AIRLINE SEAT ALLOCATION WITH MULTIPLE NESTED FARE CLASSES

S. L. BRUMELLE

University of British Columbia, Vancouver, Canada

J. I. McGILL

Queen's University, Kingston, Ontario, Canada

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This paper addresses the problem of determining optimal booking policies for multiple fare classes that share the same seating pool on one leg of an airline flight when seats are booked in a nested fashion and when lower fare classes book before higher ones. We show that a fixed-limit booking policy that maximizes expected revenue can be characterized by a simple set of conditions on the subdifferential of the expected revenue function. These conditions are appropriate for either the discrete or continuous demand cases. These conditions are further simplified to a set of conditions that relate the probability distributions of demand for the various fare classes to their respective fares. The latter conditions are guaranteed to have a solution when the joint probability distribution of demand is continuous. Characterization of the problem as a series of monotone optimal stopping problems proves optimality of the fixed-limit policy over all admissible policies. A comparison is made of the optimal solutions with the approximate solutions obtained by P. Belobaba using the expected marginal seat revenue (EMSR) method.

One of the obvious impacts of the deregulation of North American airlines has been increased price competition and the resulting proliferation of discount fare booking classes. While this has had the expected effect of greatly expanded demand for air travel, it has presented the airlines with a significant tactical planning problem— that of determining booking policies that result in optimal allocations of seats among the various fare classes. What is sought is the best tradeoff between the revenue gained through greater demand for discount seats against revenues lost when full-fare reservations requests must be turned away because of prior discount seat sales.

This problem is made more difficult by the tendency of discount fare reservations to arrive before full-fare ones. This occurs because of the nature of the customers for the respective classes (leisure travelers in the discount classes, business travelers in full fare) and because of early booking restrictions placed on the discount classes. Thus, decisions about limits to place on the number of discount fare bookings must often be made before any full-fare demand is observed. Further complications are introduced by factors such as multiple-flight passenger itineraries, interactions with other flights, cancellation and overbooking considerations, and the dynamic nature of the booking process in the long lead-time before flight departure.

Prior work on this problem has tended to fall into one of two categories. First, attempts have been made to encompass some or all of the above-mentioned complications with mathematical programming and/ or network models (Mayer 1976, Glover et al. 1982, Alstrup et al. 1986, Wollmer 1986, 1987, Dror, Trudeau and Ladany 1988). Second, elements of the problem have been studied in isolation under restrictive assumptions (Littlewood 1972, Bhatia and Parekh 1973, Richter 1982, Belobaba 1987, Brumelle et al. 1990, Curry 1990, Wollmer 1992). These studies have produced easy to apply rules that provide some insight into the nature of good solutions. Such rules are suboptimal when viewed in the context of the overall problem, but they can point the way to useful approximation methods. The present paper falls into the second category.

This paper deals with the airline seat allocation problem when multiple fare classes are booked into a common seating pool in the aircraft. The following assumptions are made:

1. *Single flight leg:* Bookings are made on the basis of a single departure and landing. No allowance is

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made for the possibility that bookings may be part of larger trip itineraries.

2. *Independent demands:* The demands for the different fare classes are stochastically independent.
3. *Low before high demands:* The lowest fare reservations requests arrive first, followed by the next lowest, etc.
4. *No cancellations:* Cancellations, no-shows and overbooking are not considered.
5. *Limited information:* The decision to close a fare class is based only on the number of current bookings.
6. *Nested classes:* Any fare class can be booked into seats not taken by bookings in lower fare classes.

The independent demands and low before high assumptions imply that at any time during the booking process the observed demands in the fare class currently being booked and in lower fare classes convey no information about future demands for higher fare classes. The limited information assumption excludes the possibility of basing a decision to close a fare class on such factors as the time remaining before the flight.

Assumptions 1–5 are restrictive when compared to the actual decision problem faced by airlines, but analysis of this simplified version can both provide insights into the nature of optimal solutions and serve as a basis for approximate solutions to more realistic versions.

The nesting of fare classes (assumption 6), which is a common practice in modern airline reservation systems, suggests the following general approach to controlling bookings: set a fixed upper limit for bookings in the lowest fare class; a second, higher limit for the total bookings in the two lowest classes, and so on up to the highest fare class. Viewed in another way, such booking limits establish *protection levels* for successive nests of higher fare classes.

The first useful result on the seat allocation problem (for two fare classes) was presented by Littlewood. He proposed that an airline should continue to reduce the protection level for class-1 (full-fare) seats as long as the fare for class-2 (discount) seats satisfied

$$f_2 \geq f_1 \Pr[X_1 > p_1], \tag{1}$$

where f_i denotes the fare or average revenue from the ith fare class, $\Pr[\cdot]$ denotes probability, X_1 is full-fare demand, and p_1 is the full-fare protection level. The intuition here is clear—accept the immediate return from selling an additional discount seat as long as the discount revenue equals or exceeds the *expected* full-fare revenue from the seat.

A continuous version of Littlewood's rule was derived in Bhatia and Parekh. Richter gave a marginal analysis which proved that (1) gives an optimal allocation (assuming certain continuity conditions).

More recently, Belobaba (1987) proposed a generalization of (1) to more than two fare classes called the Expected Marginal Seat Revenue (EMSR) method. In this approach, the protection level for the highest fare class p_1 is obtained from

$$f_2 = f_1 \Pr[X_1 > p_1]. \tag{2}$$

This is just Littlewood's rule expressed as an equation, and it is appropriate as long as it is reasonable to approximate the protection level with a continuous variable and to attribute a probability density to the demand X_1. The total protection for the two highest fare classes p_2 is obtained from

$$p_2 = p_2^1 + p_2^2, \tag{3}$$

where p_2^1 and p_2^2 are two individual protection levels determined from

$$f_3 = f_1 \Pr[X_1 > p_2^1] \tag{4}$$

and

$$f_3 = f_2 \Pr[X_2 > p_2^2]. \tag{5}$$

The protection for the three highest fare classes is obtained by summing three individual protection levels, and so on. This process is continued until nested protection levels p_k, are obtained for all classes except the lowest. The booking limit for any class k is then just $(C - p_{k-1})$, where C is the total number of seats available.

The EMSR method obtains optimal booking limits between each pair of fare classes regarded in isolation, but it does not yield limits that are optimal when all classes are considered. While the idea of comparing the expected marginal revenues from future bookings with current marginal revenues is valid, the method outlined above does not in general lead to a correct assessment of expected future revenues (except for the highest fare class). To avoid confusion, the EMSR approximation described above will henceforth be referred to as the EMSRa method.

The nonoptimality of the EMSRa approach has been reported independently by McGill (1988), Curry (1988), Wollmer (1988), and Robinson (1990). Curry (1990) derives the correct optimality conditions when demands are assumed to follow a continuous probability distribution and generalizes to the case that fare classes are nested on an origin-destination basis. Wollmer (1992) deals with the discrete demand case and provides an algorithm for computing both the optimal protection levels and the optimal expected revenue.

This paper makes the following contributions to the work on this problem:

1. The approach used (subdifferential optimization within a stochastic dynamic programming framework) admits either discrete or continuous demand distributions and obtains optimality results in a relatively straightforward manner.
2. The connection of the seat allocation problem to the theory of optimal stopping is demonstrated, and a formal proof is given that fixed-limit booking policies are optimal within the class of all policies that depend only on the observed number of current bookings.
3. We show that the optimality conditions reduce to a simple set of probability statements that clearly characterize the difference between the EMSRa solutions and the optimal ones.
4. We show with a simple counterexample that the EMSRa method can both over- or underestimate the optimal protection levels.

Specifically, we show that an optimal set of protection levels p_1^*, p_2^*, ... must satisfy the conditions

$$\delta_+ ER_k[p_k^*] \leqslant f_{k+1} \leqslant \delta_- ER_k[p_k^*] \quad \text{for each } k = 1, 2, \ldots, \tag{6}$$

where $ER_k[p_k]$ is the expected revenue from the k highest fare classes when p_k seats are protected for those classes, and δ_+ and δ_- denote the right and left derivative with respect to p_k, respectively. These conditions are just an expression of the usual first-order result—a change in p_k away from p_k^* in either direction will produce a smaller increase in expected revenues than the immediate increase of f_{k+1}. The same conditions apply whether demands are viewed as continuous random variables as in Curry (1990) or as discrete random variables as in Wollmer (1992).

It is further shown that under certain continuity conditions these optimal protection levels can be obtained by finding p_1^*, p_2^*, ... that satisfy

$$\begin{aligned} f_2 &= f_1 \Pr[X_1 > p_1^*] \\ f_3 &= f_1 \Pr[X_1 > p_1^* \cap X_1 + X_2 > p_2^*] \\ &\vdots \\ f_{k+1} &= f_1 \Pr[X_1 > p_1^* \cap X_1 + X_2 > p_2^* \\ &\quad \cap \ldots \cap X_1 + X_2 + \ldots + X_k > p_k^*]. \end{aligned} \tag{7}$$

These conditions have a simple and intuitive interpretation since, as noted in Robinson, the probability term on the right-hand side of the general equation in (7) is simply the probability that all remaining seats are solid. The first of these equations is identical to the first in the EMSRa method, so the EMSRa method does derive the optimal protection level for the highest fare class.

The paper is organized as follows. The next section presents notation and assumptions. Section 2 gives the revenue function and its directional derivatives. In the following section, concavity properties of the expected revenue function are established and results (6) and (7) are obtained. We show that when demand is integer-valued there exist integer optimal solutions that satisfy (6), and these solutions are optimal over the class of all policies that depend only on the history of the demand process. The final section provides numerical comparisons of the EMSRa and optimal solutions.

1. NOTATION AND ASSUMPTIONS

The demand for fare class k is X_k, ($k = 1, 2, \ldots$), where X_1 corresponds to the highest fare class. We assume that these demands are stochastically independent. The vector of demands is $X = (X_1, X_2, \ldots)$. Each booking of a fare class k seat generates average revenue of f_k, where $f_1 > f_2 > \ldots$.

Demands for the lowest fare class arrive first, and seats are booked for this class until a fixed time limit is reached, bookings have reached some limit, or the demand is exhausted. Sales to this fare class are then closed, and sales to the class with the next lowest fare are begun, and so on for all fare classes. It is assumed that any time limits on bookings for fare classes are prespecified. That is, the setting of such time limits is not part of the problem considered here. It is possible, depending on the airplane capacity, fares, and demand distributions that some fare classes will not be opened at all.

A *booking policy* is a set of rules which specifies at any point during the booking process whether a fare class that has not reached its time limit should be available for bookings. In general, such policies may depend on the pattern of prior demands or be randomized in some manner. Any stopping rule for fare class k which is measurable with respect to the σ-algebra generated by $[X_k \geqslant x]$ for $x = 0, 1, \ldots$ is *admissible.* However, we first restrict attention to a simpler class of booking policies, denoted by $\mathscr{P}$, that can be described by a vector of fixed protection levels $p = (p_1, p_2, \ldots)$, where p_k is the number of seats to be protected for fare classes 1–k. If at some stage in the process described above there are s seats available to be booked and there is a fare class k demand, then the seat will be booked if s is greater than the protection level p_{k-1} for the $k-1$ higher fare classes. (Restriction to this class of policies is implicit in previous research in this area except for that of Brumelle et al.) The initial number of classes that are open for any bookings is, of course, determined by setting s equal to the capacity of the aircraft or compartment. We will show formally that the class $\mathscr{P}$ contains a policy that is optimal over the class of all admissible policies.

2. THE REVENUE FUNCTION

The function $R_k[s; p; x]$ is the revenue generated by the k highest fare classes when s seats are available to satisfy all demand from these classes, when $x = (x_1, x_2, \ldots)$ is the demand vector, and $p = (p_1, p_2, \ldots)$ is the vector of protection levels. We define the revenue function recursively by

$$R_1[s; p; x] = \begin{cases} f_1 s & \text{for } 0 \leqslant s < x_1 \\ f_1 x_1 & \text{for } x_1 \leqslant s \end{cases} \tag{8}$$

$$R_{k+1}[s; p; x] = \begin{cases} R_k[s; p; x] & \text{for } 0 \leqslant s < p_k \\ (s - p_k) f_{k+1} + R_k[p_k; p; x] & \\ \qquad \text{for } p_k \leqslant s < p_k + x_{k+1} & \\ x_{k+1} f_{k+1} + R_k[s - x_{k+1}; p; x] & \\ \qquad \text{for } p_k + x_{k+1} \leqslant s, & \end{cases} \tag{9}$$

for $k = 1, 2, \ldots$.

For convenience of notation, a dummy protection level p_0 will be introduced; its value will be identically zero throughout. There is no limit to the number of fare classes or to the corresponding lengths of the protection and demand vectors; however, the revenue from the k highest fares depends only on the protection levels $(p_0, p_1, \ldots, p_{k-1})$ and the demands $(x_1, x_2, \ldots, x_k)$. The symbols p and x will be used to denote vectors of lengths which vary depending on context, as in

$$R_k[s; p; x] = R_k[s, (p_0\, p_1, \ldots, p_{k-1}); (x_1, x_2, \ldots, x_k)].$$

The objective is to find a vector p that maximizes the expected revenue $ER_k[s; p; x]$ for all k. If s is viewed as a real-valued variable, the function $ER_k[s; p; X]$ is continuous and piecewise linear on $s > 0$ and not differentiable at the points $s = p_k$. Maximization of this function can be accomplished either by treating available seats s and protection limits p as integer-valued and using arguments based on first differences, or by treating these variables as continuous and using standard tools of nonsmooth optimization. The second approach will be used in this paper because it permits greater economy of notation and terminology. Note that the demands X can be discrete or continuous in either case. In the case that demands are taken as integer-valued, both approaches are equivalent for this problem and yield the same set of integer optimal solutions. The second approach may admit additional noninteger optimal solutions, but these can easily be avoided in practice. If the demands are approximated by continuous random variables, the second approach may lead to noninteger optimal solutions. This eventuality is discussed in subsection 3.3 under implementation.

2.1. Marginal Value of an Extra Seat

This section develops the first-order properties of the revenue function. The notation and terminology used here and in what follows are consistent with Rockafellar (1970) except that they have been modified in obvious ways to handle concave rather than convex functions. Let δ_- and δ_+ denote the left and right derivatives with respect to the first argument of the revenue or expected revenue functions. Thus, $\delta_- ER_k[s; (p_0, \ldots, p_{k-1}); X]$ is the left derivative of $ER_k[.]$ with respect to s. (This slightly unconventional notation is required because s, the number of seats remaining, will sometimes be replaced by p_k when the argument is being viewed as a discretionary quantity.) For fixed p and x, the derivatives for the revenue function are easy to compute from (8) and (9) to be

$$\delta_+ R_1[s; p; x] = \begin{cases} f_1 & \text{for } s < x_1 \\ 0 & \text{for } s \geqslant x_1 \end{cases} \tag{10}$$

$$\delta_- R_1[s; p; x] = \begin{cases} f_1 & \text{for } s \leqslant x_1 \\ 0 & \text{for } s > x_1 \end{cases} \tag{11}$$

and

$$\delta_+ R_{k+1}[s; p; x] = \begin{cases} \delta_+ R_k[s; p; x] & \text{for } 0 \leqslant s < p_k \\ f_{k+1} & \text{for } p_k \leqslant s < p_k + x_{k+1} \\ \delta_+ R_k[s - x_{k+1}; p; x] & \text{for } p_k + x_{k+1} \leqslant s. \end{cases} \tag{12}$$

$$\delta_- R_{k+1}[s; p; x] = \begin{cases} \delta_- R_k[s; p; x] & \text{for } 0 < s \leqslant p_k \\ f_{k+1} & \text{for } p_k < s \leqslant p_k + x_{k+1} \\ \delta_- R_k[s - x_{k+1}; p; x] & \text{for } p_k + x_{k+1} < s. \end{cases} \tag{13}$$

Any continuous, piecewise-linear function $f[s]$ is concave on $s > 0$ if and only if the right derivative is less than or equal to the left derivative for any s. This condition can be extended to the point $s = 0$ by defining $\delta_- f[0] = +\infty$. The *subdifferential* $\delta f[s]$ is then defined for any $s \geq 0$ as the closed interval from $\delta_+ f[s]$ to $\delta_- f[s]$. Given concavity, $f[\cdot]$ will be maximized at any point s for which $0 \in \delta f[s]$.

3. OPTIMAL PROTECTION LEVELS

This section establishes the optimality within the class $\mathscr{P}$ of protection levels determined by the first-order conditions given in (6). We first consider a point in the booking process when s seats remain unbooked, fare class $k + 1$ is being booked, and the decision of whether or not to stop booking that class is to be made. That is, a decision on the value of the protection level p_k for the remaining fare classes is to be made. The following lemma establishes a condition under which concavity of the expected revenue function with respect to s is ensured, conditional on the value of X_{k+1}. This leads to an argument by induction that concavity of the conditional expected revenue function will be satisfied if (6) is satisfied for all the higher protection levels. Finally, we show that condition (6) also guarantees optimality of p_k.

Lemma 1. *If some policy p makes*

$$ER_k[s; (p_0, \ldots, p_{k+1}); X]$$

concave on $s \geq 0$ *and if* p_k^* *satisfies*

$$f_{k+1} \in \delta\, ER_k[p_k^*; (p_0, \ldots, p_{k-1}); X], \tag{14}$$

then

$$E\{R_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}$$

is concave on $s \geq 0$ *with probability* 1.

Proof. It follows from the definition of the revenue function in (9) and the hypothesized concavity of ER_k that

$$E\{R_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}$$

is continuous on $s > 0$ and concave on the three intervals $0 \leq s < p_k$, $p_k \leq s < p_k + X_{k+1}$, and $p_k + X_{k+1} \leq s$.

To complete the proof, it is enough to verify that

$$\delta_+ E\{R_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}$$
$$\leq \delta_- E\{R_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\} \tag{15}$$

at the two points $s = p_k^*$ and $s = p_k^* + X_{k+1}$. From (12) and (13) the left and right derivatives at $s = p_k^*$ are

$$\delta_- E\{R_{k+1}[p_k^*; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_k+1\}$$
$$= \delta_- ER_k[p_k^*; p; X] \tag{16}$$

and

$$\delta_+ E\{R_{k+1}[p_k^*; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\} = f_{k+1}. \tag{17}$$

By the hypothesis of the lemma, inequality (15) must be satisfied.

Again applying (12) and (13), the left and right derivatives at $s = p_k^* + X_{k+1}$ are

$$\delta_- E\{R_{k+1}[p_k^* + X_{k+1}; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}$$
$$= f_{k+1} \tag{18}$$

and

$$\delta_+ E\{R_{k+1}[p_k^* + X_{k+1}; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}$$
$$= \delta_+ ER_k[p_k; (p_0, \ldots, p_{k-1}); X]. \tag{19}$$

By the hypothesis of the lemma, inequality (15) must be satisfied at $s = p_k^* + X_{k+1}$.

Corollary 1. *If, for some* $k \in \{1, 2, \ldots\}$ *the conditions of the lemma hold, then*

$$ER_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X]$$

is concave on $s \geq 0$.

Proof. We have

$$ER_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X]$$
$$= E[E\{R_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X] \mid X_{k+1}\}].$$

It follows from the concavity of the conditional expectation on the right-hand side that

$$\delta_+ ER_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X]$$
$$\leq \delta_- ER_{k+1}[s; (p_0, \ldots, p_{k-1}, p_k^*); X].$$

(The expectation operator E and the differential operators δ_+ and δ_- can be interchanged because R_{k+1} is bounded by $f_1 s$ for all policies p and demand x.)

Theorem 1. *Let p be any policy that satisfies*

$$f_{k+1} \in \delta ER_k[p_k; (p_0, \ldots, p_{k-1}); X] \tag{20}$$

for $k = 1, 2, \ldots$. *Then* $E\{R_{k+1}[s; p; X] \mid X_{k+1}\}$ *is concave on* $s \geq 0$ *for* $k = 1, 2, \ldots$. *Moreover, it is optimal to continue the sales of fare class* $k + 1$ *while more than* p_k *seats remain unsold, and to protect* p_k *seats for the nest of the k highest fare classes.*

Proof. From (10) and (11).

$$E\{\delta_+ R_1[s;\, p;\, X] | X_1\} = f_1 I_{[X_1 > s]} \tag{21}$$

and

$$E\{\delta_- R_1[s;\, p;\, X] | X_1\} = f_1 I_{[X_1 \geqslant s]}, \tag{22}$$

where $I_{[A]} = 1$ if condition A holds, and $I_{[A]} = 0$ otherwise. Hence

$$\delta_+ E\{R_1[s;\, p;\, X] | X_1\} \leqslant \delta_- E\{R_1[s;\, p;\, X] | X_1\}.$$

Thus, $E\{R_1[s, p; X] | X_1\}$ is concave in s for any policy p, and, given condition (20), the concavity assertion in the theorem follows from Lemma 1 by induction.

To prove optimality of the protection level p_k it is necessary to examine the behavior of $ER_{k+1}[s;\, (p_0, \ldots, p_k);\, X]$ as a function of p_k for any s. Denote the left derivative, right derivative and subdifferential with respect to p_k by γ_-, γ_+, and γ, respectively.

From (9),

$$\gamma_+ R_{k+1}[s;\, p;\, x] = \begin{cases} 0 & \text{for } 0 \leqslant s \leqslant p_k \\ -f_{k+1} + \delta_+ R_k[p_k;\, p;\, x] & \text{for } p_k < s \leqslant p_k + x_{k+1} \\ 0 & \text{for } p_k + x_{k+1} < s. \end{cases} \tag{23}$$

$$\gamma_- R_{k+1}[s;\, p;\, x] = \begin{cases} 0 & \text{for } 0 < s < p_k \\ -f_{k+1} + \delta_- R_k[p_k;\, p;\, x] & \text{for } p_k \leqslant s < p_k + x_{k+1} \\ 0 & \text{for } p_k + x_{k+1} \leqslant s. \end{cases} \tag{24}$$

Recall that $R_k[p_k;\, p;\, x]$ is independent of x_{k+1}. Taking the expectations of these derivatives and reversing the order of differentiation and expectation yields for $p_k < s < p_k + x_{k+1}$

$$\gamma_+ ER_{k+1}[s;\, p;\, X] = (-f_{k+1} + \delta_+ ER_k[p_k;\, p;\, X]) \cdot \Pr[X_{k+1} \geqslant s - p_k] \tag{25}$$

$$\gamma_- ER_{k+1}[s;\, p;\, X] = (-f_{k+1} + \delta_- ER_k[p_k;\, p;\, X]) \cdot \Pr[X_{k+1} > s - p_k]. \tag{26}$$

Conditions (20), (23), (24), (25) and (26) imply

$$\gamma_+ ER_{k+1}[s;\, p;\, X] \leqslant 0 \leqslant \gamma_- ER_{k+1}[s;\, p;\, X];$$

that is, $0 \in \gamma ER_{k+1}[s;\, p;\, X]$. Also, from (25), (26) and the concavity of $ER_k[s;\, p;\, X]$ with respect to s, it follows that $ER_{k+1}[s;\, (p_0, \ldots, p_k');\, X]$ is nondecreasing over $p_k' < p_k$ and nonincreasing over $p_k' > p_k$. Thus p_k maximizes $ER_{k+1}[s;\, p;\, X]$, as required.

It has thus been established that condition (20) is sufficient for optimality of a policy p. The next theorem shows that there exist integer policies that are optimal, given that demand is integer-valued.

In what follows, the abbreviation CLBI (for Concave and Linear Between Integers) will denote that a revenue or expected revenue function is concave and piecewise linear with changes in slope only at integer values of the domain. A CLBI function has the property that the set of subdifferentials at integer points of the domain covers all real numbers between any particular right derivative and any greater left derivative. That is, a CLBI function $f(x)$ satisfies the following *covering property*:

If c is a constant that $\delta_+ f(s_2) < c < \delta_- f(s_1)$ for some $s_1 < s_2$, then there is an integer $n \in [s_1, s_2]$ such that $c \in \delta f(n)$.

Theorem 2. *If the demand random variables X_1, X_2, ... are integer-valued, there exists an optimal integer policy p^*.*

Proof. (By induction): Taking expectations with respect to X_1 in (21) and (22) yields the subdifferential

$$\delta ER_1[s; p; X] = [f_1 \Pr[X_1 > s], f_1 \Pr[X_1 \geqslant s]]. \tag{27}$$

By inspection of (8) and (27) and the fact that demand is integer-valued, $ER_1[s;\, p;\, X]$ is CLBI on $s \geqslant 0$. Furthermore, since demand is finite with probability 1, there is an s sufficiently large that

$$\delta_+ ER_1[s;\, p;\, X] = f_1 \Pr[X_1 > s] < f_2.$$

(In practice a sufficiently large s might exceed the capacity of the aircraft. However, in this case, there would be no need to find the next protection level.) Also, by definition,

$$\delta_+ ER_1[0;\, p;\, X] = +\infty > f_2.$$

Then the covering property of CLBI functions ensures the existence of an integer p_1^* that satisfies $f_2 \in \delta ER_1[p_1^*;\, p;\, X]$; that is, p_1^* satisfies the optimality condition (20) for $k = 1$.

Let $d[x]$ denote the largest integer less than or equal to x, and $u[x]$ the smallest integer greater than or equal to x. Thus, $d[x] = u[x - 1]$ when x is a noninteger, and $d[x] = u[x - 1] + 1$ when x is an integer. Taking expectations with respect to X_{k+1} in (12) and (13) yields

$$\delta_+ ER_{k+1}[s;\, p;\, X] = f_{k+1} \Pr[X_{k+1} > s - p_k] + \sum_{i=0}^{d[s-p_k]} \delta_+ ER_k[s - i;\, p;\, X] \Pr[X_{k+1} = i], \tag{28}$$

and

$$\delta_- ER_{k+1}[s; p; X] = f_{k+1}\Pr[X_{k+1} \geq s - p_k]$$

$$+ \sum_{i=0}^{u[s-p_k-1]} \delta_- ER_k[s - i; p; X]\Pr[X_{k+1} = i]. \quad (29)$$

Now suppose that $ER_k[s; p; x]$ is CLBI on $s \geq 0$ for some k, and there are integer protection levels $p_1^*, p_2^*, \ldots p_k^*$ satisfying (20). From (28) and (29), the integrality of p_k^* and X_{k+1} and the fact that $ER_k[s; p; x]$ is CLBI ensure that the left and right derivatives of $ER_{k+1}[s; p^*; X]$ are equal and constant at noninteger s and that equality can fail to hold only at integer s. That is, $ER_{k+1}[s; p^*; X]$ is CLBI. That $ER_{k+1}[s; p^*; X]$ is concave follows from Corollary 1.

By recursive application of (28) and (29), using the fact that total demand is finite with probability 1, there exists an s sufficiently large that

$$\delta_+ ER_{k+1}[s; p; X] < f_{k+2}$$

$$< \infty = \delta_- ER_{k+1}[0; p; x] \quad (30)$$

for each $k = 2, 3, \ldots$.

Property (30) together with the covering property of the subdifferentials of CLBI functions ensure that there is an integer $s = p_{k+1}^*$ satisfying $f_{k+2} \in \delta ER_k[p_{k+1}^*; p^*; X]$; that is, optimality condition (20). The existence of an optimal integer policy $p^* = (p_1^*, p_2^*, \ldots)$ follows by induction.

3.1. Monotone Optimal Stopping Problems and the Optimality of Fixed Protection Level Booking Policies

In this section, we establish that the fixed protection levels p defined by condition (20) are optimal over the set of all admissible policies, not just over the set of fixed policies $\mathscr{P}$. To this end, consider the problem of stopping bookings in fare class $k + 1$ when there are s seats remaining and $X_{k+1} \geq x_{k+1}$ has been observed, where $x_{k+1} \geq 0$.

The problem of finding an optimal policy for choosing p_k belongs to the class of stochastic optimization problems known as optimal stopping problems. It has been shown by Derman and Sacks (1960) and Chow and Robbins (1961) that optimal stopping problems defined as *monotone* have particularly simple solutions.

To check the conditions for monotonicity, we need to consider the expected gain in revenue obtained by changing the protection level for the nest of the k highest fare classes from $p_k + 1$ to p_k, given that the additional seat being released will be sold to fare class $k + 1$. Call this expected gain G_k, where

$$G_k(s; p)$$

$$= E[R_{k+1}\{s; (p_0, p_1, \ldots, p_k); X\} \mid X_{k+1} > s - p_k]$$

$$- E[R_{k+1}\{s; (p_0, p_1, \ldots, p_k + 1); X\} \mid X_{k+1} > s - p_k].$$

By (23), the gain can be rewritten as

$$G_k(s; p) = \gamma_+ E[R_{k+1}\{p_k; p; X\} \mid X_{k+1} > s - p_k]$$

$$= -f_{k+1} + \delta_+ ER_k[p_k; p; X].$$

The booking problem for fare class k will be monotone if for fixed s and $(p_1, \ldots, p_{k-1})$ the following conditions are satisfied:

1. There is a p_k^* such that the gain G_k is nonnegative for $p_k < p_k^*$ and nonpositive for $p_k \geq p_k^*$.
2. $|R[s; (p_0, p_1, \ldots, p_k + 1); X] - R[s; (p_0, p_1, \ldots, p_k); X]|$ is bounded for all p_k.

Condition 2 is trivial because the total revenue is certainly bounded by sf_1. Suppose that p^* is an integer policy satisfying the conditions in Theorem 1. Then p_k^* and $G_k[s; (p_0^*, p_1^*, \ldots, p_{k-1}^*, p_k)]$ satisfy condition 1 by Theorem 1.

If the model is monotone the expected revenue will be maximized by protecting p_k^* seats for the nest of the k highest fare classes; that is, a fixed-limit policy will be optimal for the protection level p_k.

The significance of this result in the context of airline seat allocation is that fixed protection levels defined by condition (20) will be optimal as long as no change in the probability distributions of demand is foreseen. In other words, no ad hoc adjustment of protection levels is justified unless a shift in the demand distributions is detected. In practice, one or more of the independent demands, low before high or limited information assumptions may not be satisfied, and there is the possibility that revenues can be increased by protection level adjustments in a dynamic reservations environment. The point here is that such adjustments must be properly justified, for example, the observation of a sudden rush of demand in one fare class should not lead to a protection level adjustment unless it is believed that the rush signals a genuine shift in the underlying demand distribution. For a preliminary investigation of the effects of stochastically dependent demands on the optimal policy, see Brumelle et al.

3.2. An Alternative Expression for the Optimal Protection Levels

This section presents the derivation of the expression for the optimal protection levels in terms of demands given in (7). This expression is relevant when demand distributions can be approximated by continuous distributions, and it provides the optimality conditions in a form analogous to the EMSRa approximation.

Lemma 2. *If p satisfies*

$$f_1 \Pr[X_1 > p_1 \cap X_1 + X_2 > p_2 \cap \ldots \cap X_1 + X_2 + \ldots + X_k > p_k] = f_{k+1}, \tag{31}$$

for all k, then with probability 1 *for* $k = 1, 2, \ldots$ *and* $s \geq p_k$

$$\delta_+ E[R_{k+1}[s; p; X] | X_{k+1}] = f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + X_2 + \ldots + X_k > p_k \cap X_1 + \ldots + X_{k+1} > s | X_{k+1}]. \tag{32}$$

Proof. Assume that p satisfies the hypothesis of the lemma. For $s \geq p_k$, we can obtain the following expression from (12) by taking the expectation and interchanging E and δ_+:

$$\delta_+ E\{R_{k+1}[s; p; X] | X_{k+1}\} = f_{k+1} I_{[s<p_k+X_{k+1}]} + \delta_+ E\{R_k[s - X_{k+1}; p; X] | X_{k+1}\} I_{[s \geq p_k + X_{k+1}]}. \tag{33}$$

Using (31) to substitute for f_{k+1}, the right-hand side of this expression can be rewritten as

$$f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > p_k \cap s < p_k + X_{k+1} | X_{k+1}] + \delta_+ E\{R_k[s - X_{k+1}; p; X] | X_{k+1}\} I_{[s \geq p_k + X_{k+1}]}. \tag{34}$$

For $k = 1$, using (10) and the fact that

$$[X_1 + X_2 > s \cap s \geq p_1 + X_2] \Rightarrow [X_1 > p_1],$$

(33) becomes

$$\begin{aligned} \delta_+ E\{R_2[s; p; X] | X_2\} &= f_1 \Pr[X_1 > p_1 \cap X_1 + X_2 > s \cap s < p_1 + X_2 | X_2] \\ &\quad + f_1 \Pr[X_1 > p_1 \cap X_1 + X_2 > s \cap s \geq p_1 + X_2 | X_2] \\ &= f_1 \Pr[X_1 > p_1 \cap X_1 + X_2 > s | X_2]. \end{aligned} \tag{35}$$

Thus the lemma holds for $k = 1$.

The proof is completed by induction. Using the induction hypothesis that the lemma holds for k, substitute for $\delta_+ R_k$ in the last term of (35).

$$\begin{aligned} E\{\delta_+ R_{k+1}[s; p; X] | X_{k+1}\} &= f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > p_k \cap s < p_k + X_{k+1} | X_{k+1}] \\ &\quad + f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > s - X_{k+1} \cap s - X_{k+1} \geq p_k | X_{k+1}] \\ &= f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > p_k \cap X_1 + \ldots + X_{k+1} > s | X_{k+1}], \end{aligned} \tag{36}$$

which completes the proof.

Corollary 2. *If p satisfies* (31), *then for* $s \geq p_k$

$$\delta_+ E R_{k+1}[s; p; X] = f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > p_k \cap X_1 + \ldots + X_{k+1} > s]. \tag{37}$$

Theorem 3. *If p satisfies* (31), *then p is optimal.*

Proof. By Lemma 2 if p satisfies (31), then

$$f_{k+1} = f_1 \Pr[X_1 > p_1 \cap \ldots \cap X_1 + \ldots + X_k > p_k] = \delta_+ E R_k[p_k; p; X]. \tag{38}$$

By Theorem 1, p is thus optimal.

3.3. Application of the Optimality Conditions

Condition 20 provides a concise characterization of optimal policies in terms of the subdifferential (or first differences) of the expected revenue function. Given any estimates of future demand distributions (discrete or continuous), it is easy to determine the subdifferential of the expected revenue function for fare class 1 as a function of seats remaining and then numerically identify an integer p_1^* that satisfied the optimality condition. The remaining subdifferentials and optimal protection levels can be determined in a like manner by successive applications of (20).

An alternative approach is provided by solving for the optimal protection levels given by (31) for $k = 1, 2, \ldots$. A condition which guarantees the solvability of this system of equations is that the demands have a continuous joint distribution function. If an empirical distribution for integer demand is being used, then the above equations can likely be solved to within the statistical error of the demand distribution. This approach is consistent with previous airline practice where estimated continuous demand distributions (e.g., fitted normal distributions) have been used in methods like EMSRa.

Empirical studies have shown that the normal probability distribution gives a good continuous approximation to airline demand distributions (Shlifer 1975). If normality is assumed, solutions to (31) can be obtained with straightforward numerical methods. Robinson has generalized the conditions to the case that fares are not necessarily monotonic and has proposed an efficient Monte Carlo integration scheme for finding optimal protection levels.

There is a way in which the optimality conditions (31) can be used to monitor the past performance of seat allocation decisions given historical data on seat bookings for a series of flights. For simplicity, the discussion will assume three fare classes; the method generalizes easily to an arbitrary number of classes. With three fare classes, conditions (31) can be written

$$Pr[X_1 > p_1] = \frac{f_2}{f_1} \tag{39}$$

$$\Pr[X_1 > p_1 \cap X_1 + X_2 > p_2] = \frac{f_3}{f_1}. \tag{40}$$

Given a series of past flights, the probability $\Pr[X_1 > p_1]$ can be estimated by the proportion of flights on which class-1 demand exceeded its protection level. Then (39) specifies that this proportion should be close to the ratio f_2/f_1. Similarly, (40) specifies that the proportion of flights on which both class-1 demand exceeded its protection level and the total of class-1 and 2 demands exceeded their protection level should be close to the ratio f_3/f_1. If allocation decisions are being made optimally, these conditions should be satisfied approximately in a sufficiently long series of past flights. Severe departures from these ratios would be symptomatic of suboptimal allocation decisions. The appealing aspect of this approach is its simplicity—no modeling of the demand distributions and no numerical integrations are required.

4. COMPARISON OF EMSRa AND OPTIMAL SOLUTIONS

The EMSRa method determines the optimal protection level for the full-fare class but is not optimal for the remaining fare classes. However, the EMSRa equations are particularly simple to implement because they do not involve joint probability distributions. It is thus of interest to examine the performance of the EMSRa method relative to the optimal solutions given above. Note that neither the EMSRa nor exact optimality conditions give explicit formulas for the optimal protection levels in terms of the problem parameters, so analytical comparison of the revenues produced by the two methods is difficult unless unrealistic demand distributions are assumed. Numerical comparison of the two methods can, however, give some indication of relative performance.

This section gives the results of numerical comparisons of EMSRa versus optimal solutions in a three fare-class problem. Table I presents the results of six examples in which cabin capacity is fixed at 100 seats and fares f_i are varied. Fares are expressed as proportions of full fare; thus, $f_1 = 1$ throughout. The % error revenue column gives the loss in revenues incurred from using the EMSRa method as a percentage of optimal revenues. In Table II, the fares are held constant at levels $f_3 = 0.7$ and $f_2 = 0.9$, and cabin capacity is varied.

Table I
Comparison of EMSRa Versus Optimal for Three Fare Classes

Example No.	f_3	f_2	p_1	p_2 EMSRa	p_2 Optimal	% Error Revenue
1	0.6	0.7	32	70	80	0.37
2	0.6	0.8	27	80	87	0.32
3	0.6	0.9	19	86	91	0.19
4	0.7	0.8	27	64	75	0.41
5	0.7	0.9	19	73	82	0.45
6	0.8	0.9	19	57	70	0.50

Discrete approximations to the normal probability distribution were used for all demand distributions. The nominal mean demands for fare classes 1, 2 and 3 were 40, 60 and 80, and the nominal standard deviations were 16, 24 and 32, respectively. These figures are nominal because the discretization procedure introduced small deviations from the exact parameter values. These parameters correspond to a coefficient of variation of 0.4; i.e., the standard deviation is 40% of the mean. This is slightly higher than the 0.33 that Belobaba (1987) mentions as a common airline "k factor" for total demand.

(Note that the normal distribution has significant mass below zero when the coefficient of variation is

Table II
Capacity Effects

Capacity	%Error Revenue
82	0.54
100	0.45
120	0.35
140	0.24
160	0.14

much higher than 0.4. Use of a truncated normal or other positive distribution is indicated under these circumstances.)

Remarks

In this set of examples the EMSRa method produces seat allocations that are significantly different from optimal allocations, but the loss in revenue associated is not great. Specifically:

a. In these examples, the EMSRa method consistently underestimates the number of seats that should be protected for the two upper fare classes. The discrepancy is 19% in the worst case (example 6). We will show with a counterexample that the EMSRa method is not guaranteed to underestimate in this way.
b. In the worst case the discrepancy between EMSRa and optimal solutions with respect to revenues is approximately ½%.
c. The error appears to increase as the discount fares approach the full fare; however, the sample is much too small here to justify any general conclusion of this nature.
d. The error decreases as the aircraft capacity increases. This effect is, of course, to be expected because allocation policies have less impact when the capacity is able to accommodate most of the demands.

On the basis of these examples, a decision of whether or not to use the EMSRa approach rests on whether or not a potential revenue loss on the order of ½% or less (with three fare classes) is justified by the simpler implementation of the method relative to the optimal method. Further work is needed to determine the relative performance of the EMSRa method with a larger number of fare classes or under circumstances in which dynamic adjustments of protection levels are justified.

Additional numerical analyses related to the seat allocation problem are provided in Wollmer (1992) and have been conducted by P. Belobaba and colleagues at the MIT Flight Transportation Laboratory.

4.1. EMSRa Underestimation of Protection Levels—A Counterexample

As mentioned, the EMSRa method consistently underestimated the protection level p_2 for the two upper fare classes in all the numerical trials. It is thus reasonable to conjecture that the approximation will always behave in this way. This is not true for all demand distributions, as shown by the following counterexample using exponentially distributed demands. It remains an open question whether or not the conjecture holds true for normally distributed demands.

For convenience, let the unit of demand be 100 seats, and introduce the relative fares $r_2 = f_2/f_1$ and $r_3 = f_3/f_1$. Now suppose that X_1 and X_2 follow identical, independent exponential distributions with mean 1.0 (100 seats). That is, $\Pr[X_i > x_i] = e^{-x_i}$ for $i = 1, 2$. It is not suggested that the exponential distribution has any particular merit for modeling airline demands, although it could serve as a surrogate for a severely right-skewed distribution if the need arose. Its use here is purely as a device for establishing a counterexample to a general conjecture.

Let p_i^a denote protection levels obtained with the EMSRa method. Then with the above distributional assumptions and (2)–(5), we have $p_1^a = -\ln(r_2)$, and $p_2^a = -\ln(r_3) - \ln(r_3/r_2)$.

For the optimal solutions, (7) gives $p_1 = -\ln(r_2) = p_1^a$, and

$$
\begin{aligned}
r_3 &= \Pr[X_1 > p_1 \cap X_1 + X_2 > p_2] \\
&= \Pr[X_1 > p_2] \\
&\quad + \Pr[p_1 < X_1 \leq p_2 \cap X_2 > p_2 - X_1] \\
&= e^{-p_2} + \int_{p_1}^{p_2} \Pr[X_2 > p_2 - x_1] e^{-x_1}\, dx_1 \\
&= e^{-p_2}(1 + p_2 - p_1). \qquad (41)
\end{aligned}
$$

Suppose that $r_2 = ½$ and $r_3 = ¼$. Then $p_1 \cong 0.69$ and $p_2^a \cong 2.08$ (69 and 208 seats, respectively). Given p_1, a simple line search using (41) produces the optimal $p_2 \cong 2.37$ from the equation above. Thus, for this example, the EMSRa method underestimates p_2 by 29 seats. This behavior is consistent with the conjecture.

Now suppose instead that $r_2 = 4/10$ and $r_3 = 1/10$. Then $p_1^a \cong 0.92$ and $p_2^a \cong 3.69$. In this case, however, $p_2 \cong 3.61$, and the EMSRa method *overestimates* p_2 by 8 seats. It is not difficult to show that for these demand distributions, the EMSRa method will overestimate p_2 whenever $r_2/r_3 > 3.51$, approximately.

5. SUMMARY

This paper provides a rigorous formulation of the revenue function for the multiple fare class seat allocation problem for either discrete or continuous probability distributions of demand and demonstrates conditions under which the expected revenue function is concave. We show that a booking policy that maximizes expected revenue can be characterized by a simple set of conditions on the subdifferential of the expected revenue function. These conditions are further simplified to a set of conditions relating to the probability distributions of demand for the various fare classes to their respective fares. These conditions are guaranteed to have a solution if the joint distribution of the demands is approximated by a continuous probability distribution. It is shown that the fixed protection limit policies given by these optimality conditions are optimal over the class of all policies that depend only on the history of the booking process. A numerical comparison is made of the optimal solutions with the approximate solutions yielded by the expected marginal seat revenue (EMSRa) method. A tentative conclusion on the basis of this restricted set of examples is that the EMSRa method produces seat allocations that are significantly different from optimal allocations, and the associated loss in revenue is of the order of ½%.

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