Transient Torsional Vibrations in Multiple-Inertia Systems

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Abstract—**Torque amplification in a steel mill or impact phenomena in a general elastic system can be analyzed by a similarity transformation of the linear equations of motion using their eigenvector or modal matrix. Response to initial displacement or velocity and to step function forces are developed in a way that forms the principle of a fast accurate computer program. Torque amplification is the result of ill conditioning of the modal matrix and the disturbance but is mostly related to the modal matrix and is, therefore, mostly a function of the rotor system. Steel mills and other multiple inertia systems such as motor generator sets should be analyzed by this method at an early stage of their design.**

Introduction

In RECENT years, a phenomenon called "torque amplification" has caused much difficulty in steel mills and has been the subject of intense study by many investigators [1]–[5]. Torque amplification is a phenomenon involving transient vibrations that is likely to occur in any large multiple-rotor system that is subjected to impact or abrupt changes in torque. This paper is to present an explanation of the mechanics of the phenomenon and also a corresponding mathematical method of analysis.

Reduction and Definition of a System

Fig. 1(a) might represent the rotating elements in a steel mill. These elements may be represented with adequate accuracy by a system of discrete masses interconnected by linearly-responding springs and appropriate gear ratios (Fig. 1(b)). Furthermore, the system may be transformed by the well-known methods (see [1], [9], or [11]) to the single-speed system shown in Fig. 1(c). Our further discussion will concern such idealized systems. It would also apply to more general systems which include "earth" connections, and, in fact, the "torque amplification" phenomenon might occur in any large multiple-inertia system.

Vibrations in an Elastic System

Since torque amplification is caused by transient vibrations in the rotor system, a brief review of the phenomena of elastic vibrations will clarify the discussion that follows.

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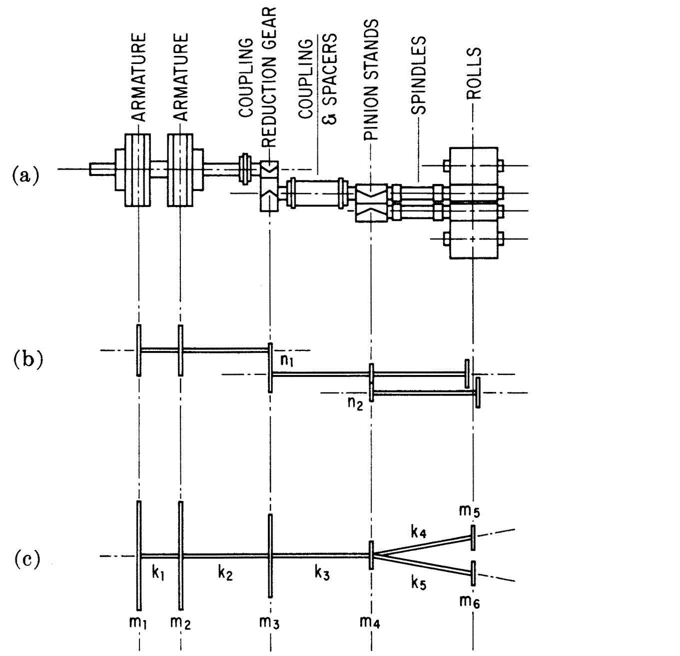


Fig. 1. (a) Rotating elements as typical finishing stand in hot-strip mill. (b) Discrete-mass and linear spring representation of rotating elements of typical mill with gear ratios of n_1:1 and n_2:1. (c) Idealized single-speed system.

The configuration of an elastic system can be defined by displacements on coordinates which are called the "degrees of freedom" of the system. The set of relative displacements that an elastic system reaches at some instant is called the "mode" of the vibration.

A simple harmonic motion that can occur in an elastic system after an appropriate disturbance is called a "natural" vibration. Most systems are capable of several natural vibrations each characterized by a "natural" frequency and a "principal" mode.

Some systems are capable of one or more motions that do not incur elastic strain and, therefore, are not vibratory and are called "rigid-body" motions. For example, the rotating elements of a steel mill can rotate as a rigid body. The number of natural vibrations plus the number of rigid-body motions equals the number of degrees of freedom.

After an event that leaves the system disturbed from its state of equilibrium a free vibration occurs. In all real systems free vibrations are transient because damping dissipates the vibratory energy. In the systems discussed here damping is slight so that it has a negligible effect on the natural frequencies, on their phase position and on the amplitude of vibration for the first few cycles.

The free vibration is the sum of one or more natural vibrations. Initially, the amplitudes of the natural vibrations, that is some multiples of the principal modes, with due consideration for phase displacement must equal in sum the initial disturbance.

Response to Impact

The events which cause torque amplification may be divided into three categories.

1) *Initial Displacement:* The event may leave the system stationary or moving as a rigid body but displaced from static equilibrium. For example, when an ingot is being rolled the mill is strained by the rolling torque but at the instant after the ingot leaves, since the rolling torque has disappeared, the mill is initially displaced from its new state of equilibrium. This event is sometimes called "snap-back."

2) *Initial Velocity:* Events of very short duration—impulsive events—can cause an initial disturbance in velocity. For example, if an ingot being rolled strikes another lying on the roll-out tables the resulting impulse will cause an abrupt change in velocity to the rolls but not to the rest of the mill and hence the system is disturbed from equilibrium.

3) *Suddenly Applied Force:* When a force is suddenly applied to the system and then maintained the system acquires a new state of equilibrium which is determined by the applied force and the steady mass accelerations of the system. For example, when an ingot enters a steel mill the system decelerates for a brief interval. During this interval, the state of equilibrium is determined by the twist in the shafts caused by the rolling torque and the opposing decelerating torques on each rotor. Since the shafts are initially unstrained the rotors are initially displaced from this new state of equilibrium.

Mathematical Prediction of Free Vibrations

Methods used to predict free vibrations and torque amplification include the following. The "integral equation" method wherein the equations of motion in integral form are solved simultaneously. The "analog" method wherein the system is modeled by an analog computer. The "differential equation" method wherein the equations of motion in differential form are integrated successively and this process is repeated for short intervals of time.

Nonlinearity can be handled by the last two methods which is their great advantage. However, they provide little insight into the nature of the phenomenon. A fourth method, which may be called the "transformation" method, that uses techniques of linear algebra, will now be presented.

Transformation of Equations of Motion

An outline will be presented here of a method of analysis that is described in detail in several works (such as [6] and [9]). Consider an elastic system such as that shown in Fig. 1(c) comprising n discrete masses, m_i, interconnected by linear springs k_k. Assume that its instantaneous configuration can be stated by the n displacements x_i of these masses, that is, the system has n degrees of freedom. A displacement x_i will incur spring forces on mass i and on masses connected to it. If k_{ij} is defined as the force on mass i caused by a unit displacement x_j of mass j then the Newton law of motion provides the following n equations:

$$\begin{aligned} -k_{11}x_1 - k_{12}x_2 - \cdots - k_{1n}x_n + F_1 &= m_1\ddot{x}_1 \\ -k_{21}x_1 - k_{22}x_2 - \cdots - k_{12n}x_n + F_2 &= m_2\ddot{x}_2 \\ &\vdots \\ -k_{n1}x_1 - k_{n2}x_2 - \cdots - k_{nn}x_n + F_n &= m_n\ddot{x}_n \end{aligned} \quad (1)$$

or in the usual matrix notation

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\} \quad (2)$$

where $[m]$ is a matrix with the diagonal $m_1, m_2, \cdots, m_n$ and zeros elsewhere. The matrix $[k]$ is shown in [6] to be symmetric according to the reciprocal theorem of Betti.

Equations (1) and (2) are systems of second-order differential equations. Consider first the complementary functions, which are solutions to

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}. \quad (3)$$

These are of the form

$$\{x\} = \{\zeta\} \sin \omega t \quad (4)$$

where the amplitudes ζ_i generally may be complex. The natural frequency ω is in general multivalued. Differentiating (4) provides

$$\{\ddot{x}\} = -\omega^2\{x\} \quad (5)$$

and substitution of (4) and (5) into (3), division by x_i, and premultiplication by $[m]^{-1}$ provides

$$[[m]^{-1}[k] - [I]\omega^2]\{\zeta\} = \{0\} \quad (6)$$

which is an eigenvalue, or characteristic value form of the problem.

Methods are described for finding the eigenvalues and eigenvectors $\{\phi\}_i$ for this particular nonsymmetric matrix in [7] and [8]. For each eigenvalue ω_i^2, we can write

$$[[m]^{-1}[k] - [I]\omega_i^2]\{\phi\}_i = \{0\}. \quad (7)$$

The eigenvector $\{\phi\}_i$ is the principal mode corresponding to natural frequency ω_i. Since (7) can be multiplied by any scalar, $\{\phi\}_i$ can be normalized for convenience so that

$$\{\phi\}_i^T\{\phi\}_i = 1. \quad (8)$$

Furthermore, for convenience, let all sets such as the equations in (7) be arranged in order of decreasing eigenvalue.

The displacements $\{x\}$ can be expressed as a linear combination of the principal modes or in matrix form, where $[\phi]$ is the matrix of eigenvectors,

$$\{x\} = [\phi]\{q\} \quad (9)$$

and differentiating twice provides

$$\{\ddot{x}\} = [\phi]\{\ddot{q}\}. \tag{10}$$

Substituting (9) and (10) into the equations of motion (2) provides

$$[m][\phi]\{\ddot{q}\} + [k][\phi]\{q\} = \{F\}. \tag{11}$$

It is shown in [6] that $[\phi]$ will reduce $[m]$ or $[k]$ to diagonal form by a similarity transformation

$$[\phi]^T[m][\phi] = [M] \tag{12}$$

$$[\phi]^T[k][\phi] = [K] \tag{13}$$

where $[M]$ and $[K]$ are diagonal matrices. Then premultiplying terms in (11) by $[\phi]^T$ provides

$$[M]\{\ddot{q}\} + [K]\{q\} = \{P\} \tag{14}$$

where

$$\{P\} = [\phi]^T\{F\}. \tag{15}$$

Since $M_{ij} = K_{ij} = 0,\ i \neq j$, the equations in (15) can be written

$$\begin{aligned} M_1\ddot{q}_1 + K_1q_1 &= P_1 \\ M_2\ddot{q}_2 + K_2q_2 &= P_2 \\ &\ \ \vdots \\ M_n\ddot{q}_n + K_nq_n &= P_n. \end{aligned} \tag{16}$$

Thus the original coupled equations of motion (2) have been transformed into a set of simple uncoupled systems by choice of a suitable set of coordinates $\{q\}$. The solutions for (16) in terms of q_i may be derived and may then be inversely transformed to find the motion in terms of x_i, the original coordinates.

It can be shown that transformation is very general. The inertias m_i and spring rates k_{ij} and displacements x_i may refer either to rotation or to straight-line motion. While $[m]$ must be symmetrical it need not be diagonal. Rigid-body motions, which are indicated by zero eigenvalues, cause no problems in the transformation. Steel mill drives always have one rigid-body motion which will be indicated by the last zero eigenvalue equation in (16) in which K_n will vanish. Hereafter, in this development, one rigid-body motion will be assumed to be possible.

Initial Condition Problems

In [10] the free vibrations resulting from initial displacements or velocities are derived though the derivation must be altered somewhat to facilitate computation. Let $\{x\}_{t=0}$ be the initial displacements from equilibrium. In the transformed systems the initial displacements are

$$\{q\}_{t=0} = [\phi]^{-1}\{x\}_{t=0}. \tag{17}$$

Solutions for motion in simple systems after such initial conditions are well known to be of the form

$$q_i = q_{i,t=0} \cos \omega_i t. \tag{18}$$

The nth system, in which $K_n = \omega_1 = 0$, is not vibratory and represents the rigid-body mode. Since shaft torques are sought, and rigid-body motion does not incur any shaft torque, the rigid-body mode can be excluded from further consideration.

Let

$$[U] = \text{diagonal matrix whose diagonal terms are } q_{i,t=0}. \tag{19}$$

Then

$$\{q\} = [U]\{\cos\} \tag{20}$$

where

$$\{\cos\} = \{\cos \omega_1 t, \cos \omega_2 t, \cdots, \cos \omega_{n-1}t, 0\} \tag{21}$$

and displacements in the original system are

$$\{x\} = [\phi][U]\{\cos\} \tag{22}$$

$$= [A]\{\cos\}. \tag{23}$$

Matrix $[A]$ contains the amplitude of each mass at each natural frequency ω_i.

Let $\{\dot{x}\}_{t=0}$ be the vector whose elements are the initial differences of the velocities of the masses from the mean velocity. In the transformed system

$$\{\dot{q}\}_{t=0} = [\phi]^{-1}\{\dot{x}\}_{t=0}. \tag{24}$$

The motion of a simple system with an initial velocity is

$$q_i = \frac{\dot{q}_{i,t=0}}{\omega_i} \sin \omega_i t, \qquad \omega_i \neq 0. \tag{25}$$

Letting $[V]$ = diagonal matrix whose diagonal terms are

$$\frac{\dot{q}_{i,t=0}}{\omega_i} \tag{26}$$

provides

$$\{x\} = [\phi]\{q\} \tag{27}$$

$$= [\phi][V]\{\sin\} \tag{28}$$

$$= [B]\{\sin\} \tag{29}$$

where

$$\{\sin\} = \{\sin \omega_1 t, \sin \omega_2 t, \cdots, \sin \omega_{n-1}t, 0\} \tag{30}$$

again assuming one rigid-body mode. Matrix $[B]$ contains the amplitudes of each mass at each natural frequency ω_i. It is convenient to eliminate the consideration of the rigid body in computation simply by causing the final columns of $[A]$, $[B]$, and $[V]$ to vanish.

Excitation by Forces

The response of the system shown in Fig. 1(c) to the external forces $\{F\}$ will depend on their temporal nature. Usually all nonzero terms in $\{F\}$ have the same temporal dependence, and, if this is the case, the terms in $\{P\}$ also will have that temporal dependence. Computation is then facilitated.

The most common problem assumes that the nonzero terms in $\{F\}$ and, therefore, in $\{P\}$, represent forces that are abruptly applied as in a step function of t. In such a case the solutions to (16) are of the form

$$q_i = \frac{P_i}{K_i}(1 - \cos \omega_i t), \qquad K_i \neq 0. \tag{31}$$

Other situations can be analyzed, for example, when F_i is a ramp function of time or a pulse. Solutions for step functions, ramp functions, and pulses are generally of the form

$$q_i = \frac{P_i}{K_i}(\psi_{ci} + \psi_{vi}), \qquad K_i \neq 0 \tag{32}$$

where ψ_{ci} is a steady term and ψ_{vi} is a vibratory term with frequency ω_i.

Let $[W]$ = the diagonal matrix whose diagonal terms are P_i/K_i, for $K_i \neq 0$. Then we can write

$$\{q\} = [W]\{\psi_c + \psi_v\} \tag{33}$$

and

$$\{x\} = [\phi][W]\{\psi_c + \psi_v\} \tag{34}$$

$$= [C_1]\{H_1\} + [C_2]\{H_2\} \tag{35}$$

where

$$\{H_2\} = \{\cos\} \text{ or } \{\sin\} \tag{36}$$

as appropriate to the nature of $\{\psi_v\}$ and

$$\{H_1\} = \{1,1,\cdots,1,0\}. \tag{37}$$

Here C_1 represents constant displacements and C_2 represents amplitudes of each mass at each natural frequency. Again, it is convenient to cause any columns in W, C_1, and C_2 corresponding to rigid-body modes, for which $K_i = 0$, to vanish.

Shaft Torques

The torques in any shaft can be calculated from the relative displacements of the two rotors that it connects. The torque for each problem must be calculated separately.

Displacements for various problems are given by (23), (29), and (35) which are of the form

$$\{x\} = [\alpha]\{\beta\} \tag{38}$$

whence one can write

$$x_i = (\alpha)_i\{\beta\} \tag{39}$$

where $(\alpha)_i$ is the vector representing the ith row of matrix $[\alpha]$. Thus the torque in the kth shaft is

$$Fel_k = k_k(x_i - x_j) \tag{40}$$

$$= k_k((\alpha)_i - (\alpha)_j)\{\beta\} \tag{41}$$

where k_k is the spring rate of the kth shaft and i and j are the two rotors which it connects. Suppose that there are n_k shafts. It is easy to construct an $n_k \times n$ matrix $[D]$ whose n_k rows are $k_k((\alpha_i) - (\alpha_j))$ and then

$$\{Fel\} = [D]\{\beta\} \tag{42}$$

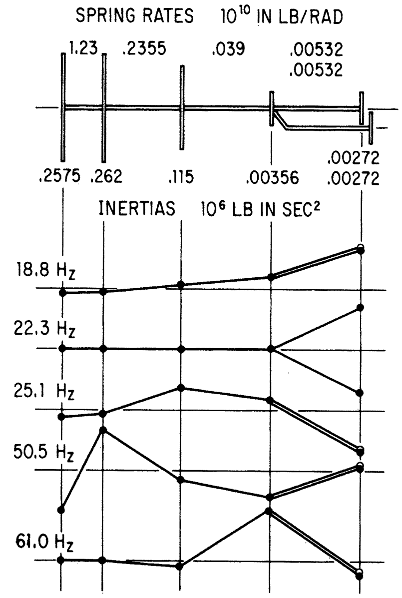


Fig. 2. System data, natural frequencies, and principle modes of example problem.

and $[D]$ contains the amplitudes of vibratory torque or the mean torque at each natural frequency for each shaft.

The main objective of the analysis is to predict the torque amplification factor (TAF), which we may define for the kth shaft as:

$$\text{TAF}_k = F\text{peak}_k/F\text{steady}_k \tag{43}$$

where $F\text{peak}_k$ is the maximum instantaneous torque and $F\text{steady}_k$ is the steady-state torque occurring on shaft k.

In principle Fel_k could be computed for various times t from (42) and the largest value in absolute magnitude selected as Fpeak.

In practice, it is adequate to add the absolute amplitudes of the variable components of the torque for initial displacement or initial velocity problems and, thereby, obtain the upper bound of magnitude of torque as an estimate of Fpeak.

For the step force case, the force prevailing after the abrupt increase implies an acceleration or deceleration. This acceleration plus the applied steady force forms the steady state and causes the shaft torques corresponding to $[C_1]\{H_1\}$ in (35). The algebraic sum of corresponding terms in $[D]$ must be added to the sum of absolute amplitudes to obtain Fpeak.

Example Problem

The first stand of an 80-in hot strip mill that was studied is taken as an example. Fig. 2 shows a reduced system representing this mill. Table I shows conditions of two problems that were considered.

The results of the eigenvalue analysis, i.e., the eigenvalues, natural frequencies, and principal modes, are listed on Table II. The principal modes are also plotted on Fig. 2. Table III lists the characteristics of the transformed systems including the transformed problem.

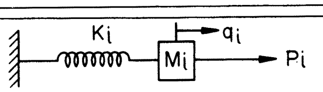
TABLE I
PROBLEM CONDITIONS

	Initial Displacement (Snap-back)		Suddenly-applied Torque (Ingot-entry)
Rotor No.	Torque in lb.	Displacement rad	in lb.
1	$-.544 \times 10^7$	-.00241	0
2	$-.544 \times 10^7$	-.00197	0
3	0	.00265	0
4	0	.0305	0
5	$.544 \times 10^7$	.1328	$.544 \times 10^7$
6	$.544 \times 10^7$	.1328	$.544 \times 10^7$

TABLE II
EIGENANALYSIS OF FINISH STAND NUMBER I

	Eigenvalues - ω^2					
	1.47×10^5	1.01×10^5	$.249 \times 10^5$	$.196 \times 10^5$	$.140 \times 10^5$	negligible
	Natural frequency - $\omega/2\pi$ Hertz					
	61.0	50.5	25.1	22.3	18.8	0
Rotor No.	Principal Modes - Eigenvectors					
1	-.00173	-.604	-.101	0	-.0201	.408
2	.00360	.667	-.0482	0	-.0142	.408
3	-.0275	-.161	.360	0	-.0385	.408
4	.977	-.383	.177	0	.198	.408
5	.150	.0925	-.643	-.707	.692	.408
6	.150	.0925	-.643	.707	.692	.408

TABLE III
TRANSFORMED SYSTEMS



Spring Rate K 10^9 in lb	Inertia M 10^4 in lb.sec^2	Torque P 10^7 in lb	Initial Displacement U 10^{-3} rad
.5311	.3611	-.163	-3.09
21.53	21.41	.1007	.0305
.5115	2.051	-.6994	-12.1
.0532	.272	.0	0
.0429	.3075	.7532	180.0
0	10.73	.4442	0

TABLE IV
TORQUE AMPLITUDES CAUSED BY INITIAL DISPLACEMENT (SNAP-BACK)

Frequency Hz	61.0	50.5	25.2	22.8	18.8	0
Shaft No.		Torque 10^7 in lb				
1	.0201	-.0731	.885	0	-1.27	0
2	-.0225	.0091	1.32	0	-2.18	0
3	.120	.0004	-.0978	0	-1.10	0
4	.0184	-.0001	-.0596	0	-.461	0
5	-.0184	-.0001	-.0596	0	-.461	0

TABLE V
TORQUE AMPLITUDES CAUSED BY SUDDENLY APPLIED TORQUE (INGOT ENTRY)

Frequency Hz	61.0	50.5	25.2	22.8	18.8	0
Shaft No.		Torque 10^7 in lb				
1	.0203	-.0476	.782	0	-1.30	0
2	-.0226	-.0060	1.160	0	-2.23	0
3	.121	-.0003	-.0865	0	-1.12	0
4	-.0185	0	-.0527	0	.473	0
5	-.0185	0	-.0527	0	.473	0

TABLE VI
PEAK TORQUES AND TAF—COMPARISON TO DIFFERENTIAL-EQUATION (DE) METHOD

	Initial Displacement (Snap-back)		Suddenly Applied Torque (Ingot-entry)		
Shaft No.	Peak Torque 10^7 in lb	TAF	Peak Torque 10^7 in lb	TAF	Peak Torque by D.E. Method 10^7 in lb
1	2.15	3.97	2.68	4.93	2.50
2	3.43	3.15	4.41	4.06	4.38
3	1.33	1.22	2.39	2.20	2.33
4	.544	1.0	1.08	1.99	1.05
5	.544	1.0	1.08	1.99	1.05

When the torques are calculated the amplitudes in Table IV for the initial displacement problem and in Table V for the suddenly-applied force problem are obtained. The peak torques and TAF for these problems are shown on Table VI.

Some TAF values are very large indicating that if this mill were built it would experience large torque amplifications.

COMPARISON WITH DIFFERENTIAL-EQUATION METHOD

The validity of the transformation method is verified by comparing its TAF with those from the differential-equation method which are also listed on Table VI. As expected, the TAF by linear transformation are slightly higher because they represent the upper bound.

The differential-equation method inherently tends to accumulate rounding errors and so it is limited to predictions over a few hundred time intervals or a few cycles of the lowest frequency in a typical steel mill problem. This is usually quite adequate.

The linear transformation method is mathematically exact, and the precise torque at any instant can be computed. However, the upper bound is an adequate prediction of the TAF for ordinary purposes.

The transformation method is strictly limited to linear problems. However, many systems are nearly linear and the transformation method will provide a useful adjunct to the differential-equation method in quasi-linear cases if only because it provides insight into the nature of the problem.

CAUSE OF TORQUE AMPLIFICATION

A close examination of the principal modes in this example reveals the cause of the large amplitudes. To simplify matters, the 22.3-Hz mode may be neglected because it obviously cannot be excited by the conditions under consideration. The modes at the second and third rotors and in the interconnecting shaft, for the 61.0-Hz, 25.2-Hz, and 18.8-Hz vibrations are interesting. They are plotted on a larger scale on Fig. 3. The 50.5-Hz vibration is conveniently ignored because it can be shown by other requirements to be small in amplitude.

The disturbances from equilibrium due to ingot leaving and due to deceleration from ingot entry are also shown

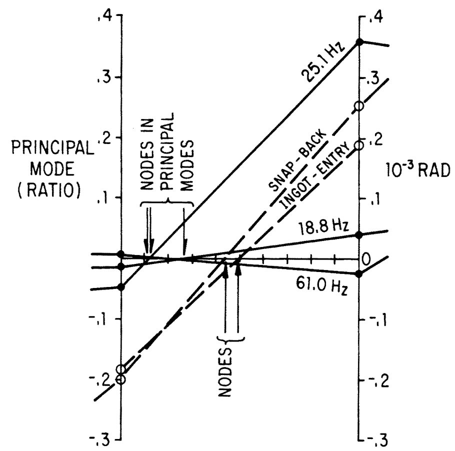


Fig. 3. Enlarged diagram comparing three principal modes with initial displacement and showing separation of the nodes.

in Fig. 3. Note that the nodes of the three principal modes are rather closely clustered and quite widely separated from the nodes of the disturbed states. Recall the earlier statement that initially the sums of the amplitudes of the natural vibrations, i.e., multiples of the principal modes, must equal the initial displacement. In the vicinity of their nodes all the principal modes are relatively small but the initial displacements are not small. Therefore, large amplitudes of the natural vibrations are required to meet the initial requirements.

In systems which do not have high torque amplifications the node in the displacement is found to lie near or within the cluster of nodes of the principal modes that are excited. This explanation of the mechanics of the system permits some generalizations.

Mathematically the requirement that the sum of amplitudes of natural vibrations must equal the initial conditions can be stated in the equation

$$[\phi]\{q\}_{t=0} = \{x\}_{t=0}. \qquad (44)$$

High torque amplification is represented mathematically by "ill-conditioning" in these linear equations. It seems proper, therefore, to speak of a physical state which displays this type of phenomenon as an "ill-conditioned" state.

Paralleling the mathematical theory (see [12, ch. 8]) ill-conditioning and, therefore, torque amplification are more a function of the system than of the disturbance. For any system some disturbances can be conceived which cause ill-conditioning as can other disturbances which do not. However, systems with narrowly clustered nodes are likely to respond violently to most disturbances and such systems may be considered ill-conditioned in their own right.

Closure

The foregoing has presented the methods of similarity transformation with modifications that facilitate computation and new methods for calculating torques as a means for analyzing the behavior of steel mills subjected to impact A computer program based on this principle can be completely general, inherently accurate, and quite fast. Most of the operations required are available as ready-made subroutines.

Torque amplification is shown to be a problem of ill-conditioning and especially ill-conditioning of the modal matrix. This implies that it is more a problem of the system than of the impact that excites it. A system which has high TAF after one event is likely to have high TAF after other events. Contrariwise, a "dead" system is likely to stay dead.

Because a linear transformation computer program is economical to use it is now practical to study preliminary proposals for steel mills to avoid badly conditioned systems before extensive design work is done. For the same reason all large multiple-rotor systems, and especially motor generator sets, should be analyzed by this method to discover and avoid unfortunate combinations of components that, when assembled, are unduly sensitive to shocks.

Future developments should include the refinements required for computation to response to ramp functions and to pulses. The latter would simulate circuit-breaker openings. Of great value would be mathematical studies that revealed the properties of ill-conditioned modal matrices.

Nomenclature

$\dot{z}$	Represents dz/dt.
$\ddot{z}$	Represents d^2z/dt^2.
z_i	ith value of z.
$\{z\}$	An ordered set of z_i as a column vector.
$[z]$	A matrix, an ordered set of vectors $\{z\}$.
(z)	An ordered set of z_i as a row vector alternatively, a function of z.
$[A]$, $[B]$, $[C_1]$, $[C_2]$	Amplitudes of displacement.
$\{\cos\}$, $\{\sin\}$, $\{H_1\}$, $\{H_2\}$	Time dependent functions of unit amplitude.
$\{F\}$	Applied torques.
$\{Fel\}$	Shaft torques.
$\{F\text{steady}\}$	Shaft torques in steady state.
$\{F\text{peak}\}$	Peak shaft torques.
k_k	Spring rate of kth shaft.
k_{ij}, $[k]$	Spring force on ith rotor due to unit deflection of jth rotor, matrix of $\|\|k_{ij}\|\|$.
K_k, $[K]$	Spring rates of transformed system, matrix of spring rates.
m_i, $[m]$	Inertia of ith rotor, matrix of rotor inertias.
M_i, $[M]$	Inertias of transformed system.
n	Number of degrees of freedom.
n_k	Number of shafts.

q_i, $\dot{q}_i$, $\ddot{q}_i$	Displacement, velocity, acceleration in a transformed system.
$\{P\}$	Applied forces in the transformed system.
t	Time.
TAF	Torque amplification factor.
$[U]$, $[V]$	Initial displacements, velocity/frequency in transformed system.
$[W]$	Applied force/K in the transformed system.
x_i, $\dot{x}_i$, $\ddot{x}_i$	Displacement, velocity, and acceleration of the ith rotor.
$[\alpha]$	$[A]$, $[B]$, $[C]$ in general.
$\{\beta\}$	$\{\cos\}$, $\{\sin\}$, $\{H_1\}$, $\{H_2\}$ in general.
$\{\phi\}_i$, $[\phi]$	ith eigenvector (ith mode), matrix of eigenvectors (modal matrix).
ζ	Amplitudes in general.
ψ	Function of time and frequency in general.
ω_i	Natural frequency (rad/s).
ω_i^2	ith eigenvalue.

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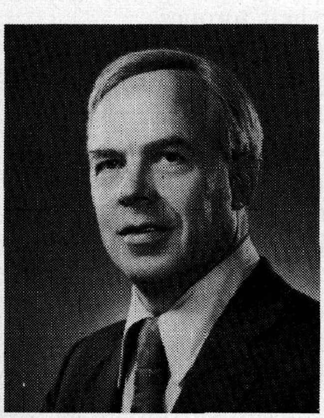

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