P1 (a) Notemos que:

$$\binom{n+2}{3} = \frac{(n+2)!}{(n-1)!\cdot 3!} = \frac{n(n+1)(n+2)}{6}$$

Con esto, queremos probar que:

$$\sum_{k=2}^{n+1} \binom{k}{2} = \frac{n(n+1)(n+2)}{6}$$

En efecto,

$$\sum_{k=2}^{n+1} \binom{k}{2} \overset{\text{por def.}}{=} \sum_{k=2}^{n+1} \frac{k!}{(k-2)!\cdot 2!} \overset{\text{simplificando}}{=} \frac{1}{2}\sum_{k=2}^{n+1} k(k-1) =$$

$$\overset{\text{cambio de índices}}{=} \frac{1}{2}\sum_{k=1}^{n} (k+1)\cdot k = \frac{1}{2}\left(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right) =$$

$$\overset{\text{sumas conocidas}}{=} \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$\overset{\text{factorizando}}{=} \frac{1}{2} n(n+1)\left(\frac{2n+1}{6} + \frac{1}{2}\right) = \frac{1}{2} n(n+1)\left(\frac{n+2}{3}\right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

(b) p.d.q: $\displaystyle\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}$

En efecto:

sumando $0 = k^2 - k^2$

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n} \frac{k^2+2k+1-k^2}{k^2(k+1)^2}$$

separando en 2 fracciones

$$= \sum_{k=1}^{n} \frac{k^2+2k+1}{k^2(k+1)^2} - \frac{k^2}{k^2(k+1)^2}$$

Simplificando

$$= \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{(k+1)^2}$$

Telescópica!

$$= \frac{1}{1} - \frac{1}{(n+1)^2}$$

q.e.d.

P2

Indicación

$$\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}\cdot k^{2} = \sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}\left[k(k-1)+k\right]$$

Separando en 2 sumas

$$= \sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}k(k-1) + \sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}k$$

Los 1eros términos son 0

$$= \sum_{k=2}^{n}\binom{n}{k}p^{k}q^{n-k}k(k-1) + \sum_{k=1}^{n}\binom{n}{k}p^{k}q^{n-k}k$$

Def. coef binomiales

$$= \sum_{k=2}^{n}\frac{n!}{(n-k)!\,k!}p^{k}q^{n-k}k(k-1) + \sum_{k=1}^{n}\frac{n!}{(n-k)!\,k!}p^{k}q^{n-k}k$$

Simplificando

$$= n\cdot(n-1)\sum_{k=2}^{n}\frac{(n-2)!}{(n-k)!\,(k-2)!}p^{k}q^{n-k} + n\cdot\sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!\,(k-1)!}p^{k}q^{n-k}$$

Def. coef binom.

$$= n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2}p^{k}q^{n-k} + n\cdot\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k}\cdot q^{n-k}$$

Cambio de índices

$$= n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}p^{(k+2)}q^{n-(k+2)} + n\sum_{k=0}^{n-1}\binom{n-1}{k}p^{(k+1)}q^{n-(k+1)}$$

Prop. de potencias

$$= n(n-1)p^{2}\sum_{k=0}^{n-2}\binom{n-2}{k}p^{k}q^{(n-2)-k} + np\sum_{k=0}^{n-1}\binom{n-1}{k}p^{k}\cdot q^{(n-1)-k}$$

Teo. del Binomio

$$= n(n-1)p^{2}(p+q)^{n-2} + np\,(p+q)^{n-1}$$

$p+q=1$

$$= np^{2}(n-1) + np = np\left(p(n-1)+1\right)$$

P3 b) En efecto

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a^{i+j} = \sum_{i=1}^{n} a^{i}\cdot \sum_{j=1}^{i} a^{j} \overset{\text{Traslación de índices}}{=} \sum_{i=1}^{n} a^{i}\cdot \sum_{j=0}^{i-1} a^{j+1}$$

$$= a\sum_{i=1}^{n} a^{i}\cdot \sum_{j=0}^{i-1} a^{j} \overset{\text{Geométrica}}{=} a\sum_{i=1}^{n} a^{i}\cdot\left(\frac{1-a^{i}}{1-a}\right)$$

$$= \frac{a}{1-a}\left(\sum_{i=1}^{n} a^{i} - \sum_{i=1}^{n} a^{2i}\right) \overset{\text{Traslación índices}}{=} \frac{a}{1-a}\left(\sum_{i=0}^{n-1} a^{i+1} - \sum_{i=0}^{n-1} a^{2i+2}\right)$$

$$= \frac{a}{1-a}\left(a\cdot\sum_{i=0}^{n-1} a^{i} - a^{2}\cdot\sum_{i=0}^{n-1}(a^{2})^{i}\right)$$

Geométricas

$$= \frac{a^{2}}{1-a}\left(\frac{1-a^{n}}{1-a} - a\cdot\frac{1-a^{2n}}{1-a^{2}}\right)$$

sólo mover cosas algebraicamente

$$= \frac{a^{2}}{(1-a)^{2}}\left(\frac{(1-a^{n})(1+a) - a\left(1-a^{2n}\right)}{(1+a)}\right)$$

$$= \frac{a^{2}}{(1-a)^{2}}\left(\frac{1-a^{n+1}-a^{n}+\cancel{a}-\cancel{a}+a^{2n+1}}{(1+a)}\right)$$

$$= \frac{a\left(a^{2n+2}-a^{n+2}-a^{n+1}+a\right)}{(a-1)^{2}(a+1)}$$

P4 (a) Usando el teo. del binomio, tenemos que:

$$(1+x)^{2n} + (1-x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k + \sum_{k=0}^{2n} (-x)^k$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} \left(x^k + (-1)^k \cdot x^k \right)$$

• Y en esta última expresión, notamos que cuando "k" toma un valor impar, el término k-esimo vale 0. Es decir,

$$\sum_{k=0}^{2n} \binom{2n}{k} \left(x^k + (-1)^k x^k \right) = \binom{2n}{0} \left(x^0 + x^0 \right) + \binom{2n}{2} \left(x^2 + x^2 \right) +$$

$$+ \binom{2n}{4} \left(x^4 + x^4 \right) + \dots + \binom{2n}{2n} \left(x^{2n} + x^{2n} \right)$$

Que corresponde a sumar sólo los términos pares. Así, podemos reescribir la suma como:

$$\sum_{k=0}^{2n} \binom{2n}{k} \left(x^k + (-1)^k \cdot x^k \right) = \sum_{i=0}^{n} \binom{2n}{2i} \left(2 \cdot x^{2i} \right)$$

$$= 2 \cdot \sum_{i=0}^{n} \binom{2n}{2i} x^{2i}$$

Es decir, tenemos que:

$$(1+x)^{2n}+(1-x)^{2n}=2\cdot\sum_{i=0}^{n}\binom{2n}{2i}x^{2i}$$

Y como esto es válido para todo $x\in\mathbb{R}$, en particular, si tomamos $x=1$:

$$2^{2n}=2\cdot\sum_{i=0}^{n}\binom{2n}{2i}$$

$$\Longrightarrow\ 2^{2n-1}=\sum_{i=0}^{n}\binom{2n}{2i}\quad,\ \forall n\geq 1.$$

Que es lo que se quería demostrar. $\blacksquare$

b) $$\sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+1)(k+2)} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!\,(k+1)(k+2)}$$

..lt. por 1

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k+2)!} \cdot \frac{(n+1)(n+2)}{(n+1)(n+2)}$$

por def. coef. binom.

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \frac{(n+2)!}{(n-k)!\,(k+2)!} = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \binom{n+2}{k+2}$$

..mbio de índices

sumando y restando los términos con k=1, k=0

$$= \frac{1}{(n+1)(n+2)} \sum_{k=2}^{n+2} \binom{n+2}{k} = \frac{1}{(n+1)(n+2)} \left(\sum_{k=0}^{n+2} \binom{n+2}{k} - \binom{n+2}{0} - \binom{n+2}{1} \right)$$

.. del ..nomio

$$= \frac{1}{(n+1)(n+2)} \left((1+1)^{n+2} - 1 - (n+2) \right)$$

$$= \frac{2^{n+2} - n - 3}{(n+1)(n+2)}$$

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