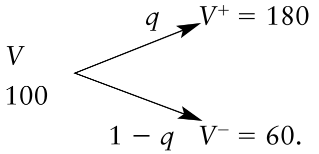
Appendix 3.1

Binomial Option Valuation

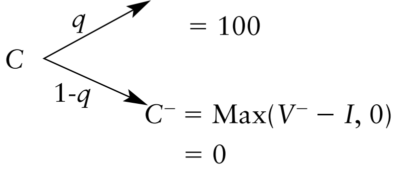
The Basic Valuation Idea: Option Replication and Risk Neutrality

The basic idea enabling the pricing of options is that one can construct a portfolio consisting of buying a particular number, N, of shares of the underlying asset (e.g., common stock) and borrowing against them an appropriate amount, $\$B$, at the riskless rate, that would exactly replicate the future returns of the option in any state of nature. Since the option and this equivalent portfolio (effectively, an appropriately levered position in the stock) would provide the same future returns, to avoid risk-free arbitrage profit opportunities they must sell for the same current value. Thus, we can value the option by determining the cost of constructing its equivalent replicating portfolio, that is, the cost of a *synthetic* or homemade option equivalent.

Suppose that the price of the underlying stock (currently at $V = \$100$) will move over the next period either up to $V^+ = 180$ (i.e., with a multiplicative up parameter, $u = 1.8$) or down to $V^- = 60$ (with a multiplicative down parameter, $d = 0.6$), with probabilities q and $(1 - q)$, respectively, that is,



The value of the option over the period would then be contingent on the price of the underlying stock. Assuming $I = \$80$ (and $r = 0.08$),



where C^+ and C^- are the values of the call option at the end of the period if the stock moves up or down, respectively.

Suppose now we construct a portfolio as described above, consisting of (*a*) buying N shares of the underlying stock at its current price, V, financed in part by (*b*) borrowing an amount of $\$B$ at the riskless interest rate (e.g, selling short Treasury bills), for a net out-of-pocket cost of $NS - B$. That is,

$$\text{Call option} \approx \text{Buy } N \text{ Shares at } V \text{ \& Borrow } \$B \text{ at } r \qquad (3.A.1)$$
$$\text{or } C \approx (NV - B).$$

After one period, we would need to repay the principal amount borrowed at the beginning (B) with interest, or $(1 + r)B$ for certain. The value of this portfolio over the next period will thus be

$$\begin{tikzcd}
& N\,V^{+} - (1 + r)B \\
\mathrm{N\,V} - B \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
& N\,V^{-} - (1 + r)B
\end{tikzcd}$$

If the portfolio is to offer the same return in each state at the end of the period as the option, then

$$\begin{tikzcd}
& N\,V^{+} - (1 + r)B = C^{+} \\
{} \arrow[ur] \arrow[dr] & \\
& N\,V^{-} - (1 + r)B = C^{-}
\end{tikzcd}$$

Solving these two equations (conditions of equal payoff) for the two unknowns, N and B, gives

$$\begin{aligned} N &= (C^+ - C^-)/(V^+ - V^-) && (3.A.2)\\ &= (100 - 0)/(180 - 60) = 0.83 \text{ shares;} \end{aligned}$$

$$\begin{aligned} B &= (V^- C^+ - V^+ C^-)/[(V^+ - V^-)(1 + r)] && (3.A.3)\\ &= (N\,V^- - C^-)/(1 + r)\\ &= (0.83 \times 60 - 0)/1.08 = \$46. \end{aligned}$$

The number of shares of the underlying asset that we need to buy to replicate one option over the next period, N, is known as the option's *delta* or *hedge ratio,* and is simply obtained in the discrete case as the difference (spread) of option prices divided by the spread of stock prices. That is, we can replicate the return to the option by purchasing N (= 0.83) shares of the underlying stock at the current price, V, and borrowing the amount $\$B$ (= \$46) at the riskless rate, r.

When substituted back into equation 3.A.1, $C = NV - \mathrm{B}$, equations 3.A.2 and 3.A.3 finally result in

$$C = [pC^+ + (1 - p)C^-]/(1 + r) \qquad (3.A.4)$$
$$= [0.4 \times 100 + 0.6 \times 0]/1.08 = \$37,$$

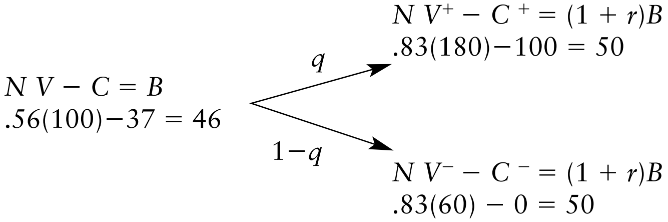
where

$$p = [(1 + r)V - V^-]/(V^+ - V^-) \qquad (3.A.5)$$
$$= [1.08 \times 100 - 60]/(180 - 60) = 0.4$$

is a transformed or *risk-neutral probability,* that is, the probability that would prevail in a risk-neutral world where investors are indifferent to risk.

Risk-Neutral Valuation

Intuitively, equation 3.A.1 can be rearranged into $NV - C = B$, that is, creating a portfolio consisting of (*a*) buying N shares of the underlying stock and (*b*) selling (writing) one call option would provide a certain amount of $(1 + r)B = \$50$ next period, regardless of whether the stock moves up or down:



Through the ability to construct such a *riskless hedge,* risk can effectively be “squeezed out” of the problem, so that investors’ risk attitudes do not matter. Therefore, we can equivalently — and more conveniently — obtain the correct option value by *pretending* to be in a *risk-neutral world* where risk is irrelevant. In such a world, all assets (including stocks, options, etc.) would earn the risk-free return, and so *expected* cash flows (weighted by the risk-neutral probabilities, p) could be appropriately discounted at the risk-free rate.

Denoting by $R^+ \equiv u - 1 = V^+/V - 1$ (= 0.80 or 80%) the return if the stock moves up (+), and by $R^- \equiv V^-/V - 1$ (= −0.40 or −40%) the down (−) return, the risk-neutral probability, p, can be alternatively obtained from the condition that the *expected* return on the stock in a risk-neutral world must equal the riskless rate, that is,

$$pR^+ + (1 - p)R^- = r.$$

Solving for p yields

$$p = (r - R^-)/(R^+ - R^-)$$
$$= [0.08 - (-0.40)]/[0.80 - (-0.40)], \text{ or} \qquad (3.A.5')$$
$$= [(1 + r) - d]/(u - d) = (1.08 - 0.6)/(1.8 - 0.6)$$
$$= 0.4.$$

Similarly, the expected return on the option must also equal the risk-free rate in a risk-neutral world, that is,

$$[pC^+ + (1 - p)C^-]/C - 1 = r,$$

resulting in above equation 3.A.4.

A number of points are worth reviewing about the above call option valuation. It provides an exact formula for the value of the option in terms of V, I, r, and the stock's volatility (spread). With no dividends, $C > V - I$, so an American call option should not be exercised early; when dividends are introduced, early exercise may be justified, however. The motivation for the pricing of the option rests with the absence of arbitrage profit opportunities, a strong economic condition.

The actual probability of up and down movements, q, does not appear in the valuation formula. Moreover, the value of the option does not depend on investors' attitudes toward risk or on the characteristics of other assets — it is priced only relative to the underlying asset, V.

The value of the option can be equivalently obtained in a risk-neutral world (since it is independent of risk preferences). Actually, p is the value probability q would have in equilibrium if investors were risk neutral. As the above valuation formula confirms, in such a risk-neutral world — where all assets are expected to earn the riskless rate of return — the current value of the option can be obtained from its expected future values (using the risk-neutral probability, p), discounted at the risk-free interest rate.

A put option can be valued similarly, except that we would need to *sell* (instead of buy) shares of the underlying stock, and *lend* (instead of borrow) at the riskless interest rate (i.e., buy government bonds), that is,

Put option ≈ *Sell* N shares at V & *Lend* \$$B$ at r.

The hedge ratio, or delta, for a put option is simply the delta of the corresponding call option minus 1, giving $0.83 - 1 = -0.17$ in the above example (with the minus sign indicating *selling,* rather than buying, 0.17 shares of the underlying stock). Applying equation 3.A.3 in the case of a similar put option where $P^- = A - V^- = 100 - 60 = 40$, the amount to lend is given by

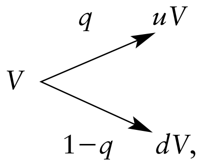
$$\begin{aligned} B &= (N\,V^- - P^-)/(1 + r) \\ &= (-0.17 \times 60 - 40)/1.08 = -\$46.3. \end{aligned} \tag{3.A.6}$$

Thus, to replicate a put option, we need to sell 0.17 shares of stock at $V = \$100$ and lend (minus sign in B) \$46.3 at the riskless rate (i.e., buy Treasury bills with that face value). Thus, the current value of the put option should be

$$P = N\,V - B = (-0.17)(100) - (-46.3) = \$29.6. \tag{3.A.7}$$

The General Multiplicative Binomial Approach

The general multiplicative binomial option pricing approach was popularized by Cox, Ross, and Rubinstein (1979). It is based on the replication argument described above, except that the underlying stock price follows a multiplicative binomial process over successive periods described by



where the stock price at the beginning of a given period, V, may increase (by a multiplicative factor u) with probability q to uV or decrease with complementary probability $(1 - q)$ to dV at the end of the period. Thus u and d represent the (continuously compounded or logarithmic) rate of return if the stock moves up or down, respectively, with $d = 1/u$. (Since riskless borrowing at the rate r is also available, to avoid riskless arbitrage profit opportunities, $u > (1 + r) > d$.)

In our earlier notation, $V^+ \equiv uV$ and $V^- \equiv dV$ with $d = 1/u$, or alternatively

$$u \equiv V^+/V = 1 + R^+, \tag{3.A.8}$$

where R^+ is the up (+) return, and

$$d \equiv V^-/V = 1 + R^-, \tag{3.A.8$'$}$$

where R^- is the down (−) return.
Thus, expressions 3.A.2, 3.A.3, 3.A.4, and 3.A.5 would now become

$$N = [C^+ - C^-]/[(u - d)V], \tag{3.A.2$'$}$$
$$B = [dC^+ - uC^-]/[(u - d)(1 + r)], \tag{3.A.3$'$}$$
$$C = [pC^+ + (1 - p)C^-]/(1 + r), \tag{3.A.4$'$}$$

and

$$\begin{aligned} p &= [(1 + r) - d]/(u - d) \\ &= (1.08 - 0.6)/(1.8 - 0.6) = 0.4. \end{aligned} \tag{3.A.5$'$}$$

This valuation procedure can be easily extended to multiple periods. If the time to expiration of the option, τ, is subdivided into n equal subintervals, each of length $h \equiv \tau/n$, and the same valuation process is repeated starting at the expiration date and working backward recursively, the general binomial pricing formula for n periods would be obtained:

$$C = \sum_{j=0}^{\mathrm{n}} \{n!/j!(n - j)!\}p^j(1 - p)^{n-j}\mathrm{Max}(u^j d^{n-j}V - I, 0) / (1 + r)^n.$$

The first part, $\{n!/j!(n - j)!\}\, p^j\, (1 - p)^{n-j}$, is the binomial distribution formula giving the probability that the stock will take j upward jumps in n steps, each with (risk-neutral) probability p. The last part, $\mathrm{Max}(u^j d^{n-j}V - I,\ 0)$, gives the value of the call option with exercise cost I at expiration conditional on the stock following j ups each by u%, and $n - j$ downs each by d% within n periods. The summation of all the possible (from $j = 0$ to n) option values at expiration, multiplied by the probability that each will occur, gives the expected terminal option value, which is then discounted at the riskless rate over the n periods.

If we let m be the minimum number of upward moves j over n periods necessary for the call option to be exercised or finish in the money (i.e., $u^m d^{n-m}V > I$, or by logarithmic transformation m is the smallest nonnegative integer greater than $\ln(I/Vd^n)/\ln(u/d)$), and break up the resulting term into two parts, then the binomial option-pricing formula can be more conveniently rewritten as

$$C = V\Phi[m; n, p'] - \{I/(1 + r)^n\}\ \Phi\ [m; n, p], \tag{3.A.9}$$

where Φ is the complementary binomial distribution function (giving the probability of at least m ups out of n steps):

$$\Phi\ [m; n, p] = \sum_{j=m}^{n} \{n!/j!(n - j)!\}\, p^j\, (1 - p)^{n-j},$$

and

$$p' \equiv [u/(1 + r)]p,$$

with p and m as defined above.

One may initially object to this discrete period-by-period binomial valuation approach, since in reality stock prices may take on more that just two possible values at the end of a given period, while actual trading in the market takes place almost continuously and not on a period-by-period basis. However, the length of a "period" can be chosen to be arbitrarily small by successive subdivisions.

As the length of a trading period, h, is allowed to become increasingly smaller (approaching 0) for a given maturity, τ, continuous trading is effectively approximated. In the continuous-time limit, as the number of periods n approaches infinity, the multiplicative binomial process approximates the log-normal distribution or *smooth* diffusion Wiener process.

By choosing the parameters {u, d, and p} so that the mean and variance of the continuously compounded rate of return of the discrete binomial process are consistent in the limit with their continuous counterparts, the stock price will become log-normally distributed and the (complementary) binomial distribution function, $\Phi[.]$, will converge to the (cumulative) standard normal distribution function, $N(.)$. Specifically, by setting

$$u = \exp(\sigma\sqrt{h}),$$

$$d = 1/u, \tag{3.A.10}$$

$$p = ½ + ½(\mu/\sigma)\sqrt{h},$$

where $\mu \equiv \ln r - ½\sigma^2$, τ is the time to option expiration, n is the number of subperiods, and $h \equiv \tau/n = dt$ is the subinterval or length of a small trading period (typically expressed as a fraction of a year). Cox, Ross, and Rubinstein (1979) show that as $n \rightarrow \infty$, $\Phi[m; n, p'] \rightarrow N(x)$, so that the above binomial formula converges to the continuous-time Black-Scholes formula:

$$C = V\,N(x) - I(1 + r)^{-\tau}\,N(x - \sigma\sqrt{\tau}), \tag{3.A.11}$$

where $x \equiv \ln(V/I(1 + r)^{-\tau})/\sigma\sqrt{\tau} + ½\sigma\sqrt{\tau}$.

For example, if $\tau = 3$ months $= 0.25$ years and $n = 12$ steps, a discrete multiplicative binomial process with $u = 1.1$ and weekly intervals ($h = \tau/n = 0.02$ years) would be consistent in the limit with a lognormal diffusion process with annual standard deviation, $\sigma = \ln(u)/\sqrt{h} = \ln(1.2)/\sqrt{0.02} = 0.66$, or 66%.