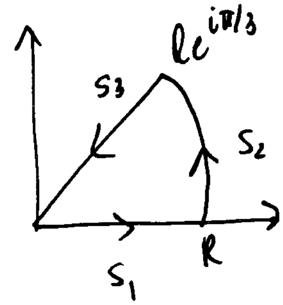
P1) a)



Calculemos $\displaystyle\oint_{S_R} \frac{z}{1+z^6}\,dz$

Notemos que $f(z) = \dfrac{p(z)}{q(z)}$ es meromorfa en $\mathbb{C}$, sus polos son las raíces de q, o sea: $z = e^{i\frac{\pi}{6}(2k+1)}$, $k \in \{0, \dots, 5\}$
(pues no son raíces de $p(z) = z$)

Notar que si $R > 1$, S_R encierra solo al polo $z = e^{i\pi/6}$

$\therefore$ Por residuos: Si $R > 1$: $\displaystyle\oint_{S_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, e^{i\pi/6})$

Además, como todas las raíces son distintas, $z = e^{i\pi/6}$ es polo simple.

$$\Rightarrow \operatorname{Res}(f, e^{i\pi/6}) = \lim_{z \to e^{i\pi/6}} \frac{(z - e^{i\pi/6})\, z}{1 + z^6} \overset{\text{L'Hopital}}{=} \lim_{z \to e^{i\pi/6}} \frac{(z - e^{i\pi/6}) + z}{6z^5}$$

$$= \frac{e^{i\pi/6}}{6e^{i5\pi/6}} = \frac{1}{6e^{i4\pi/6}} = \frac{1}{6} e^{-i2\pi/3}$$

$$\therefore \oint_{S_R} f(z)\,dz = 2\pi i \cdot \frac{1}{6} e^{-i2\pi/3} = i\frac{\pi}{3} e^{-i2\pi/3}$$

Veamos el valor en cada segmento:

En S_1: $\displaystyle\int_{S_1} \frac{z}{1+z^6}\,dz \underset{\substack{\gamma(t) = t \\ t \in [0,R]}}{\overset{\uparrow}{=}} \int_0^R \frac{t}{1+t^6}\,dt$, S_2: $\displaystyle\int_{S_2} f(z)\,dz \underset{\substack{\gamma(t) = Re^{it} \\ t \in [0, \pi/3]}}{\overset{\uparrow}{=}} \int_0^{\pi/3} \frac{Re^{it}}{1+R^6 e^{i6t}} \cdot iRe^{it}\,dt$

$$= \int_0^{\pi/3} \frac{iR^2 e^{2it}}{1+R^6 e^{i6t}}\,dt$$

S_3: $$\int_{S_3} f(z)\,dz \underset{\substack{\uparrow \\ \gamma(t)=(R-t)e^{i\pi/3} \\ t\in[0,R] \\ \gamma' = -e^{i\pi/3}}}{=} \int_0^R \frac{(R-t)e^{i\pi/3}}{1+(R-t)^6 \underbrace{e^{i2\pi}}_{1}} \cdot (-e^{i\pi/3})\,dt = -\int_0^R \frac{(R-t)e^{i2\pi/3}}{1+(R-t)^6}\,dt$$

$$= -e^{i\frac{2\pi}{3}} \int_0^R \frac{(R-t)}{1+(R-t)^6}\,dt \qquad \begin{array}{l} CV: \\ u = R-t \\ du = -dt \end{array}$$

$$= e^{i\frac{2\pi}{3}} \int_R^0 \frac{u}{1+u^6}\,du = -e^{i\frac{2\pi}{3}} \int_0^R \frac{u}{1+u^6}\,du$$

$$\therefore \oint_{S_R} f(z)\,dz = \left(\int_0^R \frac{x}{1+x^6}\,dx\right)\left(1 - e^{i\frac{2\pi}{3}}\right) + \int_{S_2} f(z)\,dz = i\frac{\pi}{3} e^{-i2\pi/3}$$

Luego, si $\lim_{R\to\infty} \int_{S_3} f(z)\,dz = 0$ Entonces: $$\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\frac{\pi i}{3} e^{-\frac{2\pi}{3}i}}{1-e^{i\frac{2\pi}{3}}} = \frac{\pi i}{3} \cdot \frac{1}{e^{i\frac{2\pi}{3}} - e^{i\frac{4\pi}{3}}}$$

$$= \frac{\pi i}{3} \cdot \frac{1}{\left(\cancel{\cos\left(\frac{2\pi}{3}\right)} + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right) - \cancel{\cos\left(\frac{4\pi}{3}\right)} - i\,\mathrm{sen}\left(\frac{4\pi}{3}\right)\right)}$$

pero $\mathrm{sen}\left(\frac{4\pi}{3}\right) = -\mathrm{sen}\left(\frac{2\pi}{3}\right)$

$$\therefore \int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\pi \cancel{i}}{3} \cdot \frac{1}{2\cancel{i}\,\mathrm{sen}\left(\frac{2\pi}{3}\right)}$$

$$= \frac{\pi}{6} \cdot \frac{1}{\mathrm{sen}\left(\frac{2\pi}{3}\right)} = \frac{\pi}{3\sqrt{3}}$$

Probemos lo pendiente:

$$\lim_{R\to\infty} \int_{S_2} f(z)\,dz = \lim_{R\to\infty} i\int_0^{\pi/3} \frac{R^2 e^{2it}}{1+R^6 e^{i6t}}\,dt$$, para ello notemos que: Si $R>1$

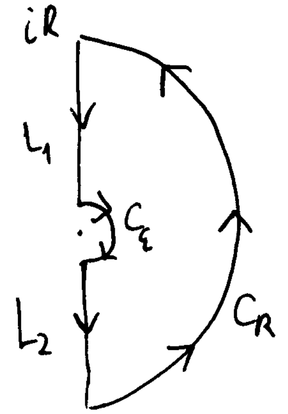
$$0 \le \left| i\int_0^{\pi/3} \frac{R^2 e^{2it}}{1+R^6 e^{6it}}\,dt \right| \le \int_0^{\pi/3} \frac{R^2 \overbrace{|e^{2it}|}^{=1}}{R^6 \underbrace{|e^{6it}|}_{1} - 1}\,dt = \frac{R^2}{R^6-1}\int_0^{\pi/3} dt = \frac{\pi}{3}\frac{R^2}{R^6-1}$$

así, $\forall R>1$:

$$\left|\int_{S_2} f(z)\,dz\right| \le \frac{\pi}{3}\,\frac{R^2}{R^6-1} \xrightarrow[\text{si } R\to\infty]{} 0 \quad , \quad \therefore \lim_{R\to\infty}\int_{S_2} f(z)\,dz = 0 .$$

y por lo tanto:
$$\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\pi}{3\sqrt{3}} .$$

b) $K_{R,\varepsilon}$



$$f(z) = \frac{(\mathrm{Log}\, z)^2}{1-z^2} \qquad K_{R,\varepsilon} = C_\varepsilon \cup C_R \cup L_1 \cup L_2 .$$

i) Sea $\varepsilon<1$ y $R>1$, es claro que f es meromorfa en $\mathbb{C}\setminus(\mathbb{R}^-\cup\{0\})$ más aun, los únicos posibles polos de f en tal dominio son las raíces positivas de $1-z^2 \Rightarrow$ solo $z=1$ es candidato a polo.

¿Lo es? Calculemos
$$\lim_{z\to1} f(z) = \lim_{z\to1}\frac{(\mathrm{Log}\, z)^2}{1-z^2} \quad \text{“}\frac{\infty}{\infty}\text{”}$$
$$\overset{L'H}{=} \lim_{z\to1}\frac{2\,\mathrm{Log}\, z\cdot\frac{1}{z}}{2z} = \lim_{z\to1}\frac{\mathrm{Log}\, z}{z^2} \underset{\text{cont en } z=1}{=} 0$$

Luego $z=1$ es singularidad evitable

$\therefore$ Por Teo. Residuos (o C-G): $\oint_{K_{\varepsilon,R}} f(z)\,dz = 0 \quad \forall R>1,\ \forall\varepsilon<1$ (al menos)

ii) Probemos primero que:

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \quad \text{Notar que} \quad \int_{C_R} f(z)\,dz \underset{\substack{\uparrow \\ \gamma(t)=Re^{it} \\ t\in[0,\pi/2] \\ \gamma'(t)=iRe^{it}}}{=} \int_0^{\pi/2} \frac{(\mathrm{Log}(Re^{it}))^2}{1-R^2e^{2it}} \cdot iRe^{it}\,dt$$

pero:

$$\overset{\text{def. de log complejo!}}{\downarrow}$$

$$\mathrm{Log}(Re^{it}) = \ln R + it$$

$$\Rightarrow (\mathrm{Log}(Re^{it}))^2 = \ln^2 R + 2i\ln R\cdot t - t^2.$$

$$\therefore \int_{C_R} f(z)\,dz = i\int_0^{\pi/2} \frac{(\ln^2 R + 2i\ln R - t^2)\,Re^{it}}{1-R^2e^{2it}}\,dt$$

$$\Rightarrow 0 \le \left|\int_{C_R} f(z)\,dz\right| \underset{\substack{\uparrow \\ \text{desig } \Delta}}{\le} \int_0^{\pi/2} \left|\frac{\ln^2 R\cdot Re^{it}}{R^2-1}\right| dt + 2\int_0^{\pi/2} \frac{|\ln R\cdot Re^{it}|}{R^2-1}\,dt + \int_0^{\pi/2} \frac{|t^2 Re^{it}|}{R^2-1}$$

$$\wedge \quad |1-R^2e^{2it}| \ge \big|1-|R^2e^{2it}|\big| = R^2-1 \quad \text{si } R>1$$

$$= \frac{\pi}{2}\left(\frac{\ln^2 R\cdot R}{R^2-1} + \frac{2\ln R\cdot R}{R^2-1} + \frac{R}{R^2-1}\cdot\frac{(\pi/2)^3}{3}\right)$$

pero, como:

$$\lim_{R\to\infty} \frac{R\cdot\ln^2 R}{R^2-1} \overset{L'H}{=} \lim_{R\to\infty} \frac{\ln^2 R + \cancel{R}\cdot 2\ln R\cdot\frac{1}{\cancel{R}}}{2R} \overset{L'H}{=} \lim_{R\to\infty} \frac{2\ln R\cdot\frac{1}{R} + \frac{2}{R}}{2}$$

$$= \lim_{R\to\infty} \frac{\ln R}{R} + \underbrace{\lim_{R\to\infty} \frac{1}{R}}_{=0} \overset{L'H}{=} \lim_{R\to\infty} \frac{\frac{1}{R}}{1} = 0.$$

$$\text{y} \quad \lim_{R\to\infty} \frac{2\ln R\cdot R}{R^2-1} \overset{L'H}{=} \lim_{R\to\infty} \frac{2\ln R + 2}{2R} \overset{L'H}{=} \lim_{R\to\infty} \frac{\frac{1}{R}}{1} = 0.$$

$$\text{y obviamente} \quad \lim_{R\to\infty} \frac{R}{R^2-1} = 0$$

de donde se deduce que (por Sandwich).

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

Veamos ahora que $\displaystyle\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = 0.$

Como estudiaremos el lím con $\varepsilon \to 0$, para acotar sup. que $\varepsilon < 1$, y haciendo las mismas cotas que antes, se tiene:

$$\left| \int_{C_\varepsilon}^{*} f(z)\,dz \right| \leq \int_0^{\pi/2} \left| \frac{\ln^2 \varepsilon \cdot \varepsilon e^{it}}{1-\varepsilon^2} \right| dt + 2\int_0^{\pi/2} \left| \frac{\ln\varepsilon \cdot \varepsilon e^{it}}{1-\varepsilon^2} \right| dt + \int_0^{\pi/2} t^2 \frac{\varepsilon\, |e^{it}|}{1-\varepsilon^2}\, dt$$

$$= \frac{\ln^2\varepsilon \cdot \varepsilon}{1-\varepsilon^2} \cdot \frac{\pi}{2} + \frac{-\ln\varepsilon \cdot \varepsilon}{1-\varepsilon^2} \cdot \pi + \frac{\varepsilon}{1-\varepsilon^2} \cdot \frac{(\pi/2)^3}{3}$$

Como $\displaystyle\lim_{\varepsilon\to 0} 1-\varepsilon^2 = 1$, para tener la conv. deseada basta ver que los lím de los numeradores son 0:

$$\lim_{\varepsilon\to 0} \ln^2\varepsilon \cdot \varepsilon = \lim_{\varepsilon\to 0} \frac{\ln^2\varepsilon}{\frac{1}{\varepsilon}} \;\; \text{“}\frac{\infty}{\infty}\text{”} \overset{L'H}{=} \lim_{\varepsilon\to 0} \frac{2\ln\varepsilon \cdot \frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = +2\lim_{\varepsilon\to 0} \frac{-\ln\varepsilon}{\frac{1}{\varepsilon}} \;\; \text{“}\frac{\infty}{\infty}\text{”}$$

$$\overset{L'H}{=} 2\cdot\lim_{\varepsilon\to 0} \frac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = 2\lim_{\varepsilon\to 0} \varepsilon = 0.$$

$$\lim_{\varepsilon\to 0} \ln\varepsilon \cdot \varepsilon = \lim_{\varepsilon\to 0} \frac{\ln\varepsilon}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{\frac{1}{\varepsilon}}{\frac{-1}{\varepsilon^2}} = -\lim_{\varepsilon\to 0} \varepsilon = 0.$$

y obviamente $\displaystyle\lim_{\varepsilon\to 0} \varepsilon = 0.$

$\therefore$ Se concluye que

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = 0 .$$

iii) Notemos que:

$$\int_{L_1} f(z)\,dz \underset{\substack{\gamma(t)= i(R-t)\\ t\in[0,R-\varepsilon]\\ \gamma'(t)=-i}}{=} \int_0^{R-\varepsilon} \frac{(\mathrm{Log}(i(R-t)))^2}{1-\underbrace{(i(R-t))^2}_{-(R-t)^2}} \cdot (-i)\,dt = -i \int_0^{R-\varepsilon} \frac{(\mathrm{Log}(i(R-t)))^2}{1+(R-t)^2}\,dt$$

$$= -i \int_0^{R-\varepsilon} \frac{\ln^2(R-t) + 2i\overbrace{\arg(i(R-t))}^{\pi/2\ \forall t}\cdot \ln(R-t) - \overbrace{\arg^2(i(R-t))}^{\pi^2/4}}{1+(R-t)^2}\,dt$$

$$= -i \int_0^{R-\varepsilon} \frac{\ln^2(R-t)}{1+(R-t)^2}\,dt + \cancel{2}\frac{\pi}{\cancel{2}} \int_0^{R-\varepsilon} \frac{\ln(R-t)}{1+(R-t)^2}\,dt + i \int_0^{R-\varepsilon} \frac{\pi^2/4}{1+(R-t)^2}\,dt$$

$$\underset{\substack{CV\\ u=R-t\\ du=-dt}}{=} -i \int_\varepsilon^R \frac{\ln^2 u}{1+u^2}\,du + \pi \int_\varepsilon^R \frac{\ln u}{1+u^2}\,du + \frac{i\pi^2}{4}\int_\varepsilon^R \frac{1}{1+u^2}\,du$$

Tomando lím $\varepsilon\to 0$ y $R\to\infty$

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{L_1} f(z)\,dz = -i \int_0^\infty \frac{\ln^2 u}{1+u^2}\,du + \pi \underbrace{\cancel{\int_0^\infty \frac{\ln u}{1+u^2}\,du}}_{0} + i\,\frac{\pi^2}{4} \underbrace{\int_0^\infty \frac{1}{1+u^2}\,du}_{\pi/2}$$

$$= -i \int_0^\infty \frac{\ln^2 u}{1+u^2}\,du + i\,\frac{\pi^3}{8}$$

Análogamente.

$$\int_{L_2} f(z)dz \underset{\uparrow}{=} \int_{\varepsilon}^{R} \frac{(\log(it))^2}{1-(it)^2} i\,dt = -i\int_{\varepsilon}^{R} \frac{\ln^2 t + 2i\overbrace{\arg(it)}^{-\pi/2\ \forall t}\ln t - \overbrace{\arg^2(it)}^{\pi^2/4}}{1+t^2}\,dt$$

$\gamma(t) = -it$

$t \in [\varepsilon, R]$

$\gamma'(t) = -i$

$$= -i\int_{\varepsilon}^{R} \frac{\ln^2 t}{1+t^2}\,dt - \pi\int_{\varepsilon}^{R} \frac{\ln t}{1+t^2}\,dt + i\frac{\pi^2}{4}\int_{\varepsilon}^{R} \frac{dt}{1+t^2}$$

Tomando lím $\varepsilon \to 0 \wedge R \to \infty$:

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{L_2} f(z)dz = -i\int_0^{\infty} \frac{\ln^2 t}{1+t^2}\,dt - \pi\cancel{\int_0^{\infty} \frac{\ln t}{1+t^2}\,dt}^{\,0} + i\frac{\pi^2}{4}\cdot\underbrace{\int_0^{\infty} \frac{dt}{1+t^2}}_{\pi/2}$$

$$= -i\int_0^{\infty} \frac{\ln^2 t}{1+t^2}\,dt + i\frac{\pi^3}{8}.$$

Juntando todo:

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \oint_{K_{R,\varepsilon}} f(z)dz \underset{a)}{=} 0 \underset{b)+c)}{=} \lim_{\substack{\varepsilon\to 0\\ R\to 0}} \underbrace{\int_{C_\varepsilon} f(z)dz}_{\to 0} + \underbrace{\int_{C_R} f(z)dz}_{\to 0} + \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$$

$$= -2i\int_0^{\infty} \frac{\ln^2 x}{1+x^2}\,dx + 2i\frac{\pi^3}{8}$$

$$\therefore\ 0 = 2i\left(\frac{\pi^3}{8} - \int_0^{\infty} \frac{\ln^2 x}{1+x^2}\,dx\right) \Rightarrow \boxed{\int_0^{\infty} \frac{\ln^2 x}{1+x^2}\,dx = \frac{\pi^3}{8}}$$

P21 b)

Pdq $\hat{g}(s,t) = e^{ist} e^{-ks^2 t}$ $k>0$ transf (en x) de

$$g(x,t) = \frac{1}{\sqrt{2kt}} e^{-(x+t)^2/4kt}$$

1/1

Sol. Definamos $h(u) = \frac{1}{\sqrt{2kt}} e^{u^2/4kt}$

Así $\forall t$:

$$g(u,t) = h(u+t) \Leftrightarrow \hat{g}(s,t) = \widehat{h(u+t)}(s)$$
$$= \hat{h}(s) e^{ist}$$

Por otro lado:

$\hat{h}(s) = \widehat{\frac{1}{\sqrt{2kt}} e^{u^2/4kt}}$, pero, si $G(u) = e^{-u^2/2}$ $u \in \mathbb{R}$ entonces:

$$\Rightarrow h(u) = \frac{1}{\sqrt{2kt}} G\left(\frac{u}{\sqrt{2kt}}\right)$$

$$\therefore \hat{h}(s) = \frac{1}{\sqrt{2kt}} \widehat{G\left(\frac{u}{\sqrt{2kt}}\right)}(s)$$

$$= \frac{1}{\sqrt{2kt}} \cdot \sqrt{2kt} \cdot \hat{G}(\sqrt{2kt}\, s) \quad (\text{Tomo } \lambda = \sqrt{2kt})$$

$$= \hat{G}(\sqrt{2kt} \cdot s) \underset{\text{ind.}}{=} e^{-\frac{s^2}{\cancel{2}} \cdot \cancel{2}kt} = e^{-s^2 kt}$$

$\therefore \hat{g}(s,t) = \hat{h}(s) e^{ist} = e^{-s^2 kt} e^{ist}$ que era lo deseado.

$\square$