6.14 A flat heater is sandwiched between two solids of equal areas (0.1 m^2) with different thermal conductivities and thicknesses. The heater operates at a uniform temperature and provides a constant power of 290 W. The external surface temperature of each solid is 300 K, and there is perfect thermal contact at each internal interface.

a) Calculate the heat flux through each solid.

b) What is the operating temperature of the heater?

Solid	Thermal Conductivity, W m^{-1} K^{-1}	Thickness, mm
A	35	60
B	9	30

$260

(1)

Sacado de: "Transport Phenomena in Material Processing"

D. R. Poirier

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7

HEAT TRANSFER AND THE ENERGY EQUATION

We have designed this chapter to introduce the reader to three interwoven topics. First, we develop differential equations in terms of temperature in space (and with time if transient conditions apply) for several simple problems, by writing energy balances for unit volumes. In order to obtain solutions, we integrate the differential equations to ascertain the temperature and arbitrary constants, and then apply boundary and initial conditions to obtain the particular solution. The general procedure is similar to that followed in Chapter 2 for obtaining the velocity profiles.

Second, several of the examples are concerned with heat transfer to and from moving fluids. We deal only with laminar convection, but this enables the reader to become involved in the fundamentals of heat transfer with convection.

Third, we bring to the reader's attention more general forms of the equation of energy, leading to Tables 7.3-7.5 which may be used in a manner similar to the general momentum equations given in Chapter 2.

7.1 HEAT TRANSFER WITH FORCED CONVECTION IN A TUBE

Consider laminar flow in a circular tube of radius R, as depicted in Fig. 7.1. If the tube and the fluid exchange heat, then clearly the fluid's temperature is a function of both the r- and z-directions. A suitable unit volume is a ring-shaped element, Δr thick and Δz high. Energy enters and leaves this ring by thermal conduction; also, a unit mass of fluid, which enters with an enthalpy, must leave with a different enthalpy. Let us now develop the energy balance for the unit volume.

Rate of energy in by conduction across surface at r	$2\pi r\, \Delta z q_r\|_r$
Rate of energy out by conduction across surface at $r + \Delta r$	$2\pi(r + \Delta r)\Delta z q_r\|_{r+\Delta r}$
Rate of energy in by conduction across surface at z	$2\pi r\, \Delta r q_z\|_z$

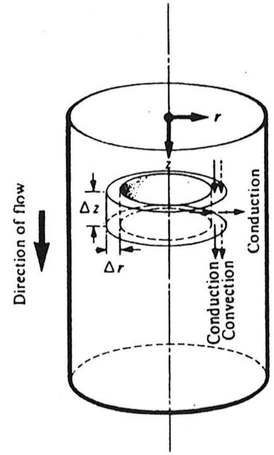


Fig 7.1 Elemental circular ring used to develop the differential energy balance for laminar tube flow.

Energy out by conduction across surface at $z + \Delta z$	$2\pi r\, \Delta r q_z\|_{z+\Delta z}$
Energy in due to fluid flow (enthalpy) across surface at z	$\rho v_z 2\pi r\, \Delta r H\|_z$
Energy out due to fluid flow across surface at $z + \Delta z$	$\rho v_z 2\pi r\, \Delta r H\|_{z+\Delta z}$

Here H is the enthalpy per unit mass, and v_z is the velocity in the z-direction. At steady state, the energy balance requires equal inputs and outputs. If we divide all terms by $2\pi\, \Delta r\, \Delta z$, we obtain

$$\frac{rq_r|_{r+\Delta r} - rq_r|_r}{\Delta r} + r\frac{q_z|_{z+\Delta z} - q_z|_z}{\Delta z} + r\rho v_z \frac{H|_{z+\Delta z} - H|_z}{\Delta z} = 0. \qquad (7.1)$$

Now Δr and Δz are allowed to approach zero.

$$\frac{\partial(rq_r)}{\partial r} + r\frac{\partial q_z}{\partial z} + r\rho v_z \frac{\partial H}{\partial z} = 0. \qquad (7.2)$$

If C_p is the heat capacity, then

$$\frac{\partial H}{\partial z} = C_p \frac{\partial T}{\partial z}. \qquad (7.3)$$

Also

$$q_r = -k(\partial T/\partial r) \quad \text{and} \quad q_z = -k(\partial T/\partial z). \qquad (7.4a,b)$$

Substituting Eqs. (7.3) and (7.4) into Eq. (7.2) yields an energy equation written in terms of temperature:

$$v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right]. \qquad (7.5)$$

We can further simplify the energy balance (Eq. (7.5)), since, except for the very slow flow of liquid metals, the term $(k/\rho C_p)\partial^2 T/\partial z^2$ is negligible even though $v_z(\partial T/\partial z)$ is not. With this assumption, Eq. (7.5) reduces to

$$v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]. \qquad (7.6)$$

Equation (7.6) contains v_z, the factor that ties together heat transfer and convection. Here we consider fully developed laminar flow; the velocity distribution is therefore parabolic, and is given by Eqs. (2.31) and (2.33):

$$v_z = 2\bar{V}_z\left[1 - \left(\frac{r}{R}\right)^2\right].$$

By including this velocity distribution, Eq. (7.6) becomes

$$2\bar{V}_z\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]. \qquad (7.7)$$

We consider the special case where a *fully developed temperature profile* exists. For any set of boundary conditions, a fully developed temperature profile exists when $(T_R - T)/(T_R - T_m)$ is a unique function of r/R, *independent of* z. Then

$$\frac{T_R - T}{T_R - T_m} = f(r/R), \qquad (7.8)$$

or

$$\frac{\partial}{\partial z}\left(\frac{T_R - T}{T_R - T_m}\right) = 0, \qquad (7.9)$$

where T_R = temperature of fluid at the wall, and T_m = mean temperature of the fluid. A fully developed temperature profile is analogous to fully developed flow. This is exemplified by Fig. 7.2, where the liquid flowing in the z-direction encounters the heated section of the tube. Over a finite interval downstream from this point, the temperature profile changes from uniform to fully developed.

For a fully developed temperature profile, an important corollary arises; namely, the heat transfer coefficient is uniform along the pipe. We realize this by employing the definition of the heat transfer coefficient based on the mean temperature of the fluid:

$$h \equiv \frac{q_R}{T_R - T_m} = -\frac{k}{R}\frac{\partial}{\partial(r/R)}\left(\frac{T_R - T}{T_R - T_m}\right)_{r=R}, \qquad (7.10)$$

where q_R is the flux evaluated at the wall ($r = R$). Because the derivative in Eq. (7.10) has a unique value at the wall, independent of z, h is therefore uniform along the pipe under the fully developed temperature conditions.

Now consider the case where q_R is uniform. This represents a uniform heat flux at the wall, and could be physically obtained by using an electric heater, depicted in Fig. 7.2. Further, since h and q_R are constant, Eq. (7.10) specifies that $T_R - T_m$ is constant, and

$$\frac{\partial T_R}{\partial z} = \frac{\partial T_m}{\partial z}. \qquad (7.11)$$

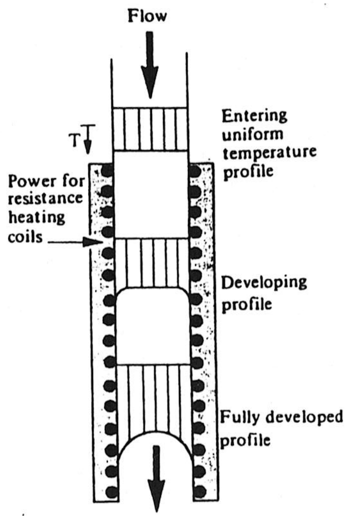


Fig. 7.2 Heating a fluid in a tube showing the development of the temperature profile.

(Note that T_R and T_m themselves are not constants.) Now expand Eq. (7.9) in a general sense where each quantity varies as follows:

$$\left[\frac{\partial T_R}{\partial z} - \frac{\partial T}{\partial z}\right] - \left[\frac{T_R - T}{T_R - T_m}\right]\left[\frac{\partial T_R}{\partial z} - \frac{\partial T_m}{\partial z}\right] = 0. \tag{7.12}$$

Then Eq. (7.11) shows that

$$\frac{\partial T_R}{\partial z} = \frac{\partial T}{\partial z} = \frac{\partial T_m}{\partial z}. \tag{7.13}$$

Equation (7.13) is important because it allows Eq. (7.7) to be integrated directly using $\partial T/\partial z = \partial T_m/\partial z$:

$$2\bar{V}_z\left[\frac{\partial T_m}{\partial z}\right]\int_{r=0}^{r} r\left[1 - \left(\frac{r}{R}\right)^2\right]dr = \frac{k}{\rho C_p}\int_{\partial T/\partial r = 0}^{\partial T/\partial r} d\left[r\frac{\partial T}{\partial r}\right]. \tag{7.14}$$

Integrating, we get

$$2\bar{V}_z\left[\frac{\partial T_m}{\partial z}\right]\frac{r}{2}\left[1 - \frac{1}{2}\left(\frac{r}{R}\right)^2\right] = \frac{k}{\rho C_p}\frac{\partial T}{\partial r}. \tag{7.15}$$

A second integration with $T = T_R$ at $r = R$ finally results in

$$T_R - T = \left[\frac{\bar{V}_z\rho C_p}{8R^2k}\right]\left[\frac{\partial T_m}{\partial z}\right](3R^4 - 4r^2R^2 + r^4). \tag{7.16}$$

Having obtained the temperature profile, we can evaluate h. From Eq. (7.10), $(T_R - T_m)$ and q_R must then be evaluated. First, we find $(T_R - T_m)$ by performing the integration:

$$T_R - T_m = \frac{\int_0^R v_z(T_R - T)2\pi r\,dr}{\int_0^R v_z 2\pi r\,dr}. \tag{7.17}$$

Second, we determine q_R by evaluating the gradient at the wall using Eq. (7.16):

$$q_R = -k\left[\frac{\partial T}{\partial r}\right]_{r=R}. \tag{7.18}$$

When these operations have been carried out, we can determine the heat transfer coefficient. The final result, with D as the diameter, is

$$\frac{hD}{k} = 4.36. \tag{7.19}$$

The dimensionless number resulting from this analysis is the *Nusselt number*. This important dimensionless number for heat flow with *forced* convection reappears as we examine other solutions and correlations. For emphasis, then, the Nusselt number is

$$\text{Nu}_\infty \equiv \frac{hD}{k}. \tag{7.20}$$

This Nusselt number is for fully developed flow and uniform heat flux with parabolic velocity profile. It is subscripted with ∞ because it represents the limiting case of a fully developed temperature profile. Many situations have been analyzed, some of which are given in Table 7.1 and others in Holman.[1]

Table 7.1 Nusselt numbers for fully developed laminar flow*

Geometry	Velocity distribution†	Condition at wall	$\text{Nu}_\infty = \frac{hD_e}{k}$ ‡
Circular tube	Parabolic	Uniform q_R	4.36
Circular tube	Parabolic	Uniform T_R	3.66
Circular tube	Slug flow	Uniform q_R	8.00
Circular tube	Slug flow	Uniform T_R	5.75
Parallel plates	Parabolic	Uniform q_R	8.23
Parallel plates	Parabolic	Uniform T_R	7.60
Triangular duct	Parabolic	Uniform q_R	3.00
Triangular duct	Parabolic	Uniform T_R	2.35

*From W. M. Rohsenow and H. Y. Choi, *Heat, Mass and Momentum Transfer*, Prentice-Hall, Englewood Cliffs, New Jersey, 1961, page 141.
†Slug flow refers to a flat velocity profile.
‡D_e is the equivalent diameter, as defined in Chapter 3.

[1] J. P. Holman, *Heat Transfer*, sixth edition, McGraw-Hill, New York, N.Y., 1986, page 281.

7.2 HEAT TRANSFER WITH LAMINAR FORCED CONVECTION OVER A FLAT PLATE

In Chapter 2, the velocity distribution, within the boundary layer over a flat plate, was determined. Here we consider the case of a plate at a different temperature than the fluid, with the plate serving to heat or cool the fluid. Just as a velocity profile continually changes with distance from the leading edge, and results in a *momentum boundary layer* which increases in thickness, there is also a changing temperature profile and development of a *thermal boundary layer* when heat transfer is involved. We depict this situation along with a unit element in Fig. 7.3. With a depth of unity perpendicular to the page, the contributions to the energy balance are:

Energy in by conduction across surface at x	$q_x\|_x \Delta y \cdot 1$
Energy out by conduction across surface at $x + \Delta x$	$q_x\|_{x+\Delta x} \Delta y \cdot 1$
Energy in by conduction across surface at y	$q_y\|_y \Delta x \cdot 1$
Energy out by conduction across surface at $y + \Delta y$	$q_y\|_{y+\Delta y} \Delta x \cdot 1$
Energy in due to fluid flow (sensible heat) across surface at x	$\rho v_x \Delta y \cdot 1 \cdot H\|_x$
Energy out due to fluid flow (sensible heat) across surface at $x + \Delta x$	$\rho v_x \Delta y \cdot 1 \cdot H\|_{x+\Delta x}$
Energy in due to fluid flow (sensible heat) across surface at y	$\rho v_y \Delta x \cdot 1 \cdot H\|_y$
Energy out due to fluid flow (sensible heat) across surface at $y + \Delta y$	$\rho v_y \Delta x \cdot 1 \cdot H\|_{y+\Delta y}$

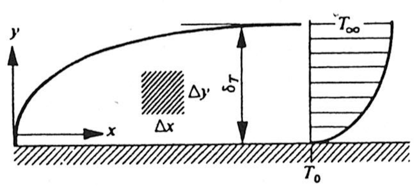


Fig. 7.3 Development of the thermal boundary layer and the temperature distribution over a flat plate.

Adding all these quantities, dividing through by $\Delta x \Delta y$, and taking the limits as $\Delta x \to 0$ and $\Delta y \to 0$, we obtain

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial(\rho v_x H)}{\partial x} + \frac{\partial(\rho v_y H)}{\partial y} = 0. \tag{7.21}$$

For constant density and conductivity, Eq. (7.21) becomes

$$\rho H\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right] + \rho\left[v_x \frac{\partial H}{\partial x} + v_y \frac{\partial H}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]. \tag{7.22}$$

Since continuity requires that $(\partial v_x/\partial x) + (\partial v_y/\partial y) = 0$ and $dH = C_p dT$, we finally obtain

$$\rho C_p\left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]. \tag{7.23}$$

The temperature gradient in the y-direction is much steeper than that in the x-direction; therefore the x-directed second derivative term may be neglected. Then Eq. (7.23) simplifies to

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{7.24}$$

Here $\alpha = k/\rho C_p$, which is called the *thermal diffusivity*. It has the same units of kinematic viscosity, which is sometimes called *momentum diffusivity*. In addition, the momentum boundary layer equation, developed in Chapter 2, is

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}. \tag{2.87}$$

Equations (7.24) and (2.87) are analogous. If $\nu = \alpha$, and if the velocity and thermal boundary conditions are similar, then the temperature and velocity profiles are exactly identical, and the thermal boundary layer δ_T equals the momentum boundary layer δ.

The method of solution for Eq. (7.24) parallels that for the velocity distribution, as given in Section 2.7.1. For the case of a uniform-temperature plate, the following boundary conditions apply

B.C.1 at $y = 0$, $T = T_0$;

B.C.2 at $y = \infty$, $T = T_\infty$;

B.C.3 at $x \le 0$, $T = T_\infty$.

We do not present the method of solution here, but Fig. 7.4 gives the temperature T as a function of y and x. The temperature is a part of the dimensionless temperature Θ on the ordinate, which includes the wall temperature T_0 and the bulk fluid temperature T_∞. Space dimensions x and y appear together on the abscissa in exactly the same way, as shown in Fig. 2.7 for describing the velocity profiles. Several curves are shown in Fig. 7.4, each for a different value of the *Prandtl number*, Pr. This number is the ratio of ν/α, and for Pr equal to unity, the Θ curve in Fig. 7.4 is exactly the same as v_x/V_∞ in Fig. 2.7. Therefore, the Prandtl number controls the similarity between the velocity profiles and the temperature profiles.

Knowing the temperature profile, we can determine the heat-transfer coefficient. From the results given in Fig. 7.4, the *local* heat transfer coefficient is

$$h_x = \frac{-k\left[\dfrac{\partial T}{\partial y}\right]_{y=0}}{T_0 - T_\infty} = 0.332k\,\mathrm{Pr}^{0.343}\left[\frac{V_\infty}{\nu x}\right]^{1/2}, \qquad \mathrm{Pr} \ge 0.6, \tag{7.25}$$

or, in terms of dimensionless numbers,

$$\mathrm{Nu}_x = 0.332\,\mathrm{Pr}^{0.343}\,\mathrm{Re}_x^{0.5}, \tag{7.26}$$

where $\mathrm{Nu}_x \equiv h_x x/k$, which is called the *local* Nusselt number. If we wish to know the *average* heat-transfer coefficient, then we can find it by averaging h_x from $x = 0$ to $x = L$:

$$h = \frac{1}{L}\int_0^L h_x\,dx = 0.664k\,\mathrm{Pr}^{0.343}\left[\frac{V_\infty}{\nu L}\right]^{1/2},$$

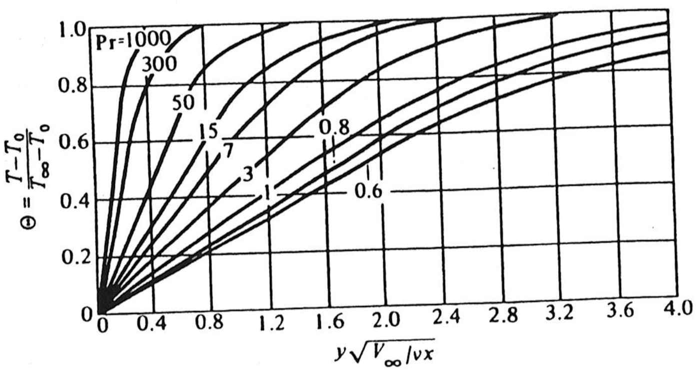


Fig. 7.4 Dimensionless temperature profiles in the laminar boundary layer over a flat plate for various Pr. (From E. Z. Pohlhausen, Z. *Angew. Math. Mech.* **1**, 115 (1921).)

or

$$\mathrm{Nu}_L = 0.664\ \mathrm{Pr}^{0.343}\ \mathrm{Re}_L^{0.5}. \tag{7.27}$$

Note the general form of either Eq. (7.26) or (7.27). We shall find in Chapter 8 that the Nusselt number is generally a function of the Reynolds and Prandtl numbers in problems of *forced convection*.

For the case of Pr = 1 ($\nu = \alpha$), the thermal boundary layer is given by

$$\frac{\delta_T}{x} = \frac{5.0}{\sqrt{V_\infty x/\nu}}, \tag{7.28}$$

which is the same as the momentum boundary layer.

At this point it is instructive to examine Prandtl numbers for various fluids; approximate values are given in Table 7.2. These numbers represent values that include several substances, and cover substantial temperature ranges. The Prandtl numbers of liquids vary significantly with temperature; however, gases show almost no variation in Pr with temperature. From Table 7.2, we see that for gases $\delta_T \cong \delta$, for common liquids $\delta_T < \delta$, and for liquid metals—due to their high thermal conductivity—$\delta_T \gg \delta$. To a close approximation, the ratio of the boundary layer thicknesses is

$$\frac{\delta_T}{\delta} = 0.975\ \mathrm{Pr}^{-1/3}, \qquad \mathrm{Pr} > 0.5. \tag{7.29}$$

Table 7.2. Typical Prandtl numbers

Substance	Range of Prandtl number (ν/α)
Common liquids (water, alcohol, etc.)	2-50
Liquid metals	0.001-0.03
Gases	0.7-1.0

Equations (7.27)-(7.29) are valid only for Pr > 0.5, and thus do not apply to liquid metals. For liquid metals with uniform wall temperatures as a boundary condition, the results are approximated by[2]

$$\mathrm{Nu}_x = \sqrt{\mathrm{Re}_x\ \mathrm{Pr}}\left[\frac{0.564}{1 + 0.90\sqrt{\mathrm{Pr}}}\right]. \tag{7.30}$$

For a uniform heat flux at the wall, we present these results.[3]

$$\mathrm{Pr} > 0.5, \qquad \mathrm{Nu}_x = 0.458\ \mathrm{Pr}^{0.343}\sqrt{\mathrm{Re}_x}\ ; \tag{7.31}$$

$$0.006 \le \mathrm{Pr} \le 0.03, \qquad \mathrm{Nu}_x = \sqrt{\mathrm{Re}_x\ \mathrm{Pr}}\left[\frac{0.880}{1 + 1.317\sqrt{\mathrm{Pr}}}\right]. \tag{7.32}$$

Example 7.1 Air at 1 atm (1.013×10^5 N m^{-2}) and 290 K flows parallel to a plate's surface at 15 m s^{-1}. The plate, 0.3 m long, is at 360 K. Assume that laminar flow is stable along the entire length.

a) Calculate the thicknesses of the velocity and thermal boundary layers 0.15 m from the leading edge of the plate.

b) Calculate the rate of heat transfer from the entire plate per 0.1 m of plate width.

Solution.

a) We calculate the thickness of the momentum boundary layer using Eq. (2.101),

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{V_\infty x/\nu}}.$$

For air, evaluating ν at an average boundary-layer temperature of $\frac{1}{2}(290 + 360) = 325$ K, the kinematic viscosity is 18.4×10^{-6} m^2 s^{-1}. Then,

$$\frac{V_\infty x}{\nu} = \frac{15\ \mathrm{m}}{\mathrm{s}}\left|\frac{0.15\ \mathrm{m}}{}\right|\frac{\mathrm{s}}{18.4 \times 10^{-6}\ \mathrm{m}^2} = 1.22 \times 10^5,$$

and

$$\delta = \frac{(0.15)(5.0)}{\sqrt{1.22 \times 10^5}} = 2.15 \times 10^{-3}\ \mathrm{m} = 2.15\ \mathrm{mm}.$$

Next, we use Eq. (7.29)

$$\frac{\delta_T}{\delta} = 0.975\ \mathrm{Pr}^{-1/3}.$$

[2] E. M. Sparrow and J. L. Gregg, *J. Aero. Sc.* **24**, 852 (1957).

[3] R. J. Nickerson and H. P. Smith, as reported in Rohsenow and Choi, *ibid.*, page 149.

The Prandtl number for air, evaluated at 325 K, is 0.703. Then

$$\delta_T = (2.15)(0.975)(0.703)^{-1/3} = 2.36 \text{ mm}.$$

b) Equation (7.27) is used for the average heat-transfer coefficient, which applies to the whole plate.

$$\mathrm{Nu}_L = (0.664)(0.703^{0.343})\sqrt{2.44 \times 10^5} = 291.$$

The thermal conductivity for air at 325 K is 28.1×10^{-3} W m^{-1} K^{-1}. Then

$$h = \frac{k}{L}\,\mathrm{Nu}_L = \frac{(28.1 \times 10^{-3})(291)}{(0.3)} = 27.2 \text{ W m}^{-2}\text{ K}^{-1},$$

and finally

$$Q = hA(T_\infty - T) = (27.2)(0.3 \times 0.1)(360 - 290) = 57.1 \text{ W}.$$

7.3 HEAT TRANSFER WITH NATURAL CONVECTION

In Sections 7.1 and 7.2 we considered heat transfer with forced convection. In forced convection, the known velocity distribution can be entered into the energy equation. The situation is more complex in problems of heat transfer with natural convection because the velocities are not known *a priori* to solving the energy equation. Hence, the velocity and temperature distributions cannot be treated as separate problems; the temperature distribution, in effect, produces the velocity distribution by causing density differences within the fluid.

Consider the vertical surface in Fig. 7.5; the surface is at T_0, and it heats the neighboring fluid whose bulk temperature is T_∞. In this situation, the velocity component v_y is quite small; the fluid moves almost entirely upward, and therefore we write the momentum equation for the x-component only. For steady forced convection over a flat plate, we ignored the gravity force, and no pressure gradient was involved. We cannot ignore these forces in free convection, and therefore the momentum equation for the present case contains these terms

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2} + g_x. \tag{7.33}$$

We apply the so-called *Boussinesq approximation* in which variations in the density of the fluid are neglected except in the buoyancy force term that drives the natural convection. Thus, in Eq. (7.33) the density is a reference density at the reference temperature, T_∞. Therefore, the inertial terms become

$$\rho_\infty \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial x} \right],$$

and the pressure gradient is approximated as

$$\frac{\partial P}{\partial x} \approx \frac{dP}{dx} = -\rho_\infty g.$$

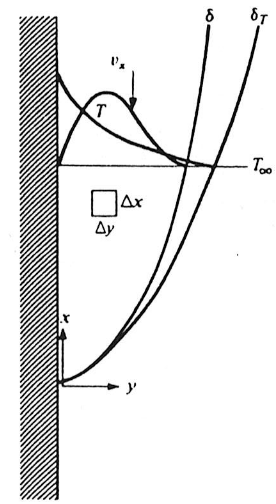


Fig. 7.5 Thermal and momentum boundary layers for vertical plate natural convection.

With $g_x = -g$, the buoyancy term is simply

$$\rho g_x = -\rho_\infty g\left[1 + \beta(T_\infty - T)\right],$$

where the volume expansion coefficient β is defined as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P.$$

The momentum balance in the x-direction then becomes

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} + g\beta(T - T_\infty), \tag{7.34}$$

which is identical with Eq. (2.87), except for the addition of the buoyancy term. Equation (7.34) shows that the momentum equation must be coupled to an appropriate energy equation, in order to treat the buoyancy term.

The energy equation applied to the control volume $\Delta x \Delta y$ in this case is identical to that for flow over a flat plate:

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{7.35}$$

This equation is coupled to Eq. (7.34) by the presence of the velocity terms. The mathematical task at hand is, therefore, to solve the coupled equations with the boundary conditions:

B.C.1	at $y = 0$,	$v_x = v_y = 0$,	$T = T_0$;
B.C.2	at $y = \infty$,	$v_x = v_y = 0$,	$T = T_\infty$.

Analytical solutions of such coupled differential equations are beyond the scope of this text; we simply present the results. First, however, let us examine a dimensional analysis approach in order to bring forth pertinent dimensionless numbers.

The problem is to determine the conditions for which the velocity profile in a natural convection situation is similar to the velocity profile in another natural convection situation. Both systems have the same boundary conditions, i.e., velocity is zero at the surface and within the bulk fluid removed from the surface. Now employ the dynamic similarity argument introduced in Section 3.1.1.

First, Eq. (7.34) is written for system 1:

$$v_{x1}\frac{\partial v_{x1}}{\partial x_1} + v_{y1}\frac{\partial v_{x1}}{\partial y_1} = \nu_1\frac{\partial^2 v_{x1}}{\partial y_1^2} + g_1\beta_1(T - T_\infty)_1. \tag{7.36}$$

System 2 is related to system 1 by geometrical and dynamic similarities expressed by the ratios:

$$K_L \equiv \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{L_1}{L_2} \qquad K_\nu = \frac{\nu_1}{\nu_2}$$

$$K_v \equiv \frac{v_{x1}}{v_{x2}} = \frac{v_{y1}}{v_{y2}} = \frac{v_1}{v_2} \qquad K_g = \frac{g_1}{g_2} \tag{7.37}$$

$$K_\beta \equiv \frac{\beta_1}{\beta_2} \qquad K_T = \frac{(T - T_\infty)_1}{(T - T_\infty)_2} = \frac{(T_0 - T_\infty)_1}{(T_0 - T_\infty)_2}.$$

Now, we replace v_{x1}, x_1, ν_1, v_{y1}, g_1, etc. in Eq. (7.36) by their equivalents in Eq. (7.37); then we write Eq. (7.34) for system 2:

$$\frac{K_v^2}{K_L} v_{x2}\frac{\partial v_{x2}}{\partial x_2} + \frac{K_v^2}{K_L} v_{y2}\frac{\partial v_{x2}}{\partial y_2} = \frac{K_\nu K_v}{K_L^2}\nu_2\frac{\partial^2 v_{x2}}{\partial y_2^2} + K_g K_\beta K_T g_2\beta_2(T - T_\infty)_2. \tag{7.38}$$

Equation (7.38) if rewritten without all the Ks would of course be valid, because it would transform back to Eq. (7.34). Hence,

$$\frac{K_v^2}{K_L} = \frac{K_\nu K_v}{K_L^2} = K_g K_\beta K_T = 1, \tag{7.39}$$

and therefore

$$\frac{v_1^2}{L_1} = \frac{v_2^2}{L_2}, \qquad \frac{\nu_1 v_1}{L_1^2} = \frac{\nu_2 v_2}{L_2^2}.$$

Thus,

$$g_1\beta_1(T_0 - T_\infty)_1 = g_2\beta_2(T_0 - T_\infty)_2. \tag{7.40a,b,c}$$

If we combine Eqs. (7.40a) and (7.40b) we get

$$\frac{v_1 L_1}{\nu_1} = \frac{v_2 L_2}{\nu_2}, \tag{7.41}$$

which are Reynolds numbers. The combination of Eqs. (7.40b) and (7.40c), yields

$$\frac{g_1\beta_1(T_0 - T_\infty)_1 L_1^2}{\nu_1 v_2} = \frac{g_2\beta_2(T_0 - T_\infty)_2 L_2^2}{\nu_2 v_2}. \tag{7.42}$$

The group of variables represented in Eq. (7.42) could be considered a dimensionless number, but by reflecting on the physical aspects, we realize that the velocity of the fluid is not an independent quantity, but that it rather depends on the buoyant force. Hence, the v's are eliminated from Eq. (7.42), and substituting their equivalents from Eq. (7.41), we obtain

$$\frac{g_1\beta_1(T_0 - T_\infty)_1 L_1^3}{\nu_1^2} = \frac{g_2\beta_2(T_0 - T_\infty)_2 L_2^3}{\nu_2^2}. \tag{7.43}$$

This dimensionless number is important in natural convection problems and is called the *Grashof number*, Gr. When buoyancy is the only driving force for convection, the velocity profile is determined entirely by the quantities in the Grashof number, and the Reynolds number is superfluous.

Recall that for forced convection, as discussed in Section 7.2, the Nusselt number is correlated in the general form

$$\text{Nu} = f(\text{Pr},\text{Re}), \qquad \textit{forced convection}.$$

Correspondingly then, for natural convection, the Nusselt number is correlated as

$$\text{Nu} = f(\text{Pr},\text{Gr}), \qquad \textit{natural convection}.$$

Returning to the complete solution of Eqs. (7.34), (7.35) and the appropriate boundary conditions, we present the velocity and temperature distributions (see Fig. 7.6). The curves show that for $\text{Pr} \le 1$, $\delta_T \cong \delta$, but for $\text{Pr} > 1$, $\delta_T < \delta$. For liquid metals, therefore, δ_T is about equal to δ in free convection as contrasted to forced convection in which $\delta_T \gg \delta$. Corresponding to the temperature profile, shown in Fig. 7.6b, the local Nusselt number is

$$\frac{\text{Nu}_x}{\sqrt[4]{\text{Gr}_x/4}} = \frac{0.676\ \text{Pr}^{1/2}}{(0.861 + \text{Pr})^{1/4}}. \tag{7.44}$$

Equation (7.44) applies for a wide range of Pr numbers ($0.00835 \le \text{Pr} \le 1000$) for laminar flow conditions, with $10^4 < \text{Gr}_x \cdot \text{Pr} < 10^{10}$.

Example 7.2 Calculate the initial heat transfer rate from a plate at 360 K, 0.3 m long × 0.1 m wide hung vertically in air at 290 K. Contrast the results with those of Example 7.1.

Solution. Equation (7.44) should be integrated to obtain the average heat transfer coefficient which can be applied to the whole plate.

In Eq. (7.44), because h_x varies as $x^{-1/4}$, then the average h equals $\frac{4}{3}h_x$. Hence Nu_L defined as hL/k is

$$\frac{\text{Nu}_L}{\sqrt[4]{\text{Gr}_L/4}} = \frac{0.902\ \text{Pr}^{1/2}}{(0.861 + \text{Pr})^{1/4}}. \tag{7.45}$$

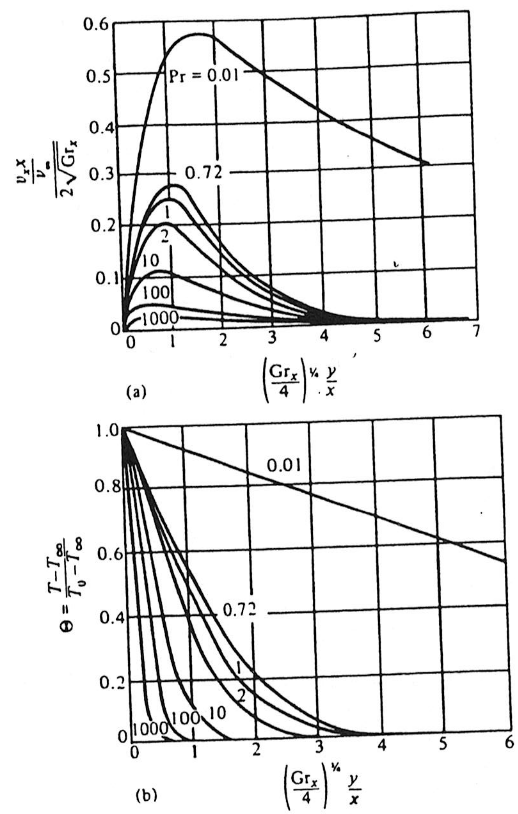


Fig. 7.6 Laminar natural convection for a vertical plate. (a) Dimensionless velocity profiles. (b) Dimensionless temperature profiles. (Calculated by S. Ostrach, *Nat. Advisory Comm. Aeronaut. Tech. Note* 2635, Feb. 1952, as presented in W. M. Rohsenow and H. Y. Choi, *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Englewood Cliffs, New Jersey, 1961, pages 155-159.)

or

$$Nu_L = 0.902 \sqrt[4]{\frac{Gr_L \cdot Pr^2}{4(0.861 + Pr)}}. \qquad (7.45a)$$

We evaluate the properties at the average boundary temperature of 325 K. For air at 325 K

$$Pr = 0.703 \quad \text{and} \quad g\beta/\nu^2 = 9.85 \times 10^7\ K^{-1}\ m^{-3}.$$

The Grashof number is

$$Gr_L = \frac{g\beta}{\nu^2}(T_0 - T_\infty)L^3 = (9.85 \times 10^7)(360 - 290)(0.3^3) = 1.86 \times 10^8.$$

Next, we calculate the product $Gr_L \cdot Pr$ to test for laminar flow conditions

$$Gr_L \cdot Pr = (1.86 \times 10^8)(0.703) = 1.31 \times 10^8.$$

Since it is between 10^4 and 10^{10}, Eq. (7.45a) is valid. When we substitute values of Gr_L and Pr into Eq. (7.45a),

$$Nu_L = 55.8,$$

from which,

$$h = Nu_L \frac{k}{L} = (55.8)\left(\frac{28.1 \times 10^{-3}}{0.3}\right) = 5.23\ W\ m^{-2}\ K^{-1}.$$

Finally, we evaluate the rate of heat transfer Q.

$$Q = h(T_\infty - T_0)A = (5.23)(360 - 290)(0.1 \times 0.3) = 11.0\ W.$$

For Example 7.1, Q was 57.1 W; that is, the rate of heat transfer for forced convection is considerably higher. This is the usual case.

It is instructive to look at special forms of Eq. (7.45a). First, if $Pr = 0.7$, then it reduces to

$$Nu_L = 0.477\ Gr^{1/4}. \qquad (7.45b)$$

It so happens that for many gases, including air, O_2, CO, He (and other inert gases), H_2 and CO_2, Pr is very close to 0.7 and practically constant for temperatures even as high as 1900 K. Thus, we can apply Eq. (7.45b) directly to gases.

Second, if $Pr \to 0$ (liquid metals), then Eq. (7.59a) reduces to

$$Nu_L = 0.936 \sqrt[4]{\frac{Gr_L \cdot Pr^2}{4}}. \qquad (7.45c)$$

7.4 HEAT CONDUCTION

We consider heat conduction through the wall of a hollow solid cylinder. Figure 7.7 depicts the situation, and also locates a suitable unit volume with a thickness Δr. From a practical point of view, we may visualize a long cylindrical shaped furnace, and it is desirable to calculate the heat loss to the surroundings. Suppose the cylinder is long enough so that end effects are negligible; in addition, the system is at steady state, so that both the inside and outside surfaces of the wall are at some fixed temperatures, T_1 and T_2, respectively. For such a system, we develop the energy equation.

Rate of energy in by conduction across surface at r $\quad 2\pi rlq_r|_r$

Rate of energy out by conduction across surface at $r + \Delta r$ $\quad 2\pi rlq_r|_{r+\Delta r}$

At steady state, these are the only terms that contribute to the energy balance. Thus

$$2\pi rlq_r|_{r+\Delta r} - 2\pi rlq_r|_r = 0.$$

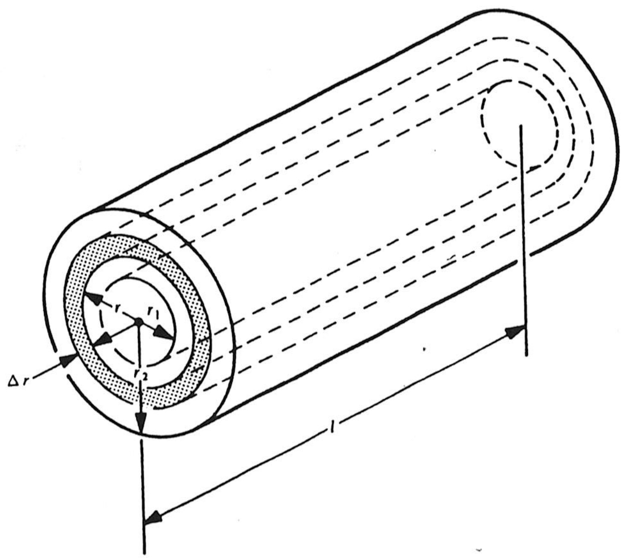


Fig. 7.7 Heat conduction through a solid cylindrical wall. The shaded area depicts the unit volume.

If we divide all terms by $2\pi l \Delta r$, and take the limit as Δr approaches zero, we obtain

$$\frac{d(rq_r)}{dr} = 0. \tag{7.46}$$

Equation (7.46) requires that

$$rq_r = C_1. \tag{7.47}$$

Note that q_r, the heat flux, is not constant in itself. Since $q_r = -k(dT/dr)$, Eq. (7.47) yields

$$-k\frac{dT}{dr} = \frac{C_1}{r}. \tag{7.48}$$

Integrating once again, we find for constant thermal conductivity that

$$T = -\frac{C_1}{k}\ln r - \frac{C_2}{k}. \tag{7.49}$$

By absorbing k in new constants, Eq. (7.49) simplifies even more to

$$T = C_3 \ln r + C_4. \tag{7.50}$$

The boundary conditions under consideration are

B.C.1 at $r = r_1$, $T = T_1$;

B.C.2 at $r = r_2$, $T = T_2$.

Determination of the constants using the boundary conditions yields the temperature distribution

$$\frac{T - T_2}{T_1 - T_2} = \frac{\ln(r/r_2)}{\ln(r_1/r_2)}, \tag{7.51}$$

and the heat flux

$$q_r = -k\frac{dT}{dr} = \frac{k}{r}\left[\frac{T_1 - T_2}{\ln(r_1/r_2)}\right]. \tag{7.52}$$

As the heat flows through the wall, it encounters larger areas, so that the flux itself decreases. The heat flow Q, however, is constant (as it must be for steady state), and is given by

$$Q = q_r(2\pi rL) = \frac{2\pi kL}{\ln(r_1/r_2)}(T_1 - T_2). \tag{7.53}$$

This problem, elementary as it is, demonstrates an interesting engineering characteristic. Suppose we use the cylindrical wall as the insulation of a furnace wall. As increasing thicknesses of insulation are added, the outside layer, because of its greater area, offers less resistance to heat flow than an inner layer of the same thickness. Thus, from a cost point of view, the expense of additional insulation can become greater than the savings associated with reduction in heat losses.

Example 7.3 As part of a proposed continuous annealing process, a rod passes through a cylindrical furnace chamber 101 mm inside diameter and 15.2 m long. The inside surface temperature of the furnace wall under operating conditions is predicted to be about 920 K and the outside surface about 310 K. If it is decided that a heat loss of 73 kW is an acceptable figure, then which of the following insulations would you use?

	k, W m^{-1} K^{-1}	Cost, \$ per m^3
Insulation *A*	0.70	350
Insulation *B*	0.35	880

Solution. Equation (7.53) can be written

$$\ln\left[\frac{r_2}{r_1}\right] = \frac{2\pi kL}{Q}(T_1 - T_2).$$

For *A*, then

$$\ln\left[\frac{r_2}{r_1}\right] = \frac{(2\pi)(0.70)(15.2)(920 - 310)}{(73 \times 10^3)} = 0.559,$$

so that with $r_1 = 50.5$ mm, we have $r_2 = 88.3$ mm. Similarly for *B*, using the ratio of conductivities, we get

$$\ln\left[\frac{r_2}{r_1}\right] = \left[\frac{0.35}{0.70}\right](0.559) = 0.280.$$

so that $r_2 = 66.8$ mm. We calculate the volume of insulation and the corresponding cost.

$$\text{Cost } A = \frac{\pi\left(88.3^2 - 50.5^2\right)\text{ mm}^2}{} \left|\frac{1\text{ m}^2}{1000^2\text{ mm}^2}\right| \frac{15.2\text{ m}}{} \left|\frac{350\ \$}{\text{m}^3}\right. = \$87.69.$$

In the same manner, Cost B = \$80.34. The obvious choice is B.

7.5 THE GENERAL ENERGY EQUATION

In Sections 7.1-7.4, we determined temperature distributions and heat fluxes for some simple systems, by developing pertinent energy balances in differential form. In this section we develop the general energy equation, which can be reduced to solve specific problems.

Consider the stationary unit volume $\Delta x \Delta y \Delta z$ in Figs. 2.4 and 2.5; we apply the law of conservation of energy to the fluid contained within this volume at any given time

$$\begin{bmatrix}\text{rate of accumulation}\\ \text{of internal and}\\ \text{kinetic energy}\end{bmatrix} = \begin{bmatrix}\text{net rate of internal}\\ \text{and kinetic energies}\\ \text{in by convection}\end{bmatrix} + \begin{bmatrix}\text{net rate of}\\ \text{heat in by}\\ \text{conduction}\end{bmatrix} - \begin{bmatrix}\text{net rate of}\\ \text{work done}\\ \text{by fluid}\end{bmatrix}. \tag{7.54}$$

This statement of the law of energy conservation is not completely general, because other forms of energy transport, e.g., radiation, and sources such as electrical Joule heating, are not included.

The rate of accumulation of internal and kinetic energy within the unit volume is simply

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}\left(\rho U + \tfrac{1}{2}\rho v^2\right), \tag{7.55}$$

where U is the internal energy per unit mass of fluid and v is the *magnitude* of the local fluid velocity.

The *net* rate of internal and kinetic energies in by convection is

$$\Delta y \Delta z\left\{v_x\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_x - v_x\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_{x+\Delta x}\right\}$$
$$+ \Delta x \Delta z\left\{v_y\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_y - v_y\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_{y+\Delta y}\right\}$$
$$+ \Delta x \Delta y\left\{v_z\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_z - v_z\left(\rho U + \tfrac{1}{2}\rho v^2\right)\Big|_{z+\Delta z}\right\}. \tag{7.56}$$

In a similar manner, the net rate of energy in by conduction is

$$\Delta y \Delta z\{q_x|_x - q_x|_{x+\Delta x}\} + \Delta x \Delta z\{q_y|_y - q_y|_{y+\Delta y}\} + \Delta x \Delta y\{q_z|_z - q_z|_{z+\Delta z}\}. \tag{7.57}$$

The work done by the fluid consists of work against gravity, work against pressure, and work against viscous forces. The rate of doing work against the three components of gravity is

$$-\rho \Delta x \Delta y \Delta z(v_x g_x + v_y g_y + v_z g_z). \tag{7.58}$$

The rate of doing work against the pressure at the six faces of the unit volume is

$$\Delta y \Delta z\left\{(Pv_x)\big|_{x+\Delta x} - (Pv_x)\big|_x\right\} + \Delta x \Delta z\left\{(Pv_y)\big|_{y+\Delta y} - (Pv_y)\big|_y\right\}$$
$$+ \Delta x \Delta y\left\{(Pv_z)\big|_{z+\Delta z} - (Pv_z)\big|_z\right\}. \tag{7.59}$$

The rate of doing work against the x-directed viscous forces is

$$\Delta y \Delta z\{\tau_{xx}v_x|_{x+\Delta x} - \tau_{xx}v_x|_x\} + \Delta x \Delta z\{\tau_{yx}v_x|_{y+\Delta y} - \tau_{yx}v_x|_y\}$$
$$+ \Delta x \Delta y\{\tau_{zx}v_x|_{z+\Delta z} - \tau_{zx}v_x|_z\}. \tag{7.60}$$

Similar expressions may be written for the work against the y- and z-directed viscous forces

$$\Delta y \Delta z\{\tau_{xy}v_y|_{x+\Delta x} - \tau_{xy}v_y|_x\} + \Delta x \Delta z\{\tau_{yy}v_y|_{y+\Delta y} - \tau_{yy}v_y|_y\}$$
$$+ \Delta x \Delta y\{\tau_{zy}v_y|_{z+\Delta z} - \tau_{zy}v_y|_z\}, \tag{7.61}$$

and

$$\Delta y \Delta z\{\tau_{xz}v_z|_{x+\Delta x} - \tau_{xz}v_z|_x\} + \Delta x \Delta z\{\tau_{yz}v_z|_{y+\Delta y} - \tau_{yz}v_z|_y\}$$
$$+ \Delta x \Delta y\{\tau_{zz}v_z|_{z+\Delta z} - \tau_{zz}v_z|_z\}. \tag{7.62}$$

Substituting all these expressions into Eq. (7.54), dividing by $\Delta x \Delta y \Delta z$, and taking the limit as Δx, Δy, and Δz approach zero, we obtain one form of the energy equation

$$\frac{\partial}{\partial t}\left(\rho U + \tfrac{1}{2}\rho v^2\right) = -\left[\frac{\partial}{\partial x} v_x\left(\rho U + \tfrac{1}{2}\rho v^2\right) + \frac{\partial}{\partial y} v_y\left(\rho U + \tfrac{1}{2}\rho v^2\right) + \frac{\partial}{\partial z} v_z\left(\rho U + \tfrac{1}{2}\rho v^2\right)\right]$$
$$- \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] + \rho\left(v_x g_x + v_y g_y + v_z g_z\right)$$
$$- \left[\frac{\partial}{\partial x} Pv_x + \frac{\partial}{\partial y} Pv_y + \frac{\partial}{\partial y} Pv_z\right]$$
$$- \left[\frac{\partial}{\partial x}\left(\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z\right) + \frac{\partial}{\partial y}\left(\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z\right)\right.$$
$$\left. + \frac{\partial}{\partial z}\left(\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z\right)\right]. \tag{7.63}$$

Table 7.5 The equation of energy in terms of the transport properties *(for Newtonian fluids of constant ρ, η, and k; note that the constancy of ρ implies that $C_v = C_p$)*

Rectangular coordinates

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$
$$+ 2\eta\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \eta\left\{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2\right.$$
$$\left. + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right\}. \quad \text{(A)}$$

Cylindrical coordinates

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$
$$+ 2\eta\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right]^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \eta\left\{\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)^2\right.$$
$$\left. + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 + \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\}. \quad \text{(B)}$$

Spherical coordinates

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right) = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] + 2\eta\left\{\left(\frac{\partial v_r}{\partial r}\right)^2\right.$$
$$\left. + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right\}$$
$$+ \eta\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2\right.$$
$$\left. + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2\right\}. \quad \text{(C)}$$

Note: The terms contained in braces { } are associated with viscous dissipation and may usually be neglected, except for systems with large velocity gradients.

Example 7.4 Refer back to Fig. 7.3 and the system described in Section 7.2. Using Table 7.4 or 7.5 derive the energy equation.

Solution. Table 7.5 is selected because the fluid has constant properties. Flow is two-dimensional in rectangular coordinates so Eq. (A) is the best choice. Before proceeding, recall that there are two velocity components v_x and v_y. Also, it is a good idea to qualitatively sketch the temperature field. Having done so, you should recognize that $T = T(x,y)$. Now, we can proceed to simplify Eq. (A) in Table 7.5. Notice that $C_v = C_p$.

$$\frac{\partial T}{\partial t} = 0 \qquad \text{because there is steady state.}$$

$$v_z\frac{\partial T}{\partial z} = 0 \qquad \text{because } v_z = 0 \text{ and } T = T(x,y).$$

$$\frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{because } T = T(x,y).$$

All terms in { } are zero because we neglect viscous heating. We are left with

$$\rho C_p\left[v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right].$$

Except for fluids with very low Pr numbers, we can ignore conduction in the direction of flow; hence

$$\frac{\partial^2 T}{\partial x^2} \simeq 0$$

and we finally write:

$$v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}.$$

PROBLEMS

7.1 For laminar flow, calculate the results given in Table 7.1 for Nu_∞ for slug flow (v_x = uniform) and uniform heat flux in a circular tube.

7.2 A liquid film at T_0 flows down a vertical wall at a higher temperature T_s. Consider heat transfer from the wall to the liquid for such contact times that the liquid temperature changes appreciably only in the immediate vicinity of the wall. (See figure on next page.)

a) Show that the energy equation can be written (state assumptions):

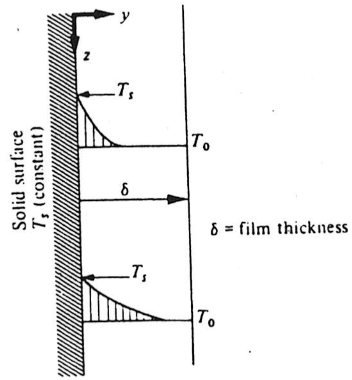
$$\rho C_p v_z\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial y^2}.$$

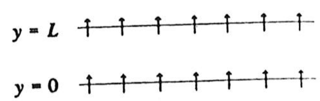
b) The energy equation contains v_z. What would you use for v_z?

c) Write appropriate boundary conditions.

(12)

7.2 (cont.)

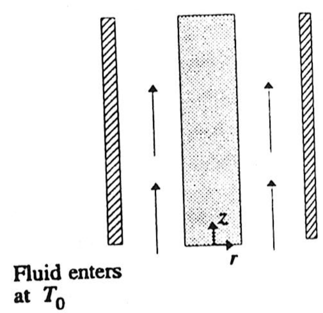




7.3 A gap of thickness L exists between two parallel plates of porous solids. Fluid is forced to flow through the bottom plate, across the gap, and then through the upper plate. Assume that the fluid flows with a constant velocity V in laminar flow with straight streamlines across the gap. The system is at steady state with the upper and lower plates at T_L and T_0, respectively. a) Write an appropriate energy equation and boundary conditions for the fluid in the gap. b) Solve for the temperature in the gap. c) Derive an equation for the heat flux across the gap.

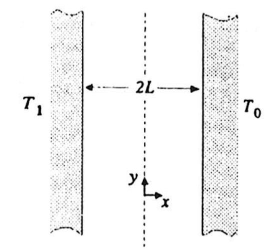
7.4 A liquid of constant density and viscosity flows upward in the annulus ($R_1 \le r \le R_2$) between two very long and concentric cylinders. Assume that both the flow and the temperature are fully developed. The inner cylinder is electrically heated and supplies a constant and uniform flux, q_i, to the liquid. The outer cylinder is maintained at a constant temperature, T_0.

a) Solve for v_z.
b) Write the energy equation and state your assumptions.
c) Write appropriate boundary conditions.



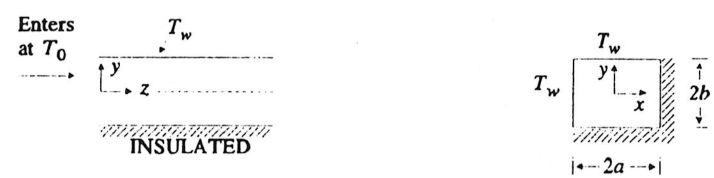
7.5 Air at 0.3 m s^{-1} and 365 K flows parallel to a flat plate at 310 K. a) Calculate the distance from the leading edge to where the momentum boundary layer thickness is 6 mm. b) At the same distance from the leading edge, what is the thermal boundary layer thickness? c) Up to the same distance from the leading edge, how much heat is transferred to the plate (one side) in 600 s, if the plate is 100 mm wide?

7.6 Consider natural convection between parallel vertical plates maintained at T_1 and T_0, respectively. Assume that the plates are very long and the convection is fully developed. For constant properties: a) Write the energy equation and boundary conditions for temperature. b) Write the momentum equation with the Boussinesq approximation and boundary conditions for velocity.



7.7 The surface temperature of a vertical plate is maintained at 390 K. At 0.24 m from the bottom of the plate, calculate the heat transfer coefficient to: a) air at 290 K; b) helium at 290 K.

7.8 Liquid metal flows through a channel with a rectangular cross-section. Two walls are perfectly insulated and two are at a constant temperature of T_w. The metal has temperature T_0 as it enters the channel, and $T_w > T_0$. Assume steady state, fully developed flow and no solidification.

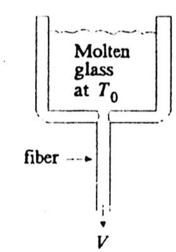


a) Write the energy equation in terms of temperature for constant thermal properties.
b) Write the boundary conditions.

7.9 Consider the creeping flow of a fluid about a rigid sphere as illustrated by Fig. 2.9. The sphere is maintained at T_0 and the fluid approaches from below with a temperature T_∞ and velocity V_∞. a) Write the energy equation which applies to the fluid in the vicinity of the sphere. Assume steady-state conditions. b) Write appropriate boundary conditions for part a). c) What other equations or results would you use in order to solve the system described by parts a) and b)?

7.10 A very long fiber of glass (radius = R) is extracted from a hole in the bottom of a crucible. It is extracted with a constant velocity V into a gas at T_∞; assume slug flow.

a) For uniform properties write the energy equation for temperature in the fiber. Do not ignore conduction in the direction of flow.
b) Write boundary conditions. [Hint: At $r = R$, the flux to the surface must equal the flux to the surrounding gas "via h."]



7.11 Starting with Eq. (7.44), derive Eq. (7.45) and define the dimensionless numbers in Eq. (7.45).

7.12

a) Determine an expression that gives the heat flow Q (W) through a solid spherical shell with inside and outside radii of r_1 and r_2, respectively.

b) Examine the results regarding what happens as the shell thickness becomes larger compared with the inside radius.

7.13 A sphere of radius R is in a motionless fluid (no forced or natural convection). The surface temperature of the sphere is maintained at T_R and the bulk fluid temperature is T_∞. a) Develop an expression for the temperature in the fluid surrounding the sphere. b) Determine the Nusselt number for this situation. Such a value would be the limiting value for the actual system with convection as the forces causing convection become very small.

7.14 For the system in Fig. 2.1 develop an expression for the temperature distribution in the falling film. Assume fully developed flow, constant properties, and fully developed temperature profile. The free liquid surface is maintained at $T = T_0$ and the solid surface at $T = T_s$, where T_0 and T_s are constants. a) Ignore viscous heating effects. b) Include viscous heating effects.

Answer b)

$$\frac{T - T_0}{T_s - T_0} = \frac{x}{\delta}\left\{1 + \frac{3}{4}\,\mathrm{Br}\left[1 - \left(\frac{x}{\delta}\right)^3\right]\right\},$$

where $\mathrm{Br} = \dfrac{\eta \bar{V}^2}{k(T_s - T_0)}$, Brinkman number.

7.15 Consider heat conduction through a plane wall of thickness Δx, and T_1 and T_2 are the surface temperatures. Derive the steady-state heat flux in terms of T_1, T_2 and Δx if the thermal conductivity varies according to

$$k = k_0(1 + aT)$$

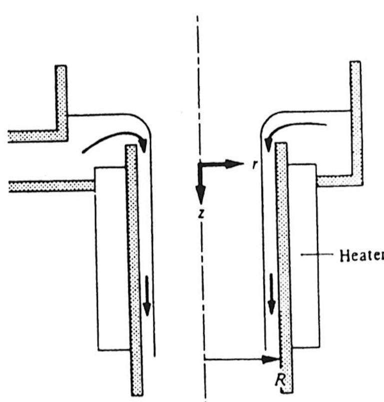
where k_0 and a are constants.

7.16 A liquid at a temperature T_0 continuously enters the bottom of a small tank, overflows into a tube, and then flows downward as a film on the inside. At some position down the tube ($z = 0$) when the flow is fully developed, the pipe heats the fluid with a uniform flux q_R. The heat loss from the liquid's surface is sufficiently small so that it may be neglected.

a) For steady-state laminar flow with constant properties, develop by shell balance or show by reducing an equation in Table 7.5 the pertinent differential energy equation that applies to the falling film.

b) Write the boundary conditions for the heat flow.

c) What other information must complement parts a) and b) in order to solve the energy equation?



8

CORRELATIONS AND DATA FOR HEAT TRANSFER COEFFICIENTS

The problems of heat flow with convection, discussed in the preceding chapter, pertain to simple systems with laminar flow. Despite the simplicity of laminar flow problems, they should not be underestimated. Many simple solutions have been applied to real systems with approximating assumptions and, besides, the simpler systems provide models for interpretation of complex systems. The more complex nature of turbulent flow and its limited accessibility to mathematical treatment requires, however, an empirical approach to heat transfer. On the other hand, the study of turbulent flow is not entirely empirical; it is possible to establish certain theoretical bases for the analyses of turbulent transfer processes and an introduction to this complex area is given in Chapter 16.

Figure 8.1 illustrates heat transfer in a bounded fluid. The fluid is artificially subdivided into three regions: the turbulent core, the transition zone, and the laminar sublayer near the surface. In the turbulent core, thermal energy is transferred rapidly due to the eddy (mixing) action of turbulent flow. Conversely, within the laminar sublayer, energy is transferred by conduction alone—a much slower process than the eddy process. In the transition zone, energy transport by both conduction and by eddies is appreciable. Hence, most of the total temperature drop between the fluid and the surface is across the laminar sublayer and the transition zone. Within the turbulent core, the temperature gradients are quite shallow.

In Chapter 3, it was convenient to define a friction factor to deal with momentum transport in fluids in contact with surfaces. Similarly, for energy transport between fluids and surfaces, it is convenient to define a heat transfer coefficient by

$$h = \frac{q_0}{T_0 - T_f} = \frac{-k(\partial T/\partial y)_0}{T_0 - T_f}, \tag{8.1}$$

where the subscript "0" refers to the respective quantities evaluated at the wall, and T_f is some temperature of the fluid. If the fluid is infinite in extent, we take T_f as the fluid temperature far removed from the surface, and designate it T_∞. If the fluid flows in a confined space, such as inside a tube, T_f is usually the *mixed mean temperature*, denoted by T_m; it is a temperature that would exist if the fluid at a particular cross section were removed and allowed to mix adiabatically.