1(x), u_2(x))$$

is a subsolution.

Problem 10. Prove Lemma 2.

[**Hint:** Let $j_k(t)$ be a sequence of nondecreasing functions such that $j_k(t) = 0$ for $t \leq 0$ and $j_k(t) = 1$ for $t \geq 1/k$. Let $\zeta \in C_0^\infty(\Omega)$, with $\zeta \geq 0$. Use $\zeta j_k(u_1 - u_2)$ and $\zeta(1 - j_k(u_1 - u_2))$ as test functions in (11).]

As a consequence of Lemma 2 we see that the sup of a **finite number** of subsolutions is again a subsolution.

LEMMA 3. *Suppose (u_j) is a sequence of subsolutions in L^∞_{loc} such that for every compact set $K \subset \Omega$,*

$$u_j \le C_K \qquad \text{on } K \quad \forall j.$$

Then

$$u(x) = \sup u_j(x)$$

is a subsolution.

Proof. For $j = 1, 2, \ldots$ the functions

$$v_j = \sup_{1 \le k \le j} u_k$$

are subsolutions. The sequence v_j is nondecreasing, and

$$u = \lim v_j.$$

The conclusion then follows easily from (9) (for the functions v_j) by passing to the limit, using dominated convergence.

The main observation now is a reduction of the case of a general family (u_α) to a countable subset.

Proof of Theorem 3. In $\Omega \times \mathbb{R}$ we consider the closed sets (relative to $\Omega \times \mathbb{R}$):

$$\text{epi } u_\alpha = \{(x, z) \in \Omega \times \mathbb{R}; z \ge u_\alpha(x)\}.$$

(The sets are closed because of the continuity of the u_α; in fact, lower semicontinuity would suffice.)

Clearly,

$$\text{epi } u = \bigcap_\alpha \text{epi } u_\alpha.$$

The set

$$U_\alpha = \{(x, z) \in \Omega \times \mathbb{R}; z < u_\alpha(x)\}$$

is open in $\mathbb{R}^n \times \mathbb{R}$ and

$$U = (\text{epi } u)^c = \cup U_\alpha.$$

CLAIM. *Any union U of open sets U_α is the union of a countable subfamily.*

This is well known but we include a

Proof of Claim. U may be written as the union of an increasing sequence of compact subsets $K_1 \subset K_2 \subset \cdots$. Each K_i is covered by a finite number of the U_α.

$$K_i = \bigcup_{\alpha \in \Lambda_i} U_\alpha, \quad \Lambda_i \text{ is a finite set.}$$

Then

$$U = \bigcup_{\alpha \in \Lambda} U_\alpha \quad \text{where } \Lambda = \cup \Lambda_j \text{ is countable.}$$

□

Returning to our situation, we may write

$$\text{epi } u = \bigcap_{\text{countable}} \text{epi } u_\alpha,$$

i.e., u is the sup of a countable subfamily of the u_α. We deduce from Lemma 3 that u is a subsolution.

□

3.2 Applications

Example 1. Consider the problem

$$(1) \qquad \begin{cases} -\Delta u = f(x,u) + g(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where

$$f(x,u) \text{ sign } u \le \alpha|u| + C \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}$$

for some constants $\alpha < \lambda_1$—the first eigenvalue of $-\Delta$ under zero Dirichlet condition—and $C \geq 0$.

THEOREM 4. *For every $g \in C^\mu(\overline{\Omega})$ and every $\varphi \in C^{2+\mu}(\partial\Omega)$ there exists at least one solution of (1). More precisely, there is a* **least solution** $\underline{u}$ *and a* **greatest solution** $\overline{u}$ *of (1).*

Proof. Without loss of generality we may suppose that $g = 0$ in Ω and that $\varphi = 0$ on $\partial\Omega$. We shall construct a subsolution $\underline{U}$ and a supersolution $\overline{U}$ for (1) such that $\underline{U} \le \overline{U}$ and such that **every** solution u of (1) satisfies $\underline{U} \le u \le \overline{U}$. The claim then follows from Theorem 1. Let $\overline{U}$ be the solution of the problem

$$\begin{aligned} \Delta\overline{U} + \alpha\overline{U} &= -C \qquad \text{in } \Omega, \\ \overline{U} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

By the maximum principle of Appendix XXX, $\overline{U} > 0$. Clearly $\overline{U}$ is a supersolution since

$$\Delta\overline{U} + f(x,\overline{U}) = -C - \alpha\overline{U} + f(x,\overline{U}) \le 0 \qquad \text{in } \Omega.$$

Moreover if u is any solution of (1) we have $u \le \overline{U}$; indeed in the domain $\tilde{\Omega} = \{x \in \Omega; u(x) > 0\}$ we have

$$\Delta u \geq -\alpha u - C$$

and hence

$$\begin{aligned} \Delta(u - \overline{U}) + \alpha(u - \overline{U}) &\geq 0 \qquad \text{in } \tilde{\Omega}, \\ -\overline{U} &\le 0 \qquad \text{on } \partial\tilde{\Omega}. \end{aligned}$$

It follows again from the maximum principle that $u \le \overline{U}$ in $\tilde{\Omega}$ and thus $u \le \overline{U}$ in Ω. Here we use the fact that the first eigenvalue for $\tilde{\Omega}$ is $\geq \lambda_1 > \alpha$. Similarly $\underline{U} = -\overline{U}$ is a subsolution such that every solution u of (1) satisfies $u \geq \underline{U}$. □

Remark 5. Consider the problem

$$(2) \qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,u) : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is a smooth function satisfying

$$(3) \qquad f(x,u) \le \alpha u + C \qquad \forall x \in \Omega, \quad \forall u \ge 0 \quad \text{with } \alpha < \lambda_1$$

and

$$f(x,0) \ge 0.$$

Then, there is a least nonnegative solution $\underline{u}$ and a greatest nonnegative solution $\overline{u}$ of (2). Indeed, one may use $\underline{U} = 0$ as a subsolution and $\overline{U}$ as defined above as a supersolution. As before, any solution u of (2) satisfies $u \le \overline{U}$. Of course, if $f(x,0) \equiv 0$ then $\underline{u} = 0$, but otherwise if $f(x,0)$ is not identically zero, then $\underline{u} > 0$ in Ω by the strong maximum principle (see Appendix XXX). When $f(x,0) \equiv 0$, it may well happen that $\overline{u} = 0$ (for instance if $f(x,u) \equiv 0$); in the next two examples we shall discuss simple assumptions which guarantee that $\overline{u} > 0$ in Ω.

Example 2. Consider the problem

$$(4) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : [0,\infty) \to \mathbb{R}$ is a smooth function with $f(0) = 0$ satisfying

$$(5) \qquad -\infty \le \limsup_{u \to +\infty} \frac{f(u)}{u} < \lambda_1$$

and

$$(6) \qquad f'(0) > \lambda_1.$$

THEOREM 5. *There exists a greatest positive solution $\bar{u}$ of (4).*

Proof. From (5) we deduce that (3) holds. Applying Remark 5 we obtain a greatest nonnegative solution $\overline{u}$ as a limit of a monotone sequence starting with the supersolution $\overline{U}$ described in Example 1. Therefore it suffices to find a subsolution $\underline{U}$ such that $0 < \underline{U} \le \overline{U}$ in Ω. One may take $\underline{U} = \varepsilon\varphi_1$ with $\varepsilon > 0$ small enough and $\varphi_1 > 0$ being the eigenfunction corresponding to λ_1. Note that $f(\varepsilon\varphi_1) \ge \varepsilon\lambda_1\varphi_1$ for $\varepsilon > 0$ small enough, by (6). Also observe that $\varepsilon\varphi_1 \le \overline{U}$ for $\varepsilon > 0$ small since $\dfrac{\partial\overline{U}}{\partial\nu} < 0$ on $\partial\Omega$ (see Appendix XXX). □

Remark 6. If the function $f(u)/u$ is **nonincreasing** on $(0,\infty)$—this happens for example if f is concave and $f(0) = 0$—then positive solutions of (4) are unique, unless $f(u) = \lambda_1 u$ in some positive neighbourhood of 0. If f is **concave** with $f(0) = 0$ and satisfies assumptions (5) and (6) then the (unique) positive solution of (4) satisfies $\lambda_1(-\Delta - f'(u)) > 0$. If the function $f(u)/u$ is **strictly decreasing** on $(0,\infty)$ then the assumptions (5) and (6) are also **necessary** for the existence of a positive solution of (4). See Problem 15.

Special Case: $f(u) = \alpha u - u^p$ with $p > 1$ and $\alpha \in \mathbb{R}$. Then (4) has a unique positive solution for every $\alpha > \lambda_1$ and no positive solution for $\alpha \leq \lambda_1$.

Remark 7. Let u be a positive solution of (4) with $f : [0, \infty) \to \mathbb{R}$ smooth and $f(0) = 0$. Then

$$\lambda_1(-\Delta - f(u)/u) = 0. \tag{7}$$

Indeed, let μ_1 be the first eigenfunction of $-\Delta - f(u)/u$ and let $\varphi_1 > 0$ be a corresponding eigenfunction. Then we have

$$-\Delta\varphi_1 - \frac{f(u)}{u}\varphi_1 = \mu_1\varphi_1. \tag{8}$$

Multiplying (8) by u and (4) by φ_1 and taking the difference, we see, on integrating, that $\mu_1 \int u\varphi_1 = 0$ and thus $\mu_1 = 0$.

Note that if the function $f(t)/t$ is strictly decreasing, then (7) is stronger than, and implies

$$\lambda_1 \left(-\Delta - f'(u)\right) > 0. \tag{9}$$

as can be seen by an argument similar to the preceding one. For the special case above, (7) asserts that

$$\int |\nabla v|^2 - (\alpha - u^{p-1})v^2 \geq 0 \quad \forall v \in H_0^1$$

while (9) just says that for some $\delta > 0$,

$$\int |\nabla v|^2 - (\alpha - pu^{p-1})v^2 \geq \delta \int v^2 \quad \forall v \in H_0^1.$$

Note also that if the function $f(t)/t$ is strictly increasing, then (7) implies that

$$\lambda_1 \left(-\Delta - f'(u)\right) < 0. \tag{10}$$

This applies for example to the case where $f(u) = u^p$ with $1 < p$. Under suitable conditions on p we will show in Part IV, Chapter 1, that problem (4) has a positive solution. Because of (10) and (6)–(7) of Section 3.1 we deduce that a solution cannot be obtained via sub and super solutions. Indeed, in Part IV the solution is obtained by variational arguments.

Example 3. Consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where λ is a positive constant and $f : [0, \infty) \to \mathbb{R}$ is a smooth function with $f(0) = 0$ satisfying **one** of the following assumptions:

either

(a) There is a constant $0 < a < \infty$ such that $f > 0$ on $(0, a)$ and $f < 0$ on $(a, +\infty)$,

or

(b) $f(u) > 0, \quad \forall u > 0$ and $\displaystyle\lim_{u \to +\infty} \frac{f(u)}{u} = 0$

[for instance $f(u) = u^q - u^p$ with $p > q > 1$ satisfies (a)].

THEOREM 6. *Assume either (a) or (b). Then, there is a constant* $0 < \lambda^\star < \infty$ *such that:*
(i) for every $\lambda > \lambda^\star$ *there exists a greatest positive solution* $\overline{u}_\lambda$ *of (11) (with* $\overline{u}_\lambda \le a$ *in case (a))*
(ii) for $\lambda < \lambda^\star$ *there exists no positive solution of (11).*
Moreover $\overline{u}_\mu > \overline{u}_\lambda$ *in* Ω *for* $\mu > \lambda \ge \lambda^\star$.

Proof. We shall only present the proof in case (a) since the argument in case (b) is very similar. For every $\lambda > 0$ the function $u \mapsto \lambda f(u)$ satisfies the assumption of Remark 5 in Example 1 and hence there is a greatest nonnegative solution $\overline{u}_\lambda$ of (11) for every $\lambda > 0$. At a maximum point x_0 of $\overline{u}_\lambda$ we have $f(\overline{u}_\lambda(x_0)) \ge 0$ and therefore $\overline{u}_\lambda(x_0) \le a$. In particular, $\overline{u}_\lambda \le a$ on Ω and $\lambda f(\overline{u}_\lambda) = -\Delta \overline{u}_\lambda$, so that $\overline{u}_\lambda$ is a subsolution for the μ-problem. Since a is a supersolution of the μ-problem there exists a solution of the μ-problem in the interval $[\overline{u}_\lambda, a]$. Thus $\overline{u}_\mu \ge \overline{u}_\lambda$.

Next, we claim that $\overline{u}_\lambda > 0$ in Ω for λ large enough. Fix any smooth function $\psi \ge 0$ with compact support in Ω, ψ not identically zero. Let φ be the solution of the problem

$$\begin{cases} -\Delta\varphi = \psi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

so that $\varphi > 0$ in Ω (see Appendix XXX) and by choosing ψ small enough we may assume that $\max_{\overline{\Omega}} \varphi < a$. Clearly $\sup_{\Omega} \psi/f(\varphi) < \infty$, and therefore, for λ large enough, we have $-\Delta\varphi \le \lambda f(\varphi)$, i.e., φ is a subsolution for (11). Hence $\overline{u}_\lambda \ge \varphi > 0$ in Ω for λ large enough.

We claim that $\overline{u}_\lambda = 0$ for $\lambda > 0$ small enough. With λ_1 the principal eigenvalue for $-\Delta$ under Dirichlet condition, fix $\lambda \ge 0$ small enough so that $\lambda f(u) < \lambda_1 u$, $\forall u > 0$. Let $\varphi_1 > 0$ be the eigenfunction of $-\Delta$ corresponding to λ_1. We have $0 = \int (\Delta \overline{u}_\lambda + \lambda f(\overline{u}_\lambda))\varphi_1 = \int (-\lambda_1 \overline{u}_\lambda + \lambda f(\overline{u}_\lambda))\varphi_1$ which implies that $\overline{u}_\lambda = 0$.

We deduce from the claims above that there is a constant $0 < \lambda^\star < \infty$ such that $\overline{u}_\lambda > 0$ for $\lambda > \lambda^\star$ and $\overline{u}_\lambda = 0$ for $\lambda < \lambda^\star$. Furthermore, by the strong maximum principle we have $\overline{u}_\mu > \overline{u}_\lambda$ in Ω for $\mu > \lambda > \lambda^\star$.

Problem 10. Prove Theorem 6 in case (b).

Remark 8. When $\lambda = \lambda^\star$ it may happen that $\overline{u}_{\lambda^\star} = 0$; this is the case for example if $f'(0) > 0$ and the function $u \mapsto f(u)/u$ is strictly decreasing on $(0, \infty)$ (see Remark 6 in Example 2). However, if in addition to the assumptions of Theorem 6 we also assume that $f'(0) = 0$ then $\overline{u}_{\lambda^\star} > 0$; moreover we shall prove in Part II, using degree theory, (and in Part IV, using variational methods) that for every $\lambda > \lambda^\star$ there exists another positive solution of (11), which is different from $\overline{u}_\lambda$.

Example 4. This is Example 5 of Chapter 1, Section 2 revisited. Consider the problem

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that
$f(x,u) \ge 0,\ f_u(x,u) \ge 0,\ f_{uu}(x,u) \ge 0 \quad \forall x \in \Omega, \quad \forall u \ge 0$, with $f(x,0)$ and $f_u(x,0)$ both not identically zero. Recall that problem (12) has a smooth curve $u(\lambda)$ of solutions such that $u(0) = 0$, defined on a maximal open interval $[0, \lambda^\star)$ with $\lambda^\star < \infty$ and that $u(\lambda) > 0$ in Ω for all $\lambda \in (0, \lambda^\star)$.

THEOREM 7. $u(\lambda)$ *is the least positive solution of (12) for every* $\lambda \in (0, \lambda^\star)$. *Moreover* $u(\lambda)$ *is the limit of the monotone iteration scheme*

$$\begin{cases} -\Delta u_{j+1} = \lambda f(x, u_j) & \textit{in } \Omega, \\ u_{j+1} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{13}$$

starting with $u_0 = 0$.

Proof. Suppose $v \geq 0$ is any solution of (12). Since f is convex with respect to u we have

$$\begin{aligned} -\Delta v = \lambda f(x,v) &\geq \lambda f(x,u(\lambda)) + \lambda f_u(x,u(\lambda))(v-u(\lambda)) \\ &= -\Delta u(\lambda) + \lambda f_u(x,u(\lambda))(v-u(\lambda)) \end{aligned}$$

and thus

$$-\Delta(v-u(\lambda)) - \lambda f_u(x,u(\lambda))(v-u(\lambda)) \geq 0. \tag{14}$$

Recall that $\lambda_1(-\Delta - \lambda f_u(x,u(\lambda))) > 0 \quad \forall \lambda \in [0,\lambda^*)$. It follows from the maximum principle (see Appendix XXX) that $v - u(\lambda) \geq 0$ and so $u(\lambda)$ is the least positive solution of (12).

Since $\underline{U} = 0$ is a subsolution and $\overline{U} = u(\lambda)$ is a supersolution the sequence (u_j) given by (13) is nondecreasing and converges to a solution u of (12) with $u \leq u(\lambda)$—thus $u = u(\lambda)$. $\square$

Example 5. Consider the problem

$$\begin{cases} -\Delta u = f(u) + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^1 convex function satisfying

$$-\infty \leq \lim_{u \to -\infty} \frac{f(u)}{u} < \lambda_1 \tag{16}$$

(by convexity the limit exists).

Set

$$K = \{g \in C^\mu(\overline{\Omega}) \text{ such that a solution of (15) exists}\}$$

and

$$P = \{h \in C^\mu(\overline{\Omega})\,;\, h \geq 0 \text{ in } \Omega\}.$$

Theorem 8. *K is a convex set and $K - P \subset K$. For every $g \in K$ there is a least solution $\underline{u}$ of (15) and it satisfies*

$$\lambda_1(-\Delta - f_u(\underline{u})) \geq 0. \tag{17}$$

More precisely, we have

$$\begin{cases} \lambda_1(-\Delta - f_u(\underline{u})) > 0 & \Leftrightarrow g \in \operatorname{Int} K, \\ \lambda_1(-\Delta - f_u(\underline{u})) = 0 & \Leftrightarrow g \in K \setminus \operatorname{Int} K. \end{cases} \tag{18}$$

If, in addition, f satisfies

$$\lambda_1 < \lim_{u \to +\infty} \frac{f(u)}{u} \leq +\infty \tag{19}$$

and $g \geq \mu =$a large positive constant, then (15) has no solution (here f need not be convex).

Special case: $f(u) = |u|^p$, $1 < p < \infty$.

Proof. To prove that K is convex, let $g_1, g_2 \in K$ with corresponding solutions u_1, u_2. Let $g = tg_1 + (1-t)g_2$, $t \in [0,1]$. then, for this g,

$$\overline{U} = tu_1 + (1-t)u_2$$

is a supersolution of (15) since

$$-\Delta\overline{U} = tf(u_1) + (1-t)f(u_2) + g \geq f(\overline{U}) + g.$$

In view of Theorem 1 it suffices to construct a subsolution $\underline{U}$ of (15) such that $\underline{U} \leq \overline{U}$. Fix $\alpha < \lambda_1$ such that $\lim_{u \to -\infty} \dfrac{f(u)}{u} < \alpha$.

We have, for some positive constant C,

$$f(u) \geq \alpha u - C \quad \forall u \leq 0.$$

Let ψ be the solution of

$$\begin{cases} \Delta\psi + \alpha\psi = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

so that, by the maximum principle, $\psi < 0$ in Ω and by the Hopf Lemma $\dfrac{\partial\psi}{\partial\nu} > 0$ on $\partial\Omega$ (see Appendix XXX).

We claim that $\underline{U} = k\psi$, with $k > 0$ large enough, is a subsolution such that $\underline{U} \leq \overline{U}$. We have $\Delta\underline{U} = k - \alpha\underline{U} = (k - C) + (C - \alpha\underline{U}) \geq k - C - f(\underline{U}) > -g - f(\underline{U})$ provided $(k - C) > |g|_{L^\infty}$. Furthermore, for k large enough, $\underline{U} \leq \overline{U}$. Thus K is convex.

Next we prove that $K - P \subset K$. Let $g \in K$ and let u be solution of (15). For any $h \in P$ we have $-\Delta u \geq f(u) + g - h$ and therefore u is a supersolution for the problem corresponding to $g - h$. Since we may always construct a subsolution $\underline{U} \leq u$ (as above) we conclude that $g - h \in K$.

We now prove that for every $g \in K$ there is a least solution $\underline{u}$ of (15) and it satisfies (17). Since $g \in K$ we have some solution v of (15). As above, for k large, $\underline{U} = k\psi$ is a subsolution (not a solution) and $\underline{U} \leq v$. By Theorem 1 and 2 there is a least solution $\underline{u}$ of (15) in the interval $[\underline{U}, v]$ and it satisfies (17).

Note that **every** solution u of (15) satisfies $u \geq \underline{U}$ (the argument is the same as in the proof of Theorem 4). Therefore the iteration (described in the proof of Theorem 1) starting with $\underline{U}$ converges to $\underline{u}$ which satisfies $\underline{u} \leq u$. This means that $\underline{u}$ is the least among **all** solutions of (15).

We now prove (18). Clearly if $\lambda_1(-\Delta - f_u(\underline{u})) > 0$, then $g \in \operatorname{Int} K$ by the Inverse Function Theorem. On the other hand, if
$\lambda_1(-\Delta - f_u(\underline{u})) = 0$ we claim that $g + \varepsilon \notin K$ for any $\varepsilon > 0$ and therefore $g \in K \setminus \operatorname{Int} K$.

Suppose not, that for some $\varepsilon > 0$ there exists a solution u_ε of the problem

$$\begin{cases} -\Delta u_\varepsilon = f(u_\varepsilon) + g + \varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We have

$$f(u_\varepsilon) - f(\underline{u}) \geq f_u(\underline{u})(u_\varepsilon - \underline{u})$$

and thus

$$-\Delta(u_\varepsilon - \underline{u}) - f_u(\underline{u})(u_\varepsilon - \underline{u}) \geq \varepsilon. \tag{20}$$

Multiplying (20) through by $\varphi_1 > 0$ satisfying

$$\begin{cases} -\Delta\varphi_1 - f_u(\underline{u})\varphi_1 = 0 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

we find a contradiction. This proves (18) in view of (17).

Finally, we prove that, under assumption (19) problem (15) has no solution when $g \geq \mu$ =a large positive constant. By (19) there exist constants $\beta > \lambda_1$ and C such that

$$f(u) \geq \beta u - C \qquad \forall u \geq 0. \tag{21}$$

Combining this with (16) we have

$$f(u) \geq \alpha u - C \qquad \forall u \in \mathbb{R} \text{ with } \alpha < \lambda_1.$$

If u is a solution of (15) with $g \geq \mu$ we have

$$-\Delta u \geq \alpha u - C + \mu.$$

Thus if $\mu > C$ we find on applying the maximum principle, that $u \geq 0$. It follows (by (21)) that

$$-\Delta u \geq \beta u - C + \mu \quad \text{in } \Omega.$$

Multiplying by φ_1 and integrating over Ω we obtain a contradiction. □

Remark 9. Further properties of the solutions of (15) are discussend in Problem 17. In particular $\underline{u}$ is the only "stable" solution of (15) in the sense that any other solution $u \neq \underline{u}$ of (15) satisfies

$$\lambda_1(-\Delta - f_u(u)) \leq 0 \quad (\text{resp. } < 0 \text{ if } f \text{ is strictly convex}).$$

Also, if $g \in K \setminus \operatorname{Int} K$ and f is strictly convex there is **exactly one** solution of (15). Under some additional assumptions we shall prove later, in Part IV, that for every $g \in \operatorname{Int} K$ problem (15) has at least two solutions.

Example 6. (Problems at resonance) Consider the problem

$$\begin{cases} \Delta u + \lambda_1 u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where $g(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that
$g_u(x,u) \geq 0 \;\; \forall x \in \Omega, \;\; \forall u \in \mathbb{R}$. Set

$$g_\pm(x) = \lim_{u\to\pm\infty} g(x,u) \quad (\text{possibly } \pm\infty).$$

As usual λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet condition and $\varphi_1 > 0$ denotes the corresponding eigenfunction normalized by $\int \varphi_1 = 1$.

THEOREM 9. *Assume*

$$\int g_{-}\varphi_1 < 0 < \int g_{+}\varphi_1. \tag{23}$$

Then, there exists a solution of (22).

Proof. Fix a constant k large enough so that

$$\int g(x, k\varphi_1)\, \varphi_1 \geq 0.$$

Solve the problem

$$\begin{cases} \Delta v + \lambda_1 v = g(x, k\varphi_1) - \int g(x, k\varphi_1)\varphi_1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that $\overline{U} = v + \mu\varphi_1$ is supersolution of (22) for μ large enough.

Indeed

$$\begin{aligned} \Delta\overline{U} + \lambda_1\overline{U} - g(x, \overline{U}) &= g(x, k\varphi_1) - \int g(x, k\varphi_1)\varphi_1 - g(x, \overline{U}) \\ &\leq g(x, k\varphi_1) - g(x, \overline{U}) \leq 0 \end{aligned}$$

provided

$$k\varphi_1 \leq \overline{U}, \tag{24}$$

and we may always choose μ large enough so that (24) holds. Similarly one can construct a subsolution $\underline{U} \leq \overline{U}$. The conclusion then follows from Theorem 1. □

Remark 10. Theorem 9 still holds if instead of (23) one makes the weaker assumption:

$$\begin{cases} \text{there exists a smooth function } u_0 \text{ such that} \\ u_0 = 0 \text{ on } \partial\Omega \text{ and } \int g(x, u_0)\varphi_1 = 0. \end{cases} \tag{25}$$

Clearly, then, (25) is a necessary and sufficient condition for the existence of a solution of (22). Note that (23) is also a necessary and sufficient condition if $g_u > 0$.

Remark 11. It is clear that in place of Δ one may consider a second order elliptic operator with variable coefficients. Furthermore, more general boundary conditions may be considered. In fact Ω might be a domain with compact closure (possibly without boundary) on a manifold. In particular, Theorem 9 applies to the following equation

$$\Delta_g u - K(x) = e^{2u} \qquad \text{on } M \tag{26}$$

where M is any compact 2-dimensional Riemannian manifold, without boundary, with metric g. Δ_g is the corresponding Laplace-Beltrami operator. Theorem 9 implies that for any function K such that $\int K < 0$ there is a unique solution of (26). A geometric consequence is that any such manifold M with metric g and Euler characteristic $\chi \geq 2$ has a metric, $\tilde{g} = e^{2u}g$, conformal to g with Gauss curvature identically -1.

3.3 Equations nonlinear in Du

We shall now present a more elaborate equation to which the method of monotone iteration applies in case one has sub and supersolutions. Consider the Dirichlet problem

$$\text{(1)}\qquad \begin{cases} -\Delta u = f(x,u,Du) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, f is a smooth function of its arguments and $\varphi \in C^{2+\mu}(\partial\Omega)$. We assume the existence of **sub and supersolutions** $\underline{U} \le \overline{U}$ in $C^2(\overline{\Omega})$ satisfying namely

$$\text{(2)}\qquad \begin{cases} \Delta\underline{U} + f(x,\underline{U},D\underline{U}) \ge 0 & \text{in } \Omega, \\ \underline{U} \le \varphi & \text{on } \partial\Omega, \end{cases}$$

and

$$\text{(3)}\qquad \begin{cases} \Delta\overline{U} + f(x,\overline{U},D\overline{U}) \le 0 & \text{in } \Omega \\ \overline{U} \ge \varphi & \text{on } \partial\Omega. \end{cases}$$

Let $M = \max(|\underline{U}|_{L^\infty}, |\overline{U}|_{L^\infty})$ and assume that $f(x,u,\xi)$ satisfies

$$\text{(4)}\qquad |f(x,u,\xi)| \le K(1+|\xi|^2) \text{ for } \underline{U}(x) \le u(x) \le \overline{U}(x),\ x \in \overline{\Omega},\ \xi \in \mathbb{R}^n.$$

THEOREM 10. *Under these conditions, problem (1) has a least solution* $\underline{u}$ *and a greatest solution* $\overline{u}$ *in the interval* $[\underline{U}, \overline{U}]$.

We may subtract φ (extended inside Ω) from $u, \underline{U}$ and $\overline{U}$ and thus suppose that $u = 0$ on $\partial\Omega$. In doing this we obtain of course a new function f but it continues to satisfy (4) with a different constant K.

The theorem will be proved by applying the method of monotone iterations to a suitably modified problem. We seek solutions u in $\underline{U} \le u \le \overline{U}$. Consider smooth mappings $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $|h(\xi)| \le 2|\xi|$. For any such h, if we have a smooth solution u in $\underline{U} \le u \le \overline{U}$ of

$$\text{(5)}\qquad \begin{cases} -\Delta u = f(x,u,h(Du)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then it satisfies

$$-\Delta u + u = a(x)(1+|Du|^2)$$

where

$$a(x) = \frac{f(x,u,h(Du)) + u}{1+|Du|^2}$$

Thus, if $\underline{U} \le u \le \overline{U}$ we see from (4) that

$$|a(x)| \le 4K + M$$

with M independent of h. According to Theorem 10 in Chapter 1 we have the estimate, taking $p > n$,

$$|u|_{C^1} \le C\|u\|_{W^{2,p}} \le \overline{C}$$

independent of our function h. We now fix a function h with the properties described and satisfying in addition, h is bounded in $\mathbb{R}^n$ and $h(\xi) = \xi$ for $|\xi| \le \max(\overline{C}, |D\underline{U}|_{L^\infty}, |D\overline{U}|_{L^\infty})$. (5) is the modified problem which we will solve. It is clear that if $\underline{U} \le u \le \overline{U}$ is a solution of it in $C^{2+\mu}(\overline{\Omega})$, then it is also a solution of the original problem (1).

Since f is smooth and h is bounded we see that for some k

$$|f_u(x,u,h(\xi))| \le k \quad \text{if } |u| \le \max(|\underline{U}|_{L^\infty}, |\overline{U}|_{L^\infty}),\ \xi \in \mathbb{R}^n.$$

Consequently, if $\underline{U} \le s \le t \le \overline{U}$ then

$$f(x,t,h(\xi)) - f(x,s,h(\xi)) + k(t-s) \ge 0. \tag{6}$$

Rewrite the problem (5) in the form

$$\begin{cases} -\Delta u + ku = f(x,u,h(Du)) + ku & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Our iteration method is based on

LEMMA 4. *Let $v \in C^\mu(\overline{\Omega})$ be a function satisfying $\underline{U} \le v \le \overline{U}$.*

The problem

$$\begin{cases} -\Delta u + ku = f(x,v,h(Du)) + kv & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{8}$$

has a unique solution $u = T(v)$ satisfying

$$\underline{U} \le u \le \overline{U}. \tag{9}$$

Furthermore

$$\underline{U} \le v_1 \le v_2 \le \overline{U} \Rightarrow T(v_1) \le T(v_2). \tag{10}$$

Proof. The uniqueness of a solution of (8) follows with the aid of the maximum principle. Existence may be proved using the continuity method (see Example 8 of Chapter 1, Section 2) for the problem: find $u = u(t)$ in $C^{2+\mu}(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u + ku = f(x,v,th(Du)) + kv, & \text{in } \Omega,\ 0 \le t \le 1, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Implicit Function Theorem yields the openness of the set of t's with a solution. One needs only establish an a priori estimate for $|u(t)|_{C^{2+\mu}}$. This is easy. For any v under consideration the right hand side is bounded and hence $\|u(t)\|_{W^{2,p}} \le C$, $1 < p < \infty$. It follows that $|u(t)|_{C^{1+\mu}} \le C$ (with a different constant C). Inserting this in the right hand side and using the Schauder estimates we obtain the desired estimate $|u(t)_{C^{2+\mu}} \le C$.

To prove (9) observe that

$$\begin{aligned}
&-\Delta(u-\underline{U})+k(u-\underline{U}) \geq f(x,v,h(Du))-f(x,\underline{U},D\underline{U})+k(v-\underline{U})\\
&= f(x,v,h(Du))-f(x,\underline{U},h(Du))+f(x,\underline{U},h(Du))-f(x,\underline{U},D\underline{U})\\
&+k(v-\underline{U}) \geq f(x,\underline{U},h(Du))-f(x,\underline{U},h(D\underline{U})) \quad \text{by (6)}\\
&= \sum a_j(x)(u-\underline{U})_j
\end{aligned}$$

by the Integral Theorem of the Mean.

Hence, by the maximum principle $u-\underline{U}\geq 0$. Similarly one finds $u-\overline{U}\leq 0$ and (9) is proved.

The proof of (10) is similar: if $T(v_1)=u_1$, $T(v_2)=u_2$, then

$$\begin{aligned}
(-\Delta+k)(u_1-u_2) &= f(x,v_1,h(Du_1))-f(x,v_2,h(Du_2))+k(v_1-v_2)\\
&\leq \sum \tilde{a}_j(x)(u_1-u_2)_j
\end{aligned}$$

by (6) and it follows that $u_1-u_2\leq 0$. Lemma 4 is proved. □

Proof of Theorem 10. Starting our iteration with $\underline{U}$ we will construct the least solution $\underline{u}$ in $[\underline{U},\overline{U}]$. If we start with $\overline{U}$ we will obtain the greatest solution $\overline{u}$. Let u_1 be the solution (given by Lemma 4) of (8) with $v=\underline{U}$. By the Lemma we see that

$$\underline{U}\leq u_1\leq \overline{U}$$

and that if u is any solution of (1) in $[\underline{U},\overline{U}]$ then

$$u_1\leq u.$$

Next, construct iteratively u_{j+1} as the solution of (8) with
$v=u_j$, $j=1,2\ldots.$ One finds inductively, as in the earlier case of monotone iterations that

$$\underline{U}\leq u_1\leq u_2\leq\cdots\leq\overline{U}$$

and $u_j\leq u$ for any solution u of (1) in $[\underline{U},\overline{U}]$.

We have

$$(-\Delta+k)u_{j+1}=f(x,u_j,h(Du_{j+1}))+ku_j,\ j=1,2\ldots.$$

The right hand sides are uniformly bounded. We have as before,

$$|u_{j+1}|_{C^1}\leq c\|u_{j+1}\|_{W^{2,p}}\leq C \quad \text{independent of } j=1,2\ldots.$$

Consequently

$$|u_{j+1}|_{C^{2+\mu}}\leq C, \qquad j=1,2\ldots.$$

Thus a subsequence of the u_j converges uniformly. Since they form a monotone sequence the whole sequence converges uniformly to some function $\underline{u}$ in $C^{2+\mu}(\overline{\Omega})$. It follows that $|u_j-\underline{u}|_{C^2}\to 0$ and so $\underline{u}$ is a solution of (1), and it is the least solution between $\underline{U}$ and $\overline{U}$.

Problem 11. Consider the Dirichlet problem

$$\begin{cases} -\Delta u = a(x)u - u^p + g(x, u, Du) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Here $p > 1$, a and g are smooth in their arguments and satisfy

$$\text{the first eigenvalue of } (-\Delta - a) \text{ is negative}$$

and, for $u \geq 0$,

$$|g(x, u, \xi)| \leq C(1 + u^q + |\xi|^2) \quad \text{for some } q < p$$

and

$$|g(x, u, \xi)| \leq C(u^2 + |\xi|^2) \quad \text{for } (u + |\xi|) \text{ small.}$$

Prove that the problem has a solution $u > 0$ in Ω.

[Hint: Use as subsolution $\varepsilon\varphi_1 + \varepsilon^{3/2}\varphi_1^2$ where φ_1 is an eigenfunction of $-\Delta - a$].