P1) $1^{ro}$ ver recta tg a $x^\alpha$ en $x=1$: 

$m_{(+1)} = (x^\alpha)'\big|_{x=1} = \alpha x^{\alpha-1}\big|_{x=1} = \alpha$.

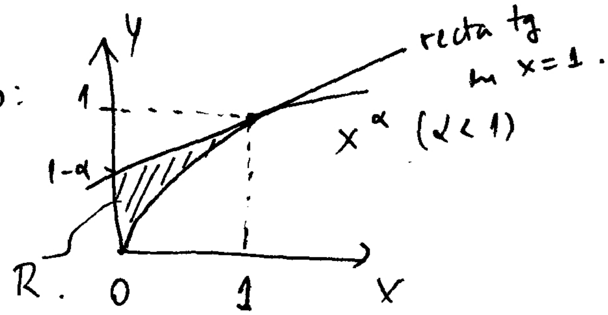
Además $f(x) = x^\alpha$ es tq $f(1) = 1$

$\Rightarrow y - y_0 = m_{(-1)}(x - x_0)$

$y - 1 = \alpha(x-1)$

$y = \alpha x + \underbrace{(1-\alpha)}_{>0}$ es la recta.

$\Rightarrow$ gráfico:




a) Área? basta ver Area recta - Area $x^\alpha$ (En $[0,1]$)

$$\Rightarrow A = \int_0^1 (\alpha x + (1-\alpha) - x^\alpha)\,dx = \alpha\frac{x^2}{2}\Big|_0^1 + (1-\alpha)x\Big|_0^1 - \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1$$

$$= \frac{\alpha}{2} + (1-\alpha) - \frac{1}{\alpha+1} = \frac{\alpha + 2(1-\alpha)}{2} - \frac{1}{\alpha+1} = \frac{2-\alpha}{2} - \frac{1}{\alpha+1}$$

$$= \frac{(2-\alpha)(\alpha+1) - 2}{2(\alpha+1)} = \frac{\alpha(1-\alpha)}{2(\alpha+1)}$$

b) ¿$V_{oy}$? De nuevo, basta ver la resta de los Vol de cada curva.

$V = V_{oy}(\text{recta}) - V_{oy}(x^\alpha)$

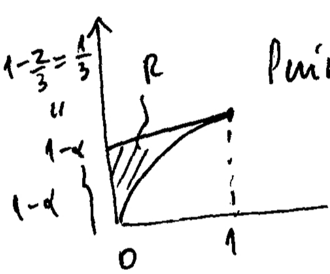
$$= 2\pi\left(\int_0^1 x\cdot(\alpha x + (1-\alpha))\,dx - \int_0^1 x^{\alpha+1}\,dx\right) = \left(\frac{\alpha}{3}x^3\Big|_0^1 + \frac{(1-\alpha)}{2}x^2\Big|_0^1 - \frac{x^{\alpha+2}}{\alpha+2}\Big|_0^1\right)2\pi$$

$$= 2\pi\left(\frac{\alpha}{3} + \frac{(1-\alpha)}{2} - \frac{1}{\alpha+2}\right) = 2\pi\left(\frac{2\alpha}{6} + \frac{3(1-\alpha)}{6} - \frac{6}{6(\alpha+2)}\right) = \left(\frac{3-\alpha}{6} - \frac{6}{6(\alpha+2)}\right)2\pi$$

$$= 2\pi\left(\frac{(3-\alpha)(\alpha+2) - 6}{6(\alpha+2)}\right) = 2\pi\left(\frac{\cancel{6} - \alpha^2 - 2\alpha + 3\alpha - \cancel{6}}{6(\alpha+2)}\right) = 2\pi\left(\frac{\alpha - \alpha^2}{6(\alpha+2)}\right) = \left(\frac{\alpha(1-\alpha)}{\cancel{6}_3(\alpha+2)}\right)\cancel{2}\pi$$

$$= \frac{\pi}{3}\frac{\alpha(1-\alpha)}{\alpha+2}$$

c) $\alpha = \frac{2}{3}$ Quiero Perím. R.




Perím $= \frac{1}{3}$ + Largo (recta) + Largo ($x^{\frac{2}{3}}$) (Largos en $(0,1)$) $\quad 2\left(\frac{1}{3}\right) = \frac{2}{3}$.

$$L(\text{rect}) = \int_0^1 \sqrt{1+\alpha^2}\,dx = \sqrt{1+\alpha^2}\Big|_{\alpha=\frac{2}{3}} = \frac{\sqrt{13}}{3}$$

$$L(\text{curva}) = \int_0^1 \sqrt{1+\alpha^2 x^{2(\alpha-1)}}\,dx = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx.$$

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$$L(x^{2/3}) = \int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}dx = \int_0^1 \sqrt{\frac{x^{2/3}+\frac{4}{9}}{x^{2/3}}}\,dx = \int_0^1 \frac{1}{x^{1/3}}\sqrt{x^{2/3}+\frac{4}{9}}\,dx.$$

Notar que: $\left(x^{\frac{2}{3}}\right)' = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3}\cdot\frac{1}{x^{1/3}}$.

$$\Rightarrow \left[\left(x^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{3}{2}}\right]' = \cancel{\frac{3}{2}}\cdot\left(x^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{3}{2}-1=\frac{1}{2}}\cdot\cancel{\frac{2}{3}}\cdot\frac{1}{x^{1/3}}$$ ✓ justo :)

$$\therefore\ L(x^{2/3})\Big|_0^1 = \left(x^{2/3}+\frac{4}{9}\right)^{3/2}\Big|_0^1 = \left(1+\frac{4}{9}\right)^{3/2} - \left(\frac{4}{9}\right)^{3/2} = \left(\frac{13}{9}\right)^{3/2} - \left(\frac{2}{3}\right)^3.$$

$$= \frac{13\sqrt{13}}{27} - \frac{8}{27} = \frac{13\sqrt{13}-8}{27}$$

$$\therefore\ P' = \frac{1}{3} + \frac{\sqrt{13}}{3} + \frac{13\sqrt{13}}{27} - \frac{8}{27} = \frac{9}{27} + \frac{9\sqrt{13}}{27} + \frac{13\sqrt{13}}{27} - \frac{8}{27} = \frac{22\sqrt{13}+1}{27}. \blacksquare$$

P3) a) Trivial.

b) $x(t) = \underbrace{e^{2t}\cos t}_{\psi(t)}$ $\quad y(t) = \underbrace{e^{2t}}_{\varphi(t)}\sin t$. $L$? si $t\in[0,2\pi]$.

$$\psi'(t) = 2e^{2t}\cos t - \operatorname{sen} t\, e^{2t} = e^{2t}(2\cos t - \operatorname{sen} t)$$

$$\varphi'(t) = e^{2t}\cos t + 2e^{2t}\operatorname{sen} t = e^{2t}(\cos t + 2\operatorname{sen} t)$$

$$L = \int_0^{2\pi}\sqrt{\psi'(t)^2+\varphi'(t)^2}\,dt = \int_0^{2\pi}$$

$$= \int_0^{2\pi}\sqrt{e^{4t}\left[4\cos^2 t - \cancel{4\operatorname{sen} t\cos t} + \operatorname{sen}^2 t + \cos^2 t + 4\operatorname{sen}^2 t + \cancel{4\cos t\operatorname{sen} t}\right]}\,dt$$

$$= \int_0^{2\pi} e^{2t}\sqrt{5}\,dt = \sqrt{5}\int_0^{2\pi} e^{2t}dt = \sqrt{5}\,\frac{e^{2t}}{2}\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi}-1\right).$$

c) Quiero $t_0$ tq $\displaystyle\int_0^{t_0}\dots = \frac{1}{2}\int_0^{2\pi}\dots$

$$\frac{\sqrt{5}}{2}\left(e^{2t_0}-1\right) = \frac{\sqrt{5}}{4}\left(e^{4\pi}-1\right)$$

$$\Rightarrow e^{2t_0} = \frac{e^{4\pi}}{2}+\frac{1}{2} \Rightarrow \ln(e^{2t_0}) = \ln\left(\frac{e^{4\pi}+1}{2}\right) \Rightarrow \cancel{2t_0 = 4\pi - \ln 2}$$

$$\cancel{\boxed{t_0 = 2\pi - \frac{\ln 2}{2}}}$$

$$\boxed{t_0 = \frac{1}{2}\ln\left(\frac{e^{4\pi}+1}{2}\right)}$$

$\boxed{P5}$ a) $\vec{r}(t) = R(t - \sin t, 1 - \cos t) \quad t \in [0, 2\pi]$.



1ro Vu long. total: $\vec{r}\,'(t) = R(1 - \cos t, \sin t)$

$$\|\vec{r}\,'(t)\| = R\sqrt{(1-\cos t)^2 + (\sin t)^2}$$
$$= R\sqrt{1 - 2\cos t + \cos^2 t + 1 + \sin^2 t}$$
$$= R\sqrt{2 - 2\cos t}$$
$$= R\sqrt{2}\cdot\sqrt{1 - \cos t}$$
$$= 2R \sin\frac{t}{2}.$$

pero: $-\sin^2 x + \cos^2 x = \cos 2x$.

$-\sin^2 x + 1 - \sin^2 x = \cos 2x$

$\sqrt{\dfrac{1 - \cos 2x}{2}} = \sqrt{\sin^2 x} = \sin x$

$\therefore \sqrt{1 - \cos x} = \sqrt{2}\sin\frac{x}{2}$

$$\therefore\ s(t) = \int_0^t 2R \sin\frac{u}{2}\,du = 2R\int_0^t \sin\frac{u}{2}\,du = 4R\left(-\cos\frac{u}{2}\right)\Big|_0^t$$
$$= 4R\left(1 - \cos\frac{t}{2}\right)$$

$\boxed{L(\text{arc}) = S(2\pi) = 8R} \quad \therefore s \in [0, 8R]$.

$\Rightarrow$ invertamos $s$:

$$s(t) = 4R\left(1 - \cos\frac{t}{2}\right) \Rightarrow \frac{s(t)}{4R} = 1 - \cos\frac{t}{2} \Rightarrow \cos\frac{t}{2} = 1 - \frac{s(t)}{4R}$$

$$\therefore\ t(s) = 2\arccos\left(1 - \frac{s}{4R}\right) \quad s \in [0, 8R]$$

$$\therefore\ \vec{\sigma}(s) = \vec{r}(t(s))$$

pero notemos que: $\cos\left(\frac{t}{2}\right) = \cos\left(\arccos\left(1 - \frac{s}{4R}\right)\right) = 1 - \frac{s}{4R}$.

$\sin t = ?$ Notar que: $\sin t = 2\sin\frac{t}{2}\cos\frac{t}{2} = 2\cos\frac{t}{2}\sqrt{1 - \cos^2\frac{t}{2}}$ ✓

$$\sin t = 2\left(1 - \frac{s}{4R}\right)\sqrt{1 - \left(1 - \frac{s}{4R}\right)^2}$$

$\sqrt{1 - \sin^2 t} = \sqrt{1 - 4\left(1-\frac{s}{4R}\right)^2\left(1 - \left(1 - \frac{s}{4R}\right)^2\right)}$

$\cos t = ?$ Notar que: $\cos t = \cancel{\cos^2 t/2 - \sin^2 t/2} = 2\cos^2\frac{t}{2} - 1$ ✓

$$\cos t = \left(1 - \frac{s}{4R}\right)^2 - 4\left(1 - \frac{s}{4R}\right)^2\left[1 - \left(1 - \frac{s}{4R}\right)^2\right] = 4\left(1 - \frac{s}{4R}\right)^4 - 3\left(1 - \frac{s}{4R}\right)^2.$$

$$= 2\left(1 - \frac{s}{4R}\right)^2 - 1. \Rightarrow 1 - \cos t = 2\left(1 - \left(1 - \frac{s}{4R}\right)^2\right) \;//$$

$$\therefore \vec{\gamma}(s) = R\left(2\arccos\left(1-\frac{s}{4R}\right) - 2\left(1-\frac{s}{4R}\right)\sqrt{1-\left(1-\frac{s}{4R}\right)^2},\ 2\left(1-\left(1-\frac{s}{4R}\right)^2\right)\right)$$

$$= 2R\left(\arccos\left(1-\frac{s}{4R}\right) - \left(1-\frac{s}{4R}\right)\sqrt{1-\left(1-\frac{s}{4R}\right)^2},\ 1-\left(1-\frac{s}{4R}\right)^2\right)$$

$s \in [0, 8R]$ ∎

b) Hélice. $\vec{r}(t) = \left(a\cos t, a\sin t, \frac{ht}{2\pi}\right) \Rightarrow \vec{r}\,'(t) = \left(-a\sin t, a\cos t, \frac{h}{2\pi}\right)$

$$\Rightarrow s(t) = \int_0^t \left(a^2\sin^2 z + a^2\cos^2 z + \frac{h^2}{4\pi^2}\right)^{1/2} dz = \int_0^t \left(a^2 + \frac{h^2}{4\pi^2}\right)^{1/2} dz$$

$$= t \cdot M$$

$L(\text{Helice}) = 2\pi \cdot M$

inverter s: $s(t) = tM \Rightarrow \frac{s}{M} = t$, $s \in [0, 2\pi \cdot M]$.

$$\Rightarrow \vec{\gamma}(s) = \left(a\cos\left(\frac{s}{M}\right), a\sin\left(\frac{s}{M}\right), \frac{h \cdot s}{2\pi M}\right) \quad s \in [0, 2\pi M]$$

$$\Leftrightarrow \vec{\gamma}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+\frac{h^2}{4\pi^2}}}\right), a\sin\left(\frac{s}{\sqrt{a^2+\frac{h^2}{4\pi^2}}}\right), \frac{h \cdot s}{2\pi\sqrt{a^2+\frac{h^2}{4\pi^2}}}\right) \quad s \in [0, 2\pi M]$$

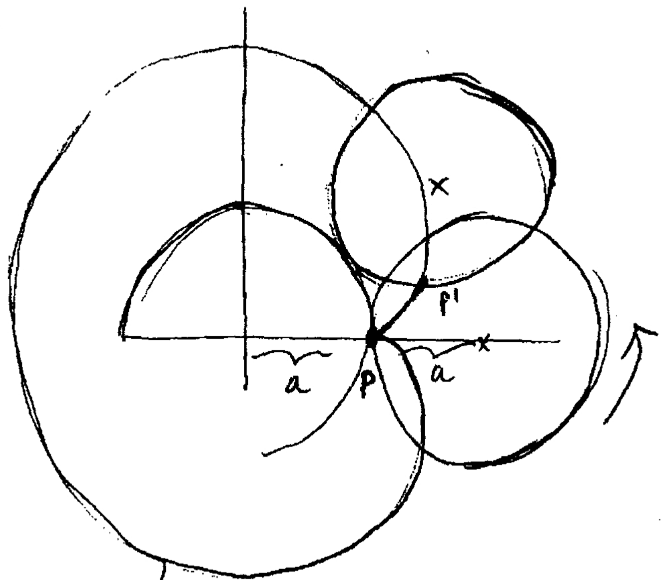
$$= \left(a\cos\left(\frac{2\pi s}{\sqrt{h^2+4\pi^2a^2}}\right), a\sin\left(\frac{2\pi s}{\sqrt{h^2+4\pi^2a^2}}\right), \frac{h \cdot s}{\sqrt{a^2 4\pi^2+h^2}}\right)$$

$$s \in \left[0, \sqrt{4\pi^2 \cdot \left(a^2+\frac{h^2}{4\pi^2}\right)}\right]$$

$$s \in \left[0, \sqrt{4\pi^2a^2+h^2}\right]$$

∎

o) idea.

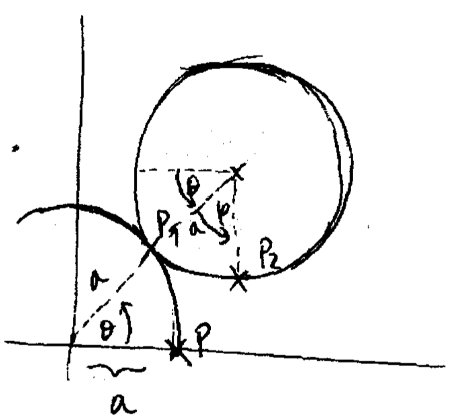


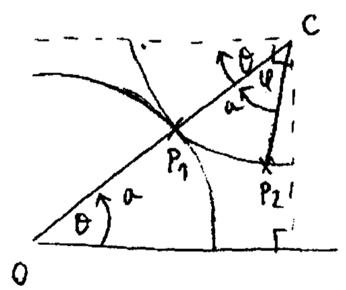

La idea es la siguiente:

Partimos con 2 circ., una de centro $(0,0)$ y la otra de centro $(2a,0)$ (ambas de radio a) queremos estudiar como evoluciona el punto P (el de tangencia), cuando la circ. de la derecha rueda sin resbalar sobre la centrada en $(0,0)$.

curva generada al seguir al punto P, la cardioide.

Para parametrizar consideremos el siguiente esquema:




Si la circunferencia no rodase (simplemente resbalase) el punto P siempre sería tangente) por lo cual al girar un ángulo θ respecto al origen se tendría el nuevo punto $P_1$. Como la circunferencia rueda sin resbalar, la nueva posición real es $P_2$ (Es por esto que aparece el ángulo φ, que está asociado al giro)

Debemos pues determinar $P_2$, claramente hay que usar coord. polares.

$$P_2(\rho,\theta) = \begin{pmatrix} x(\rho,\theta) \\ y(\rho,\theta) \end{pmatrix} = \left( \vec{OC} + \vec{CP_2} \right)$$

$$\vec{OC} = \begin{pmatrix} 2a\cos\theta \\ 2a\,\text{sen}\,\theta \end{pmatrix} \qquad \vec{CP} = \begin{pmatrix} -a\cos(\theta+\varphi) \\ -a\,\text{sen}\,(\theta+\varphi) \end{pmatrix}$$

Como la circunf. rueda sin resbalar: $a\theta = a\varphi$

$$\Rightarrow \boxed{\theta = \varphi}$$

así, $P_2 = \begin{pmatrix} 2a\cos\theta - a\cos(2\theta) \\ 2a\,\text{sen}\,\theta - a\,\text{sen}\,(2\theta) \end{pmatrix}$

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$$= \begin{pmatrix} 2a\cos\theta - 2a\cos^2\theta + a \\ 2a\,\mathrm{sen}\,\theta - 2a\,\mathrm{sen}\,\theta\cos\theta \end{pmatrix} = \begin{pmatrix} 2a\cos\theta(1-\cos\theta) + a \\ 2a\,\mathrm{sen}\,\theta(1-\cos\theta) \end{pmatrix} = \begin{pmatrix} x+a \\ y \end{pmatrix}$$

Trasladamos el origen a $(a,0)$ (Si $\theta=0$ debo estar en el origen)

$$= \begin{pmatrix} 2a\cos\theta(1-\cos\theta) \\ 2a\,\mathrm{sen}\,\theta(1-\cos\theta) \end{pmatrix} \rightsquigarrow \rho^2 = x^2+y^2$$

$$= 4a^2\cos^2\theta(1-\cos\theta)^2 + 4a^2\,\mathrm{sen}^2\theta(1-\cos\theta)^2$$

$$= 4a^2(1-\cos\theta)^2\left[\cos^2\theta + \mathrm{sen}^2\theta\right]$$

$$= 4a^2(1-\cos\theta)^2.$$

ie. $\rho(\theta) = 2a(1-\cos\theta)$ ■

~~P2) $\vec{F} = (F_1, F_2, F_3)$ campo $C^1$ en $\mathbb{R}^3$ tq $\mathrm{div}\,\vec{F} = 0 \iff \exists\, \vec{G}$ clase $C^2$ tq $\vec{F} = \mathrm{rot}\,\vec{G}$.~~

~~$\Leftarrow$) Veamos que dado que $\exists\, \vec{G}$ tq $\mathrm{rot}\,\vec{G} = \vec{F} \Rightarrow \mathrm{div}\,\vec{F} = 0$.~~

~~En efecto: $\vec{F} = \mathrm{rot}\,\vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ G_1 & G_2 & G_3 \end{vmatrix} = \hat{\imath}(\partial_y G_3 - \partial_z G_2) + \hat{\jmath}(-\partial_x G_3 + \partial_z G_1) + \hat{k}(\partial_x G_2 - \partial_y G_1)$~~

~~Notación: $\partial_{x_i} F = \dfrac{\partial F}{\partial x_i}$~~

~~Luego: $\mathrm{div}\,\vec{F} = \partial_x(\partial_y G_3 - \partial_z G_2) + \partial_y(\partial_z G_1 - \partial_x G_3) + \partial_z(\partial_x G_2 - \partial_y G_1)$~~

~~$= \partial_x\partial_y G_3 - \partial_x\partial_z G_2 + \partial_y\partial_z G_1 - \partial_y\partial_x G_3 + \partial_z\partial_x G_2 - \partial_z\partial_y G_1$~~

~~Como $G$ es $C^2$ vale el Teo. de Schwarz $\Rightarrow$ puedo cambiar el orden de las deriv.~~

~~$= \cancel{\partial_x\partial_y G_3} - \cancel{\partial_x\partial_z G_2} + \cancel{\partial_y\partial_z G_1} - \cancel{\partial_x\partial_y G_3} + \cancel{\partial_x\partial_z G_2} - \cancel{\partial_y\partial_z G_1}$~~

~~$\equiv 0$. que era lo deseado.~~

~~$\Rightarrow$) obs. $G_1$ debe ser $G_1(x,y,z) = \int_0^z F_2(x,y,s)\,ds - \int_0^y F_3(x,s,z)\,ds$.~~

~~Esta parte es simplemente calcular el rotor y usar esta consecuencia del Teo Fund del calculo: $\dfrac{d}{dx}\left(\int_0^x f(u)\,du\right) = \dfrac{d}{dx}\left(F(x) - F(0)\right) = \dfrac{d}{dx}F(x) - \dfrac{d}{dx}\overbrace{F(0)}^{\text{cte.}} = f(x) - 0 = f(x)$~~

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