Saada, Elasticity Theory and Applications
P. 18-19 (Review of matrix algebra)

1. Given

$$[a] = \begin{bmatrix} 3 & -2 & 5 \\ 6 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix} \text{ and } [b] = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 0 \\ 5 & 2 & -1 \end{bmatrix}.$$

(a) compute $[a] + [b]$ and $[a] - [b]$.
(b) Verify: $[a] + ([b] - [c]) = ([a] + [b]) - [c]$.
(c) Split $[a]$ into its symmetric and its antisymmetric parts

2. Given

$$[a] = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, [b] = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix},$$

and

$$[c] = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}, \text{ show that } [a][b] = [a][c]$$

in spite of the fact that $[b] \neq [c]$.

3. If $[b] = [a][a]'$, show that $[b] = [b]'$.

4. If $\{\bar{a}\}$ is a column matrix, show that $\{\bar{a}\}\{\bar{a}\}' = [c]$, where $[c]$ is a square matrix with the property that $[c] = [c]'$.

5. Given

$$[a] = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \{\bar{b}\} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

compute the product $[a]\{\bar{b}\}$.

6. If $[a]$ is a square matrix of order 3, show that its determinant is given by $\varepsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k} (i, j, k, = 1, 2, 3)$.

7. Write out in full the following expressions :

a) $a_{ij} x_i x_j$ b) $\delta_{ij} x_i x_j$ c) $\sigma_{ni} = \sigma_{ji}\, \ell_j$

d) $\sigma'_{ij} = \ell_{ik}\, \ell_{jm}\, \sigma_{km}$ e) $\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}$

The subscripts i, j, k, and m take the values 1, 2, and 3.

8. Find the inverse of the matrices

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 4 & 0 & 3 \end{bmatrix}.$$

P 91-94 (General Analysis of strain in Cartesian Coordinates)

1. The displacement components at the points of a body are:

$$u_1 = c_1 x_1, \quad u_2 = c_2 x_2, \quad u_3 = c_3 x_3.$$

(a) Find the components ε_{ij} of the strain matrix, and the value of the three invariants of the state of strain.
(b) What is the value of the volumetric strain ε_v?
(c) If the constants c_1, c_2, and c_3 are so small that their squares and products are negligible, show that the components of the strain matrix ε_{ij} become equal to the components of the linear strain matrix e_{ij}.

2. Solve Problem 1 for displacement components given by

$$u_1 = c_1 x_2, \quad u_2 = u_3 = 0.$$

Draw sketches showing a cubic element at a point, and with its edges parallel to the reference axes, before and after transformation.

3. Let

$$u_1 = C(2x_1 + x_2^2), \quad u_2 = C(x_1^2 - 3x_2^2), \quad u_3 = 0,$$

where $C = 10^{-2}$, be the expressions of the displacements of a certain body.
(a) Show the distorted shape of a two-dimensional element of area whose sides dx_1 and dx_2 are initially parallel to the coordinate axes; the two elements are at a point M whose coordinates are (2, 1, 0).
(b) Determine the coordinates of M after transformation.
(c) Decompose the matrix of the transformation at M into its symmetric and antisymmetric components.
(d) Find the angle of rotation and the cylindrical dilatation of the two elements dx_1 and dx_2.

4. In Problem 3, compute the strain ε_{MN} of an element MN whose direction cosines are $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. What are the principal directions and the principal strains?

5. Given the displacement components

$$u_1 = cx_1(x_2 + x_3)^2, \quad u_2 = cx_2(x_3 + x_1)^2, \quad u_3 = cx_3(x_1 + x_2)^2$$

where c is a constant:
(a) Find the components of the linear strain.
(b) Find the components of the rotation.
(c) Find the principal elongations per unit length E_1, E_2, and E_3 at a point M whose coordinates are (1, 1, 1).

6. The components of linear strain in a body are given by:

$$[e_{ij}] = \begin{bmatrix} 0 & 0 & -cx_2 \\ 0 & 0 & cx_1 \\ -cx_2 & cx_1 & 0 \end{bmatrix},$$

where c is a constant. Find the principal strains and the principal directions at the point (1, 2, 4).

7. Determine the volumetric strain ε_v for the following state of strain:

$$[\varepsilon_{ij}] = \begin{bmatrix} 0.5 & 1 & 0 \\ 1 & 2 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

Compare the result to the unit change of volume E_v, and to the first invariant.

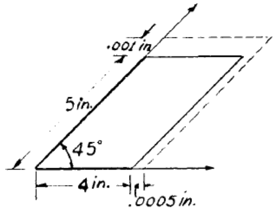


Fig. 4.6

8. A plate whose thickness is ⅛ in. is stretched as shown in Fig. 4.6. Find the principal strains, e_1, e_2, and the maximum shearing strain in the plate.

9. In a two-dimensional state of strain,

$$e_{11} = 800 \times 10^{-6}, \; e_{22} = 100 \times 10^{-6}, \; e_{12} = -800 \times 10^{-6}.$$

Find the magnitude and direction of the principal strains, e_1 and e_2, both analytically and through the use of Mohr's diagram. Draw a sketch showing the deformation of a unit square with edges initially along OX_1 and OX_2.

10. If

$$e_{11} = -800 \times 10^{-6}, \; e_{22} = -200 \times 10^{-6}, \; e_{12} = -600 \times 10^{-6},$$

show in a suitable sketch the position of the axes with which the maximum shearing strain is associated.

11. Are the following states of strain possible?

$$\begin{aligned} e_{11} &= C(x_1^2 + x_2^2) & e_{11} &= Cx_3(x_1^2 + x_2^2) \\ e_{22} &= Cx_2^2 & e_{22} &= Cx_2^2 x_3 \\ e_{12} &= 2Cx_1x_2 & e_{12} &= 2Cx_1x_2x_3 \\ e_{33} &= e_{13} = e_{23} = 0 & e_{33} &= e_{13} = e_{23} = 0 \end{aligned}$$

C is a constant.

12. Show by differentiation of the strain-displacement relations (4.10.1) that the compatibility relations (4.10.4) are necessary conditions for the existence of continuous single-valued displacements.

13. Establish by differentiation a set of compatibility relations involving both the e_{ij}' s and the ω_{ij}' s.

P p 180-182 (Analysis of Stress)

PROBLEMS

1. A stress field is given by:

$$\sigma_{11} = 20x_1^3 + x_2^2 \qquad \sigma_{12} = x_3$$
$$\sigma_{22} = 30x_1^3 + 200 \qquad \sigma_{13} = x_2^3$$
$$\sigma_{33} = 30x_2^2 + 30x_3^3 \qquad \sigma_{23} = x_1^3.$$

What are the components of the body force required to insure equilibrium?

2. The usual engineering equations for the stresses due to the bending of a circular beam are (Fig. 7.25):

$$\sigma_{11} = \frac{Mx_2}{I} \qquad \sigma_{12} = \frac{V(R^2 - x_2^2)}{3I} \qquad I = \frac{\Pi R^4}{4}$$
$$\sigma_{22} = 0 \qquad \sigma_{13} = 0$$
$$\sigma_{33} = 0 \qquad \sigma_{23} = 0$$

Do these equations satisfy equilibrium? M is the bending moment, V is the shearing force, I is the moment of inertia about a diameter of the section, and R is the radius.

3. The stress field in a continuous body is given by:

$$[\sigma_{ij}] = 10^3 \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & 4x_1 \\ 2x_2 & 4x_1 & 1 \end{bmatrix} \text{psi.}$$

Find the stress vector $\bar{\sigma}$ at a point M (1, 2, 3), acting on a plane $x_1 + x_2 + x_3 = 6$.

4. The state of stresses at a point is given by:

$$[\sigma_{ij}] = 10^2 \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \text{psi.}$$

Find the magnitude and direction of the stress vector acting on a plane whose normal has direction cosines $(1/2, 1/2, 1/\sqrt{2})$; what are the normal and tangential stresses acting on this plane?

5. In a solid circular shaft subjected to pure torsion, the stress field is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & -Cx_2 \\ 0 & 0 & Cx_1 \\ -Cx_2 & Cx_1 & 0 \end{bmatrix},$$

where C is a constant. At the point whose coordinates are (1, 2, 4), find:
(a) the principal stresses
(b) the principal directions
(c) the maximum shearing stress and the plane on which it acts.

6. At a point M of a continuous body, the components of the stress tensor are:

$$[\sigma_{ij}] = 10^3 \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix} \text{psi.}$$

(a) Find the principal stresses and the principal directions.
(b) Draw Mohr's circles, and obtain the normal and tangential stresses on a plane whose normal has direction cosines $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ with respect to the reference axes.
(c) Find the octahedral normal and shearing stresses.
(d) What are the invariants of the spherical and the deviatoric components of this stress tensor?
(e) What is the equation of the stress quadric?

7. Find the components of the stress tensor of Problem 4 in a system of coordinates whose axes have direction cosines (0, 0, 1), $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, -1/\sqrt{2}, 0)$.

8. A very thin plate is uniformly loaded as shown in Fig. 7.26. Among all the planes that are normal to the plane of the plate, which ones are the principal planes and what is the value of the stresses to which they are subjected?

9. For the following states of stress at a point, use Mohr's circle to obtain the magnitude and directions of the principal stresses:

$(a)\sigma_{11} = 4{,}000$ psi	$(b)\sigma_{11} = 14{,}000$ psi	$(c)\sigma_{11} = 12{,}000$ psi
$\sigma_{22} = 0$	$\sigma_{22} = 5{,}000$ psi	$\sigma_{22} = 5{,}000$ psi
$\sigma_{12} = 8{,}000$ psi	$\sigma_{12} = -6{,}000$ psi	$\sigma_{12} = 10{,}000$ psi
$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$	$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$	$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

10. Obtain the equations of equilibrium in the two systems of coordinates defined in Problems 1 and 2 of Chapter 6.

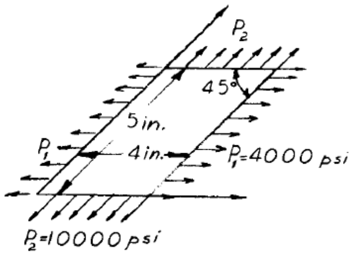


Fig. 7.26

pp 180-182 (Elastic Stress-Strain Relationships)

PROBLEMS

1. Show that the stress-strain relations for a panel (Fig. 8.7) made of orthotropic material under a condition of plane stress can be written in the following form which involves only four independent constants:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{22} & 0 \\ 0 & 0 & 2G_{12} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{bmatrix}.$$

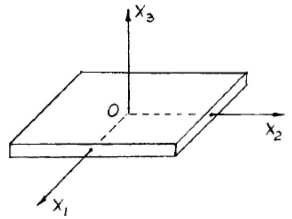


Fig. 8.7

Derive the expression of H_{11}, H_{22}, H_{12}, and G_{12} in terms of the tensor $C_{ijk\ell}$. In Fig. 8.7, the reference axes are parallel to the axes of symmetry.

2. Sometimes the components of the compliance matrix $S_{ijk\ell}$ are written in terms of the constants E, ν, and G in the following manner:

$$S_{1111} = \frac{1}{E_1}, \quad S_{1212} = \frac{1}{4G_{12}}, \quad S_{2211} = -\frac{\nu_{12}}{E_1}, \ldots,$$

where E_1 is Young's modulus in the OX_1 direction, G_{12} is the shear modulus associated with the OX_1, OX_2 directions, and ν_{12} is Poisson's ratio for the strain in the OX_2 direction caused by the stress in the OX_1 direction. The inverse of the stress-strain relations in Problem 1 is written as:

$$\begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}.$$

Determine H_{11}, H_{22}, and H_{12} in terms of E_1, E_2, and ν_{12}.

3. Find the coefficients of the matrix of the elastic coefficients in Problem 2, if the system of axes is rotated 30 degrees counterclockwise around the OX_3 axis.

4. A cubic material is a material in which the properties are the same along three orthogonal directions. Show that the matrix of the coefficients of elasticity contains three independent constants only: Choosing the coordinate axes along these directions, the compliance matrix can be written as follows:

$$\begin{bmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\ S_{1122} & S_{1111} & S_{1122} & 0 & 0 & 0 \\ S_{1122} & S_{1122} & S_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{1212} \end{bmatrix}.$$

5. Prove that in an isotropic, homogeneous, linearily elastic solid, the principal axes of the stress tensor coincide with the principal axes of the linear strain tensor.
6. Could the following stress fields be possible stress fields in an elastic solid, and, if so, under what conditions?

$\sigma_{11} = ax_1 + bx_2$	$\sigma_{11} = ax_1^2 x_2^2 + bx_1$	$\sigma_{11} = a[x_2^2 + b(x_1^2 - x_2^2)]$
$\sigma_{22} = cx_1 + dx_2$	$\sigma_{22} = cx_2^2$	$\sigma_{22} = a[x_1^2 + b(x_2^2 - x_1^2)]$
$\sigma_{12} = fx_1 + gx_2$	$\sigma_{12} = dx_1 x_2$	$\sigma_{33} = ab(x_1^2 + x_2^2)$
$\sigma_{13} = \sigma_{23} = 0$	$\sigma_{13} = \sigma_{23} = 0$	$\sigma_{12} = 2abx_1 x_2$
$\sigma_{33} = 0$	$\sigma_{33} = 0$	$\sigma_{13} = \sigma_{23} = 0.$

a, b, c, d, f, and g are constants.

7. A cube of iron whose edges are 10 in. long is subjected to a uniform pressure of 10 tons / in^2 on two opposite faces; the other faces are prevented from moving more than 0.002 in. by lateral pressure. Determine the pressures on these faces and the maximum shearing stress in the cube. $E = 30 \times 10^6$ psi and $\nu = 0.3$.
8. A cube of Duralumin, whose edges are 5 in. long, is subjected to a uniform pressure of 15,000 psi on the four faces normal to the OX_1 and OX_2 axes. The two faces normal to the OX_3 axis are restricted to a total deformation of 0.0006 in. Determine the stress σ_{33} and the change in length of the diagonal of the cube. $E = 10^7$ psi and $\nu = 0.3$.

9. In Problem 8 of Chapter 7, find the change in length of the diagonals. $E = 30 \times 10^6$ and $\nu = 0.3$.
10. A steel pulley is to be fitted tightly around a shaft. The internal diameter of the hole in the pulley is 0.998 in., while the outside diameter of the shaft is 1.000 in. The pulley will be assembled on the shaft by heating the pulley, then allowing the assembly to reach a uniform temperature. What is the temperature change required to produce a clearance of 0.001 in. for easy assembly? For steel, $\alpha = 6.0 \times 10^{-6}/°F$, $E = 30 \times 10^6$ psi, $\nu = 0.3$.
11. A weight of 20,000 lbs. is supported on two short lengths of concentric copper and steel tubes (Fig. 8.8). The thickness of these tubes is such that both tubes have a cross-sectional area of 2 in^2. Determine the amount of load carried by each tube at room temperature and when the temperature is raised 100 ° F above room temperature. For steel, $E = 30 \times 10^6$ psi, $\nu = 0.3$, $\alpha = 6 \times 10^{-6}/°F$, and for copper, $E = 17 \times 10^6$ psi, $\nu = 0.35$, $\alpha = 9.2 \times 10^{-6}/°F$. The tubes have the same length at room temperature when unloaded.
12. A prismatic bar of length l hangs under its own weight and is supported at its top by the uniform stress $\rho g l$ where ρg is the weight per unit volume (Fig. 8.9). Show that the solution,

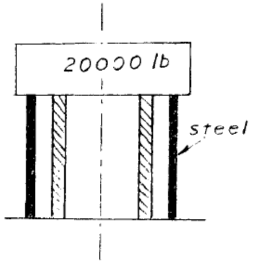


Fig. 8.8

$$\sigma_{33} = \rho g x, \quad \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0,$$

satisfies equilibrium, compatibility, and the prescribed boundary conditions. If an element at A along the OX_3 axis is fixed, find the

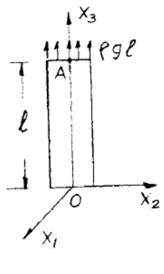


Fig. 8.9

expressions of the displacements u_1, u_2, and u_3.

13. The stress distribution in a thin disk of radius b rotating at an angular velocity ω rad./sec. is given by:

$$\sigma_{rr} = \frac{3+\nu}{8}\rho\omega^2 b^2\left(1 - \frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = \frac{3+\nu}{8}\rho\omega^2 b^2\left(1 - \frac{1+3\nu}{3+\nu}\frac{r^2}{b^2}\right)$$

$$\sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0.$$

Neglecting gravity forces, show that this solution satisfies equilibrium, compatibility, and the prescribed boundary conditions.

14. The solution of the problem of the circular shaft fixed at one end and subjected to a twisting moment at the other is given by (see Fig. 10.5):

$$u_1 = -\alpha x_2 x_3, \quad u_2 = \alpha x_1 x_3, \quad u_3 = 0.$$

What are the conditions that this solution imposes on the applied twisting moments? Is the shaft in a state of plane strain or of plane stress?

REFERENCES

[1] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, N. Y., 1970.
[2] I. N. Sneddon and D. S. Berry, "The Classical Theory of Elasticity," *Encyclopedia of Physics*, Vol. 6, Springer-Verlag, 1958.
[3] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York, N. Y., 1927.
[4] Y. C. Fung, *Foundation of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
[5] N. J. Mushkelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, 1953.
[6] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, N. Y., 1956.

pp 265-267 (Solutions of elasticity problems by potentials)

PROBLEMS

1. Given the scalar and vector potentials $\phi = x_1^2 + 2x_2^2$ and $\bar{\psi} = \rho^2 \bar{i}_3$, does the displacement field generated by ϕ and $\bar{\psi}$ satisfy Navier's equations, and, if so, what is it?
2. Find the displacements and the stresses defined by the following Lamé strain potentials:

$$\phi = A(x_1^2 - x_2^2) + 2Bx_1x_2$$

$$\phi = Cr^n \cos(n\theta).$$

3. Determine the displacements and the stresses defined by the Galerkin vectors:
   (a) $\bar{V} = C\rho^2 \bar{i}_3$
   (b) $\bar{V} = -C\rho^2 x_2 \bar{i}_1 + C\rho^2 x_1 \bar{i}_2$.
4. Find the stresses corresponding to the Lamé strain potential $\phi = C ln(\rho + x_3)$. What is the problem to which this potential furnishes a solution [1]?
5. Show that the solution of Boussinesq's problem can be obtained through a combination of a Galerkin vector $\bar{V} = B\rho \bar{i}_3$ and a Lamé strain potential $\phi = C ln(\rho + x_3)$. Show that $C = -(1 - 2\nu)B$ and $B = P/2\Pi$.
6. What are the stresses corresponding to the following Airy stress functions:

$$\phi = \frac{a}{2}x_1^2 + bx_1x_2 + \frac{c}{2}x_2^2$$

$$\phi = \frac{a}{6}x_1^3 + \frac{b}{2}x_1^2x_2 + \frac{c}{2}x_1x_2^2 + \frac{d}{6}x_2^3.$$

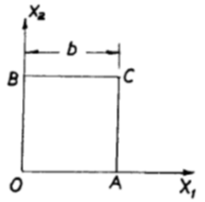


Fig. 9.6

7. A thin square plate whose sides are parallel to the OX_1 and OX_2 axes (Fig. 9.6) has in it stresses described by $\sigma_{11} = cx_2$, $\sigma_{22} = cx_1$, and possibly some shearing stresses σ_{12}. c is a constant.
   (a) Find the stress function by integration, and the most general shearing stresses which can be associated with the given σ_{11} and σ_{22}.
   (b) Obtain the strains and, by integration, deduce the expressions of the displacements u_1 and u_2.
   (c) Find the extension of the diagonal OC.
8. Show that the stress function

$$\phi = C\left[(x_1^2 + x_2^2)\tan^{-1}\frac{x_2}{x_1} - x_1x_2\right]$$

provides the solution to the problem of the semi-infinite elastic medium acted upon by a uniform pressure q on one side of the origin (Fig. 9.7).
9. Investigate what problem of plane strain is solved by the stress function $\phi = Cr\theta \sin\theta$.
10. Investigate the expression $\phi = \cos^3\theta/r$ as a possible stress function.

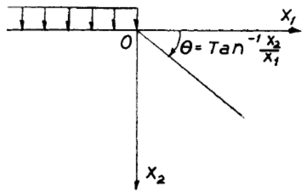


Fig. 9.7

Pp 319-320 (The Torsion Problem)

PROBLEMS

1. A circular shaft is made of an inner circular solid cylinder whose material has a shear modulus G_1 and an outer circular annulus whose material has a shear modulus G_2 (Fig. 10.38). The materials are perfectly bonded at the interface r_i and the shaft is subjected to a twisting moment M_z:

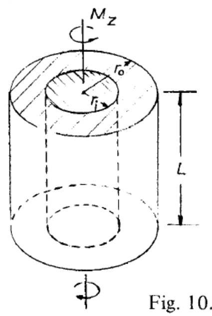


Fig. 10.38

   (a) Find the expression of the angle of rotation per unit length α.
   (b) Find the distribution of the shearing stresses $\sigma_{\theta z}$ in the cylinder and the annulus.
   (c) How much of the total twisting moment M_z does the annulus carry?

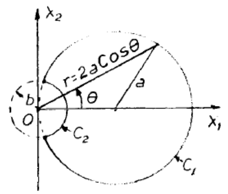


Fig. 10.39

2. Show that the Prandtl stress function

$$\phi = m(r^2 - b^2)\left(\frac{2a\cos\theta}{r} - 1\right)$$

furnishes the solution to the problem of the circular shaft with a circular groove (Fig. 10.39). Find the value of the constant m and the expressions of the stresses σ_{13} and σ_{23} on the boundaries C_1 and C_2.
3. Three bars—one with a square cross section, one with an equilateral triangle cross section, and one with a circular section—have equal cross sectional areas and are subjected to equal twisting moments. Compare the maximum shearing stresses and the torsional rigidities of the bars.
4. A steel bar having a rectangular cross section 1 in. wide and 2 in. long is subjected to a twisting moment of 1,000 lb-in. Calculate the maximum shearing stress and the shearing stress at the center of the short side. ($G = 12 \times 10^6$ psi)
5. A steel bar having a slender rectangular cross section $\frac{1}{4}$ in. wide and 6 in. long is subjected to a twisting couple of 1,500 lb-in. Find the maximum shearing stress and the angle of twist per unit length α using the exact solution of Sec. 10.7 and the approximate solution based on the membrane analogy. What is the magnitude of the error involved? ($G = 12 \times 10^6$ psi .)
6. A brass bar having the cross section shown in Fig. 10.40 is subjected to a twisting couple of 600 lb-in. This bar is 6 ft. long. What is the angle of twist per unit length and the maximum shearing stress? ($G = 4 \times 10^6$ psi .)

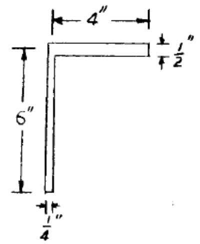


Fig. 10.40

Pp 349-351 (Thick cylinders, disks and spheres)

PROBLEMS

1. Find the ratio of thickness to internal diameter for a tube subjected to internal pressure when the pressure is equal in magnitude to $\frac{3}{4}$ of the maximum circumferential stress. If the internal diameter of the tube is 4 in., determine the increase in the external diameter when the internal pressure is 12,000 psi and the tube is prevented from changing length. ($E = 30 \times 10^6$ psi, $\nu = 0.3$.)
2. A solid bar of uniform circular section is subjected to uniform radial pressure. Show that the stress at any point in a plane section parallel to the axis of the bar is compressive and equal in magnitude to the radial stress.
3. A steel bar of 2 in. diameter is pressed into a steel sleeve so that, when assembled, the magnitude of the radial stress between the two is 2,000 psi, and that of the circumferential stress at the inside of the sleeve is 3,200 psi. Assuming a close fit and neglecting friction, determine the change of radial stress when the bar is subjected to an axial compressive load of 15,000 lb. ($\nu = 0.3$.)
4. A short steel rod of 2 in. diameter is subjected to an axial compressive load of 60,000 lb. It is surrounded by a sleeve $\frac{1}{2}$ in. thick, slightly shorter than the rod so that the load is carried only by the rod. Assuming a close fit before the load is applied and neglecting friction, find the pressure between the sleeve and the rod, and the maximum tensile stress in the sleeve. ($\nu = 0.3$.)
5. The external diameter of a steel hub is 10 in. and the internal diameter increases 0.005 in. when shrunk on to a solid steel shaft of 5 in. diameter. Find the reduction in diameter of the shaft, the radial pressure between the hub and the shaft, and the circumferential stress at the inner surface of the hub. (E 30×10^6 psi, $\nu = 0.3$.)
6. A steel cylinder of 8 in. external diameter and 6 in. internal diameter has another steel cylinder of 10 in. external diameter shrunk onto it. If the maximum tensile stress induced in the outer cylinder is 10,000 psi, find the radial compressive stress between the

cylinders. Determine the circumferential stresses at inner and outer diameter of both cylinders, and show on a diagram how these stresses vary with the radius. Calculate the necessary shrinkage allowance at the common radius. ($E = 30 \times 10^6$ psi, $\nu = 0.3$.)

7. A steel hollow sphere, whose inside diameter is 5 in., is subjected to an internal pressure of 5,000 psi. Determine the thickness of the material if the magnitude of the maximum stress is not to exceed 10,000 psi. Compare this thickness to that obtained from Eq. (11.4.25).
8. Find the expressions of the stresses and displacements for a hollow sphere subjected to an external pressure P_o and filled with an incompressible fluid such that its inner diameter does not change.
9. Determine the greatest value of the radial and circumferential stresses for a thin disk rotating at an angular velocity of 150 radians per sec.; the inner and outer radii of the disk are 6 in. and 12 in. respectively, and the mass per unit volume ρ of the material is 0.28 lb/in.^{3}. ($E = 30 \times 10^6$ psi, $\nu = 0.3$.)
10. A solid steel shaft of 8 in. diameter has a steel cylinder of 16 in. diameter shrunk onto it. The inside diameter of the cylinder prior to the shrink fit operation was 7.992 in. ($E = 30 \times 10^6$ psi, $\nu = 0.3$.)
    (a) Determine the external pressure P_o on the outside of the cylinder which is required to reduce to zero the circumferential stress at the inner surface of the cylinder.
    (b) Determine the radial pressure on the surface of contact due to shrink fit.
    (c) Find the speed of rotation to loosen the fit. ($\rho = 0.28$ lb/in.^{3}).
11. A solid steel shaft 36 in. in diameter is rotating at 200 rpm. If the shaft cannot deform longitudinally, calculate the total longitudinal thrust over a cross section due to rotational stresses. ($\rho = 0.28$ lb/in.^{3}, $E = 30 \times 10^6$ psi, $\nu = 0.3$.)
12. Show that the radial displacement in a rotating solid cylinder, whose ends are free to deform, is given by:

$$u_r = \frac{\rho\omega^2(1+\nu)(1-2\nu)}{8E(1-\nu)}\left[\frac{(3-5\nu)b^2r}{(1+\nu)(1-2\nu)} - r^3\right].$$

13. A brass rod is fitted firmly inside a steel tube whose inner and outer diameters are 1 in. and 2 in., respectively, when the materials are at a temperature of 60 ° F. If the rod and the tube are both heated to a temperature of 300° F, determine the maximum stress in the brass and in the steel. The coefficients of expansion for steel and brass are 6×10^{-6} and 10×10^{-6} per degree Fahrenheit, respectively. Young's modulus is 30×10^6 psi for steel and 12.5×10^6 psi for brass. Poisson's ratio is 0.3 for steel and 0.34 for brass.
14. Find the expression of σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} in a long hollow cylinder with fixed ends, which conducts heat in steady state according to

$$\Delta T = \frac{(\Delta T_a - \Delta T_b) ln\frac{b}{r}}{ln\frac{b}{a}}.$$

ΔT_a is a constant increase in the temperature of the inner surface of the cylinder and ΔT_b, smaller than ΔT_a, is a constant increase in the temperature of the outer surface of the cylinder.

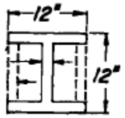
15. Find the expression of the axial stress in the cylinder of Problem 14 when the ends are free.

Hartog, Advanced Strength of Materials, Cap I

PROBLEMS

1. *a.* A 12-in. steel I beam with flanges and webb ½ in. thick is subjected to a torque of 35,000 in.-lb. Find the maximum shear stress and the twist per unit length, neglecting stress concentrations.

b. In order to reduce the stress and the angle of twist of this section, ½-in.-thick flat plates are welded onto the side of the section as shown by the dotted lines. Find the stress and the twist per unit length.



PROBLEM 1.

PROBLEM 2.

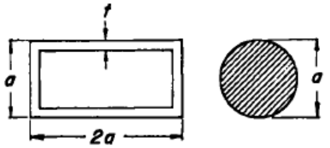
2. The three tubular sections shown all have the same wall thickness t and are made from the same width of plate, *i.e.*, they have the same circumference. Neglecting stress concentrations, find the ratio of the three shear stresses for

a. Equal twisting moments in all three cases.

b. Equal angles of twist in all three cases.

5. The specified dimensions of the cross section of a pipe were 4 in. mean diameter and 0.25 in. wall thickness. When the pipe was delivered, the wall thickness was found to vary according to the equation $t = 0.25 + 0.05 \cos \theta$ to a close enough approximation. The calculated torsional shear stress in the ideal tube was 9.500 lb/in.²

PROBLEM 5.



PROBLEM 6.

a. If subjected to the twisting moment for which it was designed, what would be the maximum shear stress in the actual tube?

b. What would be the calculated and actual angles of twist for a 5-ft length of tube?

6. A thin-walled box section of dimensions $2a \times a \times t$ is to be compared with a solid circle section of diameter a. Find the thickness t so that the two sections have

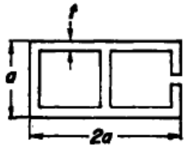
a. The same stress for the same torque.

b. The same stiffness.

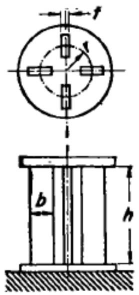
7. A two-compartment thin-walled box section with one compartment slit open has constant wall thickness t. Write formulae for

a. The stress for a given torque.

b. The stiffness, that is, M_t/θ_1.



PROBLEM 7.



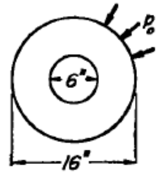
PROBLEM 8.

Cap II (Rotating Disks)

33. A solid steel shaft of 6 in. diameter has a steel cylinder of 16 in. diameter shrunk over it with a shrink allowance of 0.0005 in./in.

a. Calculate the external pressure p_0 on the outside of the cylinder which is required to reduce to zero the tangential tension at the inside of the cylinder.

b. Calculate the resultant radial pressure at the shaft surface due to the shrink fit and to that external pressure.



PROBLEM 33.

34. A steel shaft of 5 in. diameter has a steel disk shrunk on it of 25 in. diameter. The shrink allowance is 0.0008 in./in.

a. Find the radial and tangential stresses of the disk at standstill.

b. Find the rpm necessary to loosen the fit.

c. From (*a*) and (*b*) deduce quickly what the shrink pressure is at half the speed found in (*b*).

35. A steel disk of 20 in. outside diameter and 4 in. inside diameter is shrunk on a steel shaft so that the pressure between shaft and disk at standstill is 5,000 lb/sq in.

a. Assuming that the shaft does not change its dimensions because of its own centrifugal force, find the speed at which the disk is just free on the shaft.

b. Solve the problem without making the assumption *a* by considering the shaft and disk assembly as a single solid non-holed disk.

36. A steel disk of 30 in. diameter is shrunk onto a steel shaft of 3 in. diameter. The interference on the diameter is 0.0018 in.

a. Find the maximum tangential stress in the disk at standstill.

b. Find the speed in rpm at which the contact pressure is zero.

c. What is the maximum tangential stress at the speed found in (*b*)?

37. A flat steel turbine disk of 30 in. outside diameter and 6 in. inside diameter rotates at 3,000 rpm, at which speed the blades and shrouding cause a tensile rim loading of 600 lb/sq in. The maximum stress at this speed is to be 16,000 lb/sq in. Find the maximum shrinkage allowance on the diameter when the disk is put on the shaft.

38. The outward radial deflection at the outside of a thick cylinder subjected to an internal pressure p_i is

$$\frac{2p_i r_0 r_i^2}{E(r_0^2 - r_i^2)} \qquad \text{[by Eq. (41), page 54]}$$

By Maxwell's reciprocal theorem find the inward radial deflection at the inside of a thick cylinder subjected to external pressure.

39. A rotating flat disk is in a state of plane *stress*, *i.e.*, the stresses are all parallel to one plane, and the axial stress is zero. A rotating long cylinder is in a state of plane *strain*, *i.e.*, there is no distortion of the normal cross sections, but there will be an axial stress s_a. Derive the equations

$$s_r = C_1 + \frac{C_2}{r^2} - \frac{(3 - 2\mu)}{8(1 - \mu)} \rho\omega^2 r^2$$

$$s_t = C_1 - \frac{C_2}{r^2} - \frac{(1 + 2\mu)}{8(1 - \mu)} \rho\omega^2 r^2$$

which are the equivalent of Eqs. (39) (page 52), for the case of plain strain.

40. Using the results of Prob. 39, obtain the expression

$$s_a = \frac{\mu}{4(1 - \mu)} (r_0^2 - 2r^2)\rho\omega^2$$

for the axial stress in a rotating long solid cylinder with zero internal and external pressures and ends free from constraint.

41. From Eqs. (43) show that the ratio of the maximum tangential stress to the maximum radial stress for a rotating flat disk with no boundary loading is

$$\frac{2\left(r_0^2 + \frac{1 - \mu}{3 + \mu} r_i^2\right)}{(r_0 - r_i)^2}$$

Where do these maximum stresses occur?

42. A circular disk of outside and inside radii r_0 and r_i fits snugly without clearance or pressure around an incompressible core of radius r_i. It is then subjected to a compressive load of p_0 lb/sq in. uniformly distributed around the outer boundary. Develop the approximate formulae

$$s_{r_i} = -\frac{2p_0}{1 + \mu} \qquad s_{t_i} = -\frac{2\mu p_0}{1 + \mu}$$

for the radial and tangential stresses at the radius r_i, assuming that r_i is small compared with r_0.

43. A steel turbine rotor of 30 in. outside diameter, 6 in. inside diameter, and 2 in. thickness has 100 blades 6 in. long, each weighing 1 lb. Assuming no expansion of the 6 in. shaft due to its *own* centrifugal force, calculate the initial shrink allowance on the diameter so that the rotor loosens on the shaft at 3,000 rpm.

44. A disk of thickness t and outside diameter $2r_0$ is shrunk onto a shaft of diameter $2r_i$ producing a radial interface pressure p in the non-rotating condition. It is then rotated with an angular velocity ω radians/sec. If f is the coefficient of friction between disk and shaft and ω_0 is that value of the angular velocity for which the interface pressure falls to zero, show that

a. The maximum horsepower is transmitted when $\omega = \omega_0/\sqrt{3}$.

b. This maximum horsepower is equal to $0.000365 r_i^2 t f p \omega_0$, where the dimensions are pounds and inches.

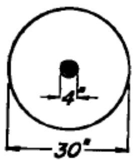
45. A steel gear is approximated by a disk 2.82 in. thick of 4 in. inside diameter and 30 in. outside diameter. The gear is shrunk onto a steel shaft with a diametral interference of 0.0024 in.; the coefficient of friction at the fit is $f = 0.3$.

a. What is the maximum horsepower which this gear can transmit?

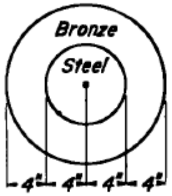
b. At what rpm should the gear run in transmitting this maximum power?

c. What should the diametral shrink interference be if the power to be transmitted is double that possible with the 0.0024-in. interference?

d. At what speed should this new gear run?



PROBLEM 45.



PROBLEM 46.

46. A bronze ring of 16 in. outside diameter is shrunk around a steel shaft of 8 in. diameter. At room temperature the shrink allowance is 0.001 in./in. (that is, 0.004 in. on the radius). Calculate

a. The temperature above room temperature to which the entire assembly must be raised in order to loosen the shrink fit.

b. The rpm at room temperature which will loosen the shrink fit.

The constants are

For steel: $E = 30 \times 10^6$ lb/sq in.; $\mu = 0.3$; $\alpha = 6.67 \times 10^{-6}$ in./in./°F; $\gamma = 0.28$ lb/cu in.

For bronze: $E = 15 \times 10^6$; $\mu = 0.3$; $\alpha = 10 \times 10^{-6}$; $\gamma = 0.33$ lb/cu in.

47. A solid cast-iron disk of 12 in. diameter has a steel rim of 16 in. outside diameter shrunk on it. If at 10,000 rpm the pressure between the rim and the disk is zero, calculate the shrink allowance used.

$$\mu = 0.3 \text{ and } \gamma = 0.283 \text{ lb/cu in. for both materials}$$

$$E_{\text{cast iron}} = 15 \times 10^6 \qquad E_{\text{steel}} = 30 \times 10^6$$

48. A steel rim of 30 in. outside diameter is shrunk on an aluminum disk of 24 in. outside diameter and 4 in. inside diameter. At standstill the normal pressure between the disk and the rim is p. Assuming no pressure between the disk and the shaft, what is the magnitude of the normal pressure between the disk and the rim when the disk is rotating at 1,800 rpm?

$$E_{\text{aluminum}} = 10 \times 10^6 \text{ lb/sq in.} \qquad \gamma = 0.095 \text{ lb/cu in.} \qquad \mu = 0.3$$

49. A steel shaft of 4 in. diameter is shrunk inside a bronze cylinder of 10 in. outside diameter. The shrink allowance is 1 part per 1,000 (that is, 0.002 in. difference between the radii). Find the tangential stress in the bronze at the inside and outside radii and the stress in the shaft.

$$E_{\text{steel}} = 30 \times 10^6 \text{ lb/sq in.} \qquad E_{\text{bronze}} = 15 \times 10^6 \text{ lb/sq in.,}$$

$\mu = 0.3$ for both metals

50. A steel shaft of 3 in. diameter has an aluminum disk shrunk on it of 10 in. outside diameter. The shrink allowance is 0.001 in./in. Calculate the rpm of rotation at which the shrink fit loosens up. Neglect the expansion of the shaft caused by rotation.

$$\mu = 0.3 \qquad E = 10 \times 10^6 \qquad \gamma = 0.095 \text{ lb/cu in.}$$

51. Show that when an aluminum disk of constant thickness and of radii r_i and r_0 is forced onto a steel shaft of radius $r_i + \delta$, the maximum stress in the disk (*i.e.*, at the inner radius) is given by

$$s_{t_i}\left[\left(\frac{1-\mu_s}{E_s} + \frac{\mu_a}{E_a}\right)\frac{r_0^2 - r_i^2}{r_0^2 + r_i^2} + \frac{1}{E_a}\right] = \frac{\delta}{r_i}$$

52. A rod of constant cross section and of length $2a$ rotates about its center in its own plane, so that each end of the rod describes a circle of radius a. Find the maximum stress in the rod as a function of the peripheral speed V. At what speed is the stress 20,000 lb/sq in. in a steel rod?

53. A thick-walled spherical shell of radii r_i and r_0 is subjected to internal or external pressure. By symmetry the principal stresses are s_r radially and s_t tangentially (the same in all tangential directions).

a. Sketch Mohr's circle for the stresses at a point.

b. Derive the equilibrium equation

$$rs_r' + 2s_r - 2s_t = 0$$

by considering the stresses on a section of an elementary shell.

c. Derive the compatibility equation by eliminating u, the radial displacement, between the expressions for the radial and tangential strains $\epsilon_r = du/dr$; $\epsilon_t = u/r$; that is, derive the equation

$$s_t(1 + \mu) + rs_t'(1 - \mu) - s_r(1 + \mu) - \mu rs_r' = 0$$

d. Combining "compatibility" with "equilibrium," obtain the differential equation $rs_r'' + 4s_r' = 0$ for s_r, and show that the solutions for s_r and s_t are

$$s_r = A + \frac{B}{r^3} \qquad s_t = A - \frac{B}{2r^3}$$

e. Show that if the internal and external pressures are p_i and p_0, we have

$$A = \frac{p_i r_i^3 - p_0 r_0^3}{r_0^3 - r_i^3} \qquad B = \frac{(p_0 - p_i) r_0^3 r_i^3}{r_0^3 - r_i^3}$$

54. *Wound Cylindrical Pressure Vessel.* Cylindrical thick-walled pressure vessels have been made by starting from a comparatively thin-walled cylinder (say 56 in. diameter and 1 in. wall thickness), to which a thin sheet (say ⅛ in. thickness) is welded all along a longitudinal line. This sheet is then wrapped around the vessel many times, under tension, so that finally the outer diameter (say 80 in.) is considerably larger than the inner one. The last wrap of the thin sheet is held in place by welding and by the end head pieces fitting over the cylinder. Assume that the tensile stress in the sheet during winding is constant $= s_0$; let $r_i =$ the inner radius of the central tube; $r_i + a =$ the outer radius of the central tube; $t =$ the thickness of the wrapping sheet, to be considered "small" calculus-wise; $r_0 =$ the outer radius of the assembly: $\rho =$ a variable radius between r_i and r_0.

a. Prove that the hoop stress locked up in the cylinder by this process is given by

$$(s_{\text{tang}})_{\text{at }\rho} = s(\rho) - \int_{r=\rho}^{r=r_0} s(\rho)\,\frac{r}{r^2 - r_i^2}\left(1 + \frac{r^2}{\rho^2}\right) dr$$

where $s(\rho) = s_0$ for $r_i + a < \rho < r_0$ and $s(\rho) = 0$ for $r_i < \rho < r_i + a$.

b. Now put an internal pressure p_0 into the vessel, which sets up a Lamé hoop-stress distribution in addition to the locked-up wrapping hoop stress. Write the condition that the *total* hoop stress at r_i is the same as that at r_0. This condition will contain as the only unknown the wrapping tension s_0.

c. Calculate the required wrapping tension s_0 for the case of $p_0 = 10{,}000$ lb/sq in., $r_i = 28$ in., $a = 1$ in., $r_0 = 40$ in.; and calculate the combined hoop stress at r_i and r_0 (which is the same value), as well as halfway between.

55. Finish the problem of page 65 of the text, by answering questions 3 and 4 of page 56 for the hyperbolic disk.

56. A disk of hyperbolic profile has diameters of 60 in. and 12 in. with corresponding disk thicknesses of 3 in. and 6 in. Find the maximum blade loading, expressed in pounds per inch of circumference permissible when the maximum stress at the bore is limited to 20,000 lb/sq in.

57. *a.* Prove that the maximum shear stress at the bore of a disk shrunk on a solid shaft of the same material, with a given interference, is independent of the shape of the disk (flat, hyperbolic, etc.)

b. Prove that the maximum shear stress at the bore of a disk shrunk on a solid shaft of the same material does not change as the speed varies from zero to the critical loosening speed (neglect the expansion of the *shaft* due to rotation).

58. A turbine blade is to be designed for constant tensile stress s_0 under the action of centrifugal force by varying the area A of the blade section. Consider the equilibrium of an element, and show that the condition is

$$\frac{A}{A_h} = e^{-\rho\omega^2(r^2 - r_h^2)/2s_0}$$

where A_h and r_h are the cross-sectional area and radius at the hub (*i.e.*, base of the blade).

59. A steel turbine rotor of 30 in. outside diameter and 4 in. inside diameter carries 100 blades, each weighing 1 lb with centers of gravity lying on a circle of 34 in. diameter. At the outside diameter of the disk its thickness must be 2 in. to accommodate the blades. The rated speed is 4,000 rpm. Assume no pressure at the bore.

a. Find the maximum stress for a disk of uniform thickness.

b. Find the maximum stress for a disk of hyperbolic profile, the thickness at the hub being 15 in. and the tip thickness being 2 in. as before.

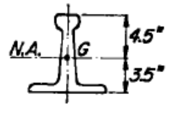
c. Find the thickness at the axis and the thickness just under the rim if a disk of constant stress (10,000 lb/sq in.) is used.

60. A turbine disk of constant stress is to be designed to suit existing blading. The design stress is to be 30,000 lb/sq in. with a maximum axial thickness of 4 in. Blading particulars are: pitch approximately 1 in.; weight of one blade and root 0.525 lb.; the center of gravity of the blades to lie at the rim of the disk; peripheral speed 1,000 ft/sec. From these data determine the wheel radius, the speed, and the disk thickness just under the blades.

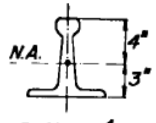
Cap V (Beams of Elastic Foundation)

91. A rail with cross section as shown rests on ballast having a modulus $k = 1{,}500$ lb/sq in. and is loaded by a single concentrated load of 40,000 lb. Find

a. The maximum rail deflection.
b. The maximum bending stress.
c. The bending moment 18 in. from the load.



PROBLEM 91.



PROBLEM 92.

92. A rail with cross section as shown rests on a ballast foundation of modulus $k = 1{,}500$ lb/sq in. The rail is subjected to two concentrated loads each of 30,000 lb, 5 ft apart. What are the maximum stress and maximum deflection of the rail?

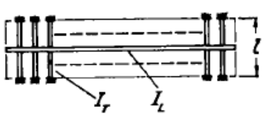
93. A long steel rail of $I = 88.5$ in.^{4} lies on a foundation of modulus $k = 1{,}500$ lb/sq in. The rail carries many concentrated loads of 30,000 lb, all equally spaced 20 ft apart along the rail. Find the deflection under the loads and also at points midway between loads.

94. A small locomotive weighing 75 tons with its weight distributed uniformly on three axles 7 ft apart runs on a light track of $k = 1{,}400$ lb/sq in., $I = 41$ in.^{4}, $Z_{min} = 15$ cu in., and $E = 30 \times 10^6$ lb/sq in. Find the maximum deflection and maximum stress produced by the locomotive in passing over the track.

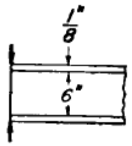
95. The semi-infinite beam of Fig. 105 (page 157) has the left end clamped instead of hinged. Show that the deflection is now given by

$$y = \frac{p_0}{k}\left[1 - F_1(\beta x)\right]$$

96. A grid work of beams is as shown. I_T, I_L are the moments of inertia of the transverse and longitudinal beams, l is the length of the transverse beams, and a is their center-to-center spacing. If the transverse beams are considered "built in" at the ends, find the value of β for the longitudinal beam considered as a beam on an elastic foundation.



PROBLEM 96.



PROBLEM 97.

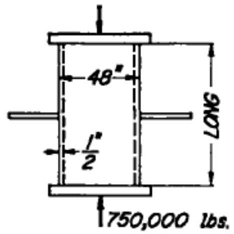
97. A bronze pipe of 6 in. diameter and ⅛ in. wall thickness is subjected to a circumferential load as shown. At what distance from the end is the diameter unchanged by the load? What is the value of the load (pounds per inch) if the diameter under it changes by 0.006 in.? $E_{bronze} = 15 \times 10^6$ lb/sq in.

98. A long thin-walled steel pipe of radius r and wall thickness t has a steel ring shrunk over it in the middle of its length. Show that if the cross section of the ring is $A = 1.56\sqrt{rt^3}$, then the shrink allowance is shared equally between the ring and the pipe (*i.e.*, the reduction in pipe diameter equals the increase in ring diameter during shrinkage).

99. A long steel pipe of 12 in. inside diameter, ⅛ in. wall thickness, with an internal pressure of 300 lb/sq in., is to have a maximum radial deflection of 0.002 in. To do this, steel rings of ½ by ½ in. square cross section are shrunk on the pipe with an interference of 1/1,000. What is the maximum ring spacing under these conditions?

100. A steel pipe of 3 ft diameter, ½ in. thickness, with an internal pressure of 500 lb/sq in., is joined to a pressure vessel. The connection is assumed to be rigid, *i.e.*, there is no expansion or angular rotation of the end of the pipe. At what distance from the end of the pipe does the pipe diameter reach its fully expanded value, and what is the maximum bending stress?

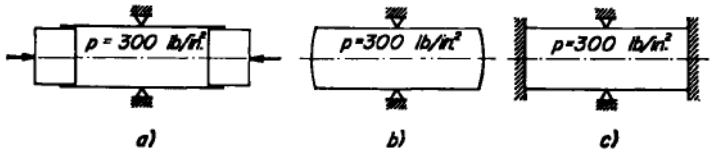
101. A long steel pipe of 48 in. outside diameter and ½ in. thickness is used in a structure as a column. At the center of the column a thin platform prevents radial expansion. Under an axial load of 750,000 lb, what are the longitudinal and tangential stresses in the outer fibers under the constraint?



PROBLEM 101.

102. A steel pipe of 30 in. internal diameter and ½ in. thickness is subjected to an internal pressure of 300 lb/sq in. A rigid circular support (assume a knife-edge) is located midway between the ends of the shell. Find the axial and tangential stresses in the outer fiber under the support for the three conditions below:

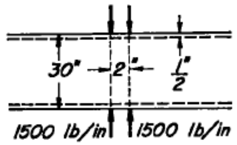
a. The pipe takes no longitudinal thrust.
b. The axial thrust of the internal pressure is taken by ends welded to the pipe.
c. The shell is rigidly supported at the ends by two fixed walls.



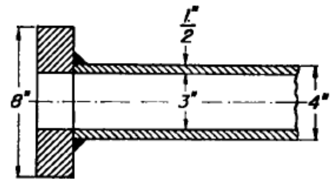
PROBLEM 102.

103. A long pipe of 30 in. outside diameter and ½ in. thickness is subjected to radial loads of 1,500 lb/in. distributed around the circumference at two sections 2 in. apart, as shown in the figure. For the section midway between the loads determine

a. The radial deflection.
b. The longitudinal and tangential stresses in the outer fiber.



PROBLEM 103.



PROBLEM 104.

104. A long steel hollow shaft of 4 in. outside diameter and ½ in. thickness rotates at 10,000 rpm and has steel flanges welded to the ends as shown. Using Eqs. (40) and (43), find the longitudinal bending stress in the shaft at the flange due to the difference in centrifugal expansion of the tube and the flange. Assume that no change of slope can take place at the end of the tube.

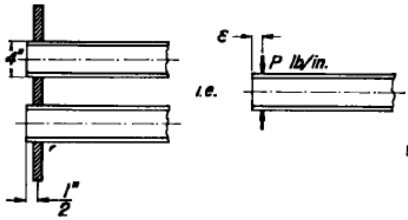
105. Steel boiler tubes of 4 in. outside diamater and ¼ in. wall thickness are full of water under 500 lb/sq in. pressure. They fit into a "header," where the radial support may be considered knife-edged, and protrude ½ in. as shown.

a. Show by superposition that the deflection under P is given by

$$y = \frac{P\beta}{2k}\left[1 + 2F_4^2(\beta\epsilon) + F_3^2(\beta\epsilon)\right]$$

where ϵ is the distance from the knife-edge to the end of the tube.

b. Apply the above to find the load per inch of circumference in this case.



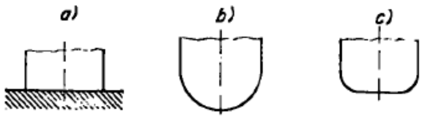
PROBLEM 105.

PROBLEM 106.

106. The body of an axial compressor rotor is constructed as shown by attaching a thin-walled hollow cylinder to two solid ends.

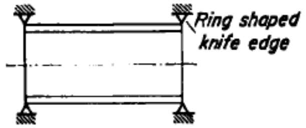
Assuming no change of slope at the ends, find the local bending stresses for 7,000 rpm, radius 6 in., and wall thickness 1 in.

107. A large 8-ft-diameter compressed-air vessel is to be made of ½-in. plate and is to carry a pressure of 100 lb/sq in. Investigate the three types of end construction shown, for local bending stresses at the discontinuity. Assume that at the joint only shear forces exist and no bending moments. Also assume that in (b) and (c) the radial gap is shared equally between the cylinder and the head. At (a) the end is attached to a "solid" foundation, (b) is a hemispherical end, and (c) a head of constant stress (page 84).



PROBLEM 107.

108. A long cylindrical shell supported as shown is raised in temperature by T degrees. Show that the maximum local bending stresses due to this expansion are given by $s = 0.588\alpha TE$, where α is the linear coefficient of thermal expansion. Thus the stresses for a given material depend only on the temperature rise and not on the shell dimensions.



PROBLEM 108.

109. A cylindrical shell with no constraints has a radial temperature difference ΔT across the wall thickness, varying linearly across the wall.

a. Show that the maximum bending stresses away from the ends due to this temperature variation are $(\alpha E\Delta T)/2(1 - \mu)$ in both the axial and tangential directions.

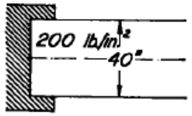
b. At the ends of the shell there are no moments. Hence the condition at the end is obtained by superimposing an end moment opposite in sign to that given by the stresses in (a). Using this, show that the increase in radius at the end is

$$u = \frac{2\beta^2}{k}\,\frac{\alpha E\,\Delta T\,t^2}{12(1-\mu)}$$

c. Find the tangential stress at the ends from (b), using the condition $u = (r/E)(s_t - \mu s_{\text{axial}})$, and to this add the stress from (a) to show that the maximum stress at the ends is 25 per cent greater than the stress at a considerable distance from the ends.

110. A steel tank of 40 ft diameter and 30 ft height is full of oil of specific gravity 0.9. The upper half of the tank is made from ¼-in. plate and the lower half from ½-in. plate. What are the values of the moment and shear force at the discontinuity?

111. A long steel pipe of 40 in. diameter and ½ in. wall thickness carries water under 200 lb/sq in. pressure. At the joints spaced "far" apart it can be considered built in, *i.e.*, no expansion or rotation occurs there. What are the longitudinal and tangential stresses at the inside and outside of the pipe at these joints?



PROBLEM 111.

112. The theory of bending of beams is governed by the second-order differential equation $EIy'' = M$, while for torsion we have the much simpler first-order equation $GI_p\theta' = -M_t$. The theory of *torsion* of a bar embedded in an elastic foundation is likewise much simpler than that of bending. Develop such a theory, and carry it to the point of finding equations corresponding to Eqs. (84) to (89) and a result corresponding to Fig. 94.

Cap VI (The Energy Method)

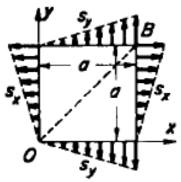
126. A square block of side a has tensile stresses in it described by $s_x = Cy$, $s_y = Cx$, and possibly some shear stresses in addition.

a. Find the stress function by integration.

b. Find the most general shear stresses which can be associated with these tensile stresses.

c. Find the displacement functions u and v, proceeding as indicated in Prob. 125.

d. Find the extension of the diagonal OB.



PROBLEM 126.

127. Using the method of Prob. 125 and the stresses obtained for Fig. 120 (page 182), show that the deflection at the center of a simply supported beam carrying uniform load w per unit length is

$$\delta_0\left[1 + \frac{3}{5}\frac{h^2}{l^2}\left(\frac{4}{5} + \frac{\mu}{2}\right)\right]$$

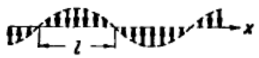
where $2l$ is the length of the beam, h the height and δ_0 the deflection given by strength of materials.

128. The stresses and deflections in rectangular beams with arbitrary load distributions along the upper or lower edge could be obtained by building up stress functions in the form of polynomials, which is mainly a matter of trial and error. A better method consists in the use of Fourier series, and for this we require the stress function for a beam with sinusoidal loading as shown in the figure. Assuming that

$$\Phi = \sin\frac{\pi x}{l}\,f(y)$$

where $f(y)$ is as yet an unknown function, show that in order to satisfy compatibility we must have

$$\Phi = \sin\frac{\pi x}{l}\left(C_1\cosh\frac{\pi y}{l} + C_2\sinh\frac{\pi y}{l} + C_3 y\cosh\frac{\pi y}{l} + C_4 y\sinh\frac{\pi y}{l}\right)$$



PROBLEM 128.

129. *a.* Apply the stress function of Prob. 128 to the case of a flat thin beam of infinite length and of height h, subjected to equal sinusoidal load distributions on both of its long sides, of half wave length l, as indicated. Solve the four integration constants by satisfying the conditions of sinusoidal normal stress and zero shear stress on these faces, and prove that the result for the normal stress *in the center line* of the beam is

$$(s_y)_{y=0} = 2A\,\frac{\dfrac{\pi h}{2l}\cosh\dfrac{\pi h}{2l} + \sin\dfrac{\pi h}{2l}}{\sinh\dfrac{\pi h}{l} + \dfrac{\pi h}{l}}$$

b. Plot the ratio $(s_y)_{y=0}/(s_y)_{y=h/2} = (s_y)_{y=0}/A$ against the ratio l/h.

Reismann, "Elasticity Theory and Applications
Pp 33-40 (Mathematical Preliminaries)

EXERCISES

1.1 (a) Write the following equation in conventional notation:

$$T_{ij,j}+F_i=0.$$

(b) Show that these equations are form-invariant under orthogonal rotations of the coordinate system.

1.2 (a) Write the following equation in conventional notation:

$$2E_{ij}=U_{i,j}-U_{j,i}+U_{k,i}U_{k,j}.$$

(b) Show that these equations are form-invariant under orthogonal rotations of the coordinate system.

1.3 Give the unabridged form of each expression listed below. If the indicial notation is used incorrectly, explain why.

(a) $A_i=B_i$

(b) $F_i=G_i+H_{ji}A_j$

(c) $U_i=V_j$

(d) $A_i=B_i+C_iD_i$

(e) $F_i=A_i+B_{ij}C_jD_j$

(f) $\phi=\dfrac{\partial F_i}{\partial x_i}$

(g) $d=\sqrt{x_ix_i}$

(h) $K=\delta_{ij}A_iB_j$.

1.4 Prove the following identities

(a) $\delta_{ii}=3$

(b) $\delta_{ij}C_{ij}=C_{ii}$

(c) $A_{ij}e_{ijk}=0$ if $A_{ij}=A_{ji}$

(d) $A_{ij}B_{kj}\delta_{ik}=A_{ij}B_{ij}$.

1.5 (a) If $f=f(x_1,x_2,x_3)$, show that $df/dt=(\partial f/\partial x_i)(dx_i/dt)$.

(b) Expand the double sum $S=A_{ij}x_ix_j$.

(c) Show that $(a_ix_i)_{,p}=a_p$.

(d) Show that $\delta_{ip}\delta_{jq}a_{pq}=a_{ij}$.

1.6 Let $S=a_{ij}x_ix_j\equiv 0$ for all values of the variables x_1,x_2,x_3. Show that $a_{ij}=-a_{ji}$.

1.7 Let $S=a_{ijk}x_ix_jx_k\equiv 0$ for all values of the variables x_1,x_2,x_3. Show that

$$a_{ijk}+a_{kij}+a_{jki}+a_{jik}+a_{kji}+a_{ikj}=0.$$

1.8 Expand the determinants in 1.6 and show that they are equal to e_{ijk}. Enumerate all possible cases.

1.9 (a) Show that $e_{ijk}a_{1i}a_{2j}a_{3k}$ is the expansion of the determinant $|a_{ij}|$ by rows.

(b) Show that $e_{ijk}a_{i1}a_{j2}a_{k3}$ is the expansion of the determinant $|a_{ij}|$ by columns.

1.10 What are the cofactors A_{ij} of each element of the determinant $|a_{ij}|\equiv a$? Give an explicit answer using conventional notation.

1.11 If $|A_{ij}|\equiv A$ is the determinant of the cofactors of the 3×3 determinant $|a_{ij}|\equiv a$, show that $A=a^2$.

1.12 Show that

$$e_{ijk}e_{pqr}a_{pqr}=a_{ijk}+a_{kij}+a_{jki}-a_{jik}-a_{kji}-a_{ikj}.$$

1.13 (a) Let

$$J=\left|\frac{\partial x_i}{\partial y_j}\right|=\begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_3}{\partial y_1}\\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_2}\\ \frac{\partial x_1}{\partial y_3} & \frac{\partial x_2}{\partial y_3} & \frac{\partial x_3}{\partial y_3}\end{vmatrix}.$$

Show that

$$J=e_{pqr}\frac{\partial x_p}{\partial y_1}\frac{\partial x_q}{\partial y_2}\frac{\partial x_r}{\partial y_3}=e_{ijk}\frac{\partial x_1}{\partial y_i}\frac{\partial x_2}{\partial y_j}\frac{\partial x_3}{\partial y_k}$$

and

$$e_{ijk}J=e_{pqr}\frac{\partial x_p}{\partial y_i}\frac{\partial x_q}{\partial y_j}\frac{\partial x_r}{\partial y_k}.$$

(b) Show that $a_{mk}A_{ml}=a\delta_{kl}$, where $a=|a_{ij}|$ and A_{ij} is the cofactor of a_{ij} (see 1.17b).

1.14 Define the two-dimensional Kronecker delta $\delta_{\alpha\beta}=1$ for $\alpha=\beta$ and $\delta_{\alpha\beta}=0$ for $\alpha\neq\beta$. Also define the two-dimensional alternator $e_{\alpha\beta}=\beta-\alpha$, where $\alpha=1,2$ and $\beta=1,2$, that is, $e_{11}=e_{22}=0$, $e_{12}=-e_{21}=1$. Show that

$$e_{\alpha\beta}=\begin{vmatrix}\delta_{\alpha1} & \delta_{\alpha2}\\ \delta_{\beta1} & \delta_{\beta2}\end{vmatrix}=\begin{vmatrix}\delta_{\alpha1} & \delta_{\beta1}\\ \delta_{\alpha2} & \delta_{\beta2}\end{vmatrix}$$

$$e_{\alpha\beta}e_{\gamma\eta}=\begin{vmatrix}\delta_{\alpha\gamma} & \delta_{\alpha\eta}\\ \delta_{\beta\gamma} & \delta_{\beta\eta}\end{vmatrix}$$

$$e_{\alpha\gamma}e_{\beta\gamma}=\delta_{\alpha\beta},\quad e_{\alpha\beta}e_{\alpha\beta}=2$$

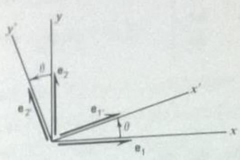
$$a=\det(a_{\alpha\beta})=e_{\alpha\beta}a_{1\alpha}a_{2\beta}=e_{\alpha\beta}a_{\alpha1}a_{\beta2}$$

$$ae_{\gamma\eta}=e_{\alpha\beta}a_{\alpha\gamma}a_{\beta\eta}=e_{\alpha\beta}a_{\gamma\alpha}a_{\eta\beta}$$

$$a=\tfrac{1}{2}e_{\alpha\beta}e_{\gamma\eta}a_{\alpha\gamma}a_{\beta\eta}.$$

1.15 Demonstrate the rule for the multiplication of 2×2 determinants. Use the notation of Exercise 1.14.

1.16 Consider a rotation of axes in the x-y plane as shown in the figure. Show that $\mathbf{e}_{\beta'}=a_{\alpha\beta'}\mathbf{e}_\alpha$, $\mathbf{e}_\beta=a_{\alpha'\beta}\mathbf{e}_{\alpha'}$ where $a_{\alpha\beta'}=a_{\beta'\alpha}=\cos(\mathbf{e}_\alpha,\mathbf{e}_{\beta'})$. Show that $a_{1'1}=a_{2'2}=\cos\theta$, $a_{1'2}=-a_{2'1}=\sin\theta$, and $a=\det(a_{\alpha\beta'})=1$.



Exercise 1.16

1.17 With reference to Exercise 1.14 and 1.16, show that $\delta_{\alpha\beta}=\delta_{\beta\alpha}$ and $e_{\alpha\beta}=-e_{\beta\alpha}$ are Cartesian tensors of order two in a two-dimensional space.

1.18 Show that in a two-dimensional space, there is a unique scalar $T=\frac{1}{2}e_{\alpha\beta}T_{\alpha\beta}$ associated with every skew-symmetric tensor of order two, so that $T_{\alpha\beta}=e_{\alpha\beta}T$ (see Exercises 1.14, 1.16, and 1.17).

1.19 (a) Use the method of Section 1.4 to show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=\begin{vmatrix}\mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D}\\ \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D}\end{vmatrix}.$$

(b) Show that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\cdot\mathbf{A})\mathbf{B}-(\mathbf{B}\cdot\mathbf{A})\mathbf{C}. \tag{1.33b}$$

1.20 If $\mathbf{D}=\mathbf{A}\times\mathbf{B}$, show that

$$D_k=e_{ijk}A_iB_j=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{vmatrix}.$$

1.21 Consider two RCS's, one primed, the other unprimed. If the direction cosines characterizing the relative rotation of the two coordinate systems are $a_{i'k}=\cos(\mathbf{e}_{i'},\mathbf{e}_k)$ (see Fig. 1.4), show that $x_k=x_{i'}a_{i'k}$. Also show that $\mathbf{e}_j=a_{i'j}\mathbf{e}_{i'}$.

Pp 75-82 (Stress)

1.22 Two coordinate systems are connected by the direction cosines $a_{i'j}$, where

$$[a_{i'j}] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

The unprimed reference is rectangular, Cartesian, and right-handed.

(a) Are the primed coordinate axes mutually orthogonal?

(b) Is the primed reference right-handed or left-handed?

1.23 Given a vector A_i and two second order tensors B_{ij} and C_{ij}, prove that:

(a) $F_{ij} = B_{ij} + C_{ij}$ is a second order tensor.

(b) $H_{ijk} = A_i B_{jk}$ is a third order tensor.

(c) B_{ii} is a scalar.

(d) $H_{ij} = A_i B_{ji}$ is a vector.

1.24 **(a)** Given that $T_{ij} N_i N_j = S$, where S is a scalar and N_i is an arbitrary vector, show that T_{ij} is a symmetric, second order tensor.

(b) Given that $T_{ij} A_i B_j = S$, where S is a scalar and A_i and B_j are independent and arbitrary vectors. Show that T_{ij} is a tensor of order two.

(c) Given the equation $T_{ijk} A_i B_j C_k = S$, where S is a scalar and A_i, B_j and C_k are independent and arbitrary vectors. Prove that T_{ijk} is a tensor of order three.

1.25 If λ is a root of the equation $|T_{ij} - \lambda A_{ij}| = 0$, show that λ is also a root of the equation $|T_{p'q'} - \lambda A_{p'q'}| = 0$, provided $T_{p'q'} = a_{p'i} a_{q'j} T_{ij}$ and $A_{p'q'} = a_{p'i} a_{q'j} A_{ij}$.

1.26 Consider the nine scalar quantities $I_{ij} = \int_v (r^2 \delta_{ij} - x_i x_j)\, dm$, where x_i are Cartesian coordinates, $r^2 = x_i x_i$, and the integration extends over the volume V of a rigid body with element of mass dm. Prove that I_{ij} is a symmetric tensor of order two, known as the inertia tensor in rigid body mechanics.

1.27 Show that the relation $T_{ij} = T_{ji}$ implies $T_{p'q'} = T_{q'p'}$ if T_{ij} is a tensor of order two. The primed RCS is obtained from the unprimed RCS by an orthogonal rotation.

1.28 Given the equation $T_{ij} = C_{ijkl} E_{kl}$ where $T_{ij} = T_{ji}$ and $E_{kl} = E_{lk}$ are symmetric tensors of order two. Show that the 81 scalars C_{ijkl} are the components of a tensor of order four with the symmetry properties $C_{ijkl} = C_{jikl} = C_{ijlk}$.

1.29 An arbitrary second order tensor can always be decomposed into the sum of a symmetric and a skew-symmetric tensor. Prove that this decomposition is unique.

1.30—*see Section 2.4.* Consider the central quadric $c_{ij} x_i x_j = K$, where $c_{ij} = c_{ji}$, x_i is a vector from the origin to the surface of the quadric, and K is a scalar. Show that c_{ij} is a symmetric tensor of order two. Set $n_i = x_i / |\mathbf{r}|$, so that $n_i n_i = 1$. Then $c_{ij} n_i n_j = K / r^2 = S$. Now extremize S subject to the constraint $n_i n_i = 1$. To facilitate a symmetrical calculation, use the method of Lagrange multipliers (see, for instance, C. R. Wylie, *Advanced Engineering Mathematics*, 4th ed., McGraw-Hill Book Co., New York, 1975, p. 595). Form the function $F = S - \lambda n_i n_i = (c_{ij} - \lambda \delta_{ij}) n_i n_j$ where λ is the Lagrange multiplier. Then

$$\frac{\partial F}{\partial n_j} = (c_{ij} - \lambda \delta_{ij}) n_i = 0$$

or

$$\begin{bmatrix} c_{11} - \lambda & c_{12} & c_{13} \\ c_{21} & c_{22} - \lambda & c_{23} \\ c_{31} & c_{32} & c_{33} - \lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

Since $|\mathbf{n}| = 1 \neq 0$, set $|c_{ij}| = 0$, so that $\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$, where $I_1 = c_{ii}$, $I_2 = \frac{1}{2}(I_1^2 - c_{ij} c_{ij})$, $I_3 = \frac{1}{3}(3 I_1 I_2 - I_1^3 + c_{ij} c_{jk} c_{ki})$. Show that $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}$ are real, and that the corresponding $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$ are mutually orthogonal (see Section 2.4). Assume that the eigenvalues are distinct and ordered: $\lambda_{(1)} > \lambda_{(2)} > \lambda_{(3)}$. Arrange for the basis $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)})$ to be right-handed, that is, require that $\mathbf{n}^{(1)} \times \mathbf{n}^{(2)} = \mathbf{n}^{(3)}$. Now label the RCS spanned by $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$ as $x_{1'}, x_{2'}, x_{3'}$, respectively, so that $x_{p'} = a_{p'i} x_i$ and $\mathbf{n}^{(1)} \equiv \mathbf{e}_{1'}, \mathbf{n}^{(2)} \equiv \mathbf{e}_{2'}, \mathbf{n}^{(3)} \equiv \mathbf{e}_{3'}$. The direction cosines are $a_{p'i} = \mathbf{n}^{(p)} \cdot \mathbf{e}_i = \cos(x_{p'}, x_i)$. Show that $n_{i'}^{(p)} = 1$ for $i' = p$ and $n_{i'}^{(p)} = 0$ for $i' \neq p$. Since $\lambda_{(i)} = c_{p'q'} n_{p'}^{(i)} n_{q'}^{(i)}$, show that $\lambda_{(1)} = c_{1'1'}$, $\lambda_{(2)} = c_{2'2'}$, $\lambda_{(3)} = c_{3'3'}$, and

EXERCISES

2.1 A solid body is known to be in a state of static equilibrium. Determine the body force vector $\mathbf{F}$ at $(x_1, x_2, x_3) = (1, 1, 1)$ if the stress field is given by

$$\tau_{11} = 10x_1^3 + x_2^2 \qquad \tau_{12} = x_3^2$$
$$\tau_{22} = 20x_1^3 + 100 \qquad \tau_{23} = x_2$$
$$\tau_{33} = 30x_2^2 + 10x_3^3 \qquad \tau_{31} = x_1^2.$$

2.2 At a point P in a solid, the stress tensor components referred to (x_1, x_2, x_3) axes are

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{bmatrix}.$$

(a) Find the stress vector $\mathbf{T}(\mathbf{n})$ acting on a plane through P with (outer) unit normal vector $\mathbf{n} = \frac{1}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2 + (1/\sqrt{2})\mathbf{e}_3$.

(b) Find $|\mathbf{T}(\mathbf{n})|$, the magnitude of the stress vector.

(c) Determine the angle between $\mathbf{T}(\mathbf{n})$ and $\mathbf{n}$.

(d) Determine N, the component of $\mathbf{T}(\mathbf{n})$ parallel to $\mathbf{n}$ (normal component).

(e) Determine S, the component of $\mathbf{T}(\mathbf{n})$ perpendicular to $\mathbf{n}$ (shear component).

2.3 Consider a stress field characterized by

$$[\tau_{ij}] = \begin{bmatrix} \tau_{11}(x_1, x_2) & \tau_{12}(x_1, x_2) & 0 \\ \tau_{12}(x_1, x_2) & \tau_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Derive the stress equations of equilibrium for the case of plane stress (see Section 5.6).

(b) Introduce the scalar function $\Phi(x_1, x_2)$ such that

$$\tau_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \qquad \tau_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}, \qquad \tau_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}.$$

Show that in this case the stress equations of equilibrium are satisfied identically provided the body force vector vanishes.

2.4 At a point P in the interior of a solid we are given the stress tensor components

$$\tau_{ij} = \begin{bmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix}$$

relative to unprimed axes. Use the stress transformation law $\tau_{p'q'} = a_{p'i} a_{q'j} \tau_{ij}$ to show that in this case the components of the stress tensor do not change with coordinate system rotations, that is, the tensor $\delta_{ij}\tau$ has the same components in all rectangular coordinate systems. The state of stress $\tau_{ij} = \tau\delta_{ij}$ is called hydrostatic tension or compression for $\tau > 0$ and $\tau < 0$, respectively.

2.5 The state of stress at a point P is given by $\tau_{ij} = -p\delta_{ij}$, where p is a scalar (hydrostatic compression). Show that the shearing stress at P vanishes on any plane through P.

2.6 Consider the case of "simple shear," in other words, the case where the only nonvanishing stress components are a single pair of shearing stresses. Select the Cartesian coordinate system such that

$$[\tau_{ij}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the magnitude and orientation of the principal stresses.

(b) Find the extreme shearing stresses and the planes on which they act.

2.7 At a point P of a solid the stress vectors $\mathbf{T}(\mathbf{n})$ and $\mathbf{T}(\mathbf{n}')$ act on the planes defined by the unit normal vectors $\mathbf{n}$ and $\mathbf{n}'$, respectively. Show that the component $\mathbf{T}(\mathbf{n})$ in the direction $\mathbf{n}'$ is equal to the component of $\mathbf{T}(\mathbf{n}')$ in the direction of $\mathbf{n}$.

2.8 Consider the stress field $\tau_{ij} = -p\delta_{ij}$, where the scalar $p = p(x_1, x_2, x_3)$. Show that the stress equations of equilibrium, in this case, assume the form

$$-\operatorname{grad} p + \mathbf{F} = \mathbf{0}$$

where $(\operatorname{grad} p)_i \equiv p_{,i} = \partial p / \partial x_i$.

2.9 Given the stress tensor

$$[\tau_{ij}] = \begin{bmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix}.$$

Prove the invariance of $\tau_{11} + \tau_{22}$ and of $\frac{1}{4}(\tau_{11} - \tau_{22})^2 + (\tau_{12})^2$ with respect to (orthogonal) rotations of the coordinate system.

2.10 Use (2.15) or (2.16) and the quotient law to prove that τ_{ij} is a tensor of order two.

2.11 Derive equations (2.21b) and (2.22) by expanding the determinant in (2.21a).

2.12 Prove that I_1, I_2 and I_3 in (2.22) are invariants with respect to axis rotations, given that $\tau'_{ij} = a_{ip}a_{jq}\tau_{pq}$. Use direct computation.

2.13 Write (2.10) in expanded notation, that is, show that (2.10) is equivalent to three equations of the type

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0. \tag{2.10a}$$

2.14 Write $(\mathbf{e}_i \times \mathbf{e}_j)\tau_{ij} = \mathbf{0}$ in unabridged notation, and show that it implies that $\tau_{ij} = \tau_{ji}$.

2.15 Write $T_k = n_i \tau_{ik}$ in unabridged notation. Compare with (2.14).

2.16 Demonstrate that the sum of the squares of the magnitudes of the stress vectors which act on mutually perpendicular planes at a point P of a solid is an invariant. In other words, show that

$$\mathbf{T}(\mathbf{e}_1)\cdot\mathbf{T}(\mathbf{e}_1) + \mathbf{T}(\mathbf{e}_2)\cdot\mathbf{T}(\mathbf{e}_2) + \mathbf{T}(\mathbf{e}_3)\cdot\mathbf{T}(\mathbf{e}_3) = I_1^2 - 2I_2$$

for any choice of the orthonormal basis system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

2.17 At a point P in a solid, the components of the stress tensor referred to x,y,z axes are given by

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} 5 & -3 & 8 \\ -3 & 7 & 1 \\ 8 & 1 & 9 \end{bmatrix} \times 10^7 \text{ pascals}.$$

Find:

(a) the three principal stresses at P.

(b) the orientation of the three principal axes at point P with respect to the (x,y,z) axes by specifying all direction cosines of the principal axes. The principal axes should form a right-handed triad.

(c) the octahedral shear and normal stresses.

(d) the orientation of the octahedral plane with respect to the x,y,z axes by specifying the direction cosines of the outward normal to the plane.

(e) the spherical and deviator parts of the given stress tensor.

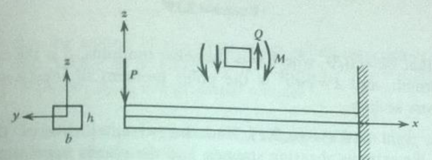
(f) the three stress invariants of the stress tensor and of the deviator tensor.

(g) the stress vector $\mathbf{T}(\mathbf{n})$ acting on a plane characterized by the outward unit normal vector $\mathbf{n} = (1/\sqrt{3})(\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z)$.

(h) the normal and shear stress components of $\mathbf{T}$ acting on the plane defined in **(g)**.

2.18 In the case of elementary beam theory, the stress field in the beam shown in the figure is given by $\tau_{xy} = \tau_{zz} = \tau_{yy} = \tau_{yz} = 0$.

$$\tau_{xx} = \frac{Mz}{I}, \qquad \tau_{xz} = \frac{6P}{bh^3}\left[\left(\frac{h}{2}\right)^2 - z^2\right].$$



Exercise 2.18

(a) Show that

$$M = Px = \int_A \tau_{xx} z \, dA$$

$$Q = P = \int_A \tau_{xz} \, dA$$

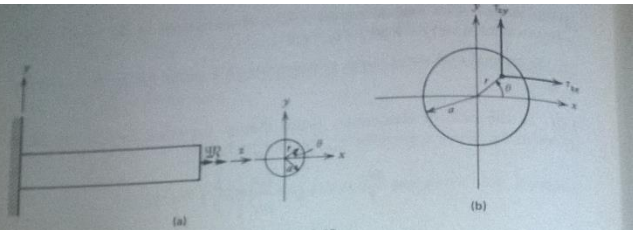
$$I = \tfrac{1}{12} bh^3.$$

(b) Show that the stress equations of equilibrium (2.10) are satisfied, provided body forces are neglected.

2.19 Consider the shaft subjected to torsion as shown in figure (*a*). The resulting torque $\mathfrak{M}$ on any cross section is given by $\mathfrak{M} = \int_A r\tau_{z\theta}\, dA$. It can be shown that the stress tensor components in the shaft (see figure (*b*)) are given by

$$[\tau_{ij}] = \begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix}.$$

We have $\tau_{zy} = \tau_{z\theta}\cos\theta$ and $-\tau_{zx} = \tau_{z\theta}\sin\theta$ so that $\tau_{z\theta} = G\beta r$.



Exercise 2.19

(a) Show that $\mathfrak{M} = G\beta J$, where G is the shear modulus, β is the angle of twist per unit length, and $J = \frac{1}{2}\pi a^4$ is the polar moment of inertia of the plane circular cross section.

(b) At the point $(r, \theta, z) = (a, 0, z)$, find the principal stresses, the principal directions, the extreme shearing stresses, and the planes upon which they act.

(c) Show that the stress field in the shaft satisfies the stress equations of equilibrium (2.10) provided body forces are neglected.

2.20 Show that the entries in Table 2.1 are solutions of (2.29). Give a full discussion.

2.21 Solve (2.34) for n_1^2, n_2^2, and n_3^2, that is, prove (2.35).

2.22 Show that (2.36) can be obtained from (2.35).

2.23 Eliminate $(2\alpha - 2\theta)$ from (2.38) and show that $(N - H)^2 + S^2 = R^2$.

2.24 **(a)** Given the stress deviator (2.47). Show that its principal directions are those of the stress tensor (2.21a). Show that its principal values are solutions of the cubic equation (2.50).

(b) Show that

$$D_3 = \tfrac{1}{3}\left(s_1^3 + s_2^3 + s_3^3\right) = s_1 s_2 s_3 \tag{2.51c}$$

where s_1, s_2, s_3 are the principal values of the stress deviator.

2.25 Express the invariants of the deviator in terms of the stress invariants, that is, prove (2.52).

2.26 By direct substitution, show that (2.53) is the solution of the cubic equation (2.50).

Pp 113-125 (Deformation and Strain)

2.27 Use (2.56), (2.55b) and (2.33) to establish the bound (2.57):

$$1 \le \frac{\sqrt{\frac{3}{2}}\,\tau_0}{\tau_{max}} \le \frac{2}{\sqrt{3}}.$$

2.28 Use (2.57) to show that

$$\sqrt{\tfrac{3}{2}}\,\tau_0 \simeq 1.08\tau_{max}.$$

What is the greatest error in this approximation?

2.29 Prove that

$$9\tau_0^2 = 2I_1^2 - 6I_2.$$

2.30 Show that the square of the shear stress acting on a plane defined by the normal unit vector $\mathbf{n}$ is given by $S^2 = n_1^2 n_2^2(\sigma_1-\sigma_2)^2 + n_2^2 n_3^2(\sigma_2-\sigma_3)^2 + n_3^2 n_1^2(\sigma_3-\sigma_1)^2$ where $\sigma_1, \sigma_2, \sigma_3$ are principal stresses.

2.31 Given the principal axis system (X, Y, Z), specify the eight unit normal vectors which define the octahedral planes.

2.32 Show that (2.41) is valid for each of the eight octahedral planes. Use the results of Exercise 2.31.

2.33 Substitute (2.28) into (2.42), and use (2.43). Show that the average value of the square of all possible shear stresses at a point is related to the octahedral shear stress by $\langle S^2 \rangle = \frac{3}{5}(\tau_0)^2$.

2.34 Show that the normal stress acting on the octahedral plane is proportional to the first stress invariant I_1, that is, show that

$$\sigma_0 = \tfrac{1}{3}(\sigma_1+\sigma_2+\sigma_3).$$

2.35 With the aid of (2.63) and (2.61), reduce (2.62) to (2.66).

2.36 Substitute (2.65) into (2.66), and apply (2.61). Show that (2.67) results.

2.37 Substitute (2.65) into (2.68), and prove the result (2.69).

2.38 Use (2.58), (2.72), and (2.73) to establish the validity of (2.74).

2.39 Provide definitions, diagrams, and analysis to establish the validity o (2.76).

2.40 Use (2.60) to prove (2.61).

2.41 Use (2.71) to prove (2.72).

2.42 If $\nabla = \mathbf{e}_i\,\partial/\partial x_i$ show that the operator $\nabla\cdot\mathbf{e}_i + \mathbf{e}_i\cdot\nabla = \partial/\partial x_i$ for Cartesian coordinates. Hence show that (2.7) can be derived from (2.58).

2.43 Show how (2.77) is obtained from (2.74).

2.44 Consider the transformation from oblate spheroidal coordinates (u, v, ϕ) to Cartesian coordinates

$$x = a\cosh u \cos v \cos\phi$$
$$y = a\cosh u \cos v \sin\phi$$
$$z = a\sinh u \sin v$$

where a is a constant.

(a) Discuss the coordinate surfaces.

(b) Derive the stress equations of equilibrium referred to oblate spheroidal coordinates (u, v, ϕ).

2.45 Consider the transformation from parabolic coordinates (ξ, η, ϕ) to Cartesian coordinates $x = \xi\eta\cos\phi$, $y = \xi\eta\sin\phi$, $z = \frac{1}{2}(\eta^2 - \xi^2)$.

(a) Discuss the coordinate surfaces.

(b) Derive the stress equations of equilibrium referred to parabolic coordinates (ξ, η, ϕ).

2.46 (a) Use the identity (1.56d) to show that

$$A_{i,i} = (\nabla\cdot\mathbf{e}_i + \mathbf{e}_i\cdot\nabla)A_i = \operatorname{div}\mathbf{A}.$$

(b) Show that

$$\mathbf{B}_{i,i} = (\nabla\cdot\mathbf{e}_i + \mathbf{e}_i\cdot\nabla)\mathbf{B}_i$$

where $\mathbf{e}_i$, $i = 1,2,3$, are mutually perpendicular unit vectors.

2.47 What is the average value of the normal stress for all possible orientations of the vector $\mathbf{n}$? That is, show that

$$\langle N \rangle = \frac{\int N\,d\Omega}{\int d\Omega} = \tfrac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \tfrac{1}{3}I_1$$

(see Section 2.8).

3.1 Repeat the analysis extending from (3.3) through (3.23) for the spatial (Eulerian) description, and establish the validity of (3.26) through (3.29).

3.2 Expand (3.19), that is, write it in conventional notation.

3.3 Expand (3.28), in other words, write it in conventional notation.

3.4 With reference to Fig. 3.3, show that

$$|d\mathbf{R}|^2 - |d\mathbf{r}|^2 = 2L_{ij}\,da_i\,da_j = 2d\mathbf{u}\cdot d\mathbf{r} + d\mathbf{u}\cdot d\mathbf{u}$$
$$|d\mathbf{R}|^2 - |d\mathbf{r}|^2 = 2E_{ij}\,dx_i\,dx_j = 2d\mathbf{u}\cdot d\mathbf{R} - d\mathbf{u}\cdot d\mathbf{u}$$

where L_{ij} and E_{ij} are the Lagrangian and Eulerian strain tensors, respectively.

3.5 Use (3.5), (3.16), and the relation $\mathbf{u} = \mathbf{R} - \mathbf{r} = \mathbf{e}_k(x_k - a_k)$ to show that

$$\mathbf{G}_i = \mathbf{e}_i + \mathbf{D}_i = \mathbf{e}_i + \frac{\partial\mathbf{u}}{\partial a_i} = \mathbf{e}_k\frac{\partial x_k}{\partial a_i}. \tag{3.30}$$

3.6 With reference to Exercises 3.1 and 3.11, show that $\mathbf{g}_i = \mathbf{e}_k \dfrac{\partial a_k}{\partial x_i}$.

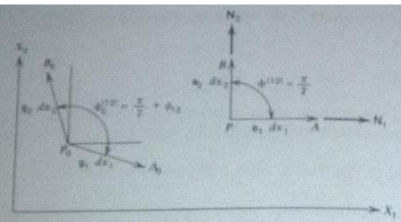
3.7 Prove that strain is a tensor of order two. Do not use the quotient rule.

3.8 Show that $2L_{ij} = G_{ij} - \delta_{ij}$, where L_{ij} is the Lagrangian strain tensor and G_{ij} is Green's deformation tensor $G_{ij} = (\partial x_k/\partial a_i)(\partial x_k/\partial a_j)$.

3.9 Show that $2E_{ij} = \delta_{ij} - g_{ij}$, where E_{ij} is the Eulerian strain tensor and g_{ij} is Cauchy's deformation tensor $g_{ij} = (\partial a_k/\partial x_i)(\partial a_k/\partial x_j)$.

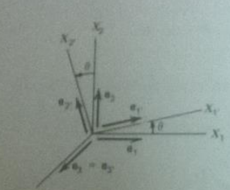
3.10 Analogous to (3.24), show that $da_i = dx_i - (\partial u_i/\partial x_j)dx_j = (\delta_{ij} - E_{ij} - \lambda_{ij})dx_j$ where E_{ij} is the Eulerian strain tensor (3.28) and where $\lambda_{ij} = \frac{1}{2}[(\partial u_i/\partial x_j) - (\partial u_j/\partial x_i) + (\partial u_k/\partial x_i)(\partial u_k/\partial x_j)]$ is the Eulerian rotation tensor.

3.11 Section 3.3 provides a geometrical and physical interpretation of the Lagrangian strain tensor components L_{ij}. With reference to the figure and to Exercises 3.1 and 3.16, provide a similar interpretation of the Eulerian strain tensor components E_{ij}.



Exercise 3.11

3.12 Consider a rotation of axes about the X_3 axis as shown in the figure.



Exercise 3.12

(a) Show that the table of direction cosines $a_{i'j} = \cos(\mathbf{e}_{i'}, \mathbf{e}_j) = \mathbf{e}_{i'}\cdot\mathbf{e}_j$ is

	X_1	X_2	X_3
$X_{1'}$	$\cos\theta$	$\sin\theta$	0
$X_{2'}$	$-\sin\theta$	$\cos\theta$	0
$X_{3'}$	0	0	1

(b) Show that in this case the equations of strain transformation (3.20) assume the form

$$L_{1'1'} = \tfrac{1}{2}(L_{11}+L_{22}) + \tfrac{1}{2}(L_{11}-L_{22})\cos 2\theta + L_{12}\sin 2\theta$$
$$L_{2'2'} = \tfrac{1}{2}(L_{11}+L_{22}) - \tfrac{1}{2}(L_{11}-L_{22})\cos 2\theta - L_{12}\sin 2\theta$$
$$L_{2'3'} = L_{23}\cos\theta - L_{13}\sin\theta$$
$$L_{3'1'} = L_{31}\cos\theta + L_{32}\sin\theta$$
$$L_{1'2'} = \tfrac{1}{2}(L_{22}-L_{11})\sin 2\theta + L_{12}\cos 2\theta$$
$$L_{3'3'} = L_{33}.$$

(c) Discuss the Mohr circle construction and its relation to the present transformation.

3.13 Discuss the concept of convected coordinates in the spatial description. Use the $\mathbf{g}_k$, $k=1,2,3$ as the local basis vectors in this case (see Exercise 3.6).

3.14 Use (3.15) and (3.18) to show the validity of (3.33a), that is, show that

$$\cos\Phi^{(12)}=\cos\left(\frac{\pi}{2}-\phi_{12}\right)=\sin\phi_{12}=\frac{2L_{12}}{(1+M_1)(1+M_2)}. \tag{3.33a}$$

With reference to (3.29) and Exercise 3.11, show that in the Eulerian representation,

$$-\cos\Phi_0^{(12)}=\cos\left(\frac{\pi}{2}+\phi_{12}\right)=\sin\phi_{12}=\frac{2E_{12}}{(1-M_1^{(e)})(1-M_2^{(e)})}$$

where

$$M_1^{(e)}=\frac{|d\mathbf{R}_1|-|d\mathbf{r}_1|}{|d\mathbf{R}_1|}, \text{ and so on.}$$

3.15 Show that the invariants of the Eulerian strain tensor are given by

$$\mathcal{E}_1=E_{11}+E_{22}+E_{33}=E_1+E_2+E_3$$

$$\mathcal{E}_2=\begin{vmatrix}E_{11}&E_{12}\\E_{21}&E_{22}\end{vmatrix}+\begin{vmatrix}E_{22}&E_{23}\\E_{32}&E_{33}\end{vmatrix}+\begin{vmatrix}E_{33}&E_{31}\\E_{13}&E_{11}\end{vmatrix}$$
$$=E_1E_2+E_2E_3+E_3E_1$$

$$\mathcal{E}_3=\begin{vmatrix}E_{11}&E_{12}&E_{13}\\E_{21}&E_{22}&E_{23}\\E_{31}&E_{32}&E_{33}\end{vmatrix}=E_1E_2E_3$$

where E_1, E_2, E_3 are principal normal strain components.

3.16 Express the Invariants $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ of the Lagrangian strain tensor (3. in terms of the invariants of the Eulerian strain tensor (Exercise 3.15), that show that

$$\mathcal{L}_1=\frac{\mathcal{E}_1-4\mathcal{E}_2+12\mathcal{E}_3}{1-2\mathcal{E}_1+4\mathcal{E}_2-8\mathcal{E}_3}$$

$$\mathcal{L}_2=\frac{\mathcal{E}_2-6\mathcal{E}_3}{1-2\mathcal{E}_1+4\mathcal{E}_2-8\mathcal{E}_3}$$

$$\mathcal{L}_3=\frac{\mathcal{E}_3}{1-2\mathcal{E}_1+4\mathcal{E}_2-8\mathcal{E}_3}.$$

3.17 Express the invariants $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ of the Eulerian strain tensor (Exercise 3.15) in terms of the invariants of the Lagrangian strain tensor (3.23), that is, show that

$$\mathcal{E}_1=\frac{\mathcal{L}_1+4\mathcal{L}_2+12\mathcal{L}_3}{1+2\mathcal{L}_1+4\mathcal{L}_2+8\mathcal{L}_3}$$

$$\mathcal{E}_2=\frac{\mathcal{L}_2+6\mathcal{L}_3}{1+2\mathcal{L}_1+4\mathcal{L}_2+8\mathcal{L}_3}$$

$$\mathcal{E}_3=\frac{\mathcal{L}_3}{1+2\mathcal{L}_1+4\mathcal{L}_2+8\mathcal{L}_3}.$$

3.18 Show that $(1+2\mathcal{L}_1+4\mathcal{L}_2+8\mathcal{L}_3)(1-2\mathcal{E}_1+4\mathcal{E}_2-8\mathcal{E}_3)=1$ where $\mathcal{L}_i$, $\mathcal{E}_i$, $i=1,2,3$, are the invariants of the Lagrangian and Eulerian strain tensors, respectively (see Exercises 3.15, 3.16, and 3.17).

3.19 Use the results of Exercise 3.18 and use (3.47) to show that the volume dilatation is given by

$$\frac{dV}{dV_0}=\frac{\rho_0}{\rho}=J=\sqrt{G}=(1-2\mathcal{E}_1+4\mathcal{E}_2-8\mathcal{E}_3)^{-1/2}.$$

Show that in the case of small strain

$$\frac{dV}{dV_0}\simeq(1-2\mathcal{E}_1)^{-1/2}\simeq 1+\mathcal{E}_1\simeq 1.$$

3.20 Show that for the case of small displacement gradients we have

(a) $|\mathbf{G}_1|\simeq 1+\dfrac{\partial u_1}{\partial a_1}$

(b) $\dfrac{\mathbf{G}_1}{|\mathbf{G}_1|}\simeq\mathbf{e}_1+\dfrac{\partial u_2/\partial a_1}{(1+(\partial u_1/\partial a_1))}\mathbf{e}_2+\dfrac{\partial u_3/\partial a_1}{(1+(\partial u_1/\partial a_1))}\mathbf{e}_3$

(c) $|d\mathbf{R}_1|\simeq da_1+\dfrac{\partial u_1}{\partial a_1}da_1$.

Derive similar relations for $|\mathbf{G}_2|$, $|\mathbf{G}_3|$, $\dfrac{\mathbf{G}_2}{|\mathbf{G}_2|}$, $\dfrac{\mathbf{G}_3}{|\mathbf{G}_3|}$, $|d\mathbf{R}_2|$, and $|d\mathbf{R}_3|$.

3.21 Draw a diagram similar to Fig. 3.8. Show the projections of the undeformed and deformed elements (1) onto the X_2-X_3 plane and (2) onto the X_3-X_1 plane.

3.22 With reference to Section 3.4 and Exercises 3.1, 3.6, and 3.11, consider the case of small strain when using the spatial description. Show that for $|M_1^{(e)}|\ll 1$ and $|\phi_{12}|\ll 1$, and so on, we have $E_{11}\simeq M_1^{(e)}$, $E_{12}\simeq\frac{1}{2}\phi_{12}$, and so forth. Provide a geometrical interpretation of these results.

3.23 Consider the rotation (in space) of the vector $\mathbf{G}_2$. Construct a drawing similar to Fig. 3.7, and denote the angle between $\mathbf{G}_2$ and $\mathbf{e}_3$ by $(\pi/2)-\theta_{23}$. Denote the angle between $\mathbf{G}_2$ and $\mathbf{e}_1$ by $(\pi/2)-\theta_{21}$. Show that for sufficiently small rotations $|\theta_{21}|\ll 1$, $|\theta_{23}|\ll 1$, and for sufficiently small relative elongations $|M_2|\ll 1$, we have $\theta_{23}\simeq\partial u_3/\partial a_2$, $\theta_{21}\simeq\partial u_1/\partial a_2$.

3.24 Consider the rotation (in space) of the vector $\mathbf{G}_3$. Construct a drawing similar to Fig. 3.7, and denote the angle between $\mathbf{G}_3$ and $\mathbf{e}_1$ by $(\pi/2)-\theta_{31}$. Denote the angle between $\mathbf{G}_3$ and $\mathbf{e}_2$ by $(\pi/2)-\theta_{32}$. Show that for sufficiently small rotations $|\theta_{32}|\ll 1$, $|\theta_{31}|\ll 1$, and for sufficiently small relative elongations $|M_3|\ll 1$, we have

$$\theta_{31}\simeq\frac{\partial u_1}{\partial a_3},\qquad \theta_{32}\simeq\frac{\partial u_2}{\partial a_3}.$$

3.25 (a) Denote by θ_{11} the angle between $\mathbf{G}_1$ and $\mathbf{e}_1$. Show that for small rotations $|\theta_{11}|\ll 1$ we have $M_1\simeq\dfrac{\partial u_1}{\partial a_1}$. Hence, in the case of small strain and small rotations, we have $L_{11}\simeq M_1\simeq\partial u_1/\partial a_1$.

(b) Show that $L_{22}\simeq M_2\simeq\partial u_2/\partial a_2$, $L_{33}\simeq M_3\simeq\partial u_3/\partial a_3$ in the case of small strain and small rotations.

3.26 Prove (3.37), that is, show that

$$|d\mathbf{A}^{(1)}|=\sqrt{G_{22}G_{33}-G_{23}^2}\,|d\mathbf{A}_0^{(1)}|.$$

3.27 Show that

$$G_{22}G_{33}-G_{23}^2=1+2(L_{22}+L_{33})+4(L_{22}L_{33}-L_{23}^2),$$

that is, prove (3.39).

3.28 Show that $\mathbf{G}_1\cdot\mathbf{G}_2\times\mathbf{G}_3=J$, where J is given by (3.2).

3.29 Use the rule for multiplying determinants to show that

$$G=\det[G_{ij}]=\det\left(\frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j}\right)$$
$$=\left[\det\left(\frac{\partial x_i}{\partial a_j}\right)\right]^2=J^2,\qquad \sqrt{G}=J>0.$$
$$g=\det[g_{ij}]=\det\left(\frac{\partial a_k}{\partial x_i}\frac{\partial a_k}{\partial x_j}\right)$$
$$=\left[\det\left(\frac{\partial a_i}{\partial x_j}\right)\right]^2.$$

Show that

$$Gg=[\det(G_{ij})][\det(g_{ij})]=1$$

so that $g=G^{-1}=J^{-2}$ and $\sqrt{g}=J^{-1}$, $JJ^{-1}=1$.

3.30 Calculate the strain invariants and the volume dilatation at a point of a solid where the strain tensor components are given by

(a)

$$[L_{ij}]=\begin{bmatrix}-0.3&0.1&0\\0.1&0.2&0.3\\0&0.3&0\end{bmatrix}.$$

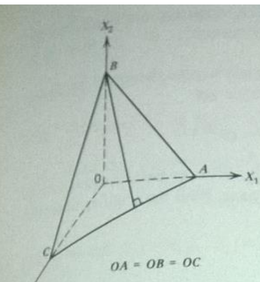
(b)

$$[L_{ij}]=\begin{bmatrix}-0.003&0.02&0\\0.02&0.04&0.005\\0&0.005&0\end{bmatrix}.$$

3.31 The components of the Lagrangian strain tensor referred to (X_1,X_2,X_3) axes (see the figure) are given by

$$[L_{ij}]=\begin{bmatrix}0.02&-0.003&0\\-0.003&0.01&0.02\\0&0.02&0.01\end{bmatrix}$$

and they are constant in the region under consideration.



Exercise 3.31

(a) Find the relative elongation of the lines AC and DB.

(b) Find the change of the initially right angle ADB.

(c) Find $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$.

(d) Find G, J, and the volume dilatation.

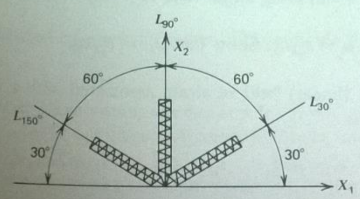
3.32 A resistance strain gage is an electromechanical device for measuring relative elongations parallel to a free surface. It is often used in an arrangement called a strain rosette which is bonded to a free surface as shown in the figure. Find the strain components L_{11}, L_{22}, L_{12} if:

(a) The strain gage rosette measures the relative elongations

$$L_{30^\circ} = 0.003,\ L_{90^\circ} = -0.003,\ L_{150^\circ} = 0.006.$$

(b) The strain gage rosette measures the relative elongations

$$L_{30^\circ} = -0.3,\ L_{90^\circ} = 0.3,\ L_{150^\circ} = 1.00.$$



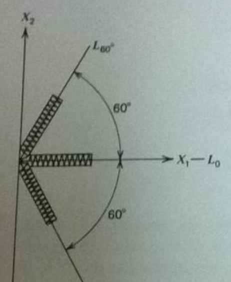
Exercise 3.32

In each case, find the angular distortion of a small cross scribed into the surface along the X_1 and X_2 axes prior to deformation.

3.33 A strain rosette is attached to a plane surface as shown in the figure. The relative elongations measured by this device are

$$L_{0^\circ} = -0.5,\ L_{60^\circ} = 0.5,\ L_{-60^\circ} = 1.0.$$

Find the strain components L_{11}, L_{12}, and L_{22}. Prior to deformation, a small cross is inscribed to coincide with the X_1-X_2 axes. Find the angular distortion of this cross after deformation.



Exercise 3.33

3.34 Given $x_i(a,t) = C_i(t) + a_k a_{k'i}(t)$, show that

$$a_k(x,t) = a_{k'i}(t)\left[x_i - C_i(t)\right].$$

3.35 Show that the infinitesimal rigid body displacement field $u_i = C_i + e_{ijk}\omega_k a_j$ results in a vanishing strain field $2\epsilon_{ij} = u_{i,j} + u_{j,i} = 0$.

3.36 Given that $\omega_{ij} = e_{ijk}\omega_k$. Show that $\omega_k = \frac{1}{2} e_{ijk}\omega_{ij}$.

3.37 Consider a (linear) field of strain associated with a simply connected region R such that

$$\epsilon_{11} = Ax_2^2, \quad \epsilon_{22} = Ax_2^2, \quad \epsilon_{12} = 2Ax_1x_2$$

$\epsilon_{33} = \epsilon_{23} = \epsilon_{31} = 0$, where $A = 0.002$. Show that it is not possible to find a single-valued, continuous displacement field which corresponds to the given strain field.

Hint: Show that $S_{33} = 2A = 0.004 \neq 0$ in (3.80a).

3.38 Show that $R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}$.

3.39 Show that the equation $R_{prst} = e_{ipr}e_{jst}S_{ij}$ is identically satisfied by

$$S_{ij} = S_{ji} = \tfrac{1}{4} e_{ikl}e_{jmn}R_{klmn}$$

and show that $S_{11} = R_{2323}$, $S_{12} = R_{2331}$, and so on.

3.40 Show that $u_{i,jk} = \epsilon_{ij,k} + \epsilon_{ik,j} - \epsilon_{kj,i}$.

3.41 Show that a given field of (linear) strain $2\epsilon_{ij} = u_{i,j} + u_{j,i}$ determines its generating displacement field only to within an arbitrary infinitesimal rigid displacement of the solid.

3.42 (a) Expand (3.77b), that is, show that

$$\begin{aligned} S_{11} &= R_{2323} & S_{12} &= R_{2331} \\ S_{22} &= R_{3131} & S_{23} &= R_{3112} \\ S_{33} &= R_{1212} & S_{31} &= R_{1223}. \end{aligned} \qquad (3.78)$$

(b) Substitute (3.68) into (3.80), and show that $S_{ij} \equiv 0$.

3.43 Show that equations (3.80) are not independent, that is, show that the equations

$$S_{11,1} + S_{12,2} + S_{13,3} = 0$$
$$S_{21,1} + S_{22,2} + S_{23,3} = 0$$
$$S_{31,1} + S_{32,2} + S_{33,3} = 0$$

are identically satisfied by any continuous, differentiable field of (linearized strain.

3.44 In the case of plane strain, $w = \partial u/\partial z = \partial v/\partial z = 0$, and therefore $\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$. What are the conditions of compatibility in this case?

3.45 Show that the strain-displacement relations $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ satisfy th compatibility conditions $S_{ij} = 0$ identically.

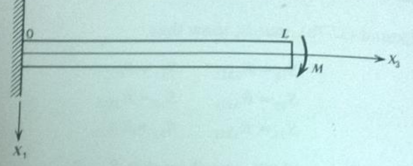
3.46 Consider the case of pure bending of a beam, as shown in the figure. The beam has stiffness EI, and it is attached to the wall at $X_3 = 0$. A moment of magnitude M is applied at $X_3 = L$. The resulting displacement field is given by

$$u_1 = \frac{M}{2EI}\left(a_3^2 + \nu a_1^2 - \nu a_2^2\right)$$
$$u_2 = \frac{M}{EI}\nu a_1 a_2$$
$$u_3 = -\frac{M}{EI}a_1 a_3.$$

(a) Find the (linear) strain and rotation fields.

(b) Discuss the boundary conditions along $a_3 = 0$.

(c) Are the compatibility conditions satisfied?



Exercise 3.46

3.47—*Cylindrical coordinates.* Show that (see (3.86))

$$2\epsilon_{r\theta} = \mathbf{e}_r \cdot \mathbf{D}_\theta + \mathbf{e}_\theta \cdot \mathbf{D}_r = \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}.$$

3.48—*Cylindrical coordinates.* Show that

$$\mathbf{D}_\theta = (\mathbf{e}_\theta \cdot \nabla)\mathbf{u} = \frac{1}{r}\frac{\partial \mathbf{u}}{\partial r} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + \mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

that is, prove the validity of (3.85b).

3.49—*Cylindrical coordinates.* By direct substitution, show that the integrability relations (3.87) are satisfied by the strain-displacement relations (3.86) referred to cylindrical coordinates.

3.50—*Spherical polar coordinates.* Show that

$$\mathbf{D}_\phi = (\mathbf{e}_\phi \cdot \nabla)\mathbf{u} = \frac{1}{r}\frac{\partial \mathbf{u}}{\partial \phi} = \frac{1}{r}\left[\mathbf{e}_r\left(\frac{\partial u_r}{\partial \phi} - u_\phi\right) + \mathbf{e}_\theta \frac{\partial u_\theta}{\partial \phi} + \mathbf{e}_\phi\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right)\right],$$

that is, prove the validity of (3.90c).

3.51—*Spherical polar coordinates.* Show that (see 3.91)

$$2\epsilon_{r\theta} = \mathbf{e}_r \cdot \mathbf{D}_\theta + \mathbf{e}_\theta \cdot \mathbf{D}_r = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \theta}.$$

3.52—*Spherical polar coordinates.* By direct substitution, show that the integrability relations (3.92) are satisfied by the strain-displacement relations (3.91) referred to spherical polar coordinates.

3.53 Derive the (linearized) strain displacement relations referred to oblate spheroidal coordinates (u, v, ϕ). See Exercise 2.44.

3.54 Derive the (linearized) strain displacement relations referred to parabolic coordinates (ξ, η, ϕ). See Exercise 2.45.

3.55 Consider the point symmetric deformation of a sphere (or spherical shell) characterized by $R = r(1+\lambda)$, where λ is a constant. The symbols r and R denote radial coordinates of the same material point before and after deformation, respectively. With reference to Fig. 2.15, show that

$$u_r = \lambda r$$
$$u_\theta = u_\phi = 0$$
$$M_r = M_\theta = M_\phi = \lambda$$
$$L_{rr} = L_{\theta\theta} = L_{\phi\phi} = \lambda(1 + \tfrac{1}{2}\lambda)$$
$$L_{r\theta} = L_{\theta\phi} = L_{\phi r} = 0$$
$$M_r^{(e)} = M_\theta^{(e)} = M_\phi^{(e)} = \frac{\lambda}{1+\lambda}$$
$$E_{rr} = E_{\theta\theta} = E_{\phi\phi} = \frac{\lambda(1+\frac{1}{2}\lambda)}{(1+\lambda)^2}$$
$$E_{r\theta} = E_{\theta\phi} = E_{\phi r} = 0$$
$$J = (1+\lambda)^3 = 1 + 3\lambda + 3\lambda^2 + \lambda^3.$$

For small strains, $|\lambda| \ll 1$. Show that in this case

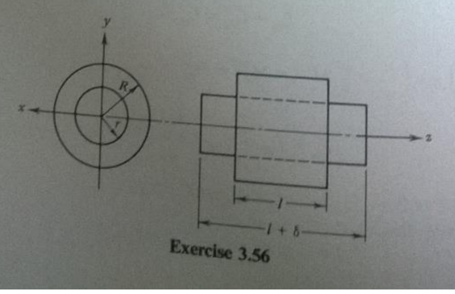
$$M_r = M_r^{(e)} = M_\theta = M_\theta^{(e)} = M_\phi = M_\phi^{(e)} = \lambda$$
$$L_{rr} = E_{rr} = L_{\phi\phi} = E_{\phi\phi} = L_{\theta\theta} = E_{\theta\theta} = \lambda$$
$$\frac{dV - dV_0}{dV_0} = J - 1 = 3\lambda = L_{rr} + L_{\theta\theta} + L_{\phi\phi} = E_{rr} + E_{\theta\theta} + E_{\phi\phi}.$$

3.56 Consider the axially symmetric deformation characterized by $R = r(1 - \nu\alpha)$, $Z = z(1+\alpha)$ where $0 < \nu < \frac{1}{2}$ and $\alpha = \delta/l$ are constants (see the figure). The quantities (r, θ, z) and (R, Θ, Z) are the cylindrical coordinates of the same material point before and after deformation. (Note: this problem is applicable to the tension test.) Show that

$$u_r = -\nu\alpha r, \quad u_\theta = 0, \quad u_z = \alpha z$$
$$M_r = M_\theta = -\nu\alpha, \quad M_z = \alpha$$
$$2L_{rr} = \nu\alpha(\nu\alpha - 2) = 2L_{\theta\theta}, \quad 2L_{zz} = \alpha(\alpha + 2)$$
$$L_{r\theta} = L_{\theta z} = L_{zr} = 0$$
$$M_r^{(e)} = M_\theta^{(e)} = -\frac{\nu\alpha}{1-\nu\alpha}, \quad M_z^{(e)} = \frac{\alpha}{1+\alpha}$$
$$2E_{rr} = 2E_{\theta\theta} = \frac{\nu\alpha(\nu\alpha - 2)}{(1-\nu\alpha)^2}, \quad 2E_{zz} = \frac{\alpha(\alpha+2)}{(1+\alpha)^2}$$
$$E_{r\theta} = E_{\theta z} = E_{zr} = 0$$
$$J = (1+\alpha)(1-\nu\alpha)^2 = 1 + (1-2\nu)\alpha + \nu(\nu-2)\alpha^2 + \nu^2\alpha^3.$$

In the case of small strain, $|\alpha| \ll 1$. Show that in this case

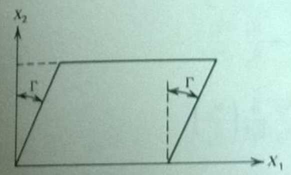
$$M_r = M_r^{(e)} = M_\theta = M_\theta^{(e)} = -\nu\alpha, \qquad M_z = M_z^{(e)} = \alpha$$
$$L_{rr} = E_{rr} = L_{\theta\theta} = E_{\theta\theta} = -\nu\alpha, \qquad L_{zz} = E_{zz} = \alpha$$
$$\frac{dV - dV_0}{dV_0} = J - 1 = (1-2\nu)\alpha.$$



Exercise 3.56

3.57 Consider the case of a solid subjected to "simple shear," such that $u_1 = a_2 \tan\Gamma$, $u_2 = u_3 = 0$, where Γ is the shear angle (see the figure). Show that

$$x_1 = a_1 + a_2\tan\Gamma, \qquad x_2 = a_2, \qquad x_3 = a_3$$
$$a_1 = x_1 - x_2\tan\Gamma, \qquad a_2 = x_2, \qquad a_3 = x_3$$
$$L_{11} = L_{33} = L_{23} = L_{31} = 0, \qquad 2L_{22} = \tan^2\Gamma$$
$$2L_{12} = \tan\Gamma$$
$$E_{11} = E_{33} = E_{23} = E_{31} = 0, \qquad 2E_{22} = -\tan^2\Gamma$$
$$2E_{12} = \tan\Gamma$$
$$M_1 = M_3 = 0, \qquad M_2 = \sqrt{1+\tan^2\Gamma} - 1$$
$$M_1^{(e)} = M_3^{(e)} = 0, \qquad M_2^{(e)} = 1 - \sqrt{1+\tan^2\Gamma}$$
$$G = J^2 = 1, \qquad \frac{dV - dV_0}{dV_0} = 0$$
$$\sin\phi_{12} = \frac{\tan\Gamma}{\sqrt{1+\tan^2\Gamma}}, \qquad \phi_{23} = \phi_3 = 0.$$



Exercise 3.57

In the case of small strain $|\Gamma| \ll 1$. Show that for this case

$$2L_{22} = 2E_{22} = 0, \qquad 2L_{12} = 2E_{12} = \Gamma$$
$$2M_2 = 0, \qquad 2M_2^{(e)} = 0, \qquad \phi_{12} = \Gamma.$$

3.58 **(a)** Show that

$$\mathbf{D}_i = \frac{\partial \mathbf{u}}{\partial a_i} = \mathbf{e}_k D_{ik}$$

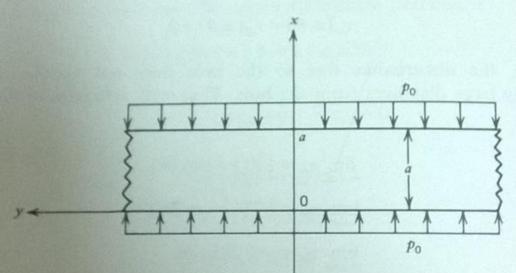
where

$$D_{ik} = \frac{\partial u_k}{\partial a_i} = \mathbf{e}_k \cdot \mathbf{D}_i.$$

Pp 198-206 (Solutions of some linear elasticity problems)

EXERCISES

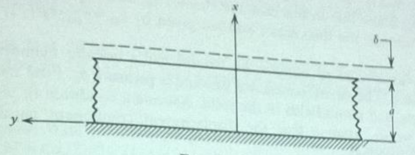
5.1 Consider a transversely constrained, elastic medium such that $v \equiv 0$ and $w \equiv 0$. The planes $x = 0$ and $x = a$ are subjected to a uniformly distributed



Exercise 5.1

pressure of magnitude p_0. Find the stress, strain and displacement fields. (See the figure.)

5.2 Consider a transversely constrained, elastic medium such that $v \equiv 0$ and $w \equiv 0$. The plane $x=0$ is fixed, and the plane which is originally at $x=a$ is displaced a distance δ in the x direction. Find the resulting fields of stress, strain, and displacement. (See the figure.)



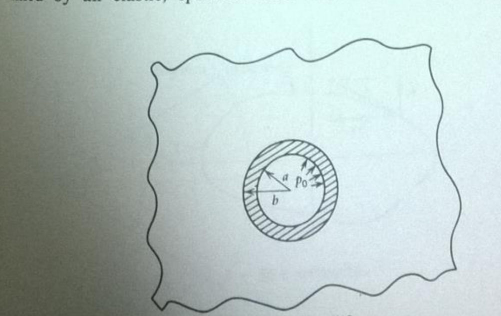
Exercise 5.2

5.3 An elastic solid of unbounded extent has a spherical cavity with radius a. The cavity contains a gas under pressure p_0. Find the displacement, stress, and strain fields in the solid.

5.4 A solid, elastic sphere is subjected to a uniform radial pressure p_0 on its surface. Find the displacement, stress, and strain fields in the sphere.

5.5 Find the stress and displacement fields in a very large solid with a spherical void centered at $r=0$. The solid is subjected to the radial stress $\tau_{rr}=T$ far from the origin. Assume point symmetry with respect to $r=0$. What is the stress concentration factor associated with $\tau_{\theta\theta}$?

5.6 An unbounded elastic medium contains a spherical cavity. The cavity is lined by an elastic, spherical shell which is bonded to the surrounding



Exercise 5.6

medium. The cavity contains a gas under pressure of magnitude p_0. (See the figure.) Investigate the stress and displacement fields in the surrounding medium as a function of the material properties of the liner. In general, the material properties of the liner are different from those of the surrounding medium.

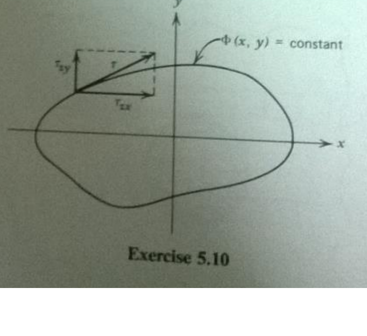
5.7 With reference to the analysis in Section 5.1, consider a thin spherical shell such that $b-a=t \ll a$ (see Fig. 5.1). The shell is filled with a gas under pressure p_0. Show that in this case the stresses $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$ are (approximately) constant through the thickness t and are given by $\tau_{\theta\theta}=\tau_{\phi\phi}=pa/2t$.

5.8 An elastic solid of unbounded extend has a circular cylindrical cavity with radius a. The cavity contains a gas under pressure p_0. Find the displacement, stress, and strain fields in the solid. Assume a condition of plane strain. In other words, assume that the displacement component parallel to the cavity axis vanishes.

5.9 A solid, elastic cylinder is subjected to a uniform radial pressure p_0 on its cylindrical surface. Find the displacement, stress, and strain fields in the cylinder. Assume a condition of plane strain, in other words, assume that the displacement component in the direction of the cylinder axis vanishes.

5.10 With reference to the torsion problem of Section 5.3, consider the curve $\Phi(x,y)=$constant, where Φ is the stress function (see the figure). Then on the curve $(\partial\Phi/\partial x)+(\partial\Phi/\partial y)(dy/dx)=0$, or, using (5.43), $dy/dx=\tau_{zy}/\tau_{zx}$, that is, at each point of the curve $\Phi(x,y)=$constant, the stress vector (τ_{zx},τ_{zy}) is tangent to the curve. These curves are called lines of shearing stress. The magnitude of the tangential stress is (see 5.43)

$$\tau=\sqrt{\tau_{zx}^2+\tau_{zy}^2}=G\beta\sqrt{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2}$$



Exercise 5.10

Show that τ is equal to the absolute value of the gradient of the surface $z=G\beta\Phi(x,y)$, and the greatest shear stress is equal to the largest gradient.

5.11 Show that $\mathfrak{M}=(\sqrt{3}/45)G\beta c^4$ for the triangular cross section in Fig. 5.6.

5.12 **(a)** Verify the correctness of equation (5.54).

(b) Show that $k=0.848$, $k_1=0.196$, and $k_2=0.231$, if $b/a=1.5$ (See (5.55), (5.56), and Table 5.1).

5.13 Show that $\mathfrak{M}=2G\beta a^4\Gamma$ for the cross section in Fig. 5.8, where $\Gamma=\frac{1}{24}(\sin 4\beta+8\sin 2\beta+12\beta)-\frac{1}{2}(b/a)^2(\sin 2\beta+2\beta)+\frac{4}{3}(b/a)^3\sin\beta-\frac{1}{4}(b/a)^4\beta$ and $2\cos\beta=b/a$.

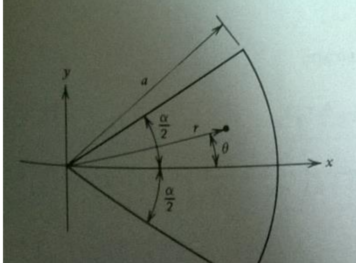
5.14—*Torsion.* With reference to Section 5.3, consider the analytic function $F(z)=\phi+i\psi=i(a/2)z^4-i\frac{1}{2}(a-1)$. Show that $\psi=(a/2)(x^4-6x^2y^2+y^4)-\frac{1}{2}(a-1)$. Show that the boundary condition (5.35) is satisfied on C, where C is characterized by the equation $x^2+y^2-a(x^4-6x^2y^2+y^4)+a-1=0$. Discuss the geometry of this cross section as a function of the parameter a. Draw the cross section for $a=0$, $a=\frac{1}{2}$, $a=\frac{1}{2}(\sqrt{2}-1)$. Find the torsional rigidity.

5.15 Show that the analytic function

$$F(z)=\frac{i}{2}\frac{z^2}{\cos\alpha}+\frac{1}{2}ia^2\sum_{n=1,3,5,\dots}^{\infty}A_n\left(\frac{z}{a}\right)^{\frac{n\pi}{\alpha}}=\phi+i\psi$$

solves the torsion problem of the sector of a circle (see Section 5.3 and the figure here). Note that $z=re^{i\theta}$, and

$$1-\frac{\cos 2\theta}{\cos\alpha}=\sum_{n=1,3,5,\dots}^{\infty}A_n\cos\frac{n\pi\theta}{\alpha}$$



Exercise 5.15

where

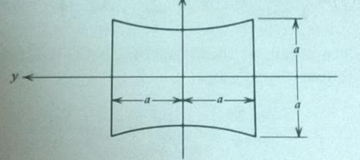
$$A_n=\frac{16a^2}{\pi^3}(-1)^{(n+1)/2}\frac{1}{n[n+(2\alpha/\pi)][n-(2\alpha/\pi)]}.$$

Find the stress and displacement field, and determine the torsional rigidity.

5.16 Perform an error analysis of the experimental method described at the end of Section 5.4 to find Poisson's ratio. What is $d\nu/d\alpha$ for steel ($\nu=\frac{1}{3}$)? Explain.

5.17 With reference to Section 5.5 and Example 5.5.A, find the displacement field in the end-loaded cantilever beam with circular cross section. Compare the stress and displacement fields of the exact solution with the corresponding results obtained from the Euler–Bernoulli beam theory, with particular emphasis upon short beams.

5.18 With reference to Section 5.5, find the stress and displacement field in an end-loaded cantilever beam, the cross section of which consists of two vertical sides $y=\pm a$ and two hyperbolas $(1+\nu)x^2-\nu y^2=a^2$ (see the figure). Use $F(z)=\phi+i\psi=a^2z$, and show that $\tau_{xz}=(PG/IE)[a^2-(1+\nu)x^2+\nu y^2]$, $\tau_{yz}\equiv 0$. Find the maximum shearing stress. Provide a full discussion.



Exercise 5.18

5.19 Show that in the case of plane strain,

(a)
$$\tau_{zz}=\nu(\tau_{xx}+\tau_{yy}) \tag{5.78}$$

(b)
$$\nabla^2(\tau_{xx}+\tau_{yy})=-\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \tag{5.80}$$

where $\frac{1}{1-\nu}=\frac{2(\lambda+G)}{\lambda+2G}$.

5.20—*Generalized plane stress.* **(a)** Verify equations (5.84) and (5.85).

(b) Show that

$$\nabla^2(\tilde{\tau}_{xx}+\tilde{\tau}_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \tag{5.89}$$

where

$$1+\nu = \frac{2(\lambda'+G)}{\lambda'+2G}, \quad \lambda' = \frac{2G\lambda}{\lambda+2G}$$

(see 5.85).

5.21 **(a)** Provide a detailed comparison between the three solutions (5.103), (5.104), and (5.105). Show that "far" from the wall, the stress, strain, and displacement fields are nearly equal for the three cases.

(b) The result in **(a)** is a consequence of Saint Venant's principle. This principle is often applied in the theory of elasticity, frequently without explicit mention. The reader is urged to consult the following references:

E. Sternberg, "On Saint-Venant's Principle," *Quarterly of Applied Mathematics*, vol. 11, 1954, pp. 393–402.

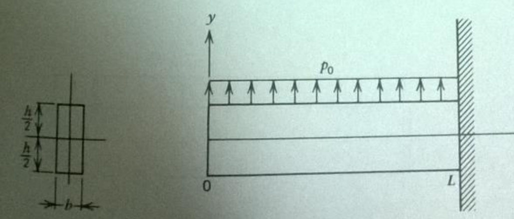
R. von Mises, "On Saint-Venant's Principle," *Bulletin of the American Mathematical Society*, vol. 51, 1945, p. 555–562.

P. M. Naghdi, "On Saint-Venant's Principle: Elastic Shells and Plates," *Journal of Applied Mechanics*, vol. 27, 1960, pp. 417–422.

5.22 Consider the cantilever beam within the framework of the generalized plane stress theory (see the figure). Note that

$$\tau_{yy}\left(x,\frac{h}{2}\right)=p_0, \quad \tau_{yy}\left(x,-\frac{h}{2}\right)=0, \quad \tau_{xy}\left(x,\pm\frac{h}{2}\right)=0$$

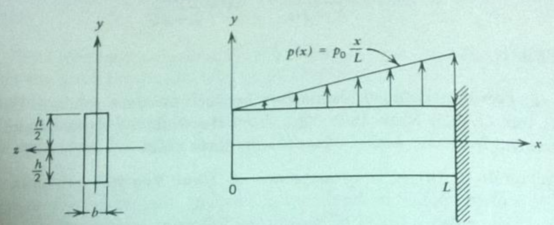
$$b\int_{-h/2}^{h/2}\tau_{xx}(0,y)\,dy = b\int_{-h/2}^{h/2} y\tau_{xx}(0,y)\,dy = 0.$$



Exercise 5.22

Assume that $\tau_{xy} = -(\partial^2\Phi/\partial x\,\partial y) = xf(y)$. Find the stress and displacement fields.

5.23 Consider the cantilever beam within the framework of the generalized plane stress theory (see the figure). Note that $\tau_{yy}(x,h/2)=p_0(x/L)$, $\tau_{yy}(x,-h/2)=0$, $\tau_{xy}(x,\pm h/2)=0$, $\tau_{xx}(0,y)=0$, and $b\int_{-h/2}^{h/2}\tau_{xy}(0,y)\,dy=0$. Assume that $\tau_{xx}=\partial^2\Phi/\partial y^2 = x^3 f_1(y)+xf_2(y)$. Find the stress and displacement fields.



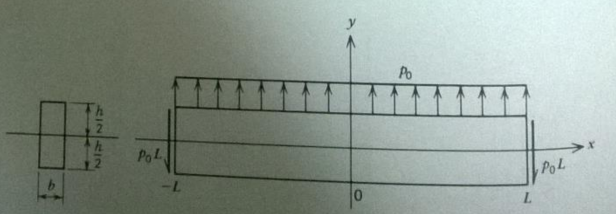
Exercise 5.23

5.24 Solve the problem of the "simply" supported beam under a uniformly distributed load within the framework of the generalized plane stress formulation (see the figure). Use $\Phi = x^2(a+dy+ey^3)+y^3(c+fy^2)$ and note that $\tau_{xy}(x,\pm h/2)=\tau_{yy}(x,-h/2)=0$, $\tau_{yy}(x,h/2)=p_0$,

$$b\int_{-h/2}^{h/2}\tau_{xx}(\pm L,y)\,dy = b\int_{-h/2}^{h/2} y\tau_{xx}(\pm L,y)\,dy = 0$$

$$b\int_{-h/2}^{h/2}\tau_{xy}(\mp L,y)\,dy = \pm p_0 L.$$

Find the stress and displacement fields.



Exercise 5.24

5.25 **(a)** Show that the stress equations of equilibrium for the generalized plane stress problem referred to polar coordinates are

$$\frac{\partial\tilde{\tau}_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tilde{\tau}_{r\theta}}{\partial\theta}+\frac{\tilde{\tau}_{rr}-\tilde{\tau}_{\theta\theta}}{r}+F_r=0$$

$$\frac{1}{r}\frac{\partial\tilde{\tau}_{\theta\theta}}{\partial\theta}+\frac{\partial\tilde{\tau}_{r\theta}}{\partial r}+\frac{2}{r}\tilde{\tau}_{r\theta}+F_\theta=0.$$

(b) Show that the stress tensor components

$$\tilde{\tau}_{rr}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$\tilde{\tau}_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2}$$

$$\tilde{\tau}_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

satisfy the stress equations of equilibrium identically, provided body forces vanish. Note that in this case $\Phi=\Phi(r,\theta)$.

5.26 In the case of generalized plane stress referred to polar coordinates, the stress-displacement equations assume the form

$$\tilde{\epsilon}_{rr}=\frac{\partial\tilde{u}}{\partial r}, \quad \tilde{\epsilon}_{\theta\theta}=\frac{\tilde{u}}{r}+\frac{1}{r}\frac{\partial\tilde{v}}{\partial\theta}, \quad 2\tilde{\epsilon}_{r\theta}=\frac{1}{r}\frac{\partial\tilde{u}}{\partial\theta}+\frac{\partial\tilde{v}}{\partial r}-\frac{\tilde{v}}{r}.$$

Eliminate $\tilde{u}$ and $\tilde{v}$ from these equations, that is, show that

$$\frac{\partial^2\tilde{\epsilon}_{\theta\theta}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\epsilon}_{rr}}{\partial\theta^2}+\frac{2}{r}\frac{\partial\tilde{\epsilon}_{\theta\theta}}{\partial r}-\frac{1}{r}\frac{\partial\tilde{\epsilon}_{rr}}{\partial r}=\frac{2}{r}\frac{\partial^2\tilde{\epsilon}_{r\theta}}{\partial r\,\partial\theta}+\frac{2}{r^2}\frac{\partial\tilde{\epsilon}_{r\theta}}{\partial\theta}.$$

Use this compatibility equation and Hooke's law

$$E\tilde{\epsilon}_{rr}=\tilde{\tau}_{rr}-\nu\tilde{\tau}_{\theta\theta}, \quad E\tilde{\epsilon}_{\theta\theta}=\tilde{\tau}_{\theta\theta}-\nu\tilde{\tau}_{rr}, \quad 2G\tilde{\epsilon}_{r\theta}=\tilde{\tau}_{r\theta}$$

as well as the results of Exercise 5.25 to show that $\nabla^4\Phi=\nabla^2(\nabla^2\Phi)=0$ wher

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and where body forces have been neglected, that is, $F_r=F_\theta=0$.

5.27 **(a)** Show that if $\Phi=\Phi(r)$ (Φ is independent of θ), then the solution $\nabla^4\Phi=0$ is $\Phi=A\ln r+Br^2\ln r+Cr^2+D$.

(b) Find the stress field in a very large plate with a circular cutout of radius a centered at $r=0$. The plate is subjected to the radial stress $\tau_{rr}=T$ far from the origin. Show that $\tau_{rr}=T[1-(a^2/r^2)]$ and $\tau_{\theta\theta}=T[1+(a^2/r^2)]$. What is the stress concentration factor associated with $\tau_{\theta\theta}$?

(c) Find the displacement field.

5.28 Solve the problem of Section 5.2 with the aid of a stress function. Use the results of Exercises 5.25 through 5.27.

5.29 Integrate equations (5.58). Show that (5.59) results if, at $(x,y,z)=(0,0,0)$ we require $u=v=w=0$, and

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=\frac{\partial v}{\partial x}=0.$$

5.30 Compare the solutions (5.103) through (5.105) to

(a) The "exact" solution in Example 5.5.C.

(b) The Timoshenko beam model (Section 6.3).

(c) The Euler–Bernoulli Beam model (Section 6.4).

5.31 Demonstrate the correctness of (5.44a).

5.32 Show that

$$D=G\int_A\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right]dA. \tag{5.44b}$$

5.33 Show that $F(z)=\phi+i\psi=[(a^2-b^2)/(a^2+b^2)](iz^2/2)$ is the complex torsion function for the elliptic cross section (see Example 5.3.A).

5.34 Show that $F(z)=\phi+i\psi=-(1/2c)(z^2-icz^2+c^2z)$ is the complex torsion function for the triangular cross section (see Example 5.3.B).

5.35 With reference to Fig. 5.4c, show that

$$\mathbf{n}=\mathbf{e}_x n_x+\mathbf{e}_y n_y, \quad \frac{d\mathbf{s}}{ds}=\mathbf{e}_x\frac{dx}{ds}+\mathbf{e}_y\frac{dy}{ds}$$

$$\mathbf{n}\cdot\mathbf{n}=1=n_x^2+n_y^2, \quad \mathbf{n}\cdot\frac{d\mathbf{s}}{ds}=n_x\frac{dx}{ds}+n_y\frac{dy}{ds}=0$$

$$\mathbf{n}\times\frac{d\mathbf{s}}{ds}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ n_x & n_y & 0\\ \frac{dx}{ds} & \frac{dy}{ds} & 0\end{vmatrix}=\mathbf{e}_z\left(n_x\frac{dy}{ds}-n_y\frac{dx}{ds}\right)$$

and for a positive traversal of C, $n_x(dy/ds)-n_y(dx/dx)>0$. Show that a unique solution is $n_x=dy/ds$; $n_y=-(dx/ds)$.

Ugural, "Mechanics of Materials"

Sections 8.6 through 8.9

8.76 An element in three-dimensional stress is subjected to stresses

$\sigma_x = 120\text{ MPa}$ $\sigma_y = 60\text{ MPa}$ $\tau_{xy} = -30\text{ MPa}$

$\sigma_z = -15\text{ MPa}$

as shown in Fig. P8.76. Using Mohr's circle, determine:

(a) The principal stresses.

(b) The absolute maximum shear stress.

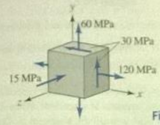


Figure P8.76

8.77 Solve Prob. 8.76 for the case in which

$\sigma_x = 10\text{ ksi}$ $\sigma_y = 0$ $\tau_{xy} = 5\text{ ksi}$ $\sigma_z = -12\text{ ksi}$

as depicted in Fig. P8.77.

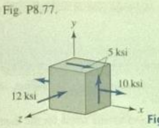


Figure P8.77

8.78 Redo Prob. 8.76 for the given state of stress

$\sigma_x = 50\text{ MPa}$ $\sigma_y = 10\text{ MPa}$ $\tau_{xy} = -40\text{ MPa}$

$\sigma_z = 25\text{ MPa}$

as shown in Fig. P8.78.

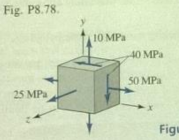


Figure P8.78

8.79 A point in a structure is subjected to a state of stress with

$\sigma_x = 15\text{ ksi}$ σ_y $\tau_{xz} = 10\text{ ksi}$

acting as shown on a three-dimensional element in Fig. P8.79. Calculate two values of σ_y for which the absolute maximum shear stress equals 13 ksi.

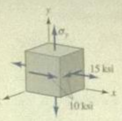


Figure P8.79

8.80 A point in a machine component is subjected to three-dimensional stress with

$\sigma_x = 150\text{ MPa}$ $\sigma_y = 30\text{ MPa}$ $\tau_{xy} = 90\text{ MPa}$ σ_z

as depicted in Fig. P8.80. Determine:

(a) The absolute maximum shear stress for $\sigma_z = 40\text{ MPa}$.

(b) The absolute maximum shear stress for $\sigma_z = -40\text{ MPa}$.

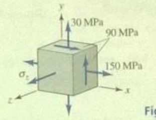


Figure P8.80

8.81 A spherical vessel of radius r and wall thickness t is submerged in water having density γ. Calculate the water depth h at which the circumferential stress in the sphere would be σ.

Assumption: A safety factor of n_s is to be used.

Given: $r = 2.5\text{ ft}$, $t = 1\frac{1}{4}\text{ in.}$, $\gamma = 62.4\text{ lb/ft}^3$, $\sigma = 4200\text{ psi}$, $n_s = 1.5$

8.82 A compressed-air tank of uniform thickness $t = 5\text{-mm}$ is subjected to an internal pressure of $p = 1.4\text{ MPa}$ (Fig. P8.82). Determine the maximum axial and circumferential stresses.

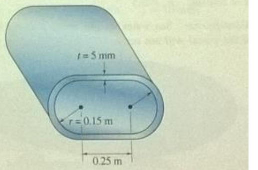


Figure P8.82

8.93 A cylindrical vessel of internal diameter 200 mm and wall thickness 2 mm has a welded helical seam angle of $\phi = 60°$ (see Fig. P8.92). The allowable tensile stress in the weld is 100 MPa. Determine the maximum value of internal pressure p and the corresponding shear stress in the weld, using:

(a) Equations (8.4).

(b) An approach based on the equilibrium equations applied to a wedge-shaped stress element.

(c) Mohr's circle.

8.94 A closed cylindrical pressure vessel of inner diameter d and wall thickness t is constructed with a helical welding that makes an angle ϕ with the longitudinal axis as depicted in Fig. P8.94. The vessel is subjected to an internal pressure p. Determine:

(a) The tangential and axial stresses.

(b) The maximum in-plane shearing stress.

(c) The absolute maximum shear stress.

(d) The normal stress σ_w and shear stress τ_w acting on planes perpendicular and parallel to the weld. Sketch the results on a properly oriented element.

Given: $d = 2r = 4\text{ ft}$, $t = \frac{1}{2}\text{ in.}$, $\phi = 55°$, $p = 500\text{ psi}$

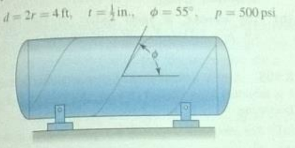


Figure P8.94

8.95 Redo Prob. 8.94 for a vessel (see Fig. P8.94) with the following given numerical values:

$d = 2r = 1.2\text{ m}$, $t = 12\text{ mm}$, $\phi = 40°$, $p = 2\text{ MPa}$

8.96 A steel boiler of inner diameter d and thickness t is welded using a spiral seam that makes an angle ϕ with respect to the axial axis (see Fig. P8.94). The boiler is subjected to an internal pressure p. Determine:

(a) The normal stress σ_w perpendicular to the weld and shear stress τ_w parallel to the weld. Sketch the results on a properly oriented element.

(b) The absolute maximum shear stress in the boiler.

Given: $d = 2r = 1.5\text{ m}$, $t = 15\text{ mm}$, $\phi = 60°$, $p = 800\text{ kPa}$

8.97 A cylindrical tank of inner diameter d and wall thickness t has a butt-welded helical seam forming an angle ϕ with the longitudinal axis, as shown in Fig. P8.97. The allowable normal stress in the weld equals σ_w. Determine the largest internal pressure p that can be applied in the tank, using Mohr's circle.

Given: $d = 2r = 1.6\text{ ft}$, $t = 8\text{ mm}$, $\phi = 55°$, $\sigma_w = 10\text{ ksi}$

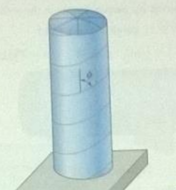


Figure P8.97

***8.98** A cylindrical tank of inner diameter d is constructed with steel plate having thickness t welded along a helix forming an angle ϕ with the longitudinal axis (Fig. P8.97). Calculate the internal pressure p that will cause a shear stress τ_w parallel to the weld, using Mohr's circle.

Given: $d = 2r = 600\text{ mm}$, $t = 10\text{ mm}$, $\phi = 70°$, $\tau_w = 40\text{ MPa}$

8.99 A cylindrical pressure tank of inner diameter d is fabricated by shaping steel plates of thickness t_c and welding the plates along helical arcs of angle ϕ with a transverse plane (Fig. P8.99). The end caps are spherical and have uniform wall thickness t_s. The maximum internal pressure in the tank is p. Calculate:

(a) The normal and absolute maximum shear stresses in the caps.

(b) The normal and absolute maximum shear stresses in the cylinder.

(c) The normal stress σ_w and shear stress τ_w acting on planes perpendicular and parallel to the weld. Sketch the results on a properly oriented element.

Given: $d = 2r = 2.4\text{ m}$, $t_c = 15\text{ mm}$, $t_s = 10\text{ mm}$, $\phi = 30°$, $p = 1.4\text{ MPa}$

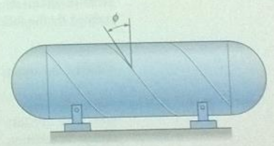


Figure P8.99

8.83 The closed-ended cylindrical steel tank seen in Fig. P8.83 has radius r and height h. It is completely filled with a liquid of density γ and is subjected to an additional internal gas pressure p. Calculate the wall thickness needed:

(a) At the top of the tank.

(b) At quarter-height.

(c) At mid-height.

Given: $p = 400\text{ kPa}$, $h = 20\text{ m}$, $r = 5\text{ m}$, $\gamma = 15\text{ kN/m}^3$

Requirement: The allowable stress in the cylinder walls is limited to 150 MPa.

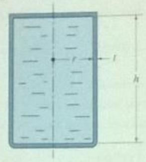


Figure P8.83

8.84 Solve Prob. 8.83, assuming that the gas pressure is $p = 100\text{ kPa}$.

8.85 Determine the required thickness t of a cylindrical vessel 3.6 ft in diameter under a pressure of $p = 150\text{ psi}$.

Given: A material strength of 25 ksi.

Assumption: A safety factor of $n_s = 1.5$ will be used.

8.86 An underwater vehicle having a living space that can be approximated as a sphere of radius r and thickness t is to operate at depth h (Fig. P8.86). It is filled with air at pressure p. The material of the sphere is steel with yield strength σ_y. The factor of safety is to be n_s. Determine the required thickness t of the sphere.

Given: $r = 2.5\text{ m}$, $p = 4\text{ MPa}$, $\sigma_y = 600\text{ MPa}$, $h = 1000\text{ m}$, $n_s = 2.5$

Assumptions: Sea water weighs about $\gamma = 10\text{ kN/m}^3$. Buckling of the vessel will not occur.

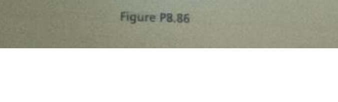


Figure P8.86

8.87 A spherical vessel of 1.6-m inner diameter is constructed by joining two hemispheres with 40 equally spaced bolts (Fig. P8.87). The vessel will operate at an internal pressure of 600 kPa.

Figure P8.87

Calculate the bolt diameter d and the vessel thickness t.

Given: The allowable stresses for the bolts and sphere wall equal 100 and 50 MPa, respectively.

8.88 A *penstock*, a pipe for conveying water (specific weight $\gamma =$ 9.81 kN/m³) to a turbine, operates at a head of 120 m. The penstock has a 0.9-m diameter and a wall thickness of t. Determine the minimum required value of t for a material strength of 100 MPa.

Assumption: A safety factor of $n_s = 1.6$ will be used.

8.89 Redo Prob. 8.88 for the case in which the allowable stress is 80 MPa.

8.90 A closed cylindrical tank fabricated of 10-mm-thick plate is subjected to an internal pressure of $p = 6\text{ MPa}$. Determine:

(a) The maximum diameter if the maximum shear stress is limited to 30 MPa.

(b) The limiting value of tensile stress for the diameter found in part (a).

8.91 A closed cylindrical vessel constructed of a thin plate is 1.2 m in diameter and subjected to an external pressure of 1 MPa. Calculate:

(a) The wall thickness t if the maximum allowable shear stress is set at 20 MPa.

(b) The corresponding maximum principal stress.

8.92 The side wall of the cylindrical steel pressure vessel has butt-welded seams (Fig. P8.92). The allowable tensile strength of the joint is 90% that of steel. Determine the maximum value of the seam angle ϕ if the tension in the steel is to be limiting.

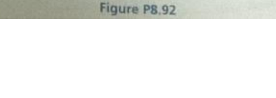


Figure P8.92

8.100 A pressure vessel of inner radius r and wall thickness t is fabricated from a welded pipe with helix angle ϕ and equipped with two end plates, as shown in Fig. P8.100. If the pressure inside the vessel is p, determine:

(a) The normal stress σ_w perpendicular to the weld.

(b) The shear stress τ_w parallel to the weld.

Given: $r = 4\text{ ft}$, $t = \frac{1}{2}\text{ in.}$, $\phi = 38°$, $p = 200\text{ psi}$

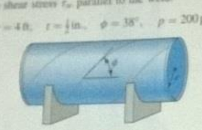


Figure P8.100

8.101 A thick-walled, closed-ended cylinder of inner radius a and outer radius b is subjected to an internal pressure p_i only (Fig. P8.101). The cylinder is made of a material with permissible tensile strength σ_{all} and shear strength τ_{all}. Calculate the allowable value of p_i.

Given: $a = 0.8\text{ m}$, $b = 1.2\text{ m}$, $\sigma_{all} = 100\text{ MPa}$, $\tau_{all} = 60\text{ MPa}$

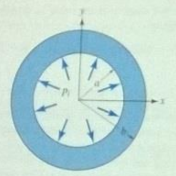


Figure P8.101

8.102 A thick-walled cylindrical steel tank of internal radius a is subjected to an internal pressure p_i (Fig. P8.101). The ultimate strength of the material in tension and compression is equal to σ_u. The factor of safety is to be n_s. Calculate the wall thickness.

Given: $a = 3\text{ ft}$, $p_i = 2\text{ ksi}$, $\sigma_{all} = 40\text{ ksi}$, $n_s = 1.5$

8.103 A thick-walled cylindrical tank of inner radius a and outer radius b is made of ASTM A-48 cast iron (see Table B.4) having an ultimate strength in tension σ_u, modulus of elasticity E, and Poisson's ratio ν. Determine the maximum radial displacement of the tank, if it is subjected to an internal pressure p_i, as shown in Fig. P8.101.

Given: $a = 0.5\text{ m}$, $b = 1\text{ m}$, $p_i = 10\text{ MPa}$, $\sigma_u = 170\text{ MPa}$, $E = 70\text{ GPa}$, $\nu = 0.3$

***8.104** A cylindrical thick-walled pipe having inner radius a and outer radius b is subjected to an internal pressure p_i (Fig. P8.101). Determine:

(a) The ratio of the wall thickness to the inner radius, if the internal pressure is one-half of the maximum tangential stress: $p_i = \frac{1}{2}(\sigma_t)_{max}$.

(b) The increase in inner radius of the pipe, if $a =$ 2 ft, $p_i = 1.2\text{ ksi}$, $E = 30 \times 10^6\text{ psi}$, and $\nu = 0.3$

8.105 A thick-walled cylinder of inner radius a and outer radius b is subjected to an internal pressure p_i as shown in Fig. P8.101. Determine:

(a) The maximum tangential and radial stresses in the cylinder.

(b) The minimum tangential and radial stresses in the cylinder.

(c) The axial stress and the maximum shear stress in the cylinder.

Given: $a = 25\text{ mm}$, $b = 75\text{ mm}$, $p_i = 35\text{ MPa}$

Shames. "Introduccion a la mecanica de solidos"
Pp 130-134 (Relaciones multidimensionales esfuerzo-deformacion

importantes ...

PROBLEMAS

6.1 |6.2| El tensor esfuerzo en un punto de un cuerpo es:

$$\begin{pmatrix} 1.000 & 2.000 & 0 \\ 2.000 & 5.000 & -2.000 \\ 0 & -2.000 & 0 \end{pmatrix} \text{lb/pulg.}^2$$

¿Qué deformaciones existen en el punto si el módulo de Poisson es 0,2 y el módulo de Young es 30×10^6 lb/pulg.²?

6.2 |6.2| Para un módulo de cizalladura de 10×10^6 lb/pulg.² y un módulo de elasticidad de 25×10^6 lb/pulg.² calcule el tensor deformación que corresponde al estado de esfuerzos siguiente:

$$\tau_{ij} = \begin{pmatrix} 1.000 & -5.000 & 0 \\ -5.000 & 500 & 500 \\ 0 & 500 & -2.000 \end{pmatrix} \text{lb/pulg.}^2$$

6.3 |6.2| Para un módulo de Poisson de 0,3 y un módulo de Young de 30×10^6 lb/pulg.², calcule el tensor deformación que corresponde al estado de esfuerzos siguiente:

$$\tau_{ij} = \begin{pmatrix} 0 & 1.000 & -2.000 \\ 1.000 & 500 & -3.000 \\ -2.000 & -3.000 & 1.000 \end{pmatrix} \text{lb/pulg.}^2$$

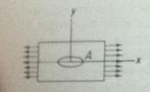


Figura 6.12

6.4 |6.2| Para determinar el factor de la concentración de esfuerzos K (Sección 4.7) producida por un agujero elíptico en una placa (Figura 6.12) sometida a esfuerzos biaxiales (ver la Sección 2.4), se utiliza un modelo de bakelita cuyo comportamiento se analiza fotoelásticamente, este método será discutido posteriormente, durante una parte del ensayo se obtuvo que el esfuerzo τ_{yy} en el vértice A de la elipse era igual a 2.000 lb/pulg.² Calcule el tensor deformación correspondiente, si $E = 5 \times 10^6$ lb/pulg.² y $\nu = 0,2$.

6.5 |6.2| Un eje cilíndrico hueco de pared delgada (Figura 6.13) está sometido a la acción de un momento torsor de 500 lb-pie y de una fuerza axial de 1.000 lbs. Calcule las deformaciones ϵ_{xx}, ϵ_{yy} y ϵ_{xy} en el punto P de la superficie del tubo si para el material $\nu = 0,3$ y $E = 30 \times 10^6$ lb/pulg.² y el plano xy es tangente a la superficie. Desprecie el efecto de la presión atmosférica.

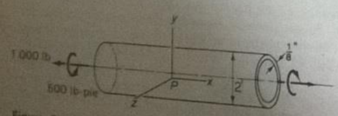


Figura 6.13

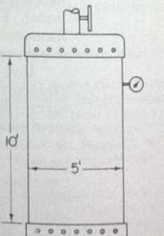


Figura 6.14

6.6 |6.2| En la Figura 6.14 se ilustra un tanque de pared delgada para almacenamiento de aire, que se mantiene a una presión manométrica de 500 lb/pulg.². El espesor de la pared es 0,25 pulg. ¿Qué valor tienen las deformaciones axial y trasversal sobre la superficie exterior del tanque y en una sección alejada de los extremos? ¿Qué cambio experimenta el diámetro del tanque en una sección intermedia?

6.7 |6.2| Durante el análisis de los esfuerzos en un codo (ver la Figura 6.15) se encuentra, mediante la utilización de extensómetros (pronto se discutirán con algún detalle), que las deformaciones en el punto A de la superficie son

$$\epsilon_{xx} = 0,00342$$
$$\epsilon_{yy} = -0,00342$$

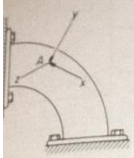


Figura 6.15

El plano xy es tangente a la superficie del codo en el punto A. ¿Qué esfuerzos normales τ_{xx}, τ_{yy} y τ_{zz} existen en A? $\nu = 0,2$ y $E = 25 \times 10^6$ lb/pulg.². ¿Qué valor tienen las deformaciones γ_{zy} y γ_{zx}?

6.8 |6.2| En la Figura 6.16 se muestra un tubo delgado doblado en ángulo recto. El diámetro exterior del tubo D es 2 pulg. y su espesor es 1/16 pulg. Al aplicar en el extremo libre una fuerza de 100 lbs. un extensómetro, que mide las deformaciones normales que se presentan en una superficie (posteriormente se discutirán en más detalle), ubicado en el punto A está orientado en la dirección x e indica que la deformación ϵ_{xx} que se induce vale 0,00346. Un segundo extensómetro ubicado en A pero orientado en la dirección z indica que $\epsilon_{zz} = 0,00137$. Si se desprecia el efecto de la presión atmosférica, ¿qué componentes tiene el tensor esfuerzo τ_{ij} en A? El módulo de Poisson es 0,25 y $E = 20 \times 10^6$ lb/pulg.². También calcule ϵ_{yy}.

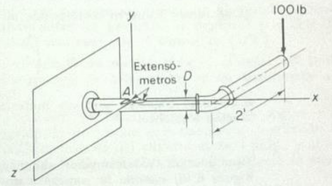


Figura 6.16

6.9 |6.2| ¿Cómo se relacionan los esfuerzos en coordenadas cilíndricas τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, $\tau_{r\theta}$, $\tau_{z\theta}$ y τ_{zr} con las deformaciones ϵ_{rr}, $\epsilon_{\theta\theta}$, ϵ_{zz}, $\epsilon_{r\theta}$, $\epsilon_{z\theta}$ y ϵ_{zr} para un material isotrópico de comportamiento elástico lineal?

6.10 |6.2| Para un cilindro de pared gruesa sometido a una presión manométrica interior p_i (ver la Figura 6.17), la teoría de la elasticidad predice que los esfuerzos radial y trasversal son:

$$\tau_{rr} = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\tau_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

¿Qué deformaciones ϵ_{rr}, $\epsilon_{\theta\theta}$ y $\epsilon_{r\theta}$ se presentan en $r = 2$ pies para un cilindro cuyos radios son $a = 1$ pie y $b = 3$ pies? Tome $E = 30 \times 10^6$ lb/pulg.², $\nu = 0,3$ y $p_i = 500$ lb/pulg.². Suponga que los extremos están libres y por tanto $\tau_{zz} = 0$.

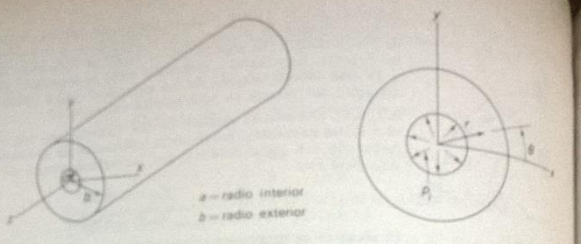


Figura 6.17

6.11 |6.3| Demuestre que en las Ecuaciones (6.18) $C_{25} = C_{24} = 0$.

6.12 |6.3| Para un material con simetría ortotrópica sus módulos elásticos corresponden a

$$C_{ij} = \begin{pmatrix} 3 & 4 & 2 & 0 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 0 \\ 7 & 6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \times 10^6 \text{ lb/pulg.}^2$$

¿Qué tensor esfuerzo corresponde al tensor deformación siguiente?

$$\epsilon_{ij} = \begin{pmatrix} 6 & 2 & 1 \\ 2 & -3 & 4 \\ 1 & 4 & 2 \end{pmatrix} \times 10^{-3}$$

6.13 |6.4| Suponga que un material posee la relación esfuerzo deformación siguiente:

$$\tau_{xx} = C(\epsilon_{xx})^{3/2}$$

¿Qué energía de deformación almacena una barra de dicho material (ver la Figura 6.18) cuando se somete a una fuerza de tensión uniforme de 500 lbs.?

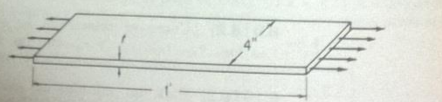


Figura 6.18

6.14 |6.4| Para el sistema mostrado en la Figura 6.19 calcule la densidad de la energía de deformación para cada uno de los miembros que están soldados entre sí y soportan una fuerza de 1.000 lbs. Utilice los datos siguientes:

$$E_1 = 30 \times 10^6 \text{ lb/pulg.}^2$$
$$\nu_1 = 0,3$$
$$E_2 = 15 \times 10^6 \text{ lb/pulg.}^2$$
$$\nu_2 = 0,2$$

¿Qué energía de deformación almacena el sistema? En sus cálculos desprecie el peso de los miembros.

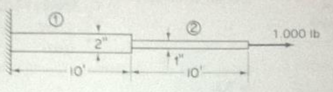


Figura 6.19

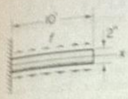


Figura 6.20

6.15 |6.4| Repita el problema anterior considerando una fuerza adicional de 2.000 lbs. que actúa en la sección que separa los dos materiales y está dirigida hacia la izquierda.

6.16 |6.4| Una fuerza distribuida $f(x) = 10x^2$ lb/pulg., en donde x se expresa en pulgadas, actúa sobre la viga mostrada en la Figura 6.20. Si la fuerza aplicada produce un estado unidimensional de esfuerzos, ¿qué energía de deformación se almacena si $E = 30 \times 10^6$ lb/pulg.² y $\nu = 0,3$? Desprecie el peso del miembro.

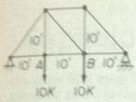


Figura 6.21

6.17 |6.4| Calcule la energía de deformación almacenada en la cercha mostrada en la Figura 6.21. Tome $E = 30 \times 10^6$ lb/pulg.² y $\nu = 0,3$. Utilizando el principio de conservación de la energía calcule el desplazamiento vertical que experimentan a causa de la carga los pasadores A y B. El área de la sección trasversal de los miembros es de 20 pulg.².

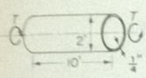


Figura 6.22

6.18 |6.4| Un cilindro de pared delgada (Figura 6.22) se somete a la acción de un torque T de 5.000 lb-pie. El espesor de la pared es $\frac{1}{4}$ pulg. y el diámetro y la longitud corresponden a 2 pulg. y 10 pulg., respectivamente. ¿Qué energía de deformación almacena el cilindro? ¿Qué ángulo de torsión θ relativo entre los extremos se produce? $G = 15 \times 10^6$ lb/pulg.². (*Ayuda*: el trabajo realizado por el torque es $\frac{1}{2}\,T\theta$.)

6.19 |6.4| Si el ángulo de torsión para el cilindro del problema anterior es 1°, ¿qué energía de deformación se ha almacenado y qué torque se ha aplicado?

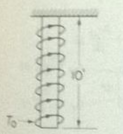


Figura 6.23

6.20 |6.4| Un cilindro de pared delgada (ver la Figura 6.23) está sometido a la acción de un torque uniformemente distribuido T_0 de 10 lb-pie/pie. El diámetro interior del cilindro es de 4 pulg., y el espesor de la pared es de 0,1 pulg. Si $G = 15 \times 10^6$ lb/pulg.², calcule la energía almacenada por deformación.

6.21 |6.4| Si en el extremo A del cilindro del problema anterior actúa también una fuerza axial de 1.000 lbs. Calcule la energía de deformación producida por el torque y la fuerza. Tome $E = 30 \times 10^6$ lb/pulg.².

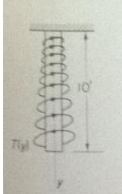


Figura 6.24

6.22 |6.4| Un torque distribuido $T(y)$ actúa sobre un cilindro de pared delgada (ver la Figura 6.24) cuyo diámetro interior es 4 pulg., el espesor de la pared es 0,1 pulg. y su longitud es 10 pies. Si $T(y) = 10y^2$ lb-pie/pie, calcule la energía de deformación almacenada por el cilindro $G = 15 \times 10^6$ lb/pulg.².

6.23 |6.4| ¿Cuál es la energía de deformación por unidad de volumen en un punto de un material elástico lineal sometido al estado de esfuerzos siguiente:

$$\begin{pmatrix} 1.000 & -500 & 2.000 \\ -500 & 2.000 & -400 \\ 2.000 & -400 & -1.000 \end{pmatrix} \text{lb/pulg.}^2$$

Tome $E = 30 \times 10^6$ lb/pulg.² y $G = 15 \times 10^6$ lb/pulg.².

6.24 [6.4] ¿Qué energía de deformación por unidad de volumen almacena en un punto un material elástico lineal sometido al estado de deformación siguiente?

$$\begin{pmatrix} 0{,}001 & -0{,}0005 & 0{,}003 \\ -0{,}0005 & 0{,}002 & -0{,}002 \\ 0{,}003 & -0{,}002 & -0{,}001 \end{pmatrix}$$

El módulo de cizalladura $G = 15 \times 10^6$ lb/pulg.², el módulo de Poisson es 0,3.

•**6.25 [6.5]** En el Problema 2.29 se propuso la obtención de las ecuaciones de equilibrio para el estado plano a partir de un elemento semejante al ilustrado en la Figura 6.25 visto desde la dirección z. Para el estado general de esfuerzos, demuestre que las ecuaciones de equilibrio en las direcciones r y θ son:

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} = 0$$

Para el análisis utilice la Figura 2.47. Utilizando la Figura 6.26 en donde se muestran solamente los esfuerzos que producen fuerzas en la dirección z, demuestre que la ecuación de equilibrio en coordenadas cilíndricas para la dirección z es

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

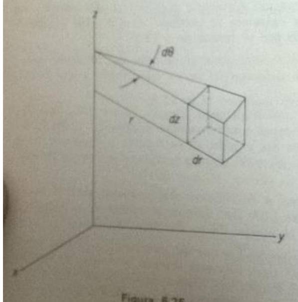


Figura 6.25

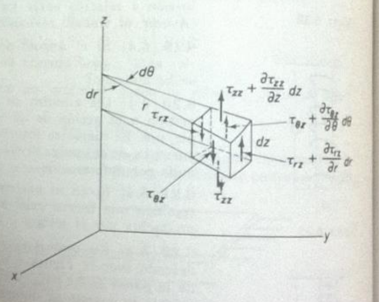


Figura 6.26

Diga si el esfuerzo cortante sobre la superficie HEC en la dirección HC es igual al esfuerzo cortante que actúa sobre la superficie $HDCB$ en la dirección HC, justifique su respuesta. Calcule el valor de estos esfuerzos cortantes y compare los resultados.

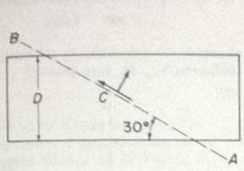


Figura 11.31

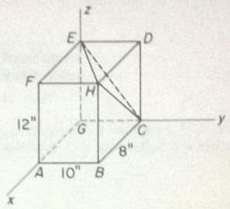


Figura 11.32

11.9 [11.2] El flujo de un fluido se describe mediante el campo de velocidad siguiente:

$$\mathbf{V} = 0{,}16x^2\,\mathbf{i} + 0{,}10y^2\,\mathbf{j} + (0{,}3y - 0{,}52z)\,\mathbf{k} \text{ pie/seg}$$

Durante el estudio de la mecánica de fluidos se demostrará que el esfuerzo en un punto en ciertos fluidos se relaciona con el campo de velocidad mediante las relaciones siguientes (ley de viscosidad de Stokes):

$$\tau_{xx} = \mu\left[2\frac{\partial V_x}{\partial x} - \frac{2}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\right] - p$$

$$\tau_{yy} = \mu\left[2\frac{\partial V_y}{\partial y} - \frac{2}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\right] - p$$

$$\tau_{zz} = \mu\left[2\frac{\partial V_z}{\partial z} - \frac{2}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\right] - p$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$

en donde p es la denominada presión hidrostática y μ es el coeficiente de viscosidad. ¿Qué valor tiene el esfuerzo normal que actúa en el punto (1, 0, 2) sobre una superficie cuya normal es

$$\boldsymbol{\epsilon}_n = 0{,}80\mathbf{i} + 0{,}60\mathbf{j}$$

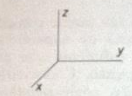
Exprese su respuesta en términos de p y μ.

11.10 [11.2] Se ha utilizado un sistema de referencia cartesiano para identificar el sistema de superficies ortogonales en un punto. Con el mismo propósito puede utilizarse cualquier sistema de coordenadas curvilíneas ortogonales. Utilizando coordenadas cilíndricas en un punto el conjunto de esfuerzos que actúan sobre las superficies ortogonales es

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} \qquad (1)$$

Pp 348-358 (Propiedades del Esfuerzo en un punto)

PROBLEMAS



11.1 [11.2] Las componentes del esfuerzo referidas a superficies ortogonales paralelas al sistema xyz en un punto son

$$\tau_{xx} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{xy} = \tau_{yx} = 200 \text{ lb/pulg.}^2$$
$$\tau_{yy} = -600 \text{ lb/pulg.}^2 \quad \tau_{xz} = \tau_{zx} = 0$$
$$\tau_{zz} = 0 \quad \tau_{yz} = \tau_{zy} = -400 \text{ lb/pulg.}^2$$

¿Qué valor tiene el esfuerzo normal en la dirección $\boldsymbol{\epsilon}$ tal que

$$\boldsymbol{\epsilon} = 0{,}11\mathbf{i} + 0{,}35\mathbf{j} + 0{,}93\mathbf{k}$$

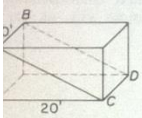


Figura 11.29

11.2 [11.2] El estado de esfuerzo τ_{xx}, τ_{yy}, τ_{zz}, τ_{xy}, τ_{xz}, τ_{yz} es uniforme a través de un cuerpo. ¿Qué implica esto para esfuerzos paralelos que actúan sobre superficies paralelas en puntos diferentes? Suponga que la distribución corresponde a

$$\tau_{xx} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{xy} = \tau_{yx} = 0$$
$$\tau_{yy} = -1.000 \text{ lb/pulg.}^2 \quad \tau_{xz} = \tau_{zx} = 500 \text{ lb/pulg.}^2$$
$$\tau_{zz} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{yz} = \tau_{zy} = -500 \text{ lb/pulg.}^2$$

¿Qué valor tiene el esfuerzo normal que actúa sobre el plano $ABCD$ del paralelepípedo de la Figura 11.29?

11.3 [11.2] Obtenga las Ecuaciones (d) del Ejemplo 2.5 a partir de la ecuación general de trasformación.

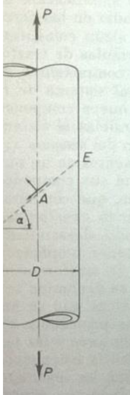


Figura 11.30

11.4 [11.2] Un cilindro de pared delgada (Figura 11.30) está sometido a la acción de una fuerza de tensión P de 1.000 lbs. El diámetro exterior del cilindro mide 3 pulg. y el espesor de la pared mide $\frac{1}{8}$ pulg. ¿Qué valor tienen los esfuerzos normal y cortante que actúan en el punto A de la sección del cilindro que se forma al cortarlo con un plano cuya orientación es $\alpha = 40°$, tal como se ilustra en el diagrama?

11.5 [11.2] Si el ángulo α que orienta al plano EF en el Problema 11.4 puede variarse, ¿qué valor debe tener para que el esfuerzo cortante adquiera su valor máximo en A? En este caso y sitio, ¿qué valor tienen los esfuerzos normal y cortante?

11.6 [11.2] En la Figura 11.31 se representa un tanque de pared delgada sometido a la acción de una presión de 500 lb/pulg.² por encima de la atmosférica. El diámetro exterior del tanque es 2 pies y el espesor de la porción cilíndrica de la pared es $\frac{1}{8}$ pulg. ¿Qué valor tienen los esfuerzos normal y cortante que actúan sobre la superficie C ubicada sobre el plano AB que está inclinado 30° con respecto al eje del tanque, tal como se ilustra en el diagrama, y alejada de los extremos? Desprecie el esfuerzo de compresión producido directamente por la presión de 500 lb/pulg.² sobre la superficie interior de la pared cilíndrica.

11.7 [11.2] Si sobre tres superficies ortogonales en un punto los tres esfuerzos normales son iguales y los esfuerzos cortantes son nulos. Demuestre que sobre cualquier superficie que contenga al punto el esfuerzo normal *también* tiene el mismo valor mencionado y el esfuerzo cortante es nulo. Este estado de esfuerzo se denomina *hidrostático* porque es el que existe en un fluido en reposo.

11.8 [11.2] En la Figura 11.32 se representa un bloque, sobre las superficies $HDBC$ y $AFEG$ los esfuerzos son $\tau_{yy} = 8.000$ lb/pulg.², $\tau_{yx} = 5.000$ lb/pulg.² y $\tau_{yz} = -1.000$ lb/pulg.². Sobre las superficies $FHAB$ y $EDCG$ los esfuerzos son $\tau_{xx} = 6.000$ lb/pulg.², $\tau_{xy} = 5.000$ lb/pulg.² y $\tau_{xz} = 3.000$ lb/pulg.². Finalmente, sobre las superficies $FEDH$ y $AGCB$ los esfuerzos son $\tau_{zz} = -6.000$ lb/pulg.², $\tau_{zx} = 3.000$ lb/pulg.² y $\tau_{zy} = -1.000$ lb/pulg.².

Estos esfuerzos se ilustran en la Figura 11.33. Suponiendo que los esfuerzos (1) corresponden a

$$\begin{pmatrix} 5.000 & 0 & -3.000 \\ 0 & 2.000 & 1.000 \\ -3.000 & 1.000 & 0 \end{pmatrix} \text{lb/pulg.}^2$$

y representan el estado de esfuerzos en el punto $r = 6$, $\theta = 30°$ y $z = 10$. ¿Qué valor tiene el esfuerzo τ_{xx} en este punto?

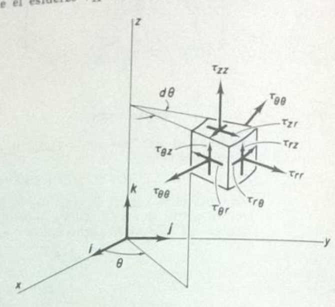


Figura 11.33

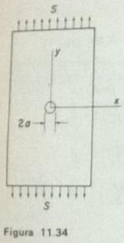


Figura 11.34

11.11 [11.2] En la Figura 11.34 se muestra una placa cargada uniformemente en sus extremos por una fuerza normal de intensidad S lb/pulg². Hay un pequeño agujero de radio a en el centro de la placa. Sin el agujero el único esfuerzo no nulo de acuerdo con el sistema de referencia xyz es

$$\tau_{yy} = S$$

Cuando existe el agujero a partir de la teoría de la elasticidad puede demostrarse que en coordenadas cilíndricas los esfuerzos no nulos corresponden a

$$\tau_{rr} = \frac{S}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(-1 + 4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau_{\theta\theta} = \frac{S}{2}\left[\left(1 + \frac{a^2}{r^2}\right) + \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \frac{S}{2}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\operatorname{sen} 2\theta$$

El esfuerzo máximo se presenta en el agujero. Demuestre que su dirección es tangente al círculo y que su valor es $3S$. Luego, la presencia de un agujero pequeño (inocente) triplica el valor del esfuerzo que existe en el caso en que no hay agujero. Este es un ejemplo de *concentración del esfuerzo* —es decir, un incremento en el esfuerzo producido por la existencia de un agujero o posiblemente, como en otros casos, de un chavetero, una entalladura, etc.

11.12 [11.2] En la situación del Problema 11.11, ¿qué valor tiene el esfuerzo cortante τ_s que actúa en el punto $r = 2$ pies y $\theta = 30°$ (ver la Figura 11.35) sobre una superficie que forma un ángulo de 45° con los ejes x y y? Tome $S = 500$ lb/pulg.^{2} y $a = 0{,}3$ pie.

11.13 [11.2] Los esfuerzos en un punto P y referidos al sistema xyz son

$$\tau_{ij} = \begin{pmatrix} 0 & 200 & 300 \\ 200 & 500 & 0 \\ 300 & 0 & 600 \end{pmatrix}$$

¿Qué valor tienen los esfuerzos $\tau_{x'x'}$ y $\tau_{x'y'}$ referidos a los ejes $x'y'z'$ que corresponden a una rotación de 30°, del sistema xyz alrededor del eje x, tal como se ilustra en la Figura 11.36?

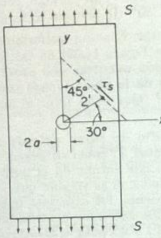


Figura 11.35

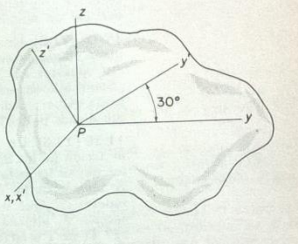


Figura 11.36

11.14 [11.2] El tensor esfuerzo en un punto referido a un sistema xyz es

$$\begin{pmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 300 \end{pmatrix}$$

¿Qué tensor esfuerzo corresponde a un sistema de ejes obtenido mediante la rotación de 45°, del sistema xyz alrededor del eje z, en sentido horario cuando se mira hacia el punto?

11.15 [11.3] Utilizando la Figura 11.11, obtenga las Ecuaciones (11.11) a partir de la ley de Newton y de la reciprocidad del esfuerzo cortante.

11.16 [11.3] Dados los esfuerzos siguientes,

$$\tau_{xx} = 500 \text{ lb/pulg.}^2$$
$$\tau_{yy} = -1.000 \text{ lb/pulg.}^2$$
$$\tau_{xy} = 200 \text{ lb/pulg.}^2$$

¿qué valor tiene el esfuerzo normal que actúa sobre una superficie cuya normal es:

$$\epsilon_1 = 0{,}707\mathbf{i} + 0{,}707\mathbf{j}$$

¿Qué vector unitario ϵ_2 corresponde a un eje perpendicular a ϵ_1 en el plano xy? ¿Qué valor tiene el esfuerzo cortante que actúa sobre las superficies normales a ϵ_1 y a ϵ_2?

11.17 [11.3] El estado de esfuerzo en un punto es

$$\tau_{xx} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{yy} = 0 \quad \tau_{xy} = -500 \text{ lb/pulg.}^2$$

¿qué valor tienen los esfuerzos que actúan sobre las superficies determinadas por los ejes $x'y'$ obtenidos al girar 30° el sistema de coordenadas alrededor del eje z en sentido antihorario cuando se mira en la dirección z hacia el origen?

11.18 [11.3] A una placa de 1 pulg. de espesor (Figura 11.37) la cargan uniformemente las fuerzas F_1 y F_2 cuyos valores son 500 y 2.000 lbs., respectivamente. ¿Qué valor tienen los esfuerzos normales a los planos diagonales? ¿Qué valor tiene el esfuerzo cortante correspondiente a un sistema de coordenadas cuyos ejes forman con las direcciones horizontal y vertical un ángulo de 30° de sentido anti-horario?

11.19 [11.3] Una fuerza F actúa uniformemente sobre los lados de una placa cuadrada (ver la Figura 11.38) en tal forma que sobre un par de lados opuestos hay tensión mientras que sobre los otros existe compresión. ¿Sobre qué superficies de la placa (vistas como líneas en el diagrama) no actúa esfuerzo normal? ¿Qué valor tiene el esfuerzo cortante que actúa sobre dichos planos?

11.20 [11.3] Un cilindro de pared delgada está sometido a torsión por la acción de los momentos T, como se ilustra en la Figura 11.39. ¿Qué valor tienen los esfuerzos que actúan en el punto H de la sección trasversal E, alejada de los extremos, si se supone que los esfuerzos no varían con el espesor t? ¿Qué valor tienen los esfuerzos que actúan en el punto G de la sección F, alejada de los extremos e inclinada 45° con respecto al plano xz? Tome $T = 50$ lb-pie, $t = 0{,}2$ pulg. y $D = 3$ pulg.

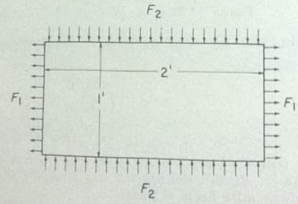


Figura 11.37

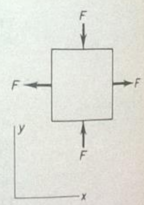


Figura 11.38

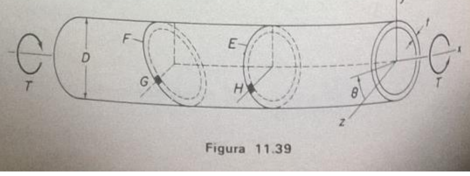


Figura 11.39

11.21 [11.3] Resuelva el Problema 11.20 en el caso en que actúa adicionalmente, en los extremos, una fuerza axial de 100 lbs.

11.22 [11.3] Se demostró que $\tau_{xx} + \tau_{yy}$ es un invariante en un punto es decir, al girar xy y convertirse en $x'y'$ se obtiene el mismo valor para la suma de los esfuerzos normales $\tau_{x'x'} + \tau_{y'y'}$. Demuestre que la cantidad siguiente también es invariante con respecto a la rotación del sistema de referencia en el caso del estado de esfuerzo plano:

$$(\tau_{xx}\tau_{yy} - \tau_{xy}^2)$$

Este es el denominado *segundo invariante tensorial* para el caso del estado de esfuerzo plano.

11.23 [11.3] Se sabe que el esfuerzo τ_{xx} en un punto vale 500 lb/pulg.^{2} y a 60° del eje x el esfuerzo $\tau_{y'y'}$ vale 1.000 lb/pulg.^{2}. Si τ_{xy} = 1.000 lb/pulg.^{2}, ¿qué valor tienen los esfuerzos τ_{yy} y $\tau_{x'x'}$?

11.24 [11.3] Dado el estado de esfuerzo siguiente

$$\tau_{xx} = -1.000 \text{ lb/pulg.}^2$$
$$\tau_{yy} = 500 \text{ lb/pulg.}^2$$
$$\tau_{xy} = 800 \text{ lb/pulg.}^2$$

¿qué valor tienen los esfuerzos principales? ¿Cuáles son los ejes principales?

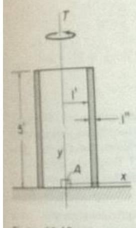


Figura 11.40

11.25 [11.3] Calcule el mínimo esfuerzo normal, en sentido algebraico, en un punto en donde $\tau_{xx} = 500$ lb/pulg.^{2}, $\tau_{yy} = -500$ lb/pulg.^{2}, y $\tau_{xy} = 1.000$ lb/pulg.^{2}. ¿Qué dirección tiene la normal al plano sobre el que actúa este esfuerzo?

11.26 [11.3] Los esfuerzos principales en un punto son 1.000 y 500 lb/pulg.^{2}. ¿Qué valor tienen los esfuerzos que actúan en la dirección de unos ejes que forman un ángulo de 30°, en sentido horario, con los ejes principales?

11.27 [11.3] Un cilindro de pared delgada tiene un extremo fijo, mientras que en el otro actúa un torque T de 50 lb-pie (ver la Figura 11.40). Si $\gamma = 450$ lb/pie^3 para el tubo y si los esfuerzos no varían con la distancia radial medida desde el eje del cilindro y son simétricos con respecto a dicho eje, ¿qué valor tiene el esfuerzo normal máximo, en sentido algebraico, que se desarrolla en el miembro? (*Sugerencia:* Considere el elemento A.)

11.28 [11.3] Para la situación del Problema 11.25, calcule el valor extremo del esfuerzo cortante y determine la dirección de los ejes correspondientes a dicho esfuerzo.

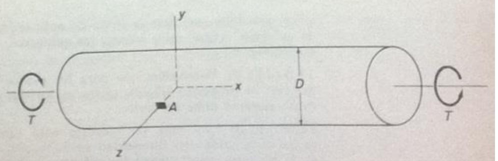


Figura 11.41

11.29 [11.3] Calcule el valor extremo del esfuerzo cortante en el Problema 11.26.

11.30 [11.3] Un tanque cilíndrico de pared delgada (Figura 11.41) está sometido a una presión interna manométrica de 10 lb/pulg.^{2} y a la acción simultánea de un torque T de 200 lb-pie. Si el diámetro exterior D = 1 pie y el espesor del cilindro es 0,1 pulg., ¿qué valor tiene el esfuerzo normal máximo en A, punto alejado de los extremos? ¿Qué valor tiene el esfuerzo cortante máximo en un sitio alejado de los extremos? (*Sugerencia:* Como el esfuerzo de compresión proviene directamente de la presión es muy pequeño con respecto a los otros esfuerzos, por tanto puede despreciarse, convirtiendo en esta forma el estado de esfuerzo, en un sitio alejado de los extremos, en un problema bidimensional.)

11.31 [11.3] Si el esfuerzo normal permisible en el Problema 11.30 es 40.000 lb/pulg.^{2}, ¿a qué presión máxima puede someterse el tanque si el torque vale 2.000 lb-pie?

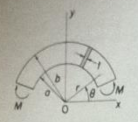


Figura 11.42

11.32 [11.3] En la Figura 11.42 se muestra una viga curva de espesor t. La viga tiene un radio interior a y radio exterior b. Un par de momentos M se aplican a los extremos de la viga en la forma indicada. La teoría de la elasticidad predice la distribución del esfuerzo siguiente:

$$\tau_{rr} = -\frac{4M}{kt}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r}\right)$$

$$\tau_{\theta\theta} = -\frac{4M}{kt}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r} + b^2 - a^2\right)$$

$$\tau_{r\theta} = 0$$

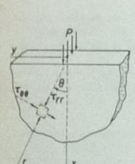


Figura 11.43

en donde $k = (b^2 - a^2)^2 - 4a^2b^2[\ln(b/a)]^2$. Examinando el diagrama, ¿qué valor predice para el esfuerzo τ_{rr} sobre las superficies circulares límites? Verifique si la solución satisface las condiciones anteriores. ¿Qué valor tiene el esfuerzo normal en $r = 3$ pulg. y $\theta = 30°$? Tome $a = 2$ pulg., b = 4 pulg., $M = 50$ lb-pie y $t = 0{,}2$ pulg. ¿Qué valor tienen los esfuerzos normal y cortante máximos?

11.33 [11.3] Una carga vertical distribuida longitudinalmente actúa sobre una placa infinita (ver la Figura 11.43). La fuerza total correspondiente a esta distribución es P. La placa queda sometida a un estado de esfuerzo plano; la teoría de la elasticidad predice los valores siguientes para las componentes del esfuerzo expresadas en coordenadas cilíndricas:

$$\tau_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

$$\tau_{\theta\theta} = 0$$

$$\tau_{r\theta} = \tau_{\theta r} = 0$$

¿Por qué debe excluirse la línea de aplicación de la carga del dominio de la solución? ¿Qué valor tienen los esfuerzos τ_{xx}, τ_{yy} y τ_{xy} en cualquier posición r, θ?

11.34 [11.3] Demuestre que para los planos normales a los ejes definidos por la Ecuación (11.18), es decir, los ejes principales, el esfuerzo cortante siempre debe ser nulo.

11.35 [11.3] Una viga en voladizo está sometida a la acción de una carga distribuida con intensidad de 50 lb/pie, como se ilustra en la Figura 11.44. ¿Qué valor tienen los esfuerzos principales en A, que está ubicado a 1 pulg. de la superficie superior en la sección intermedia de viga?

11.36 [11.3] Un eje macizo (ver la Figura 11.45) está sometido a la acción de un torque de 500 lb-pulg. y de una fuerza de tensión de 1.000 lbs. ¿Qué valor tienen los esfuerzos normal y cortante máximos?

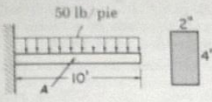


Figura 11.44

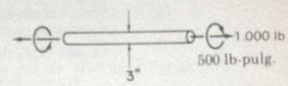


Figura 11.45

11.37 [11.4] Dibuje el círculo de Mohr para el estado de esfuerzo siguiente:

$$\tau_{xx} = -500 \text{ lb/pulg.}^2$$
$$\tau_{yy} = 500 \text{ lb/pulg.}^2$$
$$\tau_{xy} = 0 \text{ lb/pulg.}^2$$

¿Qué valor extremo tiene el esfuerzo cortante y qué dirección tiene la normal al plano sobre el que actúa este esfuerzo?

11.38 [11.4] Dibuje el círculo de Mohr para el caso en que $\tau_{xx} = 0$, $\tau_{yy} = 0$ y $\tau_{xy} = 500$ lb/pulg.^{2}. ¿Qué valor tienen los esfuerzos principales y qué ejes corresponden a estos esfuerzos?

11.39 [11.4] Dado el estado de esfuerzo siguiente,

$$\tau_{xx} = 500 \text{ lb/pulg.}^2 \quad \tau_{yy} = -800 \text{ lb/pulg.}^2 \quad \tau_{xy} = -300 \text{ lb/pulg.}^2$$

Localice sobre el círculo de Mohr los esfuerzos que actúan sobre un plano que forma un ángulo de 30° en sentido horario con el eje x.

11.40 [11.4] Para los esfuerzos

$$\tau_{xx} = 2.000 \text{ lb/pulg.}^2$$
$$\tau_{yy} = -1.000 \text{ lb/pulg.}^2$$
$$\tau_{xy} = -500 \text{ lb/pulg.}^2$$

calcule los esfuerzos correspondientes a los ejes que se obtienen al girar en sentido horario, el sistema xy un ángulo de 45°. Calcule los esfuerzos principales. Utilice el círculo de Mohr como auxiliar en los cálculos.

11.41 [11.4] Dados los esfuerzos no nulos siguientes que actúan en un punto,

$$\tau_{xx} = -3.000 \text{ lb/pulg.}^2$$
$$\tau_{yy} = 4.000 \text{ lb/pulg.}^2$$
$$\tau_{xy} = 2.000 \text{ lb/pulg.}^2$$

calcule el esfuerzo normal correspondiente a un eje que forma un ángulo de 22,5° en sentido horario con el eje x. ¿Qué valor tiene el esfuerzo cortante máximo en el punto? Utilice el círculo de Mohr como auxiliar en los cálculos.

11.42 [11.4] Dado el tensor esfuerzo

$$\tau_{ij} = \begin{pmatrix} 2.000 & 5.000 & -1.000 \\ 5.000 & -3.000 & 3.000 \\ -1.000 & 3.000 & 500 \end{pmatrix} \text{lb/pulg.}^2$$

¿qué valor tienen los esfuerzos normal máximo y mínimo en sentido algebraico que actúan sobre los planos normales al xy?

11.43 [11.4] Si los esfuerzos principales en un punto son $\tau_1 = 5.000$ lb/pulg.^{2}, $\tau_2 = 2.000$ lb/pulg.^{2} y $\tau_3 = 1.000$ lb/pulg.^{2}, dibuje los círculos de Mohr correspondientes al estado de esfuerzo en el punto. ¿Qué valor tiene el esfuerzo cortante máximo en el punto?

11.44 [11.4] Dado el estado de esfuerzo siguiente en un punto,

$$\tau_{xx} = -3.000 \text{ lb/pulg.} \quad \tau_{yy} = 4.000 \text{ lb/pulg.}^2 \quad \tau_{xy} = 1.000 \text{ lb/pulg.}^2$$

¿qué ángulo debe girarse a partir del eje x para alcanzar el eje para el que el esfuerzo normal es -2.000 lb/pulg.^{2} y el esfuerzo cortante es positivo? Utilice el círculo de Mohr como una ayuda.

11.45 [11.4] Considérese el estado de esfuerzo de la Figura 11.46. Si el sistema de referencia gira alrededor del eje z, pruebe que las ecuaciones de trasformación para $\tau_{x'x'}$, $\tau_{y'y'}$ y $\tau_{x'y'}$ corresponden a las dadas para el caso plano. Por tanto el círculo de Mohr puede utilizarse en este caso. Utilizándolo calcule el esfuerzo $\tau_{y'y'}$ que corresponde a una rotación horaria, $\theta = 30°$, del eje y alrededor del eje z. Tome

$$\tau_{xx} = 500 \text{ lb/pulg.}^2 \quad \tau_{yy} = 1.000 \text{ lb/pulg.}^2$$
$$\tau_{zz} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{xy} = 600 \text{ lb/pulg.}^2$$

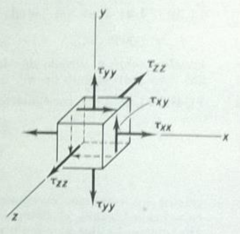


Figura 11.46

11.46 [11.5] ¿Qué esfuerzos principales corresponden al estado de esfuerzo del Problema 11.37?

11.47 [11.5] Dado el siguiente estado de esfuerzo,

$$\tau_{xx} = 0 \text{ lb/pulg.}^2 \quad \tau_{xy} = -600 \text{ lb/pulg.}^2$$
$$\tau_{yy} = 600 \text{ lb/pulg.}^2 \quad \tau_{xz} = 0 \text{ lb/pulg.}^2$$
$$\tau_{zz} = 0 \text{ lb/pulg.}^2 \quad \tau_{yz} = -300 \text{ lb/pulg.}^2$$

¿qué valor tienen los esfuerzos principales y qué dirección corresponde al esfuerzo mínimo en sentido algebraico?

11.48 [11.5] El estado de esfuerzo que existe en un punto es,

$$\tau_{xx} = 1.000 \text{ lb/pulg.}^2 \quad \tau_{yy} = 0 \quad \tau_{zz} = 500 \text{ lb/pulg.}^2$$
$$\tau_{xz} = 500 \text{ lb/pulg.}^2 \quad \tau_{xy} = \tau_{yz} = 0$$

¿Qué valor y dirección tienen los esfuerzos principales?

11.49 [11.5] Dado el tensor esfuerzo

$$\tau_{ij} = \begin{pmatrix} 5.000 & 0 & -3.000 \\ 0 & 2.000 & 1.000 \\ -3.000 & 1.000 & 0 \end{pmatrix} \text{lb/pulg.}^2$$

calcule los esfuerzos principales si uno de ellos es 2.188 lb/pulg.^{2}.

11.50 [11.5] Se sabe que uno de los esfuerzos principales para el estado de esfuerzo siguiente es 749 lb/pulg.^{2}:

$$\tau_{ij} = \begin{pmatrix} 0 & 200 & 300 \\ 200 & 500 & 0 \\ 300 & 0 & 600 \end{pmatrix} \text{lb/pulg.}^2$$

¿Qué valor tiene el esfuerzo de compresión máximo en el punto y qué dirección tiene?

11.51 [11.5] Dado el estado de esfuerzo siguiente, calcule los esfuerzos principales y la dirección del esfuerzo máximo,

$$\tau_{xx} = -800 \text{ lb/pulg.}^2 \quad \tau_{xy} = 300 \text{ lb/pulg.}^2$$
$$\tau_{yy} = 500 \text{ lb/pulg.}^2 \quad \tau_{yz} = 0$$
$$\tau_{zz} = 200 \text{ lb/pulg.}^2 \quad \tau_{xz} = 500 \text{ lb/pulg.}^2$$

11.52 [11.5] Pruebe que el segundo invariante tensorial para el esfuerzo corresponde a la suma de los menores de la diagonal principal (la diagonal constituida por los esfuerzos normales). Pruebe que el tercer invariante tensorial del esfuerzo es el determinante del tensor esfuerzo.

11.53 [11.6] Dada la información siguiente,

$$A_x = 10 \quad B_x = 3 \quad C_x = 16$$
$$A_y = -15 \quad B_y = -4 \quad C_y = 12$$
$$A_z = 6 \quad B_z = 1 \quad C_z = -3$$

Calcule $A_i C_i B_3$ y $A_i B_i A_2$.

11.54 [11.6] Utilizando los datos del Problema 11.53, calcule los términos dados por

(a) $A_i B_i C_j$
(b) $C_i B_k C_i$
(c) $A_i B_i C_j A_k$

11.55 [11.6] Exprese el producto escalar $\mathbf{A} \cdot \mathbf{B}$ en notación indicial.

11.56 [11.6] Pruebe que

$$A_{i'} = a_{i'j} A_j$$

representa la ecuación de trasformación para calcular la componente A'_i y que

$$\tau_{i'j'} = a_{i'k} a_{j'l} \tau_{kl}$$

representa la ecuación de trasformación para calcular el esfuerzo $\tau_{x'x'}$.

11.57 [11.6] El *delta de Kronecker* se define como:

$$\delta_{ij} = 0 \quad \text{si } i \neq j$$
$$\delta_{ij} = 1 \quad \text{si } i = j$$

¿Qué valor tiene $\delta_{ij}\delta_{ij}$?

11.58 [11.6] Dado el tensor esfuerzo

$$\tau_{ij} = \begin{pmatrix} 1.000 & 0 & -5.000 \\ 0 & 3.000 & 4.000 \\ -5.000 & 4.000 & -3.000 \end{pmatrix} \text{lb/pulg.}^2$$

Calcule $\tau_{ij}\delta_{ij}$.

11.59 [11.6] Pruebe que si λ es una constante y A_{ij} es tensor de segundo orden, entonces λA_{ij} es un tensor de segundo orden.

11.60 [11.6] Pruebe que si A_{ij} y B_{ij} son tensores de segundo orden, entonces $A_{ij} + B_{ij}$ es un tensor de segundo orden C_{ij}, que se obtiene mediante la suma de los elementos correspondientes de A_{ij} y B_{ij}.

11.61 [11.6] ¿Qué ecuación rige la trasformación de los tensores de cuarto orden?

•**11.62 [11.6]** ¿Cómo se expresa en notación matricial la ecuación de trasformación de los tensores de segundo orden?

11.63 [11.6] Si A_i y B_j son vectores, pruebe que $A_i B_j$ es un tensor de segundo orden.

•**11.64 [11.6]** Pruebe que las ecuaciones básicas de la elasticidad lineal para materiales isotrópicos se escriben notación indicial como:

$$\frac{\partial \tau_{ij}}{\partial x_j} + B_i = 0$$

$$\epsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\tau_{kk}\delta_{ij}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

El término δ_{ij} es el denominado delta de Kronecker (ver el Problema 11.57).

Pp 378-383 (Analisis de la deformacion en un punto)

PROBLEMAS

12.1 [12.2] Dado el campo de desplazamiento siguiente

$$\mathbf{u} = [(6x^2)\mathbf{i} + (3 + zx)\mathbf{j} + (-2 + xy)\mathbf{k}] \times 10^{-2} \text{ pies}$$

¿qué deformación normal existe en el punto (0, 1, 3) en la dirección $\mathbf{s} = 0{,}3\mathbf{i} + 0{,}4\mathbf{j} + 0{,}866\mathbf{k}$? Resuelva el problema (a) utilizando la relación $\partial u_s/\partial s = \epsilon_{ss}$ y (b) mediante una rotación de ejes.

12.2 [12.2] Dado el estado de deformación siguiente en un punto,

$$\epsilon_{ij} = \begin{pmatrix} 0{,}02 & 0{,}01 & 0{,}0 \\ 0{,}01 & -0{,}02 & 0{,}03 \\ 0{,}0 & 0{,}03 & 0{,}04 \end{pmatrix}$$

¿qué valor tiene la deformación normal en una dirección igualmente inclinada con respecto a los ejes x, y y z? ¿Qué valor tiene la deformación cortante para los ejes?

$$\epsilon_1 = 0{,}6\mathbf{i} + 0{,}8\mathbf{j}, \epsilon_2 = 0{,}4\mathbf{i} - 0{,}3\mathbf{j} + 0{,}866\mathbf{k}$$

12.3 [12.2] Dadas las deformaciones siguientes referidas al sistema xyz,

$$\begin{pmatrix} 0{,}02 & 0{,}01 & 0 \\ 0{,}01 & 0{,}03 & -0{,}04 \\ 0{,}0 & -0{,}04 & 0 \end{pmatrix}$$

¿qué valor tienen las deformaciones $\epsilon_{x'x'}$ y $\epsilon_{x'y'}$ para un sistema de referencia $x'y'z'$ que se obtiene girando el sistema original 45° alrededor del eje y en sentido horario cuando se mira desde el eje hacia el origen?

12.4 [12.2] En la Figura 12.16 se muestra una placa sometida a la acción de cargas que se distribuyen uniformemente sobre sus aristas. A consecuencia de estas cargas, las superficies AB y CD se separan 0,1 pies, mientras que BC y AD se separan 0,2 pies. El espesor original de la placa es 0,1 pies. Si la placa no experimenta cambio en el volumen, ¿qué valor tienen las deformaciones $\epsilon_{x'x'}$ y $\epsilon_{x'x'}$ correspondientes a los ejes $x'y'z'$ que forman un ángulo de 30° en sentido antihorario con el sistema de referencia xy?

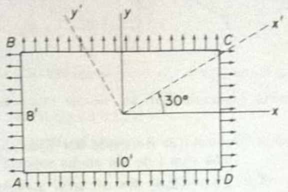


Figura 12.16

12.5 [12.2] Pruebe que los ejes principales del esfuerzo corresponden a los ejes principales de la deformación en un material que obedece a la ley de Hooke.

12.6 [12.2] El tensor esfuerzo en un punto corresponde a

$$\tau_{ij} = \begin{pmatrix} 3.000 & 4.000 & 0 \\ 4.000 & 5.000 & -2.000 \\ 0 & -2.000 & -7.000 \end{pmatrix} \text{ lb/pulg.}^2$$

Si $E = 30 \times 10^6$ lb/pulg.^{2} y $\nu = 0{,}3$, ¿qué valor tiene la deformación normal en la dirección

$$\epsilon = 0{,}6\mathbf{i} + 0{,}8\mathbf{j}$$

en el punto considerado?

12.7 [12.2] Un plástico fluye en un molde para adquirir la conformación deseada. El campo de velocidad en algún instante t en una región pequeña es

$$V = [(x^2 + y)\mathbf{i} + (zy)\mathbf{j} + (2 + zy)\mathbf{k}] \times 10^{-2}$$

¿Qué valor tienen las componentes de la velocidad de deformación $\dot{\epsilon}_{x'x'}$, $\dot{\epsilon}_{x'y'}$ correspondientes a unos ejes que se obtienen girando el sistema inicial 30° en sentido horario, cuando se mira desde el eje hacia el origen, alrededor del eje z? Calcule los valores para la posición (1, 3, 4).

12.8 [12.2] Una viga de ala ancha 18**WF**60 está conectada a un alambre cuya tensión es de 6.000 lbs (ver la Figura 12.17). En A cuya posición es

$$\mathbf{r}_A = 10\mathbf{i} + \tfrac{1}{2}\mathbf{j} \text{ pies}$$

se hace la ranura que se ilustra en el diagrama. El lado de la ranura antes de ensamblar el sistema mide 0,1 pulg. ¿Qué incremento en longitud experimentan las aristas de la ranura durante el ensamblaje? ¿Qué cambio experimentan los ángulos que forman las aristas? Tome $E = 30 \times 10^6$ lb/pulg.^{2}

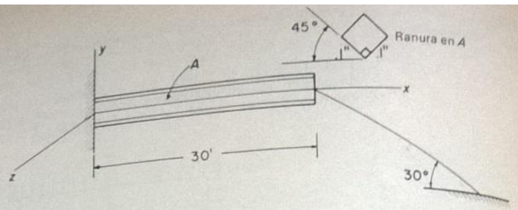


Figura 12.17

y $\nu = 0{,}3$. Recuerde del Ejemplo 8.6 que puede suponerse que el nervio de una viga I de ala ancha soporta uniformemente toda la fuerza cortante.

12.9 [12.3] En un sitio alejado de los extremos en la represa ilustrada en la Figura 12.4(b) existen las deformaciones siguientes:

$$\epsilon_{xx} = -0{,}0003$$
$$\epsilon_{yy} = -0{,}0005$$
$$\epsilon_{xy} = 0{,}0001$$

¿Qué valor tienen las deformaciones referidas a los ejes $x'y'$ que corresponden a una rotación de 30° del sistema inicial alrededor del eje z y con sentido horario cuando se mira en la dirección del eje hacia el origen?

12.10 [12.3] Determine los ejes y las deformaciones principales en el plano xy, correspondientes al estado de deformación del Problema 12.9.

12.11 [12.3] Para analizar las deformaciones en el cilindro de pared gruesa de la Figura 12.18, que está restringido en los extremos por paredes inmóviles y sometido a presión interior, es conveniente utilizar coordenadas cilíndricas. Las deformaciones correspondientes al elemento ilustrado en la Figura 12.18 son:

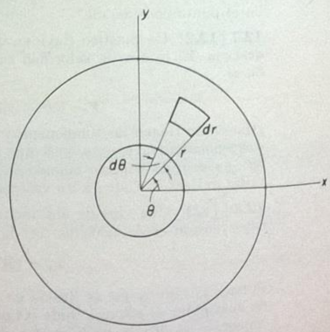


Figura 12.18

ϵ_{rr} ≡ Deformación normal en la dirección radial

$\epsilon_{\theta\theta}$ ≡ Deformación normal en la dirección trasversal

$\epsilon_{r\theta}$ ≡ Deformación cortante entre los segmentos dr y $rd\theta$ del elemento

Para un punto en el tubo ubicado en $\theta = 30°$ se conoce que

$$\epsilon_{rr} = -0{,}002$$
$$\epsilon_{\theta\theta} = 0{,}003$$
$$\epsilon_{r\theta} = 0{,}001$$

¿qué valor tienen las deformaciones ϵ_{xx} y ϵ_{yy} en el punto?

12.12 [12.3] ¿Qué valor tienen las deformaciones principales en el punto considerado en el Problema 12.11?

12.13 [12.3] En un punto de cuerpo en estado de deformación plano las deformaciones son:

$$\epsilon_{xx} = 0{,}005 \quad \epsilon_{yy} = -0{,}002 \quad \epsilon_{xy} = 0{,}001$$

¿Qué valor y dirección tienen las deformaciones principales? Dibuje el círculo de Mohr correspondiente y verifique sus resultados.

12.14 [12.3] Las deformaciones en un punto de un cuerpo en estado de deformación plano son:

$$\epsilon_{xx} = 0{,}002 \quad \epsilon_{yy} = 0 \quad \epsilon_{xy} = -0{,}003$$

¿Qué valor tienen las deformaciones correspondientes a un sistema de ejes que forman un ángulo de 30°, en sentido horario, con los ejes xy? Utilizando el círculo de Mohr verifique sus respuestas.

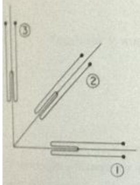


Figura 12.19

12.15 [12.4] Una roseta rectangular (Figura 12.19) mide las deformaciones siguientes sobre la superficie de un cuerpo:

Extensómetro 1 = 0,002
Extensómetro 2 = −0,001
Extensómetro 3 = −0,001

¿Qué valor y dirección tiene la deformación normal máxima en sentido algebraico en la dirección tangente a la superficie?

12.16 [12.4] Una roseta equiangular montada sobre la superficie de un cuerpo (ver la Figura 12.20) da las lecturas siguientes:

Extensómetro 1 = 0,001
Extensómetro 2 = 0,003
Extensómetro 3 = −0,002

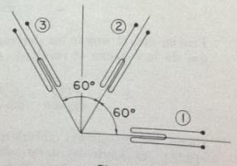


Figura 12.20

¿Qué valor y dirección tiene la deformación normal máxima en sentido algebraico en la dirección tangente a la superficie?

12.17 [**12.4**] El estado de deformación en un punto es,

$$\epsilon_{ij} = \begin{pmatrix} 0{,}002 & 0{,}003 & 0{,}001 \\ 0{,}003 & -0{,}002 & 0{,}005 \\ 0{,}001 & 0{,}005 & -0{,}004 \end{pmatrix}$$

¿qué valor tienen las deformaciones normales máxima y mínima en el plano xy?

12.18 [**12.4**] Para el estado de deformación del Problema 12.17, ¿qué valor tienen las deformaciones normales máxima y mínima en el plano yz?

12.19 [**12.4**] El estado de deformación en un punto es

$$\begin{pmatrix} 0 & 0{,}02 & 0 \\ 0{,}02 & -0{,}01 & -0{,}03 \\ 0 & -0{,}03 & 0 \end{pmatrix}$$

¿qué valor tienen las deformaciones normales máxima y mínima?

12.20 [**12.4**] La deformación en un punto es

$$\begin{pmatrix} 0{,}02 & 0 & -0{,}01 \\ 0 & -0{,}01 & 0 \\ -0{,}01 & 0 & 0{,}03 \end{pmatrix}$$

¿qué dirección tiene la deformación normal mínima en sentido algebraico?

***12.21** [**12.4**] Pruebe las relaciones deformación-desplazamiento del Capítulo 3, pueden hallarse a partir de la ecuación

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{12.21}$$

en donde u_i es monovalente y continua.

***12.22** [**12.4**] Utilizando las relaciones deformación-desplazamiento en notación indicial. Ecuación (12.21), obtenga nuevas ecuaciones calculando las derivadas parciales de segundo orden siguientes:

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} \qquad \frac{\partial^2 \epsilon_{lj}}{\partial x_k \partial x_i}$$

$$\frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} \qquad \frac{\partial^2 \epsilon_{ki}}{\partial x_l \partial x_j}$$

Pruebe que al sumar las derivadas de la izquierda y sustraerles las derivadas de la derecha el resultado es nulo. Es decir,

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{lj}}{\partial x_k \partial x_i} - \frac{\partial^2 \epsilon_{ki}}{\partial x_l \partial x_j} = 0 \tag{12.22}$$

Como las ecuaciones anteriores se obtuvieron a partir de la Ecuación (12.21), en donde las deformaciones se relacionan con un campo de desplazamiento monovalente, *deben ser satisfechas por campos de deformación compatibles*. En otros términos, son las *ecuaciones de compatibilidad* en notación indicial.

***12.23** [**12.4**] ¿Cuántas ecuaciones se representan en la Ecuación (12.22)? Sólo seis de ellas son independientes. Obtenga las Ecuaciones (3.13) a partir de la Ecuación (12.22).

Barber, "Elasticity"
Pp 23-24 (introduction)

PROBLEMS

1. Show that

$$\text{(i)}\quad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{and} \quad \text{(ii)}\quad R = \sqrt{x_i x_i}\,,$$

where $R = |\mathbf{R}|$ is the distance from the origin. Hence find $\partial R/\partial x_j$ in index notation. Confirm your result by finding $\partial R/\partial x$ in x, y, z notation.

2. Prove that the partial derivatives $\partial^2 f/\partial x^2; \partial^2 f/\partial x\partial y; \partial^2 f/\partial y^2$ of the scalar function $f(x, y)$ transform into the rotated coördinate system x', y' by rules similar to equations (1.15–1.17).

3. Show that the direction cosines defined in (1.19) satisfy the identity

$$l_{ij} l_{ik} = \delta_{jk}\,.$$

Hence or otherwise, show that the product $\sigma_{ij}\sigma_{ij}$ is invariant under coördinate transformation.

4. By restricting the indices i, j etc. to the values 1,2 only, show that the two-dimensional stress transformation relations (1.15–1.17) can be obtained from (1.22) using the two-dimensional direction cosines (1.20).

5. Use the index notation to develop concise expressions for the three stress invariants I_1, I_2, I_3 and the equivalent tensile stress σ_E.

6. Choosing a local coördinate system x_1, x_2, x_3 aligned with the three principal axes, determine the tractions on the *octahedral plane* defined by the unit vector

$$n = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}^T$$

which makes equal angles with all three principal axes, if the principal stresses are $\sigma_1, \sigma_2, \sigma_3$. Hence show that the magnitude of the resultant shear stress on this plane is $\sqrt{2}\sigma_E/3$, where σ_E is given by equation (1.31).

7. A rigid body is subjected to a small rotation $\omega_z = \Omega \ll 1$ about the z-axis. If the displacement of the origin is zero, find expressions for the three displacement components u_x, u_y, u_z as functions of x, y, z.

8. Use the index notation to develop a general expression for the derivative

$$\frac{\partial u_i}{\partial x_j}$$

in terms of strains and rotations.

9. Use the three-dimensional vector transformation rule (1.19) and the index notation to prove that the strain components (1.51) transform according to the equation

$$e'_{ij} = l_{ip} l_{jq} e_{pq}\,.$$

Hence show that the dilatation e_{ii} is invariant under coördinate transformation.

10. Find an index notation expression for the compliance tensor s_{ijkl} of equation (1.55) for the isotropic elastic material in terms of the elastic constants E, ν.

11. Show that equations (1.58–1.60, 1.64) can be written in the concise form

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\delta_{ij}\sigma_{mm}}{E}\,. \qquad (1.77)$$

Pp 32-33 (Plane Stress and Plane Strain)

PROBLEMS

1. Show that, if there are no body forces, the dilatation e must satisfy the condition

$$\nabla^2 e = 0\,.$$

2. Show that, if there are no body forces, the rotation ω must satisfy the condition

$$\nabla^2 \omega = 0\,.$$

3. One way of satisfying the compatibility equations in the absence of rotation is to define the components of displacement in terms of a potential function ψ through the relations

$$u_x = \frac{\partial\psi}{\partial x}\;;\; u_y = \frac{\partial\psi}{\partial y}\;;\; u_z = \frac{\partial\psi}{\partial z}\,.$$

Use the stress-strain relations to derive expressions for the stress components in terms of ψ.

Hence show that the stresses will satisfy the equilibrium equations in the absence of body forces if and only if

$$\nabla^2\psi = \text{constant}\,.$$

4. Plastic deformation during a manufacturing process generates a state of residual stress in the large body $z>0$. If the residual stresses are functions of z only and the surface $z=0$ is not loaded, show that the stress components $\sigma_{yz}, \sigma_{zx}, \sigma_{zz}$ must be zero everywhere.

5. By considering the equilibrium of a small element of material similar to that shown in Figure 1.2, derive the three equations of equilibrium in cylindrical polar coördinates r, θ, z.

6. In cylindrical polar coördinates, the strain-displacement relations for the 'in-plane' strains are

$$e_{rr} = \frac{\partial u_r}{\partial r}\;;\; e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\;;\; e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\,.$$

Use these relations to obtain a compatibility equation that must be satisfied by the three strains.

7. If no stresses occur in a body, an increase in temperature T causes unrestrained thermal expansion defined by the strains

$$e_{xx} = e_{yy} = e_{zz} = \alpha T\;;\; e_{xy} = e_{yz} = e_{zx} = 0\,.$$

Show that this is possible only if T is a linear function of x, y, z and that otherwise stresses must be induced in the body, regardless of the boundary conditions.

8. If there are no body forces, show that the equations of equilibrium and compatibility imply that

$$(1+\nu)\frac{\partial^2\sigma_{ij}}{\partial x_k\partial x_k} + \frac{\partial^2\sigma_{kk}}{\partial x_i\partial x_j} = 0\,.$$

9. Using the strain-displacement relation (1.51), show that an alternative statement of the compatibility condition is that the tensor

$$C_{ijkl} \equiv \frac{\partial^2 e_{ij}}{\partial x_k\partial x_l} + \frac{\partial^2 e_{kl}}{\partial x_i\partial x_j} - \frac{\partial^2 e_{il}}{\partial x_j\partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i\partial x_l} = 0\,.$$

Pp 43-44 (Equilibrium and Compatibility)

PROBLEMS

1. The plane strain solution for the stresses in the rectangular block $0<x<a, -b<y<b, -c<z<c$ with a given loading is

$$\sigma_{xx} = \frac{3Fxy}{2b^3}\;;\; \sigma_{xy} = \frac{3F(b^2-y^2)}{4b^3}\;;\; \sigma_{yy} = 0\;;\; \sigma_{zz} = \frac{3\nu Fxy}{2b^3}\,.$$

Find the tractions on the surfaces of the block and illustrate the results on a sketch of the block.

We wish to use this solution to solve the corresponding problem in which the surfaces $z=\pm c$ are traction-free. Determine an approximate corrective solution for this problem by offloading the unwanted force and moment resultants using the elementary bending theory. Find the maximum error in the stress σ_{zz} in the corrected solution and compare it with the maximum tensile stress in the plane strain solution.

2. For a solid in a state of plane stress, show that if there are body forces p_x, p_y per unit volume in the direction of the axes x, y respectively, the compatibility equation can be expressed in the form

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = -(1+\nu)\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right)\,.$$

Hence deduce that the stress distribution for any particular case is independent of the material constants and the body forces, provided the latter are constant.

3.(i) Show that the compatibility equation (2.8) is satisfied by unrestrained thermal expansion ($e_{xx} = e_{yy} = \alpha T, e_{xy} = 0$), provided that the temperature, T, is a two-dimensional harmonic function — i.e.

Pp 51-53 (Stress Function Formulation)

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0\ .$$

(ii) Hence deduce that, subject to certain restrictions which you should explicitly list, no thermal stresses will be induced in a thin body with a steady-state two-dimensional temperature distribution and no boundary tractions.

(iii) Show that an initially straight line on such a body will be distorted by the heat flow in such a way that its curvature is proportional to the local heat flux across it.

4. Find the inverse relations to equations (3.18) — i.e. the substitutions that should be made for the elastic constants E,ν in a plane strain solution if we want to recover the solution of the corresponding plane stress problem.

5. Show that in a state of plane stress without body forces, the in-plane displacements must satisfy the equations

$$\nabla^2 u_x+\left(\frac{1+\nu}{1-\nu}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)=0\ ;\ \nabla^2 u_y+\left(\frac{1+\nu}{1-\nu}\right)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)=0\ .$$

6. Show that in a state of plane strain without body forces,

$$\frac{\partial e}{\partial x}=\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial y}\ ;\ \frac{\partial e}{\partial y}=-\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial x}\ .$$

7. If a material is incompressible ($\nu=0.5$), a state of hydrostatic stress $\sigma_{xx}=\sigma_{yy}=\sigma_{zz}$ produces no strain. One way to write the corresponding stress-strain relations is

$$\sigma_{ij}=2\mu e_{ij}-q\delta_{ij}\ ,$$

where q is an unknown hydrostatic pressure which will generally vary with position. Also, the condition of incompressibility requires that the dilatation

$$e\equiv e_{kk}=0\ .$$

Show that under plain strain conditions, the stress components and the hydrostatic pressure q must satisfy the equations

$$\nabla^2 q=\operatorname{div}\boldsymbol{p}\ \text{ and }\ \sigma_{xx}+\sigma_{yy}=-2q\ ,$$

where $\boldsymbol{p}$ is the body force.

PROBLEMS

1. Newton's law of gravitation states that two heavy particles of mass m_1, m_2 respectively will experience a mutual attractive force

$$F=\frac{\gamma m_1 m_2}{R^2}\ ,$$

where R is the distance between the particles and γ is the gravitational constant. Use an energy argument and superposition to show that the force acting on a particle of mass m_0 can be written

$$\boldsymbol{F}=-\gamma m_0 \nabla V\ ,$$

where

$$V(x,y,z)=-\iiint_\Omega \frac{\rho(\xi,\eta,\zeta)d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\ ,$$

Ω represents the volume of the universe and ρ is the density of material in the universe, which will generally be a function of position (ξ,η,ζ).

Could a similar method have been used if Newton's law had been of the more general form

$$F=\frac{\gamma m_1 m_2}{R^\lambda}\ .$$

If so, what would have been the corresponding expression for V? If not, why not?

2. An ionized liquid in an electric field experiences a body force $\boldsymbol{p}$. Show that the liquid can be in equilibrium only if $\boldsymbol{p}$ is a conservative vector field. **Hint:** Remember that a stationary liquid must be everywhere in a state of hydrostatic stress.

3. An *antiplane* state of stress is one for which the only non-zero stress components are σ_{zx},σ_{zy} and these are independent of z. Show that two of the three equilibrium equations are then satisfied identically if there is no body force. Use a technique similar to that of §4.3 to develop a representation of the non-zero stress components in terms of a scalar function, such that the remaining equilibrium equation is satisfied identically.

4. If a body of fairly general axisymmetric shape is loaded in torsion, the only non-zero stress components in cylindrical polar coördinates are $\sigma_{\theta r},\sigma_{\theta z}$ and these are required to satisfy the equilibrium equation

$$\frac{\partial \sigma_{\theta r}}{\partial r}+\frac{2\sigma_{\theta r}}{r}+\frac{\partial \sigma_{\theta z}}{\partial z}=0\ .$$

Use a technique similar to that of §4.3 to develop a representation of these stress components in terms of a scalar function, such that the equilibrium equation is satisfied identically.

5.(i) Show that the function

$$\phi=y\omega+\psi$$

satisfies the biharmonic equation if ω,ψ are both *harmonic* (i.e. $\nabla^2\omega=0, \nabla^2\psi=0$).

(ii) Develop expressions for the stress components in terms of ω,ψ, based on the use of ϕ as an Airy stress function.

(iii) Show that a solution suitable for the half-plane $y>0$ subject to normal surface tractions only (i.e. $\sigma_{xy}=0$ on $y=0$) can be obtained by writing

$$\omega=-\frac{\partial\psi}{\partial y}$$

and hence that under these conditions the normal stress σ_{xx} near the surface $y=0$ is equal to the applied traction σ_{yy}.

(iv) Do you think this is a rigorous proof? Can you think of any exceptions? If so, at what point in your proof of section (iii) can you find a lack of generality?

6. The constitutive law for an orthotropic elastic material in plane stress can be written

$$e_{xx}=s_{11}\sigma_{xx}+s_{12}\sigma_{yy}\ ;\ e_{yy}=s_{12}\sigma_{xx}+s_{22}\sigma_{yy}\ ;\ e_{xy}=s_{44}\sigma_{xy}\ ,$$

where $s_{11},s_{12},s_{22},s_{44}$ are elastic constants.

Using the Airy stress function ϕ to represent the stress components, find the equation that must be satisfied by ϕ.

7. Show that if the two-dimensional function $\omega(x,y)$ is harmonic ($\nabla^2\omega=0$), the function

$$\phi=(x^2+y^2)\omega$$

will be biharmonic.

8. The constitutive law for an isotropic incompressible elastic material can be written

$$\sigma_{ij}=\overline{\sigma}\delta_{ij}+2\mu e_{ij}\ ,$$

where

$$\overline{\sigma}=\frac{\sigma_{kk}}{3}$$

represents an arbitrary hydrostatic stress field. Some soils can be approximated as an incompressible material whose modulus varies linearly with depth, so that

$$\mu=Mz$$

for the half space $z>0$.

Use the displacement function representation

$$\boldsymbol{u}=\nabla\phi$$

to develop a potential function solution for the stresses in such a body. Show that the functions $\phi,\overline{\sigma}$ must satisfy the relations

$$\nabla^2\phi=0\ ;\ \overline{\sigma}=-2M\frac{\partial\phi}{\partial z}$$

and hence obtain expressions for the stress components in terms of the single harmonic function ϕ.

If the half-space is loaded by a normal pressure

$$\sigma_{zz}(x,y,0)=-p(x,y)\ ;\ \sigma_{zx}(x,y,0)=\sigma_{zy}(x,y,0)=0\ ,$$

show that the corresponding normal surface displacement $u_z(x,y,0)$ is linearly proportional to the local pressure $p(x,y)$ and find the constant of proportionality[5].

9. Show that Dundurs' constant $\beta\to 0$ for plane strain in the limit where $\nu_1=0.5$ and $\mu_1/\mu_2\to 0$ — i.e. material 1 is incompressible and has a much lower shear modulus[6] than material 2. What is the value of α in this limit?

10. Solve Problem 3.7 for the case where there is no body force, using the Airy stress function ϕ to represent the stress components. Hence show that the governing equation is $\nabla^4\phi=0$, as in the case of compressible materials.

Pp 74-76 (Problems in Rectangular Coordinates)

PROBLEMS

1. The beam $-b<y<b$, $0<x<L$, is built-in at the end $x=0$ and loaded by a uniform shear traction $\sigma_{xy}=S$ on the upper edge, $y=b$, the remaining edges, $x=L$, $y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=L$.

2. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a shear traction $\sigma_{xy}=Sx/L$ on the lower edge, $y=-b$, the upper edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=\pm L$.

3. The beam $-b<y<b$, $0<x<L$, is built-in at the end $x=L$ and loaded by a linearly-varying compressive normal traction $p(x)=Sx/L$ on the upper edge, $y=b$, the remaining edges, $x=0$, $y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=0$.

4. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a compressive normal traction

$$p(x) = S\cos\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the lower edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem.

5. The beam $-b<y<b$, $0<x<L$, is built-in at the end $x=L$ and loaded by a compressive normal traction

$$p(x) = S\sin\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the remaining edges, $x=0$, $y=-b$ being traction-free. Use a combination of the stress function (5.97) and an appropriate polynomial to find the stress components for this problem, using the weak boundary conditions on $x=0$.

6. A large plate defined by $y>0$ is subjected to a sinusoidally varying load

$$\sigma_{yy} = S\sin\lambda x \;\; ; \;\; \sigma_{xy} = 0$$

at its plane edge $y=0$.

Find the complete stress field in the plate and hence estimate the depth y at which the amplitude of the variation in σ_{yy} has fallen to 10% of S.

Hint: You might find it easier initially to consider the case of the layer $0<y<h$, with $y=h$ traction-free, and then let $h\to\infty$.

7. The beam $-a<x<a, -b<y<b$ is loaded by a uniform compressive traction p in the central region $-a/2<x<a/2$ of both of the edges $y=\pm b$, as shown in Figure 5.5. The remaining edges are traction-free. Use a Fourier series with the appropriate symmetries to obtain a solution for the stress field, using the weak condition on σ_{xy} on the edges $x=\pm a$ and the strong form of all the remaining boundary conditions.

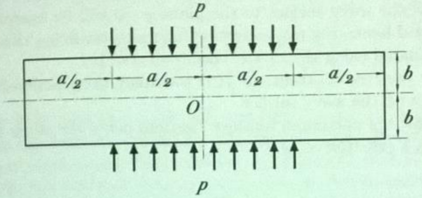


Figure 5.5

8. Use a Fourier series to solve the problem of Figure 5.4(a) in §5.2.3. Choose the terms in the series so as to satisfy the condition $\sigma_{xx}(\pm a, y)=0$ in the strong sense.

If you are solving this problem in Maple or Mathematica, compare the solution with that of §5.2.3 by making a contour plot of the difference between the truncated Fourier series stress function and the polynomial stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3) \; .$$

Examine the effect of taking different numbers of terms in the series.

9. The large plate $y>0$ is loaded at its remote boundaries so as to produce a state of uniform tensile stress

$$\sigma_{xx} = S \;\; ; \;\; \sigma_{xy} = \sigma_{yy} = 0 \; ,$$

the boundary $y=0$ being traction-free. We now wish to determine the perturbation in this simple state of stress that will be produced if the traction-free boundary had a slight waviness, defined by the line

$$y = \epsilon\cos(\lambda x) \; ,$$

where $\lambda\epsilon \ll 1$. To solve this problem

(i) Start with the stress function

$$\phi = \frac{Sy^2}{2} + f(y)\cos(\lambda x)$$

and determine $f(y)$ if the function is to be biharmonic.

(ii) The perturbation will be localized near $y=0$, so select only those terms in $f(y)$ that decay as $y\to\infty$.

(iii) Find the stress components and use the stress transformation equations to determine the tractions on the wavy boundary. Notice that the inclination of the wavy surface to the plane $y=0$ will be everywhere small if $\lambda\epsilon \ll 1$ and hence the trigonometric functions involving this angle can be approximated using $\sin(x)\approx x$, $\cos(x)\approx 1, x\ll 1$.

(iv) Choose the free constants in $f(y)$ to satisfy the traction-free boundary condition on the wavy surface.

(v) Determine the maximum tensile stress and hence the stress concentration factor as a function of $\lambda\epsilon$.

Pp 121-122 (Problems in polar coordinates)

PROBLEMS

1. A large plate with a small central hole of radius a is subjected to in-plane hydrostatic compression $\sigma_{xx}=\sigma_{yy}=-S$, $\sigma_{xy}=0$ at the remote boundaries. Find the stress field in the plate if the surface of the hole is traction-free.

2. A large rectangular plate is loaded in such a way as to generate the unperturbed stress field

$$\sigma_{xx} = Cy^2 \;\; ; \;\; \sigma_{yy} = -Cx^2 \;\; ; \;\; \sigma_{xy} = 0 \; .$$

The plate contains a small traction-free circular hole of radius a centred on the origin. Find the perturbation in the stress field due to the hole.

3. Figure 8.3 shows a thin uniform circular disk, which rotates at constant speed Ω about the diametral axis $y=0$, all the surfaces being traction-free. Determine the complete stress field in the disk.

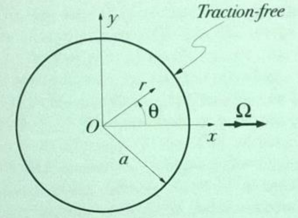


Figure 8.3: Thin disk rotating about a diametral axis.

4. A series of experiments is conducted in which a thin plate is subjected to biaxial tension/compression, σ_1, σ_2, the plane surface of the plate being traction-free (i.e. $\sigma_3=0$).

Unbeknown to the experimenter, the material contains microscopic defects which can be idealized as a sparse distribution of small holes through the thickness of the plate. Show graphically the relation which will hold at yield between the tractions σ_1, σ_2 applied to the defective plate, if the Tresca (maximum shear stress) criterion applies for the undamaged material.

5. The circular disk $0\le r<a$ is subjected to uniform compressive tractions $\sigma_{rr}=-S$ in the two arcs $-\pi/4<\theta<\pi/4$ and $3\pi/4<\theta<5\pi/4$, the remainder of the surface $r=a$ being traction-free. Expand these tractions as a Fourier series in θ and hence develop a series solution for the stress field. Use Maple or Mathematica to produce a contour plot of the Von Mises stress σ_E, using a series truncated at 10 terms.

6. A hole of radius a in a large elastic plate is loaded only by a self-equilibrated distribution of normal pressure $p(\theta)$ that varies around the circumference of the hole. By expanding $p(\theta)$ as a Fourier series in θ and using Table 8.1, show that the hoop stress $\sigma_{\theta\theta}$ at the edge of the hole is given by

$$\sigma_{\theta\theta}(a,\theta) = 2\overline{p} - p(\theta) \; ,$$

where

$$\overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)d\theta$$

is the mean value of $p(\theta)$.

Slaughter, "The Linearized theory of elasticity"
Pp 91-95 (Mathematical Preliminaries, Cylindrical and Spherical Coordinates)

Problems

2.1 For each of the following, determine whether or not the given expression is a valid indicial notation expression. If it is *valid*, identify the free indices and the dummy index pairs, determine the number of expanded equations represented and how many terms each expanded equation will have, and give the expanded equations (or, if they are too lengthy, at least demonstrate that you know what they are).

(a) $a_{ms} = b_m(c_r - d_r)$ (b) $a_{ms} = b_m(c_s - d_s)$
(c) $t_i = \sigma_{ji} n_j$ (d) $t_i = \sigma_{ji} n_i$
(e) $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$ (f) $x_i x_i = r^2$
(g) $\varepsilon_{rs} = h_r(d_s - h_s k_{rr})$ (h) $b_{ij} c_j = 3$

2.2 Show each of the following, where δ_{ij} is the Kronecker delta, ℓ_{ij} are the direction cosines, and e_{ijk} is the permutation symbol.

(a) $\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi} = 3$ (b) $\delta_{ij} e_{ijk} = 0$
(c) $\ell_{ki}\ell_{kj} = \delta_{ij}$ (d) $e_{qrs} d_q d_s = 0$

2.3 Prove the following relations between the permutation symbol e_{ijk} and the Kronecker delta δ_{ij} (**Hint:** recall that $\det[A]^{\mathsf{T}} = \det[A]$ and $\det[A]\det[B] = \det([A][B])$, and use successive contractions).

(a) $e_{ijk} = \det\begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}$ (b) $e_{ijk} e_{pqr} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$

(c) $e_{ijk} e_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$ (d) $e_{ijk} e_{ijr} = 2\delta_{kr}$
(e) $e_{ijk} e_{ijk} = 6$

2.4 Prove the following relations between the permutation symbol e_{ijk} and the determinant of the 3×3 matrix $[A]$ with elements A_{ij}.

(a) $e_{lmn} \det[A] = e_{ijk} A_{il} A_{jm} A_{kn}$
(b) $\det[A] = \frac{1}{6} e_{ijk} e_{lmn} A_{il} A_{jm} A_{kn}$

2.5 Given an orthonormal basis $\{\hat{\mathbf{e}}_i\}$ and three arbitrary vectors $\mathbf{u} = u_i \hat{\mathbf{e}}_i$, $\mathbf{v} = v_i \hat{\mathbf{e}}_i$, and $\mathbf{w} = w_i \hat{\mathbf{e}}_i$, show that

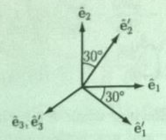
(a) $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = e_{ijk}$ (b) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = e_{ijk} u_i v_j w_k$

2.6 In the orthonormal basis $\{\hat{\mathbf{e}}_i\}$, the dyadic representations of a second-order tensor $\mathbf{A}$ and a vector $\mathbf{u}$ are

$$\mathbf{A} = 5\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 - 4\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\,, \quad \mathbf{u} = -2\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_3\,.$$

(a) Find the dyadic representation, in the orthonormal basis $\{\hat{\mathbf{e}}_i\}$, of the vector $\mathbf{v} = \mathbf{A} \cdot \mathbf{u}$.

(b) Use the tensor transformation rule to find the matrices $[\mathbf{A}]'$, $[\mathbf{u}]'$, and $[\mathbf{v}]'$ of the scalar components of $\mathbf{A}$, $\mathbf{u}$, and $\mathbf{v}$ in an orthonormal basis $\{\hat{\mathbf{e}}'_i\}$ that is obtained by rotating $\{\hat{\mathbf{e}}_i\}$ through a 30° clockwise angle about the $\hat{\mathbf{e}}_3$-direction, as shown below.



(c) Show that $[\mathbf{v}]' = [\mathbf{A}]'[\mathbf{u}]'$.

(d) Give the dyadic representations of $\mathbf{A}$, $\mathbf{u}$, and $\mathbf{v}$ in the orthonormal basis $\{\hat{\mathbf{e}}'_i\}$.

2.7 For three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{v} \cdot \mathbf{u})\mathbf{w}\,.$$

2.8 Given an arbitrary vector $\mathbf{v}$ and an arbitrary unit vector $\hat{\boldsymbol{\mu}}$, show that

$$\mathbf{v} = (\mathbf{v} \cdot \hat{\boldsymbol{\mu}})\hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}} \times (\mathbf{v} \times \hat{\boldsymbol{\mu}})$$

and give physical interpretations of $(\mathbf{v} \cdot \hat{\boldsymbol{\mu}})\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\mu}} \times (\mathbf{v} \times \hat{\boldsymbol{\mu}})$.

2.9 Given a second-order tensor $\mathbf{A}$, show that $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} = 0$ for all vectors $\mathbf{u}$ if and only if $\mathbf{A}$ is skew-symmetric.

2.10 Given arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ and arbitrary second-order tensors $\mathbf{A}$ and $\mathbf{B}$, show that

(a) $(\mathbf{u} \cdot \mathbf{A}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{A} \cdot \mathbf{v})$
(b) $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{u} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{u})$
(c) $\mathbf{u} \cdot (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{u} \cdot \mathbf{A}) \cdot \mathbf{B}$

2.11 Given arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ and arbitrary second-order tensors $\mathbf{A}$ and $\mathbf{B}$, show that

(a) $(\mathbf{A} \cdot \mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}} \cdot \mathbf{A}^{\mathsf{T}}$
(b) $(\mathbf{A} \cdot \mathbf{u}) \cdot (\mathbf{B} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{A}^{\mathsf{T}} \cdot \mathbf{B}) \cdot \mathbf{v}$
(c) $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$
(d) $(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}} \equiv \mathbf{A}^{-\mathsf{T}}$

2.12 If, in an orthonormal basis $\{\hat{\mathbf{e}}_i\}$, A_{ij} are the scalar components of a skew-symmetric second-order tensor $\mathbf{A}$, then the vector $\mathbf{b}$ with scalar components given by

$$b_i = \frac{1}{2} e_{ijk} A_{kj}$$

is called the *axial vector of* $\mathbf{A}$.

(a) Show that the scalar components of $\mathbf{A}$ are given in terms of the scalar components of $\mathbf{b}$ by $A_{pq} = e_{qpi} b_i$.

(b) Give the matrix of scalar components of $\mathbf{A}$ in terms of the scalar components of $\mathbf{b}$.

(c) Show that, for an arbitrary vector $\mathbf{c}$, $\mathbf{A} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c}$.

2.13 Show that, in an orthonormal basis $\{\hat{\mathbf{e}}_i\}$, the determinant of a second-order tensor $\mathbf{A}$ is equal to the determinant of its matrix of scalar components $[\mathbf{A}]$, that is, that

$$\det \mathbf{A} \equiv \frac{[(\mathbf{A} \cdot \mathbf{u}) \times (\mathbf{A} \cdot \mathbf{v})] \cdot (\mathbf{A} \cdot \mathbf{w})}{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}} = \det[\mathbf{A}]\,,$$

where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are *arbitrary* vectors.

2.14 Given, in an orthonormal basis, that

$$\widetilde{N}_{\mathbf{A}}(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \lambda) = A_{ij}\hat{\mu}_i\hat{\mu}_j - \lambda(\hat{\mu}_i\hat{\mu}_i - 1)$$

and $A_{ij} = A_{ji}$, show that the conditions $\partial\widetilde{N}_{\mathbf{A}}/\partial\hat{\mu}_k = 0$ lead to the equations

$$A_{kj}\hat{\mu}_j = \lambda\hat{\mu}_k\,.$$

(**Hint:** note that $\partial\hat{\mu}_i/\partial\hat{\mu}_k = \delta_{ik}$.)

2.15 Given a nonsingular second-order tensor $\mathbf{F}$, show that $\mathbf{C} = \mathbf{F}^{\mathsf{T}} \cdot \mathbf{F}$ is symmetric and positive definite. Why is it necessary that $\mathbf{F}$ be nonsingular?

2.16 Prove the *Cayley-Hamilton theorem*, which says that a symmetric second-order tensor $\mathbf{A}$ satisfies its own characteristic equation, that is, that

$$\mathbf{A}^3 - I_{\mathbf{A}}\mathbf{A}^2 + II_{\mathbf{A}}\mathbf{A} - III_{\mathbf{A}}\mathbf{I} = 0\,,$$

where $I_{\mathbf{A}}$, $II_{\mathbf{A}}$, and $III_{\mathbf{A}}$ are the principal invariants of $\mathbf{A}$.

2.17 Consider a nonsingular second-order tensor $\mathbf{A}$ and let $\mathbf{B} = \mathbf{A}^{-1}$. Show that the principal invariants of $\mathbf{B}$ can be expressed in terms of the principal invariants of $\mathbf{A}$ by

$$I_{\mathbf{B}} = \frac{II_{\mathbf{A}}}{III_{\mathbf{A}}}\,, \quad II_{\mathbf{B}} = \frac{I_{\mathbf{A}}}{III_{\mathbf{A}}}\,, \quad III_{\mathbf{B}} = \frac{1}{III_{\mathbf{A}}}\,.$$

[**Hint:** recall that $\det(\mathbf{B} - \lambda\mathbf{I}) = \lambda^3 - I_{\mathbf{B}}\lambda^2 + II_{\mathbf{B}}\lambda - III_{\mathbf{B}}$ and $\det(\alpha\mathbf{C}) = \alpha^3 \det\mathbf{C}$ and note that $(\mathbf{B} - \lambda\mathbf{I}) = \mathbf{A}^{-1}(\mathbf{I} - \lambda\mathbf{A})$.]

2.18 For the function $F = a_{ij}x_ix_j$, where a_{ij} are constants, derive expressions for

(a) $\dfrac{\partial F}{\partial x_r}$ (b) $\dfrac{\partial^2 F}{\partial x_r \partial x_s}$

2.19 Given an arbitrary scalar field ζ, an arbitrary vector field $\mathbf{v}$, and the position vector $\mathbf{x}$, show

(a) $\nabla(\nabla^2\zeta) = \nabla^2(\nabla\zeta)$ (b) $\nabla(\mathbf{v}\cdot\mathbf{x}) = \mathbf{v} + \mathbf{x}\cdot(\nabla\mathbf{v})$

(c) $\nabla\cdot(\nabla\times\mathbf{v}) = 0$ (d) $\nabla\cdot(\nabla^2\mathbf{v}) = \nabla^2(\nabla\cdot\mathbf{v})$

(e) $\nabla\cdot(\zeta\mathbf{v}) = \zeta\nabla\cdot\mathbf{v} + (\nabla\zeta)\cdot\mathbf{v}$ (f) $\nabla\times(\nabla^2\mathbf{v}) = \nabla^2(\nabla\times\mathbf{v})$

(g) $\nabla^2(\zeta\mathbf{x}) = 2\nabla\zeta + \mathbf{x}\nabla^2\zeta$ (h) $\nabla^2(\mathbf{v}\cdot\mathbf{x}) = 2\nabla\cdot\mathbf{v} + \mathbf{x}\cdot\nabla^2\mathbf{v}$

(i) $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

2.20 Given the dyad $\mathbf{ab}$ formed by two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$, show that

$$\nabla^2(\mathbf{ab}) = (\nabla^2\mathbf{a})\mathbf{b} + \mathbf{a}(\nabla^2\mathbf{b}) + 2\nabla\mathbf{a}\cdot(\nabla\mathbf{b})^{\mathsf{T}}\,.$$

2.21 Let $R \equiv |\mathbf{x}|$ be the distance from the origin. Show each of the following first by using a Cartesian coordinate system, as in Example 2.4, and then by using a spherical coordinate system.

(a) $\nabla(R^n\mathbf{x}) = R^n\mathbf{I} + nR^{n-2}\mathbf{xx}$ (b) $\nabla\cdot(R^n\mathbf{x}) = (n+3)R^n$

(c) $\nabla\cdot(R^n\mathbf{xx}) = (n+4)R^n\mathbf{x}$ (d) $\nabla^2(R^n\mathbf{x}) = n(n+3)R^{n-2}\mathbf{x}$

2.22 Consider the vector field $\mathbf{h} = \alpha R^{-2}\hat{\mathbf{e}}_R$, where α is a constant, R is the distance from the origin, and $\hat{\mathbf{e}}_R$ is the spherical coordinate base vector pointing out from the origin. Determine the gradient $\nabla\mathbf{h}$, the divergence $\nabla\cdot\mathbf{h}$, and the curl $\nabla\times\mathbf{h}$ of this vector field.

2.23 Recall that, in a Cartesian coordinate system, the curl of a vector field $\mathbf{v} = v_i\hat{\mathbf{e}}_i$ is given in terms of its scalar components by $\nabla\times\mathbf{v} = e_{ijk}v_{j,i}\,\hat{\mathbf{e}}_k$ and the curl of a second-order tensor $\mathbf{S} = S_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$ is given in terms of its scalar components by $\nabla\times\mathbf{S} = e_{ijk}S_{rj,i}\,\hat{\mathbf{e}}_k\hat{\mathbf{e}}_r$. Determine the curl of a third-order tensor $\mathbf{T} = T_{ijk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k$ in terms of its scalar components.

2.24 Given, in a Cartesian coordinate system, a second-order tensor field $\boldsymbol{\varepsilon} = \varepsilon_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$, derive a dyadic notation expression for $\nabla\times(\nabla\times\boldsymbol{\varepsilon})$ in terms of the scalar components of $\boldsymbol{\varepsilon}$. Assuming that $\boldsymbol{\varepsilon}$ is symmetric, determine explicitly the six independent scalar equations, in a Cartesian coordinate system, that are represented by the tensor notation equation $\nabla\times(\nabla\times\boldsymbol{\varepsilon}) = \mathbf{0}$.

2.25 It was shown in Example 2.5 that $\nabla\times(\nabla\phi) = \mathbf{0}$, where ϕ is an arbitrary scalar field. Prove that $\nabla\times(\nabla\mathbf{T}) = \mathbf{0}$ for an arbitrary tensor field $\mathbf{T}$ of *any* order. [**Hint:** use the definition (2.4.36) with $\mathbf{T}\to\nabla\mathbf{T}$, show that $\mathbf{a}\cdot(\nabla\mathbf{T}) = \nabla(\mathbf{a}\cdot\mathbf{T})$ since $\mathbf{a}$ is a constant, and establish the proof by a process of induction.]

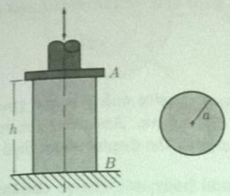
Pp 250-254 (Linearized Elasticity Problems)

Problems

6.1 A circular cylindrical material sample of radius a and height h is placed into a servo-hydraulic testing machine as shown below. The two surfaces of the testing machine that come into contact with the sample at A and B are called *platens*. These platens are made of a very stiff alloy that may be assumed to be rigid. The platen at B is held stationary and an active control circuit is integrated into the hydraulic actuator so that the vertical motion of the platen at A can be prescribed. Suppose that the platen at A is first adjusted until it is just in contact with the sample, and then an additional downward motion δ of the platen is prescribed. Define a coordinate system and give a complete statement of the boundary conditions for the sample, assuming

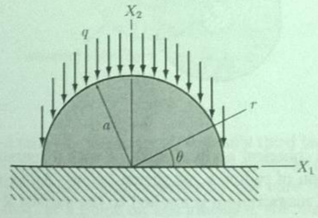
(a) that the sample is ideally bonded to the platens at A and B

(b) that the platens at A and B are frictionless (e.g., some sort of ideal lubricant has been applied).



Note: do *not* attempt to solve the corresponding boundary value problem.

6.2 Consider the infinitely long cylindrical body, with semicircular cross section, illustrated below. In Cartesian coordinates, the body occupies the region $X_1^2 + X_2^2 \leq a^2$, $X_2 \geq 0$, $-\infty < X_3 < \infty$. In cylindrical coordinates, it occupies the region $r \leq a$, $0 \leq \theta \leq \pi$, $-\infty < z < \infty$. A uniform vertical traction of magnitude q acts on the semicircular surface of the body and the base is fixed to a rigid support.



(a) Give a complete statement of the boundary conditions for this problem using the Cartesian coordinate system.

(b) Give a complete statement of the boundary conditions for this problem using the cylindrical coordinate system.

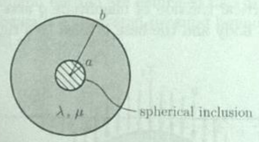
Note: do *not* attempt to solve the corresponding boundary value problem.

6.3 Consider the linear elastic deformation of a homogeneous sphere of radius a due to the mutual gravitational attraction of its parts. In a spherical coordinate system, with the origin at the center of the sphere, the body force field within such a self-gravitating sphere is

$$\mathbf{f} = -\frac{\rho g R}{a}\,\hat{\mathbf{e}}_R\,,$$

where ρ is the mass density and g is the gravitational acceleration at the surface of the sphere. Assuming the surface of the sphere is traction-free, determine the displacement field in the sphere.

6.4 Consider a spherical body, with Lamé constants λ and μ, that has embedded within it a concentric spherical *inclusion*. Initially, when the radius of the inclusion is a, the spherical body is stress-free and its outer radius is b. Suppose some event occurs that triggers a transformation in the inclusion that causes its radius to increase by a known amount κa. (This is *not* what the change in radius of the inclusion would be if it were isolated, but the actual change in radius of the inclusion while it is embedded in the spherical body.) Determine the change in the outer radius of the spherical body caused by this expansion of the inclusion.



6.5 Consider a spherical body of radius b with a concentric, rigid spherical inclusion of radius a, as shown below. An external hydrostatic pressure of magnitude p is applied to the surface of the body.

(a) Determine the displacement and stress fields in the body.

(b) In the limiting case of a rigid spherical inclusion in an infinite body under remote hydrostatic pressure (i.e., as $b \to \infty$), show that the stress concentration factor for the rigid inclusion is

$$\text{s.c.f.} = 1 + \frac{4\mu}{3K}\,,$$

where μ is the shear modulus and K is the bulk modulus.

6.6 Conside
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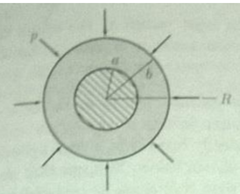
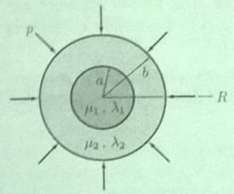


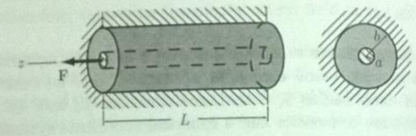
6.6 Consider a composite sphere composed of a solid spherical core of radius a, with Lamé constants μ_1 and λ_1, and an outer spherical shell of outer radius b, with Lamé constants μ_2 and λ_2. The core and shell are concentric and a spherical coordinate system is defined with the origin at their center. Assume that the core and shell are *ideally bonded* at their *interface* at $R = a$, which means that the displacement field $\mathbf{u}$ is continuous across the interface.



(a) Recall the generalization of Newton's third law, which says that the traction vector exerted by the shell on the core is the negative of the traction vector exerted by the core on the shell. Note also that, at a point on the interface, the unit outward normal to the core is the negative of the unit outward normal to the shell. What consequences does this have for continuity of the scalar components of stress (in spherical coordinates) across the interface?

(b) The composite sphere is subject to an external, hydrostatic pressure of magnitude p. Assuming that the spherical symmetry of the problem is preserved in the solution, use the semi-inverse method to find the displacement and stress fields in the composite sphere. Note that, given the symmetry assumption, the displacement must vanish at the center of the core. The answer will involve separate expressions for the core and shell that satisfy the continuity conditions at the interface.

6.7 A hollow circular cylinder has inner radius a, outer radius b, and length L. The outer surface of the hollow cylinder is fixed and its inner surface is ideally bonded to a rigid cylindrical core of radius a and length L, as shown below. Suppose that an axial force $\mathbf{F} = F\hat{\mathbf{e}}_z$ is applied to the rigid core along its centroidal axis.



(a) Find the resulting (rigid-body) axial displacement of the core and the work done by the applied force $\mathbf{F}$, by assuming a displacement field in the hollow cylinder of the form $u_r = u_\theta = 0$ and $u_z = U(r)$.

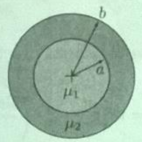
(b) Is this the exact solution to the problem, or is it a solution only in the sense of Saint-Venant's principle? Explain.

Pp 327-329 (Torsion of non-circular Cylinders)

Problems

8.1 Determine the Prandtl stress function for torsion of a hollow circular cylinder of inner radius a and outer radius b, in terms of the shear modulus μ and the twist per unit length α.

8.2 An axisymmetric composite cylinder is composed of a solid inner shaft, of radius a and shear modulus μ_1, and an outer sleeve of outer radius b and shear modulus μ_2. The shaft and sleeve are ideally bonded at their interface and the composite cylinder is subjected to an applied torque T.



(a) Determine the distribution of stress within the composite cylinder in terms of the twist per unit length α.

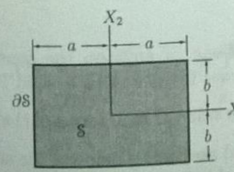
(b) Find an expression for the twist per unit length α in terms of the applied torque T.

(c) How much of the total torque T is carried by the outer sleeve?

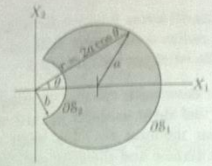
8.3 Consider torsion of a cylinder with the rectangular cross section shown below. Explain why a function of the form

$$\phi = m\left(\frac{X_1^2}{a^2} - 1\right)\left(\frac{X_2^2}{b^2} - 1\right)$$

cannot be used as a Prandtl stress function for this cross section.



8.4 Consider the circular shaft with a circular keyway whose cross section is shown below.



(a) Show that the Prandtl stress function,

$$\phi = m(r^2 - b^2)\left(\frac{2a\cos\theta}{r} - 1\right),$$

may be used to describe the torsion for this cross section and determine the value of the constant m in terms of the twist per unit length α.

(b) Assuming that the maximum shear traction $\tau_{\max}$ on the cross section occurs at the point $(r, \theta) = (b, 0)$ on the boundary, compute the value of $\tau_{\max}$ in the limit as $b \to 0$ and compare it with the maximum shear traction for torsion of a solid circular cylinder of radius a (the two maximum shear tractions are different).

8.5 Consider the Prandtl stress function

$$\phi = m(a^2 - X_1^2 + b^2X_2^2)(a^2 + b^2X_1^2 - X_2^2),$$

where $|b| < 1$.

(a) What restrictions (if any) on the constants a, b, and m are required in order for this to be a valid Prandtl stress function?

(b) Determine the value of m in terms of the twist per unit length α.

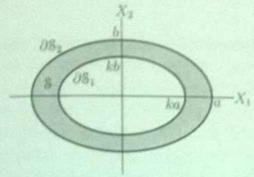
(c) Determine and sketch the shape of the cross section of the cylinder for which this Prandtl stress function gives the torsion solution. Why was it necessary to have $|b| < 1$?

8.6 Consider the hollow cylinder shown below whose cross section is bounded by two concentric, similar ellipses ∂S_1 and ∂S_2:

$$\partial S_1 = \left\{ (X_1, X_2) \mid \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = k^2 \right\},$$

$$\partial S_2 = \left\{ (X_1, X_2) \mid \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = 1 \right\},$$

where a, b, and k are constants and $0 < k < 1$.



(a) Show that the Prandtl stress function,

$$\phi = m\left(\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} - 1\right),$$

may be used to describe the torsion for this cross section and determine the value of the constant m in terms of the twist per unit length α.

(b) Find the torque T as a function of the twist per unit length and determine the torsion constant $\bar{J}$. **Hint:** the moments of inertia for a region S with elliptic boundary

$$\partial S = \left\{ (X_1, X_2) \mid \frac{X_1^2}{c^2} + \frac{X_2^2}{d^2} = 1 \right\}$$

are

$$I_1 \equiv \int_S X_2^2\,dA = \frac{\pi c d^3}{4}, \quad I_2 \equiv \int_S X_1^2\,dA = \frac{\pi c^3 d}{4}.$$