Pauta Aux 8 MF3204

P1) $[\sigma_{ij}] = \begin{pmatrix} 0 & 0 & -cx_2 \\ 0 & 0 & cx_1 \\ -cx_2 & cx_1 & 0 \end{pmatrix} \wedge \underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

a) $[\sigma_{ij}] = \begin{pmatrix} 0 & 0 & -2c \\ 0 & 0 & c \\ -2c & c & 0 \end{pmatrix}$

* Calculando los invariantes

1-) $I_1 = tr(\underset{\sim}{\sigma}) = 0$

2-) $\underset{\sim}{\sigma}^2 = \underset{\sim}{\sigma}\,\underset{\sim}{\sigma} = \begin{pmatrix} 4c^2 & -2c^2 & 0 \\ -2c^2 & c^2 & 0 \\ 0 & 0 & 5c^2 \end{pmatrix} \wedge tr(\underset{\sim}{\sigma}^2) = 10$

$\rightarrow I_2 = \frac{1}{2}\left(\cancel{(tr(\underset{\sim}{\sigma}))^2} - tr(\underset{\sim}{\sigma}^2)\right) = -5c^2$

3-) $I_3 = det(\underset{\sim}{\sigma}) = 0$

* La ec. característica queda como:

$$-\lambda^3 + \lambda^2 \cancel{I_1} - \lambda I_2 + \cancel{I_3} = 0$$

$$-\lambda^3 - \lambda(-5c^2) = 0$$

$$\boxed{\lambda(\lambda^2 - 5c^2) = 0}$$

$\therefore \lambda_1 = 0 \wedge \lambda_{2,3} = \frac{0 \pm \sqrt{20c^2}}{2} = \pm \frac{2c\sqrt{5}}{2}$

$$\lambda_1 = 0 \;;\; \lambda_2 = c\sqrt{5} \;;\; \lambda_3 = -c\sqrt{5}$$

b) La ec. característica viene de $(\underset{\sim}{I} - \lambda_1 \underset{\sim}{\Omega})\, \underset{\sim}{n_1} = 0$. Luego

$\boxed{\lambda_1 = 0}$

$$\begin{pmatrix} 0 & 0 & -2c \\ 0 & 0 & c \\ -2c & c & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} -2c n_3 &= 0 \quad (1) \\ c n_3 &= 0 \quad (2) \\ -2c n_1 + c n_2 &= 0 \quad (3) \end{aligned}}$$

* De (1), (2) $\rightsquigarrow n_3 = 0$. De (3)

$$\boxed{n_2 = 2 n_1} \quad (4)$$

* Imponemos que los vectores sean unitarios

$$n_1^2 + \underbrace{n_2^2}_{(2n_1)} + n_3^2 = 1 \Rightarrow n_1^2 + (2n_1)^2 = 1$$

$$\therefore n_1 = 1/\sqrt{5} \wedge n_2 = 2/\sqrt{5} \Rightarrow \boxed{\underset{\sim}{n}^1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \frac{1}{\sqrt{5}}}$$

$\boxed{\lambda_1 = c\sqrt{5}}$

$$\begin{pmatrix} -c\sqrt{5} & 0 & -2c \\ 0 & -c\sqrt{5} & c \\ -2c & c & -c\sqrt{5} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} -c\sqrt{5}\, n_1 - 2c n_3 &= 0 \quad (1) \\ -c\sqrt{5}\, n_2 + c n_3 &= 0 \quad (2) \\ -2c n_1 + c n_2 - c\sqrt{5}\, n_3 &= 0 \quad (3) \end{aligned}}$$

• De (1) $\Rightarrow \boxed{n_3 = -\frac{\sqrt{5}}{2} n_1}$ (4) . ~~De (2)~~ $\Rightarrow$

• De (2) $\Rightarrow -c\sqrt{3}\,n_2 + c\left(-\frac{\sqrt{3}}{2}\right)n_1 = 0$

$$\boxed{\therefore n_2 = -\frac{1}{2}n_1} \quad (5)$$

• La ec. (3) es L.D. de las otras. Imponiendo que $\hat{n}$ sea unitario

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$$n_1^2 - \left(-\frac{1}{2}n_1\right)^2 + \left(-\frac{\sqrt{3}}{2}n_1\right)^2 = 1$$

$$\Rightarrow \boxed{n_1 = \sqrt{\tfrac{2}{3}} \quad n_2 = -\frac{1}{\sqrt{10}} \wedge n_3 = -\frac{\sqrt{2}}{2}} \qquad \therefore \underset{\sim}{\hat{n}}^{2} = \begin{pmatrix} \sqrt{2/3} \\ -\sqrt{1/10} \\ -\sqrt{1/2} \end{pmatrix}$$

$\underline{\lambda_3 = -c\sqrt{3}}$

$$\begin{pmatrix} c\sqrt{3} & 0 & -2c \\ 0 & c\sqrt{3} & c \\ -2c & c & c\sqrt{3} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} c\sqrt{3}\,n_1 - 2cn_3 &= 0 \quad (1) \\ c\sqrt{3}\,n_2 + cn_3 &= 0 \quad (2) \\ -2cn_1 + cn_2 + c\sqrt{3}\,n_3 &= 0 \quad (3) \end{aligned}}$$

• (3) es L.D. • De (1) $\rightarrow n_3 = \frac{\sqrt{3}}{2}n_1$

• De (2) $\rightarrow n_2 = -\frac{1}{2}n_1$

• De $\hat{n}$ unitario $\rightarrow n_1^2 + n_2^2 + n_3^2 = 0$

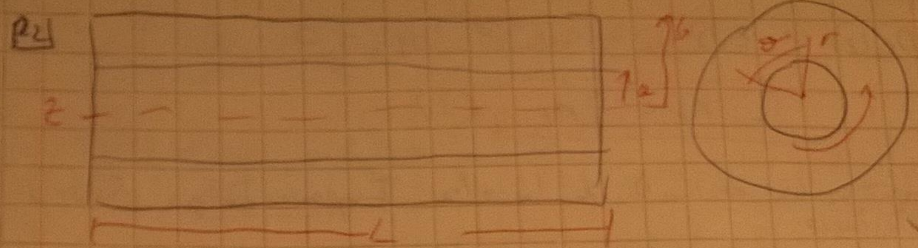
$$\therefore n_1 = \sqrt{\tfrac{2}{5}} \Rightarrow n_2 = -\sqrt{1/10} \wedge n_3 = \sqrt{1/2}$$

$$\therefore \underset{\sim}{\hat{n}}^{3} = \begin{pmatrix} \sqrt{2/5} \\ -\sqrt{1/10} \\ \sqrt{1/2} \end{pmatrix}$$

c) Esf. corte max

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} \Big/ \left.\begin{aligned} \sigma_{max} &= c\sqrt{3} \\ \sigma_{min} &= 0 \end{aligned}\right\} \boxed{\tau_{max} = c\frac{\sqrt{3}}{2}}$$

P2



• Tubo no se alarga en z → $U_z = 0$

• ~~No hay cambio de radio → $U_r = 0$~~

• $U_\theta = U_\theta(r)$ (Simetría axial por torsión homogénea

$U_r = U_r(r)$

⇒ Debemos encontrar $U_\theta(r)$

1-) Ec. de equilibrio:

i) $\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \rho b_r = 0$

ii) $\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + 2\frac{T_{r\theta}}{r} + \rho b_\theta = 0$

iii) $\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r} + \rho b_z = 0$

• Suposiciones: - No hay esf. en z ⇒ $T_{zi} = T_{iz} = 0$

• ~~Simetría axial~~ - Simetría axial, sólo hay dependencia en r

$$\frac{\partial T_{ij}}{\partial \theta} = \frac{\partial T_{ij}}{\partial z} = 0$$

- Fzas de cuerpo nulas ⇒ $\rho b_i = 0$

• Luego:

$$\boxed{\begin{aligned} &i)\ \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0 \\ &ii)\ \frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} = 0 \end{aligned}}$$

2) ~~Ecuaciones constitutivas~~; Estado de deformación - despl:

$$\varepsilon_{rr} = \frac{\partial U_r}{\partial r} \;;\; \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} \;;\; \varepsilon_{zz} = \frac{\partial U_z}{\partial z}$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z}\right) \;;\; \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right)$$

• Bajo las suposiciones de simetría axial e invariales:

(iii)

$= \frac{U_r}{r}$; $\varepsilon_{rr} = \frac{\partial U_r}{\partial r}$ (iv), $\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)$ (v), Resto $= 0$

) Ecs. constitutivas:

$$\cancel{\varepsilon_{ij} = \frac{(1+\nu)}{E} T_{ij} - \frac{\nu}{E} T_{kk}\delta_{ij}}$$

$$T_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij}$$

$T_{rr} = 2\mu\,\varepsilon_{rr} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta})$

$$\hookrightarrow \boxed{T_{rr} = 2\mu \frac{\partial U_r}{\partial r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)} \text{(vi)}$$

$\Rightarrow T_{\theta\theta} = 2\mu\, \varepsilon_{\theta\theta} + \cancel{\ } \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta})$

$$\boxed{T_{\theta\theta} = 2\mu \frac{U_r}{r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)} \quad (vii)$$

$$\Rightarrow \boxed{T_{r\theta} = 2\mu\, \varepsilon_{r\theta} = \mu\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)} \quad (viii)$$

- Ahora tenemos todas las ecs. necesarias. De (iii)

$$\frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} = 0 \rightarrow \boxed{T_{r\theta} = \frac{C_0}{r^2}}$$

- Reemplazando (viii) en la ec. anterior

$$\mu\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) = \frac{C_0}{r^2}$$

$$\cancel{\frac{\partial U_\theta}{\partial r} = \frac{C_0}{\mu}\frac{1}{r^2} + \frac{U_0}{}}$$

$$\frac{\partial U_\theta}{\partial r} - \frac{1}{r}U_\theta = \frac{C_0}{\mu}\frac{1}{r^2}$$

Soln $\boxed{U_\theta(r) = C_1 r - \frac{C_0}{2\mu}\cdot\frac{1}{r}}$

- C.B. • $U_\theta(b) = 0$ (Empotramiento)

$$\rightarrow C_1 b - \frac{C_0}{2\mu b} = 0 \Rightarrow \boxed{C_1 = \frac{C_0}{2\mu b^2}}$$

• $U_\theta(a) = U_0$ (Despl. conocido)

$$U_0 = C_1 a - \frac{C_0}{2\mu a} = \frac{C_0 a}{2\mu b^2} - \frac{C_0}{2\mu a}$$

$$\rightsquigarrow \frac{C_0}{2\mu}\left(\frac{a}{b^2} - \frac{1}{a}\right) = U_0 \Rightarrow \boxed{C_0 = \frac{2\mu U_0 a b^2}{a^2 - b^2}}$$

$$\boxed{\therefore U_\theta = \frac{U_0 a}{a^2-b^2} r - \frac{U_0 b^2}{a^2-b^2}\frac{1}{r}} \quad C_1 = \frac{U_0 a}{a^2-b^2}$$

• De (i)

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0$$

• Reemplazando (vi) y (vii) en la ec. anterior:

$$\frac{\partial}{\partial r}\left(2\mu\frac{\partial U_r}{\partial r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)\right) + \frac{1}{r}\left(2\mu\frac{\partial U_r}{\partial r} + \lambda\cancel{\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)}\right.$$

$$\left. - 2\mu\frac{U_r}{r} - \lambda\cancel{\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)}\right) = 0$$

$$\frac{\partial}{\partial r}\left(2\mu\frac{\partial U_r}{\partial r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)\right) + \frac{1}{r}\left(2\mu\frac{\partial U_r}{\partial r} - 2\mu\frac{U_r}{r}\right) = 0$$

$$\Rightarrow 2\mu\frac{\partial^2 U_r}{\partial r^2} + \lambda\frac{\partial^2 U_r}{\partial r^2} + \lambda\left(U_r\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) + \lambda\frac{\partial U_r}{\partial r}\cdot\frac{1}{r} + \frac{1}{r}2\mu\frac{\partial U_r}{\partial r} - 2\mu\frac{U_r}{r^2} = 0$$

$$(2\mu+\lambda)\frac{\partial^2 U_r}{\partial r^2} + (2\mu+\lambda)\frac{1}{r}\frac{\partial U_r}{\partial r} - (2\mu+\lambda)\frac{U_r}{r^2} = 0$$

$$\boxed{\therefore \frac{d^2 U_r}{dr^2} + \frac{1}{r}\frac{dU_r}{dr} - \frac{U_r}{r^2} = 0}$$

Sol.n) $u_r(r) = \frac{C_3}{r} + C_4 r$

* $u_r(b) = 0 \Rightarrow \frac{C_3}{b} + C_4 b = 0$

* $u_r(a) = 0 \Rightarrow \frac{C_3}{a} + C_4 a = 0$

$\boxed{\therefore C_3 = C_4 = 0} \Rightarrow \boxed{u_r(r) = 0}$

∴ Estado final de desplazamiento

$$\boxed{\underset{\sim}{u}(r) = \begin{pmatrix} u_r(r) \\ u_\theta(r) \\ u_z(r) \end{pmatrix} = \begin{pmatrix} 0 \\ u_\theta(r) \\ u_z(r) \end{pmatrix}}$$