DC Circuits
Challenge Problem Solutions

Problem 1:
A battery of emf ε has internal resistance R_i, and let us suppose that it can provide the emf to a total charge Q before it expires. Suppose that it is connected by wires with negligible resistance to an external (load) with resistance R_L.

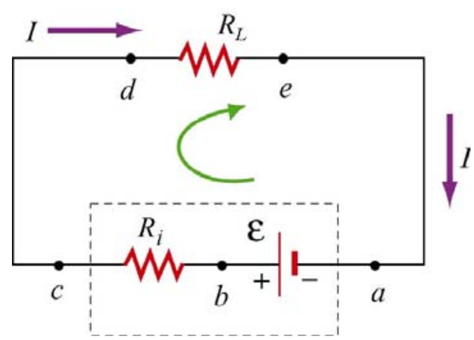
a) What is the current in the circuit?

b) What value of R_L maximizes the current extracted from the battery, and how much chemical energy is generated in the battery before it expires?

c) What value of R_L maximizes the total power delivered to the load, and how much energy is delivered to the load before it expires? How does this compare to the energy generated in the battery before it expires?

d) What value for the resistance in the load R_L would you need if you want to deliver 90% of the chemical energy generated in the battery to the load? What current should flow? How does the power delivered to the load now compare to the maximum power output you found in part c)?

Problem 1 Solution:
(a)



The Kirchoff loop law (the sum of the voltage differences across each element around a closed loop is zero) yields

$$\varepsilon - I\,R_i - I\,R_L = 0\,.$$

Solving for the current we find that

$$I = \frac{\varepsilon}{R_i + R_L}.$$

(b) The current is maximized when $R_L = 0$.

The chemical energy generated in the battery is given by

$$U_{emf} = \int_0^{\Delta t} \varepsilon I dt = \varepsilon I \Delta t$$

During this time interval, the battery delivers a charge

$$Q = \int_0^{\Delta t} I dt = I \Delta t.$$

Therefore the chemical energy generated is

$$U_{emf} = \varepsilon I \Delta t = \varepsilon I \frac{Q}{I} = \varepsilon Q$$

This result is independent of the current and only depends on the charge Q that is transferred across the EMF. So for all the following parts, this quantity is the same.

All of this chemical energy is dissipated into thermal energy due to the internal resistance of the battery to the flow of current. When the battery stops delivering current, the battery will reach thermal equilibrium with the surroundings and this thermal energy will flow into the surroundings.

(c) The power delivered to the load is

$$P_L = I^2 R_L = \left(\frac{\varepsilon}{R_i + R_L} \right)^2 R_L.$$

We can maximize this by considered the derivative with respect to R_L:

$$\frac{dP_L}{dR_L} = \varepsilon^2 \left(\left(\frac{1}{R_i + R_L} \right)^2 - 2R_L \left(\frac{1}{R_i + R_L} \right)^3 \right) = 0.$$

Solve this equation for R_L:

$$\left(\frac{1}{R_i+R_L}\right)^2 = 2R_L\left(\frac{1}{R_i+R_L}\right)^3,$$

$$R_i + R_L = 2R_L,$$

$$R_L = R_i.$$

The current is then

$$I = \frac{\varepsilon}{R_i+R_L} = \frac{\varepsilon}{2R_i}.$$

The power delivered to the load is

$$P_{L,\max} = I^2 R_L = \left(\frac{\varepsilon}{2R_i}\right)^2 R_i = \frac{1}{4}\frac{\varepsilon^2}{R_i}$$

The energy delivered to the load is then

$$U_L = I\ R_L\ Q = \frac{\varepsilon}{2R_i}R_i\ Q = \frac{\varepsilon Q}{2} = \frac{1}{2}U_{chem}.$$

So exactly half the chemical energy is delivered to the load.

(d) Even though we maximized the power delivered to the load in part cc), we are wasting one half the chemical energy. Suppose you want to waste only 10% of the chemical energy. What current should flow?

$$U_L = 0.9\,U_{chem} = 0.9\,\varepsilon Q = I'\ R_L\ Q.$$

This implies that

$$I'\ R_L = \frac{\varepsilon}{R_i+R_L}R_L = 0.9\varepsilon.$$

This is satisfied when

$$R_L = 9R_i.$$

So the current is

$$I' = \frac{\varepsilon}{10R_i} \quad .$$

The power output is then

$$P_L = I'^2 R_L = \left(\frac{\varepsilon}{10R_i}\right)^2 9R_i = \frac{9}{25}\left(\frac{1}{4}\frac{\varepsilon^2}{R_i}\right) = \frac{9}{25} P_{L,\max} \, .$$

So we waste 10% of the energy and still maintain 36% of the maximum power output.

Problem 2:

AAA, AA, … D batteries have an open circuit voltage (EMF) of 1.5 V. The difference between different sizes is in their lifetime (total energy storage). A AAA battery has a life of about 0.5 A-hr while a D battery has a life of about 10 A-hr. Of course these numbers depend on how quickly you discharge them and on the manufacturer, but these numbers are roughly correct. One important difference between batteries is their internal resistance – alkaline (now the standard) D cells are about 0.1Ω.

Suppose that you have a multi-speed winch that is 50% efficient (50% of energy used does useful work) run off a D cell, and that you are trying to lift a mass of 60 kg (hmmm, I wonder what mass that would be). The winch acts as load with a variable resistance R_L that is speed dependent.

a) Suppose the winch is set to super-slow speed. Then the load (winch motor) resistance is much greater than the battery's internal resistance and you can assume that there is no loss of energy to internal resistance. How high can the winch lift the mass before discharging the battery?

b) To what resistance R_L should the winch be set in order to have the battery lift the mass at the fastest rate? What is this fastest rate (m/sec)? HINT: You want to maximize the power delivery to the winch (power dissipated by R_L).

c) At this fastest lift rate how high can the winch lift the mass before discharging the battery?

d) Compare the cost of powering a desk light with D cells as opposed to plugging it into the wall. Does it make sense to use rechargeable batteries? Residential electricity costs about $0.1/kwh.

Problem 2 Solutions:

(a)This is just a question of energy. The battery has an energy storage of (1.5 V)(10 A-hr) = 15 W-hr or 54 kJ. So it can lift the mass:

$$U = mgh \Rightarrow \quad h = \frac{U}{mg} = \frac{54\,\text{kJ}\cdot\frac{1}{2}}{(60\text{ kg})(9.8\text{ m/s})} = \boxed{46\text{ m}}$$

The factor of a half is there because the winch is only 50% efficient.

(b)First we need to determine how to maximize power delivery. If a battery V is connected to two resistances, r_i (the internal resistance) and R, the load resistance, the power dissipated in the load is:

$$P = I^2 R = \left(\frac{V_0}{R+r_i}\right)^2 R = V_0^2 \frac{R}{(R+r_i)^2}$$

We want to maximize this by varying R:

$$\frac{dP}{dR} = \frac{d}{dR}\left(V_0^2 R\left(R+r_i\right)^{-2}\right) = V_0^2\left[\left(R+r_i\right)^{-2} - 2R\left(R+r_i\right)^{-3}\right] = 0$$

Multiply both sides by $V_0^{-2}\left(R+r_i\right)^3$: $\left[\left(R+r_i\right)-2R\right] = r_i - R = 0 \quad \Rightarrow \quad \boxed{R = r_i}$

So, to get the fastest rate of lift (most power dissipation in the winch) we need the winch resistance to equal the battery internal resistance, $R_L = r_i = 0.1\ \Omega$.

Using this we can get the lift rate from the power:

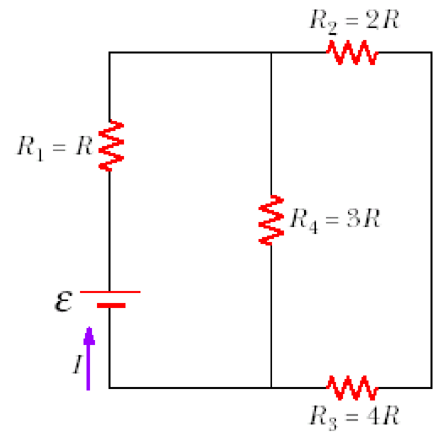
$$P = I^2 R_L = \left(\frac{V_0}{R_L + r_i}\right)^2 R_L = \frac{V_0^2}{4r_i} = \overbrace{2}^{50\%\,eff} \frac{d}{dt}\left(mgh\right) \Rightarrow v = \frac{dh}{dt} = \frac{V_0^2}{8r_i mg}$$

Thus we find a list rate of $\boxed{v = 4.8\ \text{mm/s}}$

(c)This is just part a over again, except now we waste half the energy in the internal resistor, so the winch will only rise half as high, to $\boxed{23\ \text{m}}$

d)
A D cell has a battery life of 10 A-hr, meaning a total energy storage of (1.5 V)(10 A-hr) = 15 Watt-hrs. We could convert that to about 50 kJ but Watt-hours are a useful unit to use because electricity is typically charged by the kW-hour so this will make comparison easier. A D battery costs about $1 (you can pay more, but why?) So D batteries cost about $1/0.015 kwh or $70/kwh.
Residential electricity costs about $0.1/kwh. So the battery is nearly three orders of magnitude more expensive. It definitely makes sense to use rechargeable batteries – even though the upfront cost is slightly more expensive you will get it back in a couple recharges. As for your desk light, or anything that can run on batteries or wall power, plug it in. If it is 60 Watts, for every hour you pay only 0.6¢ with wall power but run through $4 in D batteries.

Problem 3:



Four resistors are connected to a battery as shown in the figure. The current in the battery is I, the battery emf is $\mathcal{E}$, and the resistor values are $R_1 = R$, $R_2 = 2R$, $R_3 = 4R$, $R_4 = 3R$.

(a) Rank the resistors according to the potential difference across them, from largest to smallest. Note any cases of equal potential differences.

(b) Determine the potential difference across each resistor in terms of $\mathcal{E}$.

(c) Rank the resistors according to the current in them, from largest to smallest. Note any cases of equal currents.

(d) Determine the current in each resistor in terms of I.

(e) If R_3 is increased, what happens to the current in each of the resistors?

(f) In the limit that $R_3 \to \infty$, what are the new values of the current in each resistor in terms of I, the original current in the battery?

Problem 3 Solutions:

(a)Resistors 2 and 3 can be combined (in series) to give $R_{23} = R_2 + R_3 = 2R + 4R = 6R$. R_{23} is in parallel with R_4 and the equivalent resistance R_{234} is

$$R_{234} = \frac{R_{23}R_4}{R_{23} + R_4} = \frac{(6R)(3R)}{6R + 3R} = 2R$$

Since R_{234} is in series with R_1, the equivalent resistance of the whole circuit is $R_{1234} = R_1 + R_{234} = R + 2R = 3R$. In series, potential difference is shared in proportion to the resistance, so R_1 gets 1/3 of the battery voltage ($\Delta V_1 = \varepsilon / 3$) and R_{234} gets 2/3 of the battery voltage ($\Delta V_{234} = 2\varepsilon / 3$). This is the potential difference across R_4 ($\Delta V_4 = 2\varepsilon / 3$), but R_2 and R_3 must share this voltage: 1/3 goes to R_2 ($\Delta V_2 = (1/3)(2\varepsilon / 3) = 2\varepsilon / 9$) and 2/3 to R_3 ($\Delta V_3 = (2/3)(2\varepsilon / 3) = 4\varepsilon / 9$). The ranking by potential difference is $\Delta V_4 > \Delta V_3 > \Delta V_1 > \Delta V_2$.

(b)As shown from the reasoning above, the potential differences are

$$\Delta V_1 = \frac{\varepsilon}{3}, \quad \Delta V_2 = \frac{2\varepsilon}{9}, \quad \Delta V_3 = \frac{4\varepsilon}{9}, \quad \Delta V_4 = \frac{2\varepsilon}{3}$$

(c)All the current goes through R_1, so it gets the most ($I_1 = I$). The current then splits at the parallel combination. R_4 gets more than half, because the resistance in that branch is less than in the other branch. R_2 and R_3 have equal currents because they are in series. The ranking by current is $I_1 > I_4 > I_2 = I_3$.

(d) R_1 has a current of *I*. Because the resistance of R_2 and R_3 in series ($R_{23} = R_2 + R_3 = 2R + 4R = 6R$) is twice that of $R_4 = 3R$, twice as much current goes through R_4 as through R_2 and R_3. The current through the resistors are
$I_1 = I, \quad I_2 = I_3 = \frac{I}{3}, \quad I_4 = \frac{2I}{3}$

(e)Since

$$R_{1234} = R_1 + R_{234} = R_1 + \frac{R_{23}R_4}{R_{23} + R_4} = R_1 + \frac{(R_2 + R_3)R_4}{R_2 + R_3 + R_4}$$

increasing R_3 increases the equivalent resistance of the entire circuit. The current in the circuit, which is the current through R_1, decreases. This decreases the potential difference across R_1, increasing the potential difference across the parallel combination. With a larger potential difference the current through R_4 is increased. With more current going through R_4, and less in the circuit to start with, the current through R_2 and R_3 must decrease. Thus,

I_4 increases and I_1, I_2, and I_3 decrease

(f)If R_3 has an infinite resistance, it blocks any current from passing through that branch and the circuit effectively is just R_1 and R_4 in series with the battery. The circuit now has an equivalent resistance of $R_{14} = R_1 + R_4 = R + 3R = 4R$. The current in the circuit drops to 3/4 of the original current because the resistance has increased by 4/3. All this current passes through R_1 and R_4, and none passes through R_2 and R_3. Therefore,

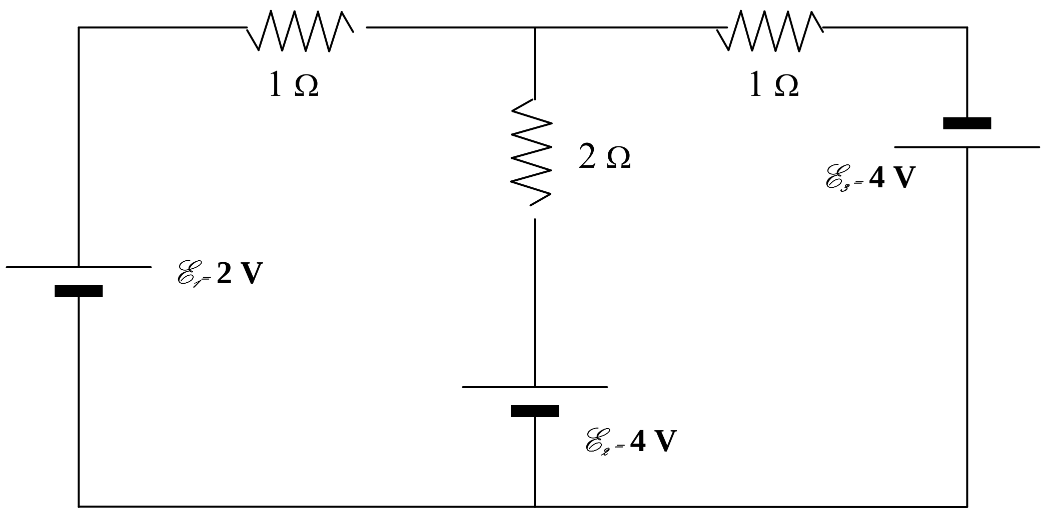
$$I_1 = \frac{3I}{4}, \; I_2 = I_3 = 0, \; I_4 = \frac{3I}{4}$$

Problem 4:

In the circuit below, you can neglect the internal resistance of all batteries.

(a) Calculate the current through each battery

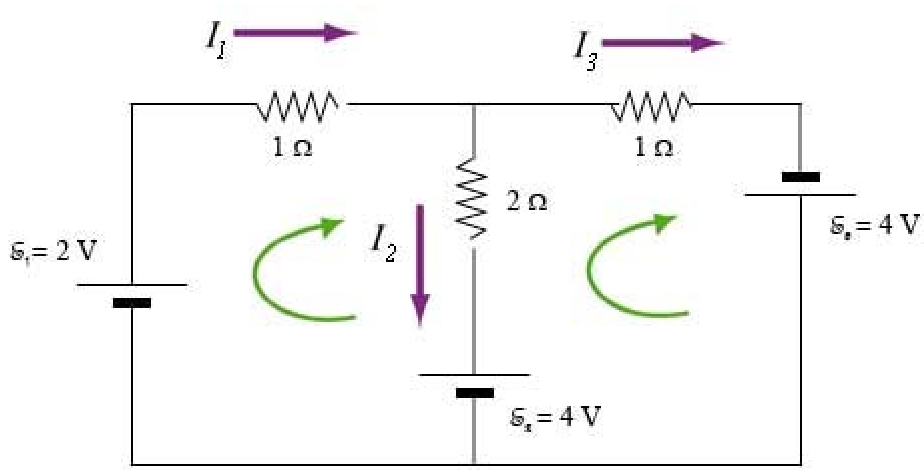
(b) Calculate the power delivered or used (specify which case) by each battery



Problem 4 Solutions:

(a) Calculate the current through each battery.

We begin by choosing currents in every branch and travel directions in the two loops as shown below.



Current conservation is given by the condition that the current into a junction of branches is equal to the current that leaves that junction:

$$I_1 = I_2 + I_3 .$$

The two loop laws for the voltage differences are:

$$2\text{ V} - (I_1)(1\,\Omega) - (I_2)(2\,\Omega) - 4\text{ V} = 0 .$$

$$-(I_3)(1\,\Omega) + 4\text{ V} + 4\text{ V} + (I_2)(2\,\Omega) = 0 .$$

Strategy: Solve the first loop law for I_1 in terms of I_2. Solve the second loop law for I_3 in terms of I_2. Then substitute these results into the current conservation and solve for I_2. Then determine I_1 and I_3.

The first loop law becomes

$$I_1 = -2\text{ A} - 2I_2 .$$

The second loop law becomes

$$I_3 = 8\text{ A} + 2I_2 .$$

Current conservation becomes

$$-2\text{ A} - 2I_2 = I_2 + 8\text{ A} + 2I_2 .$$

Solve for I_2:

$$I_2 = -2\text{ A} .$$

Note that the negative sign means the I_2 is flowing in a direction opposite the direction indicated by the arrow. This means that battery 2 is supplying current.

Solve for I_1:

$$I_1 = -2\text{ A} - 2(-2\text{ A}) = 2\text{ A}$$

Solve for I_3:

$$I_3 = 8\text{ A} + 2(-2\text{ A}) = 4\text{ A} .$$

(b) Calculate the power delivered or used (specify which case) by each battery.

The power delivered by battery 1 is $P_1 = (\mathcal{E}_1)(I_1) = (2\text{ V})(2\text{ A}) = 4\text{ W}$.

The power delivered by battery 2 is $P_2 = (\mathcal{E}_2)(I_2) = (4\text{ V})(2\text{ A}) = 8\text{ W}$.

The power delivered by battery 3 is $P_3 = (\mathcal{E}_3)(I_3) = (4\text{ V})(4\text{ A}) = 16\text{ W}$.

The total power delivered by the batteries is 28 W

Check: The power delivered to the resistors:
The power delivered to resistor 1 (in left branch) $P_1 = (I_1^{\ 2})(R_1) = (2\text{ A})^2(1\,\Omega) = 4\text{ W}$.
The power delivered to resistor 2 (in center branch) $P_2 = (I_2^{\ 2})(R_2) = (2\text{ A})^2(2\,\Omega) = 8\text{ W}$.
The power delivered to resistor 3 (in right branch) $P_3 = (I_3^{\ 2})(R_3) = (4\text{ A})^2(1\,\Omega) = 16\text{ W}$.
The total power delivered to the resistors is also 28 W.