2014

• Energía de Deformación y Teorema de Castigliano

↳ Energía por esf. axial: $U_{tr} = \int_0^L \frac{1}{2E} \frac{P^2}{A} dx$

↳ Energía por flexión: $U_{fl} = \int_0^L \frac{1}{2E} \frac{M(x)^2}{I_z} dx$

↳ Energía por torsión: $U_{tor} = \int_0^L \frac{1}{2G} \frac{T^2}{J} dx$

↳ Energía por corte: $U_{cor} = \int_0^L \frac{3}{5GA} V(x)^2 dx$
(Sección rectangular)

Energía de deformación total U_T es la suma de las energías

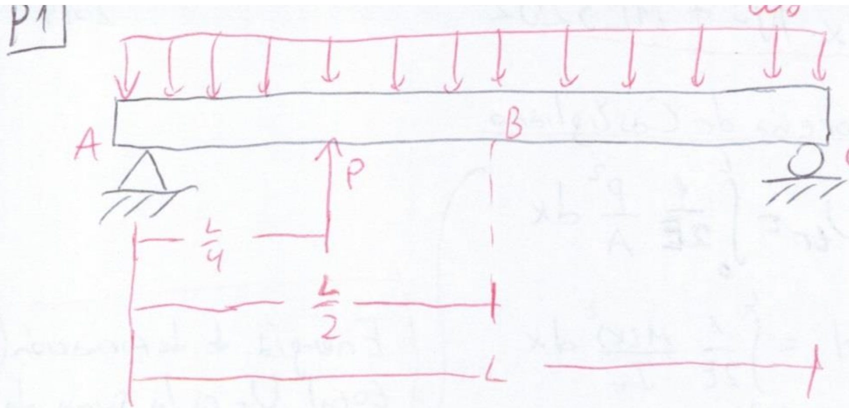
• Teorema de Castigliano

* Deflexión en el punto i: $\delta_i = \frac{\partial U_T}{\partial F_i}$ (F_i: Fza. aplicada en el punto i)

* Ángulo de rotación en i: $\theta_i = \frac{\partial U_T}{\partial C_i}$ (C_i: Momento aplicado en el punto i)

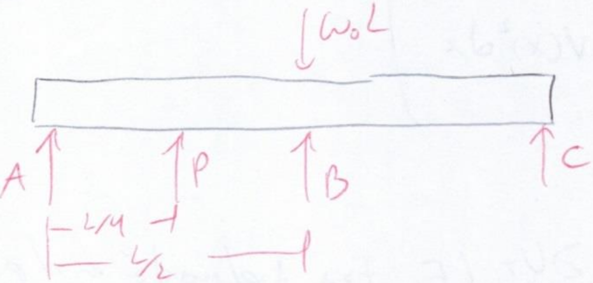
* Si no hay fza./momento aplicado en el punto de interés, se genera uno virtual, y luego de la derivación (Aplicación del Teorema de Castigliano) se hace igual a cero.

P1



* En general se puede despreciar la energía por corte si $L \gg h$ (Alto). Si no se dice nada, es preferible calcularla

• Para aplicar Castigliano, debemos "Inventar" una fuerza en B. Haciendo DCL:



• $\sum_A M_z = 0$

↳ $\frac{PL}{4} + \frac{BL}{2} + CL - \frac{w_0 L^2}{2} = 0$

$$\boxed{\therefore C = \frac{w_0 L}{2} - \frac{P}{4} - \frac{B}{2}}$$

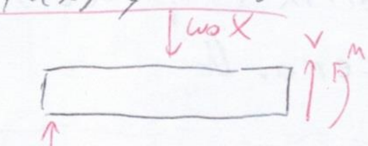
• $\sum F_y = 0 \Rightarrow A + P + B + C - w_0 L = 0 \Rightarrow A = w_0 L - P - B - C$

$$= w_0 L - \frac{w_0 L}{2} + \frac{P}{4} + \frac{B}{2} - P - B$$

$$\boxed{\therefore A = \frac{w_0 L}{2} - \frac{3P}{4} - \frac{B}{2}}$$

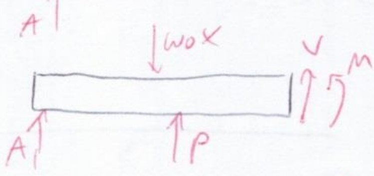
• Calculando $M(x)$ y $V(x)$:

$0 < x < \frac{L}{4}$



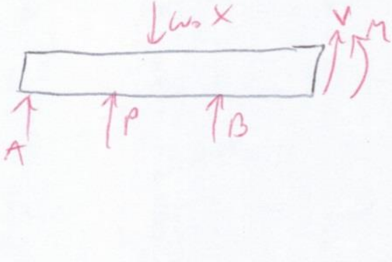
$$\boxed{\begin{aligned} V(x) &= w_0 x - A \\ M(x) &= Ax - w_0 \frac{x^2}{2} \end{aligned}}$$

$\frac{L}{4} < x < \frac{L}{2}$



$$\boxed{\begin{aligned} V(x) &= w_0 x - A - P \\ M(x) &\; Ax - w_0 \frac{x^2}{2} + P\left(x - \frac{L}{4}\right) \end{aligned}}$$

$\frac{L}{2} < x < L$



$$\boxed{\begin{aligned} V(x) &= w_0 x - A - P - B \\ M(x) &= Ax - w_0 \frac{x^2}{2} + P\left(x - \frac{L}{4}\right) + B\left(x - \frac{L}{2}\right) \end{aligned}}$$

• Calculando las energías por flexión y por corte, y aplicando Castigliano

$$U_{fl} = \frac{1}{2EI}\int_0^L M(x)^2 dx \Rightarrow \delta_{fl} = \frac{\partial U_{fl}}{\partial B} = \frac{1}{EI}\int_0^L M(x)\frac{\partial M(x)}{\partial B}dx$$

$$U_{cor} = \frac{3}{5GA}\int_0^L V(x)^2 dx \Rightarrow \delta_{cor} = \frac{\partial U_{cor}}{\partial B} = \frac{6}{5GA}\int_0^L V(x)\frac{\partial V(x)}{\partial B}dx$$

$$* \; M(x) = \begin{cases} Ax - w_0\frac{x^2}{2} \\ Ax - w_0\frac{x^2}{2} + P(x-\frac{L}{4}) \\ Ax - w_0\frac{x^2}{2} + P(x-\frac{L}{4}) + B(x-\frac{L}{2}) \end{cases} \Rightarrow \frac{\partial M(x)}{\partial B} = \begin{cases} \frac{\partial A}{\partial B}x \\ \frac{\partial A}{\partial B}x \\ \frac{\partial A}{\partial B}x + (x-\frac{L}{2}) \end{cases}$$

$$* \; V(x) = \begin{cases} w_0x - A \\ w_0x - A - P \\ w_0x - A - P - B \end{cases} \Rightarrow \frac{\partial V(x)}{\partial B} = \begin{cases} -\frac{\partial A}{\partial B} \\ -\frac{\partial A}{\partial B} \\ -\frac{\partial A}{\partial B} - 1 \end{cases}$$

$\rightsquigarrow$ Dado que $A = \frac{w_0L}{2} - \frac{3P}{4} - \frac{B}{2} \Rightarrow \frac{\partial A}{\partial B} = -\frac{1}{2}$

$$\Rightarrow \delta_{fl} = \frac{1}{EI}\left[\int_0^{L/4} -\frac{x}{2}\left(Ax - w_0\frac{x^2}{2}\right)dx + \int_{L/4}^{L/2} -\frac{x}{2}\left(Ax - w_0\frac{x^2}{2} + P(x-\frac{L}{4})\right) + \int_{L/2}^{L}\left(-\frac{x}{2} + (x-\frac{L}{2})\right)\left(Ax - w_0\frac{x^2}{2} + P(x-\frac{L}{4}) + B(x-\frac{L}{2})\right)\right]$$

• Calculando las integrales, evaluándolas y simplificando (y haciendo $B=0$)

$$\boxed{\delta_{fl} = \frac{1}{EI}\left(-\frac{5}{384}w_0L^4 + \frac{11}{768}PL^3\right)}$$

• Por otro lado

$$\Rightarrow \delta_{cor} = \frac{6}{5GA}\left[\int_0^{L/4}\frac{1}{2}(w_0x - A)dx + \int_{L/4}^{L/2}\frac{1}{2}(w_0x - A - P)dx + \int_{L/2}^{L} -\frac{1}{2}(w_0x - A - P - B)dx\right.$$

$$= \frac{6}{5GA}\left(-\frac{w_0L^2}{4} - \cancel{\frac{BL}{4}}^{\,0} + \frac{PL}{8}\right)$$

$$\boxed{\therefore \delta_{cor} = \frac{6}{5GA}\left(-\frac{w_0L^2}{4} + \frac{PL}{8}\right)}$$

• Finalmente, dado que la deflexión en ese punto debe ser cero, se tiene que

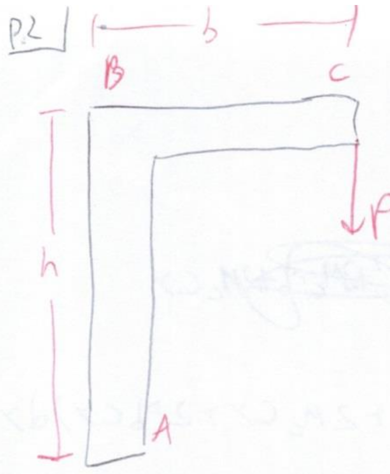
$$\delta_{fl} + \delta_{cor} = \delta_T = 0$$

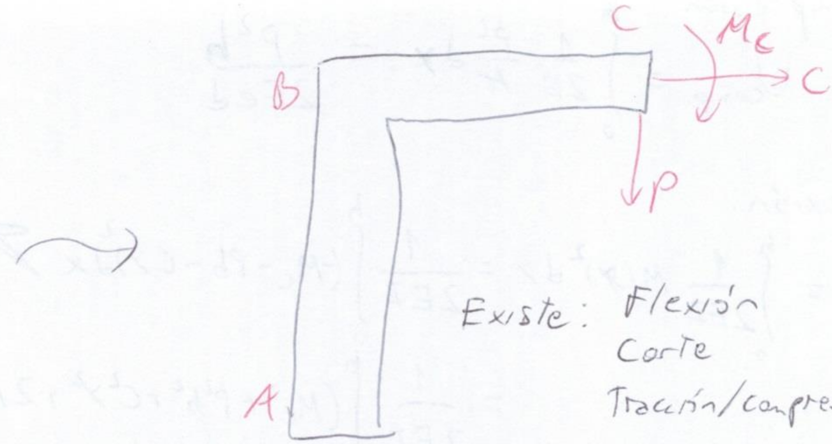
$$\frac{1}{EI}\left(-\frac{5}{384}w_0L^4 + \frac{11}{768}PL^3\right) + \frac{6}{5GA}\left(-\frac{w_0L^2}{4} + \frac{PL}{8}\right) = 0$$

$$P\left(\frac{11}{768}\frac{L^3}{EI} + \frac{3}{20}\frac{L}{GA}\right) = \left(\frac{5}{384}\frac{L^4}{EI} + \frac{3}{10}\frac{L^2}{GA}\right)w_0$$

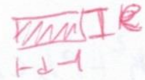
$$\boxed{P = 2\left(\frac{25L^3GA + 576LEI}{55L^2GA + 576EI}\right)}$$

P.2

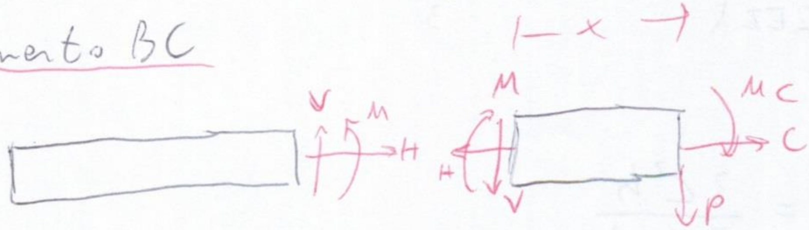




Existe: Flexión
Corte
Tracción/compresión



- Segmento BC



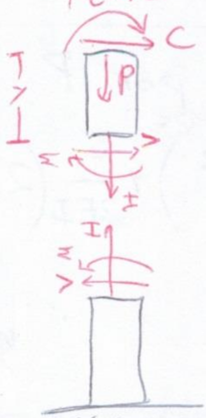
$$\boxed{\begin{aligned} H &= C \\ V &= -P \\ M &= -Px - M_C \end{aligned}}$$

→ Tracción
$$U_{tr} = \int_0^b \frac{1}{2E}\frac{C^2}{A}\,dx = \frac{C^2 b}{2EA} = \frac{C^2 b}{2Eed}$$

→ Flexión
$$U_f = \int_0^b \frac{M(x)^2}{2EI}\,dx = \frac{1}{2EI}\int_0^b (-Px - M_C)^2\,dx = \frac{1}{2EI}\left[\frac{P^2b^3}{3} + \frac{M_C P b^2}{} + M_C^2 b\right]$$

→ Corte
$$U_{cor} = \int_0^b \frac{3}{5GA} V(x)^2\,dx = \frac{3P^2 b}{5GA} = \frac{3P^2 b}{5Ged}$$

- Segmento AB



$$\boxed{\begin{aligned} H &= -P \\ V &= -C \\ M &= -M_C - Pb - Cy \end{aligned}}$$

~> Compresión

$$U_{comp} = \int_0^h \frac{1}{2E}\frac{P^2}{A}\,dy = \frac{P^2 h}{2Eed}$$

~> Flexión:

$$U_{Fl} = \int_0^h \frac{1}{2EI} M(y)^2\,dy = \frac{1}{2EI}\int_0^h (-M_c - Pb - Cy)^2\,dx$$

$$= \frac{1}{2EI}\int_0^h \left(M_c^2 + P^2b^2 + C^2y^2 + 2M_cPb + 2M_cCy + 2PbCy\right)dy$$

$$= \frac{1}{2EI}\left(M_c^2h + P^2b^2h + \frac{C^2h^3}{3} + 2M_cPbh + M_cCh^2 + PbCh^2\right)$$

~> Corte:

$$U_{cor} = \frac{3}{5GA}\int_0^h V(y)^2\,dy = \frac{3C^2h}{5Ged}$$

• Luego, la energía total de deformación está dada por

$$\boxed{U_T = U_{tr} + U_{comp} + U_{Fl}^{AB} + U_{Fl}^{BC} + U_{cor}^{AB} + U_{cor}^{BC}}$$

$$\boxed{\begin{aligned} U_T = {} & \frac{C^2b}{2Eed} + \frac{P^2h}{2Eed} + \frac{1}{2EI}\left(\frac{P^2b^3}{3} + M_cPb^2 + M_c^2b\right) \\ & + \frac{1}{2EI}\left(M_c^2h + P^2b^2h + \frac{C^2h^3}{3} + 2M_cPbh + M_cCh^2 + PbCh^2\right) \\ & + \frac{3P^2b}{5Ged} + \frac{3C^2h}{5Ged} \end{aligned}}$$

• Deflexión vertical δ_y en C ~> Producida por P

$$\left(\frac{\partial U_T}{\partial P}\right)_{\substack{M_c=0\\C=0}} = \frac{Ph}{Eed} + \frac{1}{2EI}\left(\frac{2}{3}Pb^3 + \cancel{M_c}b^2\right) + \frac{1}{2EI}\left(2Pb^2h + 2\cancel{M_c}bh + \cancel{b}Ch^2\right)$$

$M_c = 0$ $\quad$ $M_c = 0$ $\quad C = 0$

$$+ \frac{6Pb}{5Ged}$$

$$\doteq \frac{Ph}{Eed} + \frac{Pb^3}{3EI} + \frac{Pb^2h}{EI} + \frac{6Pb}{5Ged} \quad / \; I_z = \frac{ed^3}{12}$$

$$\therefore \left(\frac{\partial U_T}{\partial P}\right)_{\substack{M_c=0\\ C=0}} = \delta_y = \frac{P}{Eed}\left(h + \frac{4b^3}{d^2} + \frac{12b^2h}{d^2}\right) + \frac{6Pb}{5Ged}$$

$$\boxed{\therefore \delta_y = \frac{4P}{Eed}\left(\frac{h}{4} + \frac{b^3+3b^2h}{d^2}\right) + \frac{6}{5}\frac{Pb}{Ged}}$$

• Deflexión horizontal δ_x en $C \rightsquigarrow$ Producido por C

$$\left(\frac{\partial U_T}{\partial C}\right)_{\substack{M_c=0\\ C=0}} = \frac{\cancel{Cb}}{Eed}^{C=0} + \frac{\cancel{Ch^3}}{3EI}^{C=0} + \frac{\cancel{M_ch^2}}{2EI}^{M_c=0} + \frac{Pbh^2}{2EI} + \frac{\cancel{6Ch}}{5Ged}^{C=0}$$

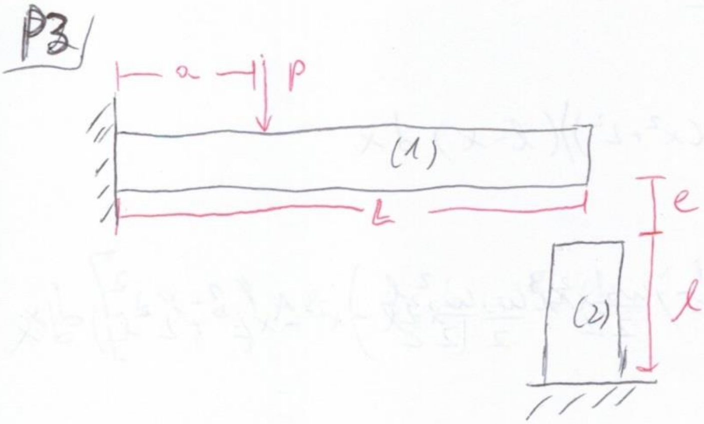
$$\boxed{\therefore \delta_x = \frac{6Pbh^2}{Eed^3}}$$

• ~~Deflexión~~ Ángulo de rotación en $C \longrightarrow$ Producido por M_c

$$\left(\frac{\partial U_T}{\partial M_c}\right)_{\substack{M_c=0\\ C=0}} = \frac{Pb^2}{2EI} + \frac{\cancel{M_cb}}{EI}^{M_c=0} + \frac{\cancel{M_ch}}{EI}^{M_c=0} + \frac{Pbh}{EI} + \frac{\cancel{Ch^2}}{2EI}^{M_c=0}$$

$$\Rightarrow \theta_c = \frac{Pb^2}{2EI} + \frac{Pbh}{EI} \Rightarrow \boxed{\theta_c = \frac{6Pb}{Eed^3}(b+2h)}$$

P3



E_1, I_z, E_2, A_2

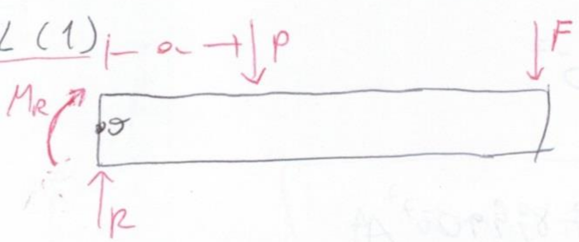
(Considerar sólo energía por flexión)

i) Dado que nos piden deflexión y NO hay una fuerza en el punto solicitado, inventaremos una y luego la anularemos.

ii) Dos casos: Barra (1) toca o no toca a (2)

* Barra (1) no toca a (2)

• DCL (1)



• $\sum F_y = 0$

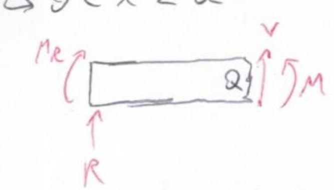
$$\boxed{R = P + F}$$

• $\sum_o M_z = 0$

$$\boxed{M_R = -(a \cdot P + L \cdot F)}$$

• Cálculo de fuerzas internas $M(x)$:

↳ $0 < x < a$

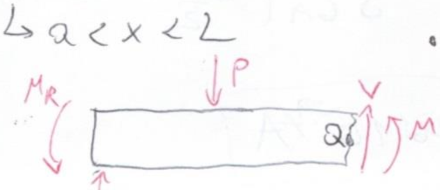


$\cdot \sum_a M_z = 0 \Rightarrow M(x) = M_R + Rx$

$$M(x) = -aP - LF + Px + Fx$$

$$\boxed{M(x) = F(x-L) + P(x-a)}$$

↳ $a < x < L$



$\cdot \sum_a M_z = 0 \Rightarrow M(x) = M_R + Rx - (x-a)P$

$$= -a\cancel{P} - LF - \cancel{Px} + \cancel{a}P + \cancel{Px} + Fx$$

$$\boxed{M(x) = F(x-L)}$$

- <u>Teorema de Castigliano:</u>

$$\delta_{Fl} = \frac{\partial U_{Fl}}{\partial F} = \frac{1}{2EI}\int_0^L 2M(x)\cdot\frac{dM(x)}{dF}\,dx = \frac{1}{EI}\int_0^L M(x)\,\frac{dM(x)}{dF}\,dx$$

$$* \frac{dM(x)}{dF} = \begin{cases} (x-L) & 0<x<a \\ (x-L) & a<x<L \end{cases}$$

$$\Rightarrow \delta_{Fl} = \frac{1}{EI}\left(\int_0^a \big(F(x-L)+P(x-a)\big)(x-L)\,dx + \int_a^L F(x-L)^2\,dx\right)$$

$$= \frac{1}{EI}\left(\int_0^a F(x^2-2Lx+L^2)+P(x^2-x(a+L)+aL + \int_a^L F(x^2-2Lx+L^2)\,dx\right)$$

$$= \frac{1}{EI}\left(\cancel{F\left(\frac{a^3}{3}-La^2+L^2a\right)}+P\left(\frac{a^3}{3}-\frac{a^2}{2}(a+L)+a^2L\right)+\cancel{(F\cdot(\dots))}\Big|_a^L\right.$$

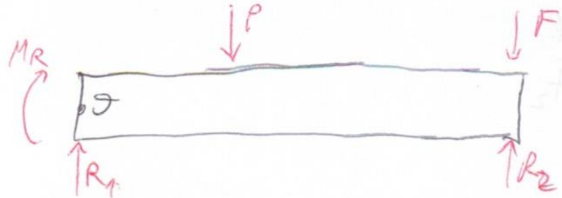
$*$ Dado que $F=0$

$$\delta_{Fl} = P\left(\frac{a^3}{3}-\frac{a^3}{2}-\frac{a^2L}{2}+a^2L\right)\cdot\frac{1}{EI}$$

$$\boxed{\delta_{Fl} = \frac{P}{6EI}(3L-a)\,a^2}$$

$*$ <u>Barra (1) si toca a (2)</u>: Aplicaremos teo. de Castigliano en ambos casos

- <u>DCL (1)</u>



- $\sum F_y = 0$

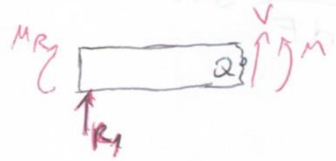
$$\boxed{R_1 = P + F - R_2}$$

- $\sum_{\vartheta} M_z = 0$

$$\boxed{M_R = -a\cdot P - FL + R_2 L}$$

- Cálculo de $M(x)$

$\hookrightarrow 0 < x < a$

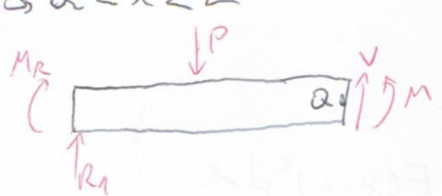


$\cdot \sum_a M_z = 0 \Rightarrow M(x) = M_R + R_1 x$

$= -aP - FL + R_2 L + Px + Fx - R_2 x$

$$\boxed{M(x) = P(x-a) + (F - R_2)(x-L)}$$

$\hookrightarrow a < x < L$



$\cdot \sum_a M_z = 0 \Rightarrow M(x) = M_R + R_1 x - P(x-a)$

$$\boxed{M(x) = (F - R_2)(x-L)}$$

- Castigliano

$* \dfrac{dM(x)}{dF} = x - L \;,\; \forall x \in [0, L]$

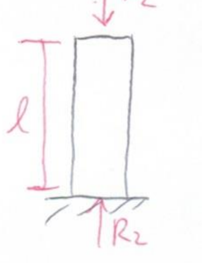
$$\Rightarrow \delta_{F1} = \frac{1}{EI}\int_0^a \left(P(x-a) + (F-R_2)(x-L)\right)(x-L)\,dx + \int_a^L (F-R_2)(x-L)^2\,dx$$

$* F \equiv 0$

$$= \frac{1}{EI}\left(P\left(\frac{a^3}{3} - \frac{a^2}{2}(a+L) + a^2 L\right) + \cancel{R_2\frac{(a-L)^3}{3}} - \frac{R_2 a^3}{3} - \cancel{R_2\frac{(a-L)^3}{3}}\right)$$

$$\boxed{\delta_{F1} = \frac{a^2 L P}{2} - \frac{a^3 P}{6} - \frac{R_2 a^3}{3}}$$

- DCL (2)



$*$ Energía por compresión: $U_T = \dfrac{R_2^2\, l}{2E_2 A_2}$

$\Rightarrow$ Castigliano:

$$\delta_T = \frac{\partial U_T}{\partial R_2} = \frac{R_2\, l}{E_2 A_2}$$

* Viga (1) se deflecta e más la que se acorta (2):

$$\delta_{fl} = e + \delta_T$$

$$\frac{a^2 P L}{2} - \frac{a^3 P}{6} - \frac{R_2 a^3}{3} = e + \frac{R_2 l}{E_2 A_2}$$

$$\Rightarrow R_2 \left(\frac{a^3}{3} + \frac{l}{E_2 A_2} \right) = \frac{P a^2}{6}(3L - a) + e$$

$$\boxed{\therefore R_2 = \frac{P a^2 E_2 A_2 (3L - a)}{3l + a^3 E_2 A_2} + \left(\frac{e}{\frac{a^3}{3} + \frac{l}{E_2 A_2}} \right)}$$