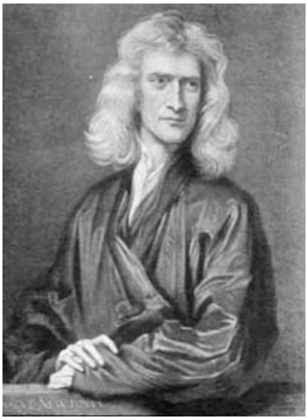
Chapter 2

Newton, Lagrange and Hamilton's Treatments of the Rigid Body

2.1 Newton

2.1.1 Newtonian form of free rigid rotation



Isaac Newton

Definition 2.1.1 *In **free rigid rotation**, a body undergoes rotation about its centre of mass and the pairwise distances between all points in the body remain fixed.*

Definition 2.1.2 *A system of coordinates in free rigid motion is stationary in the rotating orthonormal basis called the **body frame**, introduced by Euler [Eu1758].*

The orientation of the orthonormal body frame $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ relative to a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ fixed in space depends smoothly on time $t \in \mathbb{R}$.

In the fixed spatial coordinate system, the body frame is seen as the moving frame

$$(O(t)\mathbf{E}_1,\, O(t)\mathbf{E}_2,\, O(t)\mathbf{E}_3)\,,$$

where $O(t) \in SO(3)$ defines the attitude of the body relative to its reference configuration according to the following matrix multiplication on its three unit vectors:

$$\mathbf{e}_a(t) = O(t)\mathbf{E}_a\,, \quad a = 1, 2, 3. \tag{2.1.1}$$

Here the unit vectors $\mathbf{e}_a(0) = \mathbf{E}_a$ with $a = 1, 2, 3$, comprise at initial time $t = 0$ an orthonormal basis of coordinates and $O(t)$ is a special ($\det O(t) = 1$) orthogonal ($O^T(t)O(t) = Id$) 3×3 matrix. That is, $O(t)$ is a continuous function defined along a curve parameterised by time t in the special orthogonal matrix group $SO(3)$. At the initial time $t = 0$, we may take $O(0) = Id$, without any loss.

As its orientation evolves according to (2.1.1), each basis vector in the set,

$$\mathbf{e}(t) \in \{\mathbf{e}_1(t),\, \mathbf{e}_2(t),\, \mathbf{e}_3(t)\},$$

preserves its (unit) length,

$$\begin{aligned} 1 &= |\mathbf{e}(t)|^2 := \mathbf{e}(t)\cdot\mathbf{e}(t) := \mathbf{e}(t)^T\mathbf{e}(t) = (O(t)\mathbf{E})^T O(t)\mathbf{E} \\ &= \mathbf{E}^T O^T(t)O(t)\mathbf{E} = \mathbf{E}^T(Id)\mathbf{E} = |\mathbf{E}|^2\,, \end{aligned} \tag{2.1.2}$$

which follows because $O(t)$ is orthogonal; that is, $O^T(t)O(t) = Id$.

The basis vectors in the orthonormal frame $\mathbf{e}_a(0) = \mathbf{E}_a$ define the initial orientation of the set of rotating points with respect to some choice of fixed spatial coordinates at time $t = 0$. Each point $\mathbf{r}(t)$ in the subsequent rigid motion may be represented in either fixed, or rotating coordinates as,

$$\mathbf{r}(t) = r_0^A(t)\mathbf{e}_A(0) \quad \text{in the fixed basis}\,, \tag{2.1.3}$$

$$\phantom{\mathbf{r}(t)} = r^a\mathbf{e}_a(t) \quad \text{in the rotating basis}\,. \tag{2.1.4}$$

The fixed basis is called the ***spatial frame*** and rotating basis is the ***body frame***. The constant components r^a of a position vector relative to the rotating basis are related to its initial spatial position as $r^a = O^a_A(0)r^A_0(0)$. (This is simply $r^a = \delta^a_A r^A_0(0)$ for the choice $O(0) = Id$ that the two coordinate bases are initially aligned.) The components of any vector $\mathbf{J}$ in the fixed (spatial) frame are related to those in the moving (body) frame by the mutual rotation of their axes in (2.1.1) at any time. That is,

$$\mathbf{J} = \mathbf{e}_a(0)J^a_{space}(t) = \mathbf{e}_a(t)J^a_{body} = O(t)\mathbf{e}_a(0)J^a_{body}\,, \tag{2.1.5}$$

or equivalently, as in equation (2.1.1),

$$\mathbf{J}_{space}(t) = O(t)\mathbf{J}_{body}\,. \tag{2.1.6}$$

Lemma 2.1.3 *The velocity $\dot{\mathbf{r}}(t)$ of a point $\mathbf{r}(t)$ in free rigid rotation depends linearly on its position relative to the centre of mass.*

Proof. In particular, $\mathbf{r}(t) = r^a O(t)\mathbf{e}_a(0)$ implies

$$\dot{\mathbf{r}}(t) = r^a\dot{\mathbf{e}}_a(t) = r^a\dot{O}(t)\mathbf{e}_a(0) =: r^a\dot{O}O^{-1}(t)\mathbf{e}_a(t) =: \widehat{\omega}(t)\mathbf{r}\,, \tag{2.1.7}$$

which is linear. ■

Remark 2.1.4 (Wide-hat notation)
The wide-hat notation $(\widehat{\cdot})$ denotes a skew-symmetric 3×3 matrix. There is no danger of confusing wide-hat notation $(\widehat{\cdot})$ with narrow-hat notation $(\hat{\cdot})$, which denotes unit vector (or, later, unit quaternion).

Lemma 2.1.5 (Skew symmetry)
*The **spatial angular velocity matrix** $\widehat{\omega}(t) = \dot{O}O^{-1}(t)$ in (2.1.7) is skew-symmetric, i.e.,*

$$\widehat{\omega}^T = -\,\widehat{\omega}\,.$$

Proof. Being orthogonal, the matrix $O(t)$ satisfies $OO^T = Id$. This implies that $\widehat{\omega}$ is skew-symmetric,

$$\begin{aligned} 0 = (OO^T)^{\,\cdot} &= \dot{O}O^T + O\dot{O}^T = \dot{O}O^T + (\dot{O}O^T)^T \\ &= \dot{O}O^{-1} + (\dot{O}O^{-1})^T = \widehat{\omega} + \widehat{\omega}^T\,. \end{aligned}$$

■

Remark 2.1.6 *The skew 3×3 real matrices form a closed linear space under addition.*

Definition 2.1.7 (Commutator product of skew matrices) *The commutator product of two skew matrices $\widehat{\omega}$ and $\widehat{\xi}$ is defined as the skew matrix product,*

$$[\widehat{\omega},\, \widehat{\xi}] := \widehat{\omega}\,\widehat{\xi} - \widehat{\xi}\,\widehat{\omega}\,. \tag{2.1.8}$$

Remark 2.1.8 *This commutator product is again a skew 3×3 real matrix.*

Definition 2.1.9 (Basis set for skew matrices) *Any 3×3 antisymmetric matrix $\widehat{\omega}^T = -\,\widehat{\omega}$ may be written as a linear combination of the following three linearly independent basis elements for the 3×3 skew matrices:*

$$\widehat{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \widehat{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \widehat{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is, the element $\widehat{J}_a$ for this choice of basis has matrix components $(\widehat{J}_a)_{bc} = -\epsilon_{abc}$, where ϵ_{abc} is the totally antisymmetric tensor with $\epsilon_{123} = +1$, $\epsilon_{213} = -1$, $\epsilon_{113} = 0$, etc.

Lemma 2.1.10 (Commutation relations) *The skew matrix basis $\widehat{J}_a$ with $a = 1, 2, 3$, satisfies the* ***commutation relations****,*

$$[\,\widehat{J}_a,\, \widehat{J}_b\,] := \widehat{J}_a\widehat{J}_b - \widehat{J}_b\widehat{J}_a = \epsilon_{abc}\widehat{J}_c\,. \tag{2.1.9}$$

Proof. This may be verified by a direct calculation $[\,\widehat{J}_1,\, \widehat{J}_2\,] = \widehat{J}_3$, etc. ∎

Remark 2.1.11 *The closure of the basis set of skew-symmetric matrices under the commutator product gives the linear space of skew-symmetric matrices a* ***Lie algebra structure****. The constants ϵ_{abc} in the commutation relations among the skew 3×3 matrix basis elements are called the* ***structure constants*** *and the corresponding Lie algebra is called $so(3)$. This also means the abstract $so(3)$ Lie* *algebra may be represented by skew 3×3 matrices, which is a great convenience, as we shall see for example in section 4.2. The Lie algebra $so(3)$ may also be defined as the tangent space to the Lie group $SO(3)$ at the identity, as discussed in Appendix B.*

Theorem 2.1.12 (Hat map)
The components of any 3×3 skew matrix $\widehat{\omega}$ may be identified with the corresponding components of a vector $\boldsymbol{\omega} \in \mathbb{R}^3$.

Proof. In the basis (2.1.9), one writes the linear invertible relation,

$$\widehat{\omega} = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix} = \omega^a \widehat{J}_a =: \boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}}\,, \tag{2.1.10}$$

for $a = 1, 2, 3$. This is a one-to-one invertible map, i.e., an ***isomorphism***, between 3×3 skew-symmetric matrices and vectors in $\mathbb{R}^3$.
■

Remark 2.1.13 *The superscript hat $(\,\widehat{}\,)$ applied to a vector identifies that vector in $\mathbb{R}^3$ with a 3×3 skew-symmetric matrix. For example, the unit vectors in the Cartesian basis set, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are associated with the basis elements $\widehat{J}_a$ for $a = 1, 2, 3$, in equation (2.1.9) by $\widehat{J}_a = \widehat{\mathbf{e}}_a$, or in matrix components,*

$$(\widehat{\mathbf{e}}_a)_{bc} = -\delta^d_a \epsilon_{dbc} = -\epsilon_{abc} = (\mathbf{e}_a \times)_{bc}\,.$$

Remark 2.1.14 *The last equality in the definition of the hat map in equation (2.1.10) introduces the convenient notation $\widehat{\boldsymbol{J}}$ that denotes the basis for the 3×3 skew-symmetric matrices $\widehat{J}_a$ with $a = 1, 2, 3$, as a vector of matrices.*

Definition 2.1.15 (Hat map for angular velocity vector)
*The relation $\widehat{\omega} = \boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}}$ in equation (2.1.10) identifies the skew-symmetric 3×3 matrix $\hat{\omega}(t)$ with the **angular velocity vector** $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ whose components $\omega^c(t)$, with $c = 1, 2, 3$, are given by*

$$(\dot{O}O^{-1})_{ab}(t) = \widehat{\omega}_{ab}(t) = -\,\epsilon_{abc}\,\omega^c(t)\,. \tag{2.1.11}$$

*Equation (2.1.11) defines the matrix components of the **hat map for angular velocity**.*

Remark 2.1.16 *Equivalently, the hat map in equation (2.1.10) is defined by the identity,*

$$\widehat{\omega}\,\boldsymbol{\lambda} = \boldsymbol{\omega} \times \boldsymbol{\lambda} \quad \textit{for all} \quad \boldsymbol{\omega}, \boldsymbol{\lambda} \in \mathbb{R}^3.$$

Thus, we may write $\widehat{\boldsymbol{\omega}} = \widehat{\omega} = \boldsymbol{\omega}\times$ *to identify the vector* $\boldsymbol{\omega} \in \mathbb{R}^3$ *with the skew* 3×3 *matrix* $\widehat{\omega} \in so(3)$*.*

Proposition 2.1.17 *The* 3×3 *skew matrices*

$$\widehat{\omega} = \boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}} \quad \textit{and} \quad \widehat{\lambda} = \boldsymbol{\lambda} \cdot \widehat{\boldsymbol{J}}\,,$$

associated with the vectors $\boldsymbol{\omega}$ *and* $\boldsymbol{\lambda}$ *in* $\mathbb{R}^3$ *satisfy the commutation relation*

$$[\,\widehat{\omega},\,\widehat{\lambda}\,] = \boldsymbol{\omega} \times \boldsymbol{\lambda} \cdot \widehat{\boldsymbol{J}} =: (\boldsymbol{\omega} \times \boldsymbol{\lambda})\widehat{\ }\,, \tag{2.1.12}$$

where $\boldsymbol{\omega} \times \boldsymbol{\lambda}$ *is the vector product in* $\mathbb{R}^3$*.*

Proof. Formula (2.1.9) implies the result, by

$$\begin{aligned} [\,\widehat{\omega},\,\widehat{\lambda}\,] &= [\,\boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}},\,\boldsymbol{\lambda} \cdot \widehat{\boldsymbol{J}}\,] = [\,\omega^a \widehat{J}_a,\,\lambda^b \widehat{J}_b\,] \\ &= \omega^a \lambda^b [\,\widehat{J}_a,\,\widehat{J}_b\,] = \omega^a \lambda^b \epsilon_{abc} \widehat{J}_c = \boldsymbol{\omega} \times \boldsymbol{\lambda} \cdot \widehat{\boldsymbol{J}}\,. \end{aligned}$$

∎

Remark 2.1.18 *According to Proposition 2.1.17, the* ***hat map*** $\widehat{\ }:$ $(\mathbb{R}^3, \times) \mapsto (so(3), [\,\cdot\,,\,\cdot\,])$ *allows the velocity in space (2.1.7) of a point at* $\mathbf{r}$ *undergoing rigid-body motion to be expressed equivalently either by a skew-matrix multiplication, or as a vector product. That is,*

$$\dot{\mathbf{r}}(t) =: \widehat{\omega}(t)\mathbf{r} =: \boldsymbol{\omega}(t) \times \mathbf{r}\,. \tag{2.1.13}$$

Hence, free rigid motion of a point displaced by $\mathbf{r}$ *from the centre of mass is a rotation in space of* $\mathbf{r}$ *about the time dependent angular velocity vector* $\boldsymbol{\omega}(t)$*. Accordingly,* $d|\mathbf{r}|^2/dt = 2\mathbf{r} \cdot \dot{\mathbf{r}} = 0$*, and the displacement distance is preserved,* $|\mathbf{r}|(t) = |\mathbf{r}|(0)$*.*

Kinetic energy of free rigid rotation

The kinetic energy for N particles of masses m_j, $j = 1, 2, \ldots, N$ mutually undergoing free rigid rotation is computed in terms of the angular velocity as

$$\begin{aligned} K &= \frac{1}{2}\sum_{j=1}^{N} m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j \\ &= \frac{1}{2}\sum_{j=1}^{N} m_j (\boldsymbol{\omega} \times \mathbf{r}_j) \cdot (\boldsymbol{\omega} \times \mathbf{r}_j) \\ &=: \frac{1}{2}\langle\!\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle\!\rangle \,. \end{aligned}$$

Definition 2.1.19 (Symmetric mass-weighted pairing)
The kinetic energy induces a symmetric mass-weighted pairing

$$\langle\!\langle \,\cdot\,,\,\cdot\, \rangle\!\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \,,$$

defined for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ *as,*

$$\langle\!\langle \mathbf{a}, \mathbf{b} \rangle\!\rangle := \sum_j m_j (\mathbf{a} \times \mathbf{r}_j) \cdot (\mathbf{b} \times \mathbf{r}_j) =: \mathbb{I}\,\mathbf{a} \cdot \mathbf{b} \,. \tag{2.1.14}$$

Definition 2.1.20 (Moment of inertia tensor)
The mass-weighted pairing, or inner product in (2.1.14)

$$\langle\!\langle \mathbf{a}, \mathbf{b} \rangle\!\rangle = \mathbb{I}\,\mathbf{a} \cdot \mathbf{b} \,,$$

defines the symmetric ***moment of inertia tensor*** $\mathbb{I}$ *for the particle system.*

Definition 2.1.21 (Angular momentum of rigid motion)
The ***angular momentum*** *is defined as the derivative of the kinetic energy with respect to angular velocity. In the present case*

with (2.1.14), this produces the linear relation,

$$\begin{aligned}\mathbf{J} = \frac{\partial K}{\partial \boldsymbol{\omega}} &= -\frac{1}{2}\sum_{j=1}^{N} m_j \mathbf{r}_j \times \Big(\mathbf{r}_j \times \boldsymbol{\omega}\Big) \\ &= \frac{1}{2}\sum_{j=1}^{N} m_j \Big(|\mathbf{r}_j|^2 Id - \mathbf{r}_j \otimes \mathbf{r}_j\Big)\Big)\boldsymbol{\omega} \\ &=: \mathbb{I}\,\boldsymbol{\omega}\,, \end{aligned} \tag{2.1.15}$$

where $\mathbb{I}$ is the moment of inertia tensor defined by the symmetric pairing in (2.1.14).

Conservation of angular momentum in free rigid rotation

In *free* rigid rotation no external torques are applied, so the angular momentum $\mathbf{J}$ is conserved. In the fixed basis $\mathbf{J} = J_0^A(t)\mathbf{e}_A(0)$ and this conservation law is expressed as,

$$0 = \frac{d\mathbf{J}}{dt} = \frac{dJ_0^A}{dt}\mathbf{e}_A(0)\,, \tag{2.1.16}$$

so each component J_0^A for $A = 1, 2, 3$, of angular momentum in the spatial frame is separately conserved. In the rotating basis $\mathbf{J} = J^a(t)\mathbf{e}_a(t)$ and angular momentum conservation becomes

$$\begin{aligned} 0 = \frac{d\mathbf{J}}{dt} &= \frac{dJ^a}{dt}\mathbf{e}_a(t) + J^a\frac{d\mathbf{e}_a(t)}{dt} \\ &= \frac{dJ^a}{dt}\mathbf{e}_a(t) + \boldsymbol{\omega} \times J^a\mathbf{e}_a(t) \\ &= \Big(\frac{dJ^a}{dt} + (\boldsymbol{\omega}\times\mathbf{J})^a\Big)\mathbf{e}_a(t)\,. \end{aligned} \tag{2.1.17}$$

Consequently, the components J^a for $a = 1, 2, 3$, of angular momentum in the body frame satisfy the quadratically nonlinear *system* of equations (2.1.17) with $\mathbf{J} = \mathbb{I}\,\boldsymbol{\omega}$.

Lemma 2.1.22 (Space vs body dynamics)
Upon denoting $\mathbf{J}_{space} = (J_0^1,\, J_0^2,\, J_0^3)$ in the fixed basis $\mathbf{e}_A(0)$ and $\mathbf{J}_{body} = (J^1,\, J^2,\, J^3)$ in the time-dependent basis $\mathbf{e}_a(t)$, with

$$\mathbf{J}_{space} = O(t)\mathbf{J}_{body}\,, \tag{2.1.18}$$

one may summarise the two equivalent sets of equations (2.1.16) and (2.1.17) as,

$$\frac{d\mathbf{J}_{space}}{dt} = 0 \quad and \quad \frac{d\mathbf{J}_{body}}{dt} + \big(\mathbb{I}^{-1}\mathbf{J}_{body}\big) \times \mathbf{J}_{body} = 0 \,. \tag{2.1.19}$$

Proof. The time derivative of relation (2.1.18) gives

$$\begin{aligned}
\frac{d\mathbf{J}_{body}}{dt} &= \frac{d}{dt}\Big(O^{-1}(t)\mathbf{J}_{space}(t)\Big) \\
&= -\,O^{-1}\dot{O}\Big(O^{-1}(t)\mathbf{J}_{space}(t)\Big) + O^{-1}\,\underbrace{\frac{d\mathbf{J}_{space}}{dt}}_{\text{vanishes}} \\
&= -\,\widehat{\omega}_{body}\mathbf{J}_{body} = -\,\boldsymbol{\omega}_{body} \times \mathbf{J}_{body}\,,
\end{aligned} \tag{2.1.20}$$

with $\widehat{\omega}_{body} := O^{-1}\dot{O} = \boldsymbol{\omega}_{body}\times$ and $\boldsymbol{\omega}_{body} = \mathbb{I}^{-1}\mathbf{J}_{body}$, which defines the ***body angular velocity***. This is the usual heuristic derivation of the dynamics for body angular momentum. ■

Remark 2.1.23 (Darwin, Coriolis and centifugal forces)

Many elementary mechanics texts make the following points about the various ***noninertial forces*** *that arise in a rotating frame. For any vector* $\mathbf{r}(t) = r^a(t)\mathbf{e}_a(t)$ *the body and space time derivatives satisfy the first time-derivative relation, as in (2.1.17),*

$$\begin{aligned}
\dot{\mathbf{r}}(t) &= \dot{r}^a\mathbf{e}_a(t) + r^a\dot{\mathbf{e}}_a(t) \\
&= \dot{r}^a\mathbf{e}_a(t) + \boldsymbol{\omega} \times r^a\mathbf{e}_a(t) \\
&= \big(\dot{r}^a + \epsilon^a_{bc}\omega^b r^c\big)\mathbf{e}_a(t) \\
&=: \big(\dot{r}^a + (\boldsymbol{\omega} \times \mathbf{r})^a\big)\mathbf{e}_a(t)\,.
\end{aligned} \tag{2.1.21}$$

Taking a second time-derivative in this notation yields

$$\begin{aligned}
\ddot{\mathbf{r}}(t) &= \big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}} \times \mathbf{r})^a + (\boldsymbol{\omega} \times \dot{\mathbf{r}})^a\big)\mathbf{e}_a(t) + \big(\dot{r}^a + (\boldsymbol{\omega} \times \mathbf{r})^a\big)\dot{\mathbf{e}}_a(t) \\
&= \big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}} \times \mathbf{r})^a + (\boldsymbol{\omega} \times \dot{\mathbf{r}})^a\big)\mathbf{e}_a(t) + \Big(\boldsymbol{\omega} \times \big(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}\big)\Big)^a\mathbf{e}_a(t) \\
&= \Big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}} \times \mathbf{r})^a + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}})^a + (\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r})\Big)^a\mathbf{e}_a(t)\,.
\end{aligned}$$

Newton's 2nd Law for the evolution of the position vector $\mathbf{r}(t)$ *of a particle of mass* m *in a frame rotating with time-dependent angular velocity* $\boldsymbol{\omega}(t)$ *becomes*

$$\mathbf{F}(\mathbf{r}) = m\big(\,\ddot{\mathbf{r}} + \underbrace{\dot{\boldsymbol{\omega}}\times\mathbf{r}}_{Darwin} + \underbrace{2(\boldsymbol{\omega}\times\dot{\mathbf{r}})}_{Coriolis} + \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}_{centrifugal}\,\big) \qquad (2.1.22)$$

- *The Darwin force is usually small; so, it is often neglected.*
- *Only the Coriolis force depends on the velocity in the moving frame. The Coriolis force is very important in large-scale motions on Earth. For example, pressure balance with Coriolis force dominates the (geostrophic) motion of the weather systems that comprise the climate.*
- *The centrifugal force is important, for example, in obtaining orbital equilibria in gravitationally attracting systems.*

Remark 2.1.24 *The space and body angular velocities differ by*

$$\widehat{\omega}_{body} := O^{-1}\dot{O} \quad versus \quad \widehat{\omega}_{space} := \dot{O}O^{-1} = O\widehat{\omega}_{body}O^{-1}\,.$$

Namely, $\widehat{\omega}_{body}$ *is left invariant under* $O \to RO$ *and* $\widehat{\omega}_{space}$ *is right-invariant under* $O \to OR$*, for any choice of matrix* $R \in SO(3)$*. This means that neither angular velocity depends on initial orientation.*

Remark 2.1.25 *The angular velocities* $\widehat{\omega}_{body} = O^{-1}\dot{O}$ *and* $\widehat{\omega}_{space} = \dot{O}O^{-1}$ *are respectively the left and right translations to the identity of the tangent matrix* $\dot{O}(t)$ *at* $O(t)$*. These are called the left and right tangent spaces of* $SO(3)$ *at its identity.*

Remark 2.1.26 *Equations (2.1.19) for free rigid rotations of particle systems are prototypes of Euler's equations for the motion of a rigid body.*

2.1.2 Newtonian form of rigid-body motion

In describing rotations of a rigid body, for example a solid object occupying a spatial domain $\mathcal{B} \subset \mathbb{R}^3$, one replaces the mass-weighted sums over points in space in the previous definitions of dynamical quantities for free rigid rotation, by volume integrals weighted with a mass density as a function of position in the body. That is,

$$\sum_j m_j \to \int_{\mathcal{B}} d^{\,3}\mathbf{X}\,\rho(\mathbf{X})\,,$$

where $\rho(\mathbf{X})$ is the mass density at a point $\mathbf{X} \in \mathcal{B}$ fixed inside the body, as measured in coordinates whose origin is at the centre of mass.

Example 2.1.27 (Kinetic energy of a rotating solid body) *The kinetic energy of a solid body rotating about its centre of mass is given by*

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\,|\dot{\mathbf{x}}(\mathbf{X},t)|^2\, d^{\,3}\mathbf{X}\,, \tag{2.1.23}$$

where the ***spatial path*** *in $\mathbb{R}^3$ of a point $\mathbf{X} \in \mathcal{B}$ in the rotating body is given by*

$$\mathbf{x}(\mathbf{X},t) = O(t)\mathbf{X} \in \mathbb{R}^3 \quad \textit{with} \quad O(t) \in SO(3)\,.$$

The time derivative of this rotating motion yields the ***spatial velocity****,*

$$\dot{\mathbf{x}}(\mathbf{X},t) = \dot{O}(t)\mathbf{X} = \dot{O}O^{-1}(t)\mathbf{x} =: \widehat{\omega}(t)\mathbf{x} =: \boldsymbol{\omega}(t)\times\mathbf{x}\,, \tag{2.1.24}$$

as in equation (2.1.7) for free rotation.

Kinetic energy and angular momentum of a rigid body

The kinetic energy (2.1.23) of a rigid body rotating about its centre of mass may be expressed in the ***spatial frame*** in analogy to equation (2.1.14) for free rotation,

$$K = \frac{1}{2}\int_{O(t)\mathcal{B}} \rho(O^{-1}(t)\mathbf{x})\,|\boldsymbol{\omega}(t)\times\mathbf{x}|^2\, d^{\,3}\mathbf{x}\,. \tag{2.1.25}$$

However, its additional time dependence makes this integral unwieldy. Instead, one takes advantage of the preservation of scalar products by rotations to write

$$|\dot{\mathbf{x}}|^2 = |O^{-1}\dot{\mathbf{x}}|^2\,,$$

and one computes as in equation (2.1.24) the ***body velocity***

$$O^{-1}\dot{\mathbf{x}}(\mathbf{X},t) = O^{-1}\dot{O}(t)\mathbf{X} =: \widehat{\Omega}(t)\mathbf{X} =: \boldsymbol{\Omega}(t)\times\mathbf{X}\,. \tag{2.1.26}$$

Here skew symmetry $\widehat{\Omega}^T = -\widehat{\Omega}$ of the matrix

$$\widehat{\Omega} = O^{-1}\dot{O} = \widehat{\omega}_{body}\,,$$

follows because the matrix O is orthogonal, that is, $OO^T = Id$. Skew symmetry of $\widehat{\Omega}$ allows one to introduce the ***body angular velocity vector*** $\boldsymbol{\Omega}(t)$ whose components Ω_i, with $i = 1, 2, 3$, are given in body coordinates by

$$(O^{-1}\dot{O})_{jk} = \widehat{\Omega}_{jk} = -\,\Omega_i\epsilon_{ijk}\,. \tag{2.1.27}$$

In terms of body angular velocity vector $\boldsymbol{\Omega}(t)$ the kinetic energy of a rigid body becomes

$$K = \frac{1}{2}\int_{\mathcal{B}}\rho(\mathbf{X})\,|\boldsymbol{\Omega}(t)\times\mathbf{X}|^2\,d^3\mathbf{X} =: \frac{1}{2}\Big\langle\!\!\Big\langle \boldsymbol{\Omega}(t),\boldsymbol{\Omega}(t)\Big\rangle\!\!\Big\rangle\,, \tag{2.1.28}$$

where $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ is a mass-weighted symmetric pairing, defined for any two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ as the following integration over the body,

$$\Big\langle\!\!\Big\langle \mathbf{a},\mathbf{b}\Big\rangle\!\!\Big\rangle := \int_{\mathcal{B}}\rho(\mathbf{X})(\mathbf{a}\times\mathbf{X})\cdot(\mathbf{b}\times\mathbf{X})\,d^3\mathbf{X}\,. \tag{2.1.29}$$

Definition 2.1.28 (Moment of inertia tensor)
The mass-weighted pairing, or inner product in (2.1.29)

$$\Big\langle\!\!\Big\langle \mathbf{a},\mathbf{b}\Big\rangle\!\!\Big\rangle = \mathbb{I}\mathbf{a}\cdot\mathbf{b}\,,$$

defines the symmetric ***moment of inertia tensor*** $\mathbb{I}$ *for the rigid body.*

Exercise. By definition, $\mathbb{I}$ is constant in the body frame. What is its time dependence in the spatial frame? ★

Remark 2.1.29 (Principal axis frame)
The moment of inertia tensor becomes diagonal,

$$\mathbb{I} = \operatorname{diag}(I_1, I_2, I_3)\,,$$

upon aligning the body reference coordinates with its ***principal axis frame****. In the principal axis coordinates of $\mathbb{I}$, the kinetic energy (2.1.28) takes the elegant form,*

$$K = \frac{1}{2}\,\langle\!\langle\,\mathbf{\Omega}\,,\,\mathbf{\Omega}\,\rangle\!\rangle = \frac{1}{2}\,\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\,(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)\,.$$

Definition 2.1.30 (Body angular momentum)
The ***body angular momentum*** *is defined as the derivative of the kinetic energy (2.1.28) with respect to body angular velocity. This produces the linear relation,*

$$\begin{aligned}\mathbf{\Pi} = \frac{\partial K}{\partial \mathbf{\Omega}} &= -\int_{\mathcal{B}} \rho(\mathbf{X})\,\mathbf{X}\times\Big(\mathbf{X}\times\mathbf{\Omega}(t)\Big)\,d^{\,3}\mathbf{X} \\ &= \bigg(\int_{\mathcal{B}} \rho(\mathbf{X})\,\Big(|\mathbf{X}|^2 Id - \mathbf{X}\otimes\mathbf{X}\Big)\,d^{\,3}\mathbf{X}\bigg)\mathbf{\Omega}(t) \\ &= \mathbb{I}\mathbf{\Omega}\,. \end{aligned} \tag{2.1.30}$$

This $\mathbb{I}$ is the continuum version of the moment of inertia tensor defined for particle systems in Definition 2.1.14. It is called the moment of inertia tensor of the rigid body.

Remark 2.1.31 *In general, the body angular momentum vector $\mathbf{\Pi}$ is not parallel to the body angular velocity vector $\mathbf{\Omega}$. Their misalignment is measured by $\mathbf{\Pi}\times\mathbf{\Omega}\neq 0$.*

Angular momentum conservation

The rigid body rotates freely along $O(t)$ in the absence of any externally applied forces or torques, so Newton's 2nd Law implies that the motion of the rigid body conserves total angular momentum when expressed in the fixed space coordinates in $\mathbf{R}^3$. This conservation law is expressed in the spatial frame as

$$\frac{d\boldsymbol{\pi}}{dt} = 0\,,$$

where $\boldsymbol{\pi}(t)$ is the ***angular momentum vector in space***. The angular momentum vector in space $\boldsymbol{\pi}(t)$ is related to the angular momentum vector in the body $\mathbf{\Pi}(t)$ by the mutual rotation of their axes at any time. That is, $\boldsymbol{\pi}(t) = O(t)\mathbf{\Pi}(t)$. Likewise, the ***angular velocity vector in space*** satisfies $\boldsymbol{\omega}(t) = O(t)\mathbf{\Omega}(t)$. The angular velocity vector in space is related to its corresponding angular momentum vector by

$$
\begin{aligned}
\boldsymbol{\pi}(t) = O(t)\mathbf{\Pi}(t) &= O(t)\mathbb{I}\mathbf{\Omega}(t) \\
&= \big(O(t)\mathbb{I}O^{-1}(t)\big)\,\boldsymbol{\omega}(t) =: \mathbb{I}_{space}(t)\boldsymbol{\omega}(t)\,.
\end{aligned}
$$

Thus, the ***moment of inertia tensor in space*** $\mathbb{I}_{space}(t)$ transforms as a symmetric tensor,

$$
\mathbb{I}_{space}(t) = O(t)\mathbb{I}O^{-1}(t)\,, \tag{2.1.31}
$$

so it is *time-dependent* and the relation of the spatial angular velocity vector $\boldsymbol{\omega}(t)$ to the motion of the rigid body may be found by using (2.1.26), as

$$
\dot{\mathbf{x}}(\mathbf{X},t) = \dot{O}(t)\mathbf{X} =: \dot{O}O^{-1}(t)\mathbf{x} =: \widehat{\omega}(t)\mathbf{x} =: \boldsymbol{\omega}(t)\times\mathbf{x}\,. \tag{2.1.32}
$$

As expected, the motion in space of a point at $\mathbf{x}$ within the rigid body is a rotation by the time-dependent angular velocity $\boldsymbol{\omega}(t)$. We may formally confirm the relation of the spatial angular velocity vector to the body angular velocity vector $\boldsymbol{\omega}(t) = O(t)\mathbf{\Omega}(t)$ by using (2.1.26) in the following calculation:

$$
\begin{aligned}
\mathbf{\Omega}(t)\times\mathbf{X} &= \widehat{\Omega}(t)\mathbf{X} = O^{-1}\dot{O}\mathbf{X} \\
&= O^{-1}\widehat{\omega}(t)\mathbf{x} = O^{-1}(\boldsymbol{\omega}(t)\times\mathbf{x}) = (O^{-1}\boldsymbol{\omega}(t)\times O^{-1}\mathbf{x})\,.
\end{aligned}
$$

Consequently, the conservation of spatial angular momentum in the absence of external torques implies

$$
\begin{aligned}
\frac{d\boldsymbol{\pi}}{dt} &= \frac{d}{dt}\Big(O(t)\mathbf{\Pi}\Big) \\
&= O(t)\Big(\frac{d\mathbf{\Pi}}{dt} + O^{-1}\dot{O}\,\mathbf{\Pi}\Big) \\
&= O(t)\Big(\frac{d\mathbf{\Pi}}{dt} + \widehat{\Omega}\,\mathbf{\Pi}\Big) \\
&= O(t)\bigg(\frac{d\mathbf{\Pi}}{dt} + \ \mathbf{\Omega}\times\mathbf{\Pi}\bigg) = 0\,.
\end{aligned}
$$

Hence, the body angular momentum satisfies ***Euler's equations*** for a rigid body,

$$\frac{d\mathbf{\Pi}}{dt} + \mathbf{\Omega} \times \mathbf{\Pi} = 0\,. \tag{2.1.33}$$

Remark 2.1.32 (Body angular momentum equation)
Viewed in the moving frame, the rigid body occupies a fixed domain $\mathcal{B}$, so its moment of inertia tensor in that frame $\mathbb{I}$ is constant. Its body angular momentum vector $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega}$ evolves according to (2.1.33) by rotating around the body angular velocity vector $\mathbf{\Omega} = \mathbb{I}^{-1}\mathbf{\Pi}$. That is, conservation of spatial angular momentum $\boldsymbol{\pi}(t) = O(t)\mathbf{\Pi}(t)$ relative to a fixed frame implies the body angular momentum $\mathbf{\Pi}$ appears constant in a frame rotating with the body angular velocity $\mathbf{\Omega} = \mathbb{I}^{-1}\mathbf{\Pi}$.

Proposition 2.1.33 (Conservation laws)
The dynamics of equation (2.1.33) conserves both the square of the body angular momentum $|\mathbf{\Pi}|^2$ and the kinetic energy $K = \mathbf{\Omega}\cdot\mathbf{\Pi}/2$.

Proof. These two conservation laws may be verified by direct calculations:

$$\begin{aligned}
\frac{d|\mathbf{\Pi}|^2}{dt} &= 2\mathbf{\Pi}\cdot\frac{d\mathbf{\Pi}}{dt} = 2\mathbf{\Pi}\cdot\mathbf{\Pi}\times\mathbf{\Omega} = 0\,,\\
\frac{d(\mathbf{\Omega}\cdot\mathbf{\Pi})}{dt} &= 2\mathbf{\Omega}\cdot\frac{d\mathbf{\Pi}}{dt} = 2\mathbf{\Omega}\cdot\mathbf{\Pi}\times\mathbf{\Omega} = 0\,,
\end{aligned}$$

where one uses symmetry of the moment of inertia tensor in the second line. ∎

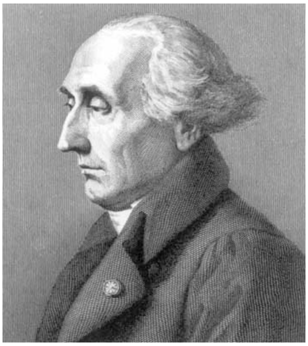
Remark 2.1.34 (Reconstruction formula)
Having found the evolution of $\mathbf{\Pi}(t)$ and thus $\mathbf{\Omega}(t)$ by solving (2.1.33), one may compute the net angle of rotation $O(t)$ in body coordinates from the skew-symmetric angular velocity matrix $\widehat{\Omega}$ in (2.1.27) and its defining relation,

$$\dot{O}(t) = O\widehat{\Omega}(t)\,.$$

Solving this linear differential equation with time-dependent coefficients yields the paths of rotations $O(t) \in SO(3)$. Having these, one may finally construct the trajectories in space taken by points $\mathbf{X}$ in the body $\mathcal{B}$ given by $\mathbf{x}(\mathbf{X}, t) = O(t)\mathbf{X} \in \mathbb{R}^3$.

2.2 Lagrange

2.2.1 The principle of stationary action



J. L. Lagrange

In Lagrangian mechanics, a mechanical system in a ***configuration space*** with generalised coordinates and velocities,

$$q^a,\, \dot{q}^a\,, \quad a = 1, 2, \dots, 3N\,,$$

is characterised by its Lagrangian $L(q(t), \dot{q}(t))$ – a smooth, real-valued function. The motion of a Lagrangian system is determined by the principle of stationary action, formulated using the operation of variational derivative.

Definition 2.2.1 (Variational derivative)
*The **variational derivative** of a functional $S[q]$ is defined as its linearisation in an arbitrary direction δq in the configuration space. That is, $S[q]$ is defined as*

$$\delta S[q]; = \lim_{s\to 0} \frac{S[q+s\delta q]-S[q]}{s} = \frac{d}{ds}\Big|_{s=0} S[q+s\delta q] =: \Big\langle \frac{\delta S}{\delta q}\,, \delta q \Big\rangle\,,$$

where the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is obtained in the process of linearisation.

Theorem 2.2.2 (The principle of stationary action)
The Euler-Lagrange equations,

$$\big[\,L\,\big]_{q^a} := \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = 0\,, \tag{2.2.1}$$

*follow from stationarity of the **action integral**, S, defined as the integral over a time interval $t \in (t_1\,, t_2)$*

$$S := \int_{t_1}^{t_2} L(q, \dot{q})\, dt\,. \tag{2.2.2}$$

*Then the **principle of stationary action**,*

$$\delta S = 0\,,$$

implies $[\,L\,]_{q^a}=0$, *for variations* δq^a *that vanish at the endpoints in time.*

Proof. Applying the variational derivative in Definition 2.2.1 to the action integral in (2.2.2) yields

$$\begin{aligned}
\delta S[q] &= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^a}\,\delta q^a+\frac{\partial L}{\partial \dot q^a}\,\delta \dot q^a\right)dt\\
&= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^a}-\frac{d}{dt}\frac{\partial L}{\partial \dot q^a}\right)\delta q^a\,dt+\left[\frac{\partial L}{\partial \dot q^a}\,\delta q^a\right]_{t_1}^{t_2}\\
&=: \Big\langle -\,[\,L\,]_{q^a}\,,\delta q^a\Big\rangle =: \Big\langle \frac{\delta S}{\delta q}\,,\delta q\Big\rangle = 0\,. \qquad (2.2.3)
\end{aligned}$$

Here one integrates by parts and applies the condition that the variations δq^a vanish at the endpoints in time. Because the variations δq^a are otherwise arbitrary, one concludes that the Euler-Lagrange equations (2.2.1) are satisfied. ■

Remark 2.2.3 *The principle of stationary action is sometimes also called **Hamilton's principle**.*

Example 2.2.4 (Geodesic motion in a Riemannian space) *The Lagrangian for the motion of a free particle in a Riemannian space is its kinetic energy with respect to the Riemannian metric,*

$$L(q,\,\dot q)=\frac{1}{2}\dot q^b g_{bc}(q)\dot q^c\,.$$

The Lagrangian in this case has partial derivatives,

$$\frac{\partial L}{\partial \dot q^a}=g_{ac}(q)\dot q^c \quad \textit{and} \quad \frac{\partial L}{\partial q^a}=\frac{1}{2}\frac{\partial g_{bc}(q)}{\partial q^a}\dot q^b\dot q^c\,.$$

Consequently, its Euler-Lagrange equations $[\,L\,]_{q^a}=0$ *are*

$$\begin{aligned}
[\,L\,]_{q^a} &:= \frac{d}{dt}\frac{\partial L}{\partial \dot q^a}-\frac{\partial L}{\partial q^a}\\
&= g_{ae}(q)\ddot q^{\,e}+\frac{\partial g_{ae}(q)}{\partial q^b}\dot q^b\dot q^e-\frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot q^b\dot q^e=0\,.
\end{aligned}$$

Symmetrising the middle term and contracting with co-metric g^{ca} satisfying $g^{ca}g_{ae} = \delta^c_e$ yields

$$\ddot{q}^{\,c} + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = 0\,, \tag{2.2.4}$$

where Γ^c_{be} are the ***Christoffel symbols****, given in terms of the metric by*

$$\Gamma^c_{be}(q) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(q)}{\partial q^b} + \frac{\partial g_{ab}(q)}{\partial q^e} - \frac{\partial g_{be}(q)}{\partial q^a}\right]. \tag{2.2.5}$$

These Euler-Lagrange equations are the ***geodesic equations*** *of a free particle moving in a Riemannian space.*

2.3 Noether's theorem

2.3.1 Lie symmetries & conservation laws

Emmy Noether

Recall from Definition 1.2.3 that a Lie group depends smoothly on its parameters. (See Appendix B for more details.)

Definition 2.3.1 (Lie symmetry)
A smooth transformation of variables $\{t, q\}$ depending on a single parameter s defined by

$$\{t, q\} \mapsto \{\,\bar{t}(t, q, s),\, \overline{q}(t, q, s)\}\,,$$

that leaves the action $S = \int L\,dt$ invariant is called a ***Lie symmetry of the action****.*

Theorem 2.3.2 (Noether's theorem) *Each Lie symmetry of the action for a Lagrangian system defined on a manifold M with Lagrangian L corresponds to a constant of the motion – [No1918].*

Example 2.3.3 *Suppose the variation of the action in (2.2.3) vanishes ($\delta\mathcal{S} = 0$) because of a Lie symmetry which does* ***not*** *preserve the endpoints. Then on solutions of the Euler-Lagrange equations,* *the endpoint term must vanish for another reason. For example, if the Lie symmetry* leaves time invariant*, so that*

$$\{t, q\} \mapsto \{\, t, \, \overline{q}(t, q, s)\} \,,$$

then the endpoint term must vanish,

$$\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} = 0 \,.$$

Hence, the quantity

$$A(q, \dot{q}, \delta q) = \frac{\partial L}{\partial \dot{q}^a} \, \delta q^a$$

is a ***constant of motion*** *for solutions of the Euler-Lagrange equations. In particular, if* $\delta q^a = c^a$ *for constants* c^a*,* $a = 1, \dots, n$*, that is, for spatial translations in* n *dimensions, then the quantities* $\partial L / \partial \dot{q}^a$ *(the corresponding momentum components) are constants of motion.*

Remark 2.3.4 *This result first appeared in Noether [No1918]. See e.g. [Ol2000, SaCa1981] for good discussions of the history, framework and applications of Noether's theorem. We shall see in a moment that Lie symmetries that reparameterise time may also yield constants of motion.*

2.3.2 Infinitesimal transformations of a Lie group

Definition 2.3.5 (Infinitesimal Lie transformations) *Consider the Lie group of transformations*

$$\{t, q\} \mapsto \{\, \overline{t}(t, q, s), \, \overline{q}(t, q, s)\} \,,$$

and suppose the identity transformation is arranged to occur for $s = 0$*. The derivatives with respect to the group parameters* s *at the identity,*

Sophus Lie

$$\begin{aligned} \tau(t, q) &= \frac{d}{ds}\bigg|_{s=0} \overline{t}(t, q, s) \,, \\ \xi^a(t, q) &= \frac{d}{ds}\bigg|_{s=0} \overline{q}^a(t, q, s) \,, \end{aligned}$$

are called the ***infinitesimal transformations*** *of the action of a Lie group on the time and space variables.*

Thus, at linear order in a Taylor expansion in the group parameter s one has

$$\overline{t} = t + s\tau(t,q)\,, \quad \overline{q}^a = q^a + s\xi^a(t,q)\,, \tag{2.3.1}$$

where τ and ξ^a are functions of coordinates and time, but do not depend on velocities. Then, to first order in s the velocities of the transformed trajectories are computed as,

$$\frac{d\overline{q}^a}{d\overline{t}} = \frac{\dot{q}^a + s\dot{\xi}^a}{1 + s\dot{\tau}} = \dot{q}^a + s(\dot{\xi}^a - \dot{q}^a\dot{\tau})\,, \tag{2.3.2}$$

where order $O(s^2)$ terms are neglected and one defines the total time derivatives

$$\dot{\tau} \equiv \frac{\partial\tau}{\partial t}(t,q) + \dot{q}^b\frac{\partial\tau}{\partial q^b}(t,q) \quad \textit{and} \quad \dot{\xi}^a \equiv \frac{\partial\xi^a}{\partial t}(t,q) + \dot{q}^b\frac{\partial\xi^a}{\partial q^b}(t,q)\,.$$

We are now in a position to prove Noether's Theorem 2.3.2.
Proof. The variation of the action corresponding to the Lie symmetry with infinitesimal transformations (2.3.1) is

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a + L\frac{d\delta t}{dt}\Big)\,dt \\
&= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^a}\xi^a + \frac{\partial L}{\partial \dot{q}^a}(\dot{\xi}^a - \dot{q}^a\dot{\tau}) + L\dot{\tau}\Big)\,dt \\
&= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a}\Big)(\xi^a - \dot{q}^a\tau) + \frac{d}{dt}\Big(L\tau + \frac{\partial L}{\partial \dot{q}^a}(\xi^a - \dot{q}^a\tau)\Big)\,dt \\
&= \int_{t_1}^{t_2} [\,L\,]_{q^a}(\xi^a - \dot{q}^a\tau) + \frac{d}{dt}\bigg[\frac{\partial L}{\partial \dot{q}^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot{q}^a}\,\dot{q}^a - L\Big)\tau\bigg]\,dt\,.
\end{aligned}$$

Thus, stationarity $\delta S = 0$ and the Euler-Lagrange equations $[\,L\,]_{q^a} = 0$ imply

$$0 = \bigg[\frac{\partial L}{\partial \dot{q}^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot{q}^a}\,\dot{q}^a - L\Big)\tau\bigg]_{t_1}^{t_2}\,,$$

so that the quantity

$$\begin{aligned} C(t,q,\dot q) &= \frac{\partial L}{\partial \dot q^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot q^a}\,\dot q^a - L\Big)\tau & (2.3.3)\\ &\equiv \langle\, p\,,\,\delta q\,\rangle - E\,\delta t\,, & (2.3.4)\end{aligned}$$

has the same value at every time along the solution path. That is, $C(t,q,\dot q)$ is a constant of the motion. ■

Remark 2.3.6 *The abbreviated notation in equation (2.3.4) for δq and δt is standard. If δt is absent and δq is a constant (corresponding to translations in space) then $\delta S = 0$ implies that the canonically conjugate momentum p is conserved for solutions of the Euler-Lagrange equations $[\,L\,]_q = 0$.*

Exercise. Show that ***conservation of energy results from Noether's theorem*** if, in Hamilton's principle, the variations are chosen as

$$\delta q(t) = \frac{d}{ds}\Big|_{s=0} q(\,\bar t(t,q,s))\,,$$

corresponding to symmetry of the Lagrangian under spacetime dependent transformations of time along a given curve, so that $q(t)\to q(\,\bar t(t,q,s))$ with $\bar t(t,q,0)=t$.

★

Answer. Under reparameterisations of time along the curve

$$q(t)\to q(\,\bar t(t,q,s))\,,$$

the action $S=\int_{t_1}^{t_2} L(q,\dot q)\,dt$, changes infinitesimally according to

$$\delta S = \Big[\Big(L(q,\dot q) - \frac{\partial L}{\partial \dot q}\,\dot q\Big)\delta t\Big]_{t_1}^{t_2}\,,$$

with variations in position and time defined by

$$\delta q(t) = \frac{d}{ds}\Big|_{s=0} q(\,\bar t(t,q,s)) = \dot q(t)\delta t \quad \text{and} \quad \delta t = \frac{d\bar t(t,q,s)}{ds}\Big|_{s=0}\,.$$

For translations in time, δt is a constant and stationarity of the action $\delta S = 0$ implies that the energy

$$E(t,q,\dot q) \equiv \frac{\partial L}{\partial \dot q^a}\,\dot q^a - L\,, \tag{2.3.5}$$

is a constant of motion along solutions of the ELd'A equations. ▲

> **Exercise.** (Euclidean group)
> Find the infinitesimal transformations of $SE(3)$ – that is, calculate its tangent vectors at the identity – by using its 4×4 matrix representation. ★

Answer. The 4×4 matrix representation of tangent vectors for $SE(3)$ at the identity is found by first computing the derivative of a general group element $(O(s), s\mathbf{r}_0)$ along the group path with parameter s and bringing the result back to the identity at $s=0$,

$$\left[\begin{pmatrix} O(s) & s\mathbf{r}_0 \\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix} O'(s) & \mathbf{r}_0 \\ 0 & 0\end{pmatrix}\right]_{s=0} = \begin{pmatrix} O^{-1}(0)O'(0) & O^{-1}(0)\mathbf{r}_0 \\ 0 & 0\end{pmatrix}$$
$$=: \begin{pmatrix} \widehat{\Xi} & \mathbf{r}_0 \\ 0 & 0\end{pmatrix}.$$

The quantity $\widehat{\Xi} = O^{-1}(s)O'(s)|_{s=0}$ is a 3×3 skew-symmetric matrix, since O is a 3×3 orthogonal matrix. Thus, $\widehat{\Xi}$ may be written using the ***hat map***, as in equation (2.1.11),

$$\widehat{\Xi} = \begin{pmatrix} 0 & -\Xi_3 & \Xi_2 \\ \Xi_3 & 0 & -\Xi_1 \\ -\Xi_2 & \Xi_1 & 0\end{pmatrix}, \tag{2.3.6}$$

in terms of a vector $\boldsymbol{\Xi} \in \mathbb{R}^3$ with components Ξ_i, with $i = 1,2,3$. Infinitesimal rotations are expressed by the vector cross product,

$$\widehat{\Xi}\mathbf{r} = \boldsymbol{\Xi}\times\mathbf{r}\,. \tag{2.3.7}$$

The matrix components of $\widehat{\Xi}$ may also be written in terms of the components of the vector $\boldsymbol{\Xi}$ as

$$\widehat{\Xi}_{jk} = \left(O^{-1}\frac{dO}{ds}\right)_{jk}\bigg|_{s=0} = -\,\Xi_i\epsilon_{ijk}\,,$$

where ϵ_{ijk} with $i, j, k = 1, 2, 3$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$, $\epsilon_{213} = -1$, etc. One may compute directly for a fixed vector $\mathbf{r}$,

$$\frac{d}{ds} e^{s\widehat{\Xi}} \mathbf{r} = \widehat{\Xi} e^{s\widehat{\Xi}} \mathbf{r} = \boldsymbol{\Xi} \times e^{s\widehat{\Xi}} \mathbf{r} \,.$$

Consequently, one may evaluate at $s = 0$,

$$\frac{d}{ds} e^{s\widehat{\Xi}} \mathbf{r} \bigg|_{s=0} = \widehat{\Xi} \mathbf{r} = \boldsymbol{\Xi} \times \mathbf{r} \,.$$

This expression recovers the expected result in (2.3.7) in terms of the exponential notation. It means the quantity $\mathbf{r}(s) = \exp(s\widehat{\Xi})\mathbf{r}$ describes a finite, right-handed rotation of the initial vector $\mathbf{r} = \mathbf{r}(0)$ by the angle $s|\boldsymbol{\Xi}|$ around the axis pointing in the direction of $\boldsymbol{\Xi}$.

▲

Remark 2.3.7 (Properties of the hat map)
The hat map arises in the infinitesimal rotations,

$$\widehat{\Xi}_{jk} = (O^{-1} dO/ds)_{jk}|_{s=0} = -\,\Xi_i \epsilon_{ijk} \,.$$

The hat map is an isomorphism:

$$(\mathbb{R}^3, \times) \mapsto (\mathfrak{so}(3), [\,\cdot\,,\,\cdot\,]\,) \,.$$

That is, the hat map identifies the composition of two vectors in $\mathbb{R}^3$ *using the cross product with the commutator of two skew-symmetric* 3×3 *matrices. Specifically, we write for any two vectors* $\mathbf{Q}, \boldsymbol{\Xi} \in \mathbb{R}^3$*,*

$$-\,(\mathbf{Q} \times \boldsymbol{\Xi})_k = \epsilon_{klm} \Xi^{\,l} Q^m = \widehat{\Xi}_{km}\, Q^m \,.$$

That is,

$$\boldsymbol{\Xi} \times \mathbf{Q} = \widehat{\Xi}\, \mathbf{Q} \quad \textit{for all} \quad \boldsymbol{\Xi},\, \mathbf{Q} \in \mathbb{R}^3 \,.$$

The following formulas may be easily verified for $\mathbf{P}, \mathbf{Q}, \boldsymbol{\Xi} \in \mathbb{R}^3$*:*

$$\begin{aligned} (\mathbf{P} \times \mathbf{Q})\widehat{\ } &= \left[\widehat{P}, \widehat{Q}\right], \\ \left[\widehat{P}, \widehat{Q}\right]\boldsymbol{\Xi} &= (\mathbf{P} \times \mathbf{Q}) \times \boldsymbol{\Xi}\,, \\ \mathbf{P} \cdot \mathbf{Q} &= -\,\frac{1}{2}\,\mathrm{trace}\left(\widehat{P}\widehat{Q}\right). \end{aligned}$$

Remark 2.3.8 *The commutator of infinitesimal transformation matrices given by the formula,*

$$\left[\begin{pmatrix} \widehat{\Xi}_1 & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} \widehat{\Xi}_2 & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} \widehat{\Xi}_1 \widehat{\Xi}_2 - \widehat{\Xi}_2 \widehat{\Xi}_1 & \widehat{\Xi}_1 \mathbf{r}_2 - \widehat{\Xi}_2 \mathbf{r}_1 \\ 0 & 0 \end{pmatrix} ,$$

provides a matrix representation of $se(3)$, the Lie algebra of the Lie group $SE(3)$. In vector notation, this becomes

$$\left[\begin{pmatrix} \boldsymbol{\Xi}_1 \times & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} \boldsymbol{\Xi}_2 \times & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} (\boldsymbol{\Xi}_1 \times \boldsymbol{\Xi}_2) \times & \boldsymbol{\Xi}_1 \times \mathbf{r}_2 - \boldsymbol{\Xi}_2 \times \mathbf{r}_1 \\ 0 & 0 \end{pmatrix} .$$

Remark 2.3.9 *The $se(3)$ matrix commutator yields*

$$\left[(\widehat{\Xi}_1 \, , \, \mathbf{r}_1) \, , \, (\widehat{\Xi}_2 \, , \, \mathbf{r}_2) \right] = \left(\widehat{\Xi}_1 \widehat{\Xi}_2 - \widehat{\Xi}_2 \widehat{\Xi}_1 \, , \, \widehat{\Xi}_1 \mathbf{r}_2 - \widehat{\Xi}_2 \mathbf{r}_1 \right) ,$$

which is the classic expression for the Lie algebra of a semidirect-product Lie group.

> **Exercise.** (Infinitesimal Galilean transformations)
> From their finite transformations in Definition 1.2.1 and Exercise 1.3 compute the infinitesimal transformations of the Galilean group under composition of first rotations, then boosts, then translations in space and time. ★

Answer. The composition of translations, $g_1(s\mathbf{r}_0, st_0)$, Galilean boosts $g_3(s\mathbf{v}_0)$ and rotations $g_2(O(s))$ acting on a velocity-space-time point $(\mathbf{v}, \mathbf{r}, t)$ is given by

$$g_1 g_3 g_2(\mathbf{v}, \mathbf{r}, t) = \Big(O(s)\mathbf{v} + s\mathbf{v}_0, O(s)\mathbf{r} + ts\mathbf{v}_0 + s\mathbf{r}_0, t + st_0 \Big) .$$

One computes the infinitesimal transformations as

$$\begin{aligned} \tau &= \frac{dt}{ds}\bigg|_{s=0} = t_0 \, , \\ \boldsymbol{\xi} &= \frac{d\mathbf{r}}{ds}\bigg|_{s=0} = \mathbf{r}_0 + \mathbf{v}_0 t + \boldsymbol{\Xi} \times \mathbf{r} \, , \\ \dot{\boldsymbol{\xi}} - \mathbf{v}\dot{\tau} &= \frac{d\mathbf{v}}{ds}\bigg|_{s=0} = \mathbf{v}_0 + \boldsymbol{\Xi} \times \mathbf{v} \, . \end{aligned}$$

The infinitesimal velocity transformation may also be computed from equation (2.3.2).

Consequently, the infinitesimal transformation by the Galilean group of a function $F(t, \mathbf{r}, \mathbf{v})$ is given by operation of the following vector field, obtained as the first term in a Taylor series,

$$\begin{aligned} &\frac{d}{ds}\bigg|_{s=0} F(t(s), \mathbf{r}(s), \mathbf{v}(s)) \qquad (2.3.8) \\ &\quad = t_0 \frac{\partial F}{\partial t} + \big(\mathbf{r}_0 + \mathbf{v}_0 t + \boldsymbol{\Xi} \times \mathbf{r}\big) \cdot \frac{\partial F}{\partial \mathbf{r}} + \big(\mathbf{v}_0 + \boldsymbol{\Xi} \times \mathbf{v}\big) \cdot \frac{\partial F}{\partial \mathbf{v}} \,. \end{aligned}$$

The finite transformations of the translations, $g_1(s\mathbf{r}_0, st_0)$, boosts $g_3(s\mathbf{v}_0)$ and rotations $g_2(O(s))$ acting on a space-time point as

$$(\mathbf{r}(s), t(s)) = g_1(s\mathbf{r}_0, st_0) g_3(s\mathbf{v}_0) g_2(O(s))(\mathbf{r}, t) \,,$$

are obtained by integrating the characteristic curves of this vector field from the identity $s = 0$ at which $(\mathbf{r}(0), t(0)) = (\mathbf{r}, t)$.

▲

Exercise. (Galilean Lie symmetries)
Since the Galilean transformations form a Lie group, one may expect them to be a source of Lie symmetries of the action. Compute the corresponding Noether conservation laws. ★

Answer. As we have already seen, symmetries under space and time translations imply conservation of linear momentum and energy, respectively. Likewise, symmetry under rotations implies angular momentum conservation. Finally, symmetry under Galilean boosts implies conservation of centre-of-mass momentum. These classical statements may all be proved explicitly from Noether's theorem and the infinitesimal transformations of the Galilean group. ▲

Exercise. (Galilean infinitesimal transformations)
Verify the infinitesimal transformations of $Gal(3)$ – that is, calculate its tangent vectors at the identity – by using its 5×5 matrix representation in Exercise 1.3. ★

Answer. The 5×5 matrix representation of tangent vectors for $Gal(3)$ at the identity is found by first computing the derivative of a general group element $(O(s), s\mathbf{v}_0, s\mathbf{r}_0, st_0)$ along the group path with parameter s and bringing the result back to the identity at $s = 0$,

$$
\begin{aligned}
&\left[\begin{pmatrix} O(s) & s\mathbf{v}_0 & s\mathbf{r}_0 \\ 0 & 1 & st_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} O'(s) & \mathbf{v}_0 & \mathbf{r}_0 \\ 0 & 0 & t_0 \\ 0 & 0 & 0 \end{pmatrix} \right]_{s=0} \\
&\quad = \begin{pmatrix} O^{-1}(s)O'(s) & O^{-1}(s)\mathbf{v}_0 & O^{-1}(s)(\mathbf{r}_0 - \mathbf{v}_0 t_0) \\ 0 & 0 & t_0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg|_{s=0} \\
&\quad = \begin{pmatrix} \widehat{\Xi} & \mathbf{v}_0 & \mathbf{r}_0 - \mathbf{v}_0 t_0 \\ 0 & 0 & t_0 \\ 0 & 0 & 0 \end{pmatrix} =: \left(\widehat{\Xi},\, \mathbf{v}_0,\, \mathbf{r}_0,\, t_0 \right),
\end{aligned}
$$

in terms of the 3×3 skew-symmetric matrix $\widehat{\Xi} = O^{-1}(s)O'(s)|_{s=0}$ and the Galilean shift parameters $\mathbf{v}_0$, $\mathbf{r}_0$, t_0. ▲

Exercise. (Galilean Lie algebra commutators)
Verify the commutation relation

$$
\begin{aligned}
&\Big[(\widehat{\Xi}_1,\, \mathbf{v}_1,\, \mathbf{r}_1,\, t_1),\, (\widehat{\Xi}_2,\, \mathbf{v}_2,\, \mathbf{r}_2,\, t_2) \Big] = \\
&\left(\Big[\widehat{\Xi}_1,\, \widehat{\Xi}_2 \Big],\, \widehat{\Xi}_1 \mathbf{v}_2 - \widehat{\Xi}_2 \mathbf{v}_1,\, \widehat{\Xi}_1(\mathbf{r}_2, \mathbf{v}_2, t_2) - \widehat{\Xi}_2(\mathbf{r}_1, \mathbf{v}_1, t_1),\, 0 \right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\widehat{\Xi}_1(\mathbf{r}_2, \mathbf{v}_2, t_2) - \widehat{\Xi}_2(\mathbf{r}_1, \mathbf{v}_1, t_1) \\
&\quad = \left(\widehat{\Xi}_1(\mathbf{r}_2 - \mathbf{v}_2 t_2) + \mathbf{v}_1 t_2 \right) - \left(\widehat{\Xi}_2(\mathbf{r}_1 - \mathbf{v}_1 t_1) + \mathbf{v}_2 t_1 \right).
\end{aligned}
$$

Show that this commutation relation is equivalent to the commutation relation for the vector field representation in equation (2.3.8).

★

2.4 Lagrangian form of rigid-body motion

In the absence of external torques, ***Euler's equations*** (2.1.33) for rigid-body motion in principal axis coordinates are:

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3, \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1, \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2, \end{aligned} \tag{2.4.1}$$

or, equivalently,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}\,, \tag{2.4.2}$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector and I_1, I_2, I_3 are the moments of inertia in the principal axis frame of the rigid body. We ask whether these equations may be expressed using Hamilton's principle on $\mathbb{R}^3$. For this, we will need to define the variational derivative of a functional $S[(\boldsymbol{\Omega}]$.

Definition 2.4.1 (Variational derivative)
The variational derivative of a functional $S[(\boldsymbol{\Omega}]$ is defined as its linearisation in an arbitrary direction $\delta\boldsymbol{\Omega}$ in the vector space of body angular velocities. That is,

$$\delta S[\boldsymbol{\Omega}] := \lim_{s\to 0} \frac{S[\boldsymbol{\Omega} + s\delta\boldsymbol{\Omega}] - S[\boldsymbol{\Omega}]}{s} = \frac{d}{ds}\Big|_{s=0} S[\boldsymbol{\Omega} + s\delta\boldsymbol{\Omega}] =: \Big\langle \frac{\delta S}{\delta \boldsymbol{\Omega}}\,, \delta\boldsymbol{\Omega} \Big\rangle\,,$$

where the new pairing, also denoted as $\langle\,\cdot\,,\,\cdot\,\rangle$, is between the space of body angular velocities and its dual, the space of body angular momenta.

Theorem 2.4.2 (Euler's rigid-body equations)
Euler's rigid-body equations are equivalent to Hamilton's principle

$$\delta S(\boldsymbol{\Omega}) = \delta \int_a^b l(\boldsymbol{\Omega})\, dt = 0, \tag{2.4.3}$$

in which the Lagrangian $l(\mathbf{\Omega})$ appearing in the ***action integral*** *$S(\mathbf{\Omega}) = \int_a^b l(\mathbf{\Omega})\,dt$ is given by the kinetic energy in principal axis coordinates,*

$$l(\mathbf{\Omega}) = \frac{1}{2}\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\rangle = \frac{1}{2}\,\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right), \quad (2.4.4)$$

and variations of $\mathbf{\Omega}$ are restricted to be of the form

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Xi}} + \mathbf{\Omega}\times\mathbf{\Xi}\,, \qquad (2.4.5)$$

where $\mathbf{\Xi}(t)$ is a curve in $\mathbb{R}^3$ that vanishes at the endpoints in time.

Proof. Since $l(\mathbf{\Omega}) = \frac{1}{2}\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\rangle$, and $\mathbb{I}$ is symmetric, one obtains

$$\begin{aligned}
\delta\int_a^b l(\mathbf{\Omega})\,dt &= \int_a^b \Big\langle \mathbb{I}\mathbf{\Omega}, \delta\mathbf{\Omega}\Big\rangle\,dt \\
&= \int_a^b \Big\langle \mathbb{I}\mathbf{\Omega}, \dot{\mathbf{\Xi}} + \mathbf{\Omega}\times\mathbf{\Xi}\Big\rangle\,dt \\
&= \int_a^b \left[\Big\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega}, \mathbf{\Xi}\Big\rangle + \Big\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\times\mathbf{\Xi}\Big\rangle\right] dt \\
&= \int_a^b \Big\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega} + \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega}, \mathbf{\Xi}\Big\rangle\,dt,
\end{aligned}$$

upon integrating by parts using the endpoint conditions,

$$\mathbf{\Xi}(a) = 0 = \mathbf{\Xi}(b)\,.$$

Since $\mathbf{\Xi}$ is otherwise arbitrary, (2.4.3) is equivalent to

$$-\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) + \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega} = 0,$$

which recovers Euler's equations (2.4.1) in vector form. ∎

Proposition 2.4.3 (Derivation of the restricted variation) *The restricted variation in (2.4.5) arises via the following steps.*

1. *Vary the definition of body angular velocity,* $\widehat{\Omega} = O^{-1}\dot{O}$.
2. *Take the time derivative of the variation,* $\widehat{\Xi} = O^{-1}O\,'$.
3. *Use equality of cross derivatives,* $O^{\cdot\,\prime} = d^2O/dtds = O\,'^{\cdot}$.
4. *Apply the hat map.*

Proof. One computes directly that

$$
\begin{aligned}
\widehat{\Omega}\,' &= (O^{-1}\dot{O})\,' = -\,O^{-1}O\,'O^{-1}\dot{O} + O^{-1}O^{\cdot\,\prime} = -\,\widehat{\Xi}\widehat{\Omega} + O^{-1}O^{\cdot\,\prime}\,, \\
\widehat{\Xi}^{\cdot} &= (O^{-1}O\,')^{\cdot} = -\,O^{-1}\dot{O}O^{-1}O\,' + O^{-1}O\,'^{\cdot} = -\,\widehat{\Omega}\widehat{\Xi} + O^{-1}O\,'^{\cdot}\,.
\end{aligned}
$$

On taking the difference, the cross derivatives cancel and one finds a variational formula equivalent to (2.4.5),

$$
\widehat{\Omega}\,' - \widehat{\Xi}^{\cdot} = \Big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\Big] \quad \text{with} \quad [\,\widehat{\Omega}\,,\,\widehat{\Xi}\,] := \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,. \tag{2.4.6}
$$

Under the bracket-relation (5.1.1) for the hat map, this equation recovers the vector relation (2.4.5) in the form,

$$
\boldsymbol{\Omega}\,' - \dot{\boldsymbol{\Xi}} = \boldsymbol{\Omega} \times \boldsymbol{\Xi}\,. \tag{2.4.7}
$$

Thus, Euler's equations for the rigid body in $T\mathbb{R}^3$,

$$
\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}\,, \tag{2.4.8}
$$

do follow from the variational principle (2.4.3) with variations of the form (2.4.5) derived from the definition of body angular velocity $\widehat{\Omega}$.

■

Exercise. The body angular velocity is expressed in terms of the spatial angular velocity by $\boldsymbol{\Omega}(t) = O^{-1}(t)\boldsymbol{\omega}(t)$. Consequently, the kinetic energy Lagrangian in (2.4.4) transforms as

$$
l(\boldsymbol{\Omega}) = \frac{1}{2}\,\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbb{I}_{space}(t)\boldsymbol{\omega} =: l_{space}(\boldsymbol{\omega})\,,
$$

where $\mathbb{I}_{space}(t) = O(t)\mathbb{I}O^{-1}(t)$ as in (2.1.31).

Show that Hamilton's principle for the action

$$S(\boldsymbol{\omega}) = \int_a^b l_{space}(\boldsymbol{\omega})\, dt\,,$$

yields conservation of spatial angular momentum

$$\boldsymbol{\pi} = \mathbb{I}_{space}(t)\boldsymbol{\omega}(t)\,.$$

Write this as a geodesic equation in the form (2.2.4).

★

Remark 2.4.4 (Reconstruction of $O(t) \in SO(3)$)
The Euler solution is expressed in terms of the time-dependent angular velocity vector in the body, $\boldsymbol{\Omega}$. The body angular velocity vector $\boldsymbol{\Omega}(t)$ yields the tangent vector $\dot{O}(t) \in T_{O(t)}SO(3)$ along the integral curve in the rotation group $O(t) \in SO(3)$ by the relation,

$$\dot{O}(t) = O(t)\widehat{\Omega}(t)\,, \tag{2.4.9}$$

where the left-invariant skew-symmetric 3×3 matrix $\widehat{\Omega}$ is defined by the hat map (2.1.27),

$$(O^{-1}\dot{O})_{jk} = \widehat{\Omega}_{jk} = -\,\Omega_i \epsilon_{ijk}\,. \tag{2.4.10}$$

*Equation (2.4.9) is the **reconstruction formula** for $O(t) \in SO(3)$.*

Once the time dependence of $\boldsymbol{\Omega}(t)$ and hence $\widehat{\Omega}(t)$ is determined from the Euler equations, solving formula (2.4.9) as a linear differential equation with time-dependent coefficients yields the integral curve $O(t) \in SO(3)$ for the orientation of the rigid body.

2.4.1 Hamilton-Pontryagin constrained variations

Formula (2.4.6) for the variation $\widehat{\Omega}$ of the skew-symmetric matrix

$$\widehat{\Omega} = O^{-1}\dot{O}\,,$$

may be imposed as a constraint in Hamilton's principle and thereby provide a variational derivation of Euler's equations (2.1.33) for rigid-body motion in principal axis coordinates. This constraint is incorporated into the matrix Euler equations, as follows.

Proposition 2.4.5 (Matrix Euler equations)
Euler's rigid-body equation may be written in matrix form as,

$$\frac{d\Pi}{dt} = -\,\big[\,\widehat{\Omega}\,,\,\Pi\,\big] \quad \textit{with} \quad \Pi = \mathbb{I}\widehat{\Omega} = \frac{\delta l}{\delta\widehat{\Omega}}\,, \tag{2.4.11}$$

for the Lagrangian $l(\widehat{\Omega})$ given by

$$l = \frac{1}{2}\Big\langle\,\mathbb{I}\widehat{\Omega}\,,\,\widehat{\Omega}\,\Big\rangle\,. \tag{2.4.12}$$

Here, the bracket

$$\big[\,\widehat{\Omega}\,,\,\Pi\big] := \widehat{\Omega}\Pi - \Pi\widehat{\Omega}\,, \tag{2.4.13}$$

denotes the commutator and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the ***trace pairing****, e.g.,*

$$\Big\langle\,\Pi\,,\,\widehat{\Omega}\,\Big\rangle =: \frac{1}{2}\,\mathrm{trace}\Big(\Pi^T\,\widehat{\Omega}\Big)\,. \tag{2.4.14}$$

Remark 2.4.6 *Note that the symmetric part of Π does not contribute in the pairing and if set equal to zero initially, it will remain zero.*

Proposition 2.4.7 (Constrained variational principle)
The matrix Euler equations (2.4.11) are equivalent to stationarity $\delta S = 0$ of the following ***constrained action***

$$\begin{aligned} S(\widehat{\Omega}, O, \dot{O}, \Pi) &= \int_a^b l(\widehat{\Omega}, O, \dot{O}, \Pi)\,dt \qquad (2.4.15)\\ &= \int_a^b \Big[l(\widehat{\Omega}) + \langle\,\Pi\,,\,(O^{-1}\dot{O} - \widehat{\Omega}\,)\,\rangle\Big]\,dt\,. \end{aligned}$$

Remark 2.4.8 *The integrand of the constrained action in (2.4.15) is similar to the formula for the Legendre transform, but its functional dependence is different. This variational approach is related to the classic* ***Hamilton-Pontryagin principle*** *for control theory in [YoMa2007]. It is also used in [BoMa2007] to develop algorithms for geometric numerical integrations of rotating motion.*

Proof. The variations of S in formula (2.4.15) are given by

$$\begin{aligned} \delta S &= \int_a^b \bigg\{ \Big\langle \frac{\delta l}{\delta \widehat{\Omega}} - \Pi \,,\, \delta\widehat{\Omega} \Big\rangle \\ &\qquad + \Big\langle \delta\Pi \,,\, (O^{-1}\dot{O} - \Omega) \Big\rangle + \Big\langle \Pi \,,\, \delta(O^{-1}\dot{O}) \Big\rangle \bigg\} \, dt \,, \end{aligned}$$

where

$$\delta(O^{-1}\dot{O}) = \widehat{\Xi}^{\,\cdot} + [\,\widehat{\Omega} \,,\, \widehat{\Xi}\,] \,, \tag{2.4.16}$$

and $\widehat{\Xi} = (O^{-1}\delta O)$ from equation (2.4.6).

Substituting for $\delta(O^{-1}\dot{O})$ into the last term of δS produces

$$\begin{aligned} \int_a^b \Big\langle \Pi \,,\, \delta(O^{-1}\dot{O}) \Big\rangle \, dt &= \int_a^b \Big\langle \Pi \,,\, \widehat{\Xi}^{\,\cdot} + [\,\widehat{\Omega} \,,\, \widehat{\Xi}\,] \Big\rangle \, dt \\ &= \int_a^b \Big\langle -\Pi^{\,\cdot} - [\,\widehat{\Omega} \,,\, \Pi\,] \,,\, \widehat{\Xi} \Big\rangle \, dt \,, \end{aligned}$$

where one uses the cyclic properties of the trace operation for matrices,

$$\mathrm{trace}(\Pi^T \,\widehat{\Xi}\,\widehat{\Omega}) = \mathrm{trace}(\widehat{\Omega}\,\Pi^T\,\widehat{\Xi}) \,. \tag{2.4.17}$$

Thus, stationarity of the Hamilton-Pontryagin variational principle implies the following set of equations

$$\frac{\delta l}{\delta \widehat{\Omega}} = \Pi \,, \quad O^{-1}\dot{O} = \widehat{\Omega} \,, \quad \Pi^{\,\cdot} = -[\,\widehat{\Omega} \,,\, \Pi\,] \,. \tag{2.4.18}$$

■

Remark 2.4.9 (Interpreting the formulas in (2.4.18))
The first formula in (2.4.18) defines the angular momentum matrix

Π *as the* ***fibre derivative*** *of the Lagrangian with respect to the angular velocity matrix* $\widehat{\Omega}$. *The second formula is the reconstruction formula (2.4.9) for the solution curve* $O(t) \in SO(3)$, *given the solution* $\widehat{\Omega}(t) = O^{-1}\dot{O}$. *And the third formula is Euler's equation for rigid-body motion in matrix form.*

2.4.2 Manakov's formulation of the $SO(n)$ rigid body

Proposition 2.4.10 (Manakov [Man1976])
Euler's equations for a rigid body on $SO(n)$ *take the matrix commutator form,*

$$\frac{dM}{dt} = [\,M\,,\,\Omega\,] \quad with \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,, \tag{2.4.19}$$

where the $n \times n$ *matrices* M, Ω *are skew-symmetric (forgoing superfluous hats) and* $\mathbb{A}$ *is symmetric.*

Proof. Manakov's commutator form of the $SO(n)$ rigid-body equations (2.4.19) follows as the Euler-Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l\,dt$ for the Lagrangian

$$l = -\frac{1}{2}\mathrm{tr}(\Omega\mathbb{A}\Omega)\,,$$

where $\Omega = O^{-1}\dot{O} \in so(n)$ and the $n \times n$ matrix $\mathbb{A}$ is symmetric. Taking matrix variations in Hamilton's principle yields

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\,\Big(\delta\Omega\,(\mathbb{A}\Omega + \Omega\mathbb{A})\Big)\,dt = -\frac{1}{2}\int_a^b \mathrm{tr}\,\Big(\delta\Omega\,M\Big)\,dt\,,$$

after cyclically permuting the order of matrix multiplication under the trace and substituting $M := \mathbb{A}\Omega + \Omega\mathbb{A}$. Using the variational formula (2.4.16) for $\delta\Omega$ now leads to

$$\delta S \;=\; -\frac{1}{2}\int_a^b \mathrm{tr}\,\Big((\Xi^{\cdot} + \Omega\Xi - \Xi\Omega)M\Big)\,dt\,.$$

Integrating by parts and permuting under the trace then yields the equation

$$\delta S \;=\; \frac{1}{2}\int_a^b \mathrm{tr}\,\Big(\Xi\,(\,\dot{M} + \Omega M - M\Omega\,)\Big)\,dt\,.$$

Finally, invoking stationarity for arbitrary Ξ implies the commutator form (2.4.19). ■

2.4.3 Matrix Euler-Poincaré equations

Manakov's commutator form of the rigid-body equations recalls much earlier work by Poincaré [Po1901], who also noticed that the matrix commutator form of Euler's rigid-body equations suggests an additional mathematical structure going back to Sophus Lie's theory of groups of transformations depending continuously on parameters. In particular, Poincaré [Po1901] remarked that the commutator form of Euler's rigid-body equations would make sense for any Lie algebra, not just for $so(3)$. The proof of Manakov's commutator form (2.4.19) by Hamilton's principle makes contact with Poincaré's generalisation of Euler's equations in the following Theorem.

Theorem 2.4.11 (Matrix Euler-Poincaré equations) *The Euler-Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l(\Omega)\, dt$ may be expressed in matrix commutator form*

$$\frac{dM}{dt} = [\, M \,,\, \Omega\,] \quad \textit{with} \quad M = \frac{\delta l}{\delta \Omega}\,, \tag{2.4.20}$$

for any Lagrangian $l(\Omega)$, where $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$ and $\mathfrak{g}$ is the matrix Lie algebra of any matrix Lie group G.

Proof. The proof here is the same as the proof of Manakov's commutator formula via Hamilton's principle, modulo replacing $O^{-1}\dot{O} \in so(n)$ by $g^{-1}\dot{g} \in \mathfrak{g}$. ■

Remark 2.4.12 *Poincaré's observation leading to the Matrix Euler-Poincaré equation (2.4.20) was reported in two pages with no references [Po1901]. The proof above shows that the Matrix Euler-Poincaré equations possess a natural variational principle. Note that if $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, then $M = \delta l/\delta\Omega \in \mathfrak{g}^*$, where the dual is defined in terms of the matrix trace pairing.*

Exercise. Retrace the proof of the variational principle for the Euler-Poincaré equation replacing the left-invariant quantity $g^{-1}\dot{g}$ by the right-invariant quantity $\dot{g}g^{-1}$.

★

2.4.4 Manakov's integration of the $SO(n)$ rigid body

Manakov [Man1976] observed that equations (2.4.19) may be "deformed" into

$$\frac{d}{dt}(M + \lambda A) = [(M + \lambda A), (\Omega + \lambda B)]\,, \tag{2.4.21}$$

where A, B are also $n \times n$ matrices and λ is a scalar constant parameter. For these deformed rigid-body equations on $SO(n)$ to hold for any value of λ, the coefficient of each power must vanish.

- The coefficent of λ^2 is

  $$0 = [A, B]\,.$$

  Therefore, A and B must commute. For this, let them be constant and diagonal:

  $$A_{ij} = \mathrm{diag}(a_i)\delta_{ij}\,, \quad B_{ij} = \mathrm{diag}(b_i)\delta_{ij} \quad \text{(no sum)}.$$

- The coefficent of λ is

  $$0 = \frac{dA}{dt} = [A, \Omega] + [M, B]\,.$$

  Therefore, by antisymmetry of M and Ω,

  $$(a_i - a_j)\Omega_{ij} = (b_i - b_j)M_{ij}\,,$$

  which implies that

  $$\Omega_{ij} = \frac{b_i - b_j}{a_i - a_j} M_{ij} \quad \text{(no sum)}.$$

  Hence, angular velocity Ω is a linear function of angular momentum, M.

- Finally, the coefficent of λ^0 recovers the Euler equation,

$$\frac{dM}{dt} = [M, \Omega]\,,$$

  but now with the restriction that the moments of inertia are of the form,

$$\Omega_{ij} = \frac{b_i - b_j}{a_i - a_j} M_{ij} \quad \text{(no sum)}.$$

  This relation turns out to possess only five free parameters for $n = 4$.

Under these conditions, Manakov's deformation of the $SO(n)$ rigid-body equation into the commutator form (2.4.21) implies for every non-negative integer power K that

$$\frac{d}{dt}(M + \lambda A)^K = [(M + \lambda A)^K, (\Omega + \lambda B)]\,.$$

Since the commutator is antisymmetric, its trace vanishes and K conservation laws emerge, as

$$\frac{d}{dt}\mathrm{tr}\,(M + \lambda A)^K = 0\,,$$

after commuting the trace operation with time derivative. Consequently,

$$\mathrm{tr}\,(M + \lambda A)^K = \text{constant}\,,$$

for each power of λ. That is, all the coefficients of each power of λ are constant in time for the $SO(n)$ rigid body. Manakov [Man1976] proved that these constants of motion are sufficient to completely determine the solution for $n = 4$.

Remark 2.4.13 *This result generalises considerably. For example, Manakov's method determines the solution for all the algebraically solvable rigid bodies on $SO(n)$. The moments of inertia of these bodies possess only $2n-3$ parameters. (Recall that in Manakov's case for $SO(4)$ the moment of inertia possesses only five parameters.)*

> **Exercise.** Try computing the constants of motion $\mathrm{tr}(M+\lambda A)^K$ for the values $K = 2, 3, 4$. Hint: Keep in mind that M is a skew-symmetric matrix, $M^T = -M$, so the trace of the product of any diagonal matrix times an odd power of M vanishes. ★

Answer. The traces of the powers $\mathrm{trace}(M + \lambda A)^K$ are given by

$$
\begin{aligned}
\boxed{\text{n=2}} &: \quad \mathrm{tr}\, M^2 + 2\lambda \mathrm{tr}\,(AM) + \lambda^2 \mathrm{tr}\, A^2\,, \\
\boxed{\text{n=3}} &: \quad \mathrm{tr}\, M^3 + 3\lambda \mathrm{tr}\,(AM^2) + 3\lambda^2 \mathrm{tr}\, A^2 M + \lambda^3 \mathrm{tr}\, A^3\,, \\
\boxed{\text{n=4}} &: \quad \mathrm{tr}\, M^4 + 4\lambda \mathrm{tr}\,(AM^3) + \lambda^2 (2\mathrm{tr}\, A^2 M^2 + 4\mathrm{tr}\, AMAM) \\
&\qquad + \lambda^3 \mathrm{tr}\, A^3 M + \lambda^4 \mathrm{tr}\, A^4\,.
\end{aligned}
$$

The number of conserved quantities for $n = 2, 3, 4$ are, respectively, one ($C_2 = \mathrm{tr}\, M^2$), one ($C_3 = \mathrm{tr}\, AM^2$) and two ($C_4 = \mathrm{tr}\, M^4$ and $I_4 = 2\mathrm{tr}\, A^2 M^2 + 4\mathrm{tr}\, AMAM$). ▲

> **Exercise.** How do the Euler equations look on $so(4)^*$ as a matrix equation? Is there an analog of the hat map for $so(4)$? Hint: The Lie algebra $so(4)$ is locally isomorphic to $so(3) \times so(3)$. ★

2.5 Hamilton

The ***Legendre transform*** of Lagrangian (2.4.4) in the variational principle (2.4.3) for Euler's rigid-body dynamics (2.4.8) on $\mathbb{R}^3$ will reveal its well-known Hamiltonian formulation.

Definition 2.5.1 (Legendre transformation)
*The Legendre transformation $\mathbb{F}l : \mathbb{R}^3 \to \mathbb{R}^{3*} \simeq \mathbb{R}^3$ is defined by the **fibre derivative**,*

$$
\mathbb{F}l(\Omega) = \frac{\delta l}{\delta \Omega} = \Pi\,.
$$

The Legendre transformation defines the ***body angular momentum*** by the variations of the rigid-body's reduced Lagrangian with respect to the body angular velocity. For the Lagrangian in (2.4.3), the $\mathbb{R}^3$ components of the body angular momentum are

$$\Pi_i = I_i \Omega_i = \frac{\partial l}{\partial \Omega_i}\,, \quad i = 1, 2, 3. \tag{2.5.1}$$

Remark 2.5.2 *This is also how body angular momentum was defined in the Newtonian setting. See Definition 2.1.30.*

Exercise. Express the Lagrangian (2.4.4) in terms of the matrices $O(t)$ and $\dot{O}(t)$. Show that this Lagrangian is left-invariant under $(O,\, \dot{O}) \mapsto (RO,\, R\dot{O})$ for any orthogonal matrix $R^T = R^{-1}$. Compute the Euler-Lagrange equations for this Lagrangian in geodesic form (2.2.4).

★

Exercise. Compute the Legendre transformation and pass to the canonical Hamiltonian formulation using the Lagrangian $l(\Omega) = L(O\,,\, \dot{O})$ and the following definitions of the canonical momentum and Hamiltonian,

$$P = \frac{\partial L(O\,,\, \dot{O})}{\partial \dot{O}} \quad \text{and} \quad H(P\,,\, O) = \Big\langle P\,,\, \dot{O} \Big\rangle - L(O\,,\, \dot{O})\,,$$

in combination with the chain rule for $\Omega = O^{-1}\dot{O}$. ★

2.5.1 Hamiltonian form of rigid-body motion

Definition 2.5.3 (Poisson bracket)
A ***Poisson bracket operation*** $\{\cdot\,,\,\cdot\}$ *is defined as possessing the following properties.*

1. *It is* ***bilinear****,*
2. ***skew-symmetric****,* $\{F\,,\,H\} = -\,\{H\,,\,F\}$*,*

3. *satisfies the **Leibnitz rule** (product rule),*

$$\{FG\,,\,H\} = \{F\,,\,H\}G + F\{G\,,\,H\}\,,$$

for the product of any two functions F and G on M, and

4. *satisfies the **Jacobi identity***

$$\{F\,,\,\{G\,,\,H\}\} + \{G\,,\,\{H\,,\,F\}\} + \{H\,,\,\{F\,,\,G\}\} = 0\,, \quad (2.5.2)$$

for any three functions F, G and H on M.

Remark 2.5.4 *This definition of Poisson bracket does not require it to be the standard canonical bracket in position q and conjugate momentum p, although it does include that case, as well.*

Definition 2.5.5 (Dynamical systems in Hamiltonian form) *A dynamical system on a manifold M*

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x})\,, \quad \mathbf{x} \in M\,,$$

*is said to be in **Hamiltonian form**, if it can be expressed as*

$$\dot{\mathbf{x}}(t) = \{\mathbf{x}, H\}\,, \quad \textit{for} \quad H : M \mapsto \mathbb{R}\,,$$

*in terms of a **Poisson bracket** operation $\{\cdot\,,\,\cdot\}$ among smooth real functions $\mathcal{F}(M) :\ M \mapsto \mathbb{R}$ on the manifold M,*

$$\{\cdot\,,\,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \mapsto \mathcal{F}(M)\,,$$

so that $\dot{F} = \{F\,,\,H\}$ for any $F \in \mathcal{F}(M)$.

2.5.2 Lie-Poisson Hamiltonian rigid-body dynamics

Let

$$h(\mathbf{\Pi}) := \langle \Pi, \Omega \rangle - l(\mathbf{\Omega})\,, \quad (2.5.3)$$

where the pairing $\langle \cdot\,,\,\cdot \rangle : \mathbb{R}^{3*} \times \mathbb{R}^3 \to \mathbb{R}$ is understood as the vector dot product on $\mathbb{R}^3$

$$\langle \Pi, \Omega \rangle := \mathbf{\Pi} \cdot \mathbf{\Omega}\,. \quad (2.5.4)$$

Hence, one finds the expected expression for the rigid-body Hamiltonian

$$h = \frac{1}{2}\,\mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi} := \frac{\Pi_1^2}{2I_1} + \frac{\Pi_2^2}{2I_2} + \frac{\Pi_3^2}{2I_3}\,. \tag{2.5.5}$$

The Legendre transform $\mathbb{F}l$ for this case is a diffeomorphism, so one may solve for

$$\frac{\partial h}{\partial \mathbf{\Pi}} = \mathbf{\Omega} + \left\langle \Pi\,,\,\frac{\partial \Omega}{\partial \mathbf{\Pi}}\right\rangle - \left\langle \frac{\partial l}{\partial \Omega}\,,\,\frac{\partial \Omega}{\partial \mathbf{\Pi}}\right\rangle = \mathbf{\Omega}\,, \tag{2.5.6}$$

upon using the definition of angular momentum $\Pi = \partial l/\partial\Omega$ in (2.5.1). In $\mathbb{R}^3$ coordinates, the relation (2.5.6) expresses the body angular velocity as the derivative of the reduced Hamiltonian with respect to the body angular momentum, namely,

$$\frac{\partial h}{\partial \mathbf{\Pi}} = \mathbf{\Omega}\,.$$

Hence, the reduced Euler-Lagrange equations for l may be expressed equivalently in angular momentum vector components in $\mathbb{R}^3$ and Hamiltonian h as:

$$\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) = \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega} \Longleftrightarrow \dot{\mathbf{\Pi}} = \mathbf{\Pi}\times\frac{\partial h}{\partial \mathbf{\Pi}} := \{\mathbf{\Pi}, h\}\,.$$

This expression suggests we introduce the following ***rigid-body Poisson bracket*** on functions of the $\mathbf{\Pi}$'s:

$$\{f,h\}(\mathbf{\Pi}) := -\,\mathbf{\Pi}\cdot\left(\frac{\partial f}{\partial \mathbf{\Pi}}\times\frac{\partial h}{\partial \mathbf{\Pi}}\right). \tag{2.5.7}$$

For the Hamiltonian (2.5.5), one checks that the Euler equations in terms of the rigid-body angular momenta,

$$\begin{aligned}
\dot\Pi_1 &= \left(\frac{1}{I_3}-\frac{1}{I_2}\right)\Pi_2\Pi_3\,,\\
\dot\Pi_2 &= \left(\frac{1}{I_1}-\frac{1}{I_3}\right)\Pi_3\Pi_1\,,\\
\dot\Pi_3 &= \left(\frac{1}{I_2}-\frac{1}{I_1}\right)\Pi_1\Pi_2\,,
\end{aligned} \tag{2.5.8}$$

that is, the equations,

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}^{-1}\mathbf{\Pi}\,, \tag{2.5.9}$$

are equivalent to

$$\dot{f} = \{f, h\}\,, \quad \text{with} \quad f = \mathbf{\Pi}\,.$$

The Poisson bracket proposed in (2.5.7) is an example of a ***Lie-Poisson bracket***. It satisfies the defining relations of a Poisson bracket for a number of reasons, not least because it is an example of a ***Nambu bracket***, to be discussed next.

2.5.3 Nambu's $\mathbb{R}^3$ Poisson bracket

The rigid-body Poisson bracket (2.5.7) is a special case of the Poisson bracket for functions of $\mathbf{x} \in \mathbb{R}^3$ introduced in [Na1973],

$$\{f, h\} = -\,\nabla c \cdot \nabla f \times \nabla h\,. \tag{2.5.10}$$

This bracket generates the motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h\,. \tag{2.5.11}$$

For this bracket the motion takes place along the intersections of level surfaces of the functions c and h in $\mathbb{R}^3$. In particular, for the rigid body, the motion takes place along intersections of angular momentum spheres $c = |\mathbf{x}|^2/2$ and energy ellipsoids $h = \mathbf{x} \cdot \mathbb{I}\mathbf{x}$. (See the cover illustration of [MaRa1994].)

> **Exercise.** Consider the $\mathbb{R}^3$ Nambu bracket
>
> $$\{f, h\} = -\,\nabla c \cdot \nabla f \times \nabla h\,. \tag{2.5.12}$$
>
> Let $c = \mathbf{x}^T \cdot \mathbb{C}\mathbf{x}/2$ be a quadratic form on $\mathbb{R}^3$, and let $\mathbb{C}$ be the associated symmetric 3×3 matrix. Show by direct computation that this Nambu bracket satisfies the Jacobi identity. ★

Exercise. Find the general conditions on the function $\mathbf{c}(\mathbf{x})$ so that the $\mathbb{R}^3$ bracket

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h\,,$$

satisfies the defining properties of a Poisson bracket. Is this $\mathbb{R}^3$ bracket also a derivation satisfying the Leibnitz relation for a product of functions on $\mathbb{R}^3$? If so, why?

★

Answer. The bilinear skew-symmetric Nambu $\mathbb{R}^3$ bracket yields the divergenceless vector field,

$$X_{c,h} = \{\,\cdot\,,\, h\} = (\nabla c \times \nabla h) \cdot \nabla \quad \text{with} \quad \operatorname{div}(\nabla c \times \nabla h) = 0\,.$$

Divergenceless vector fields are derivative operators that satisfy the Leibnitz product rule. They also satisfy the Jacobi identity for any choice of C^2 functions c and h. Hence, the Nambu $\mathbb{R}^3$ bracket is a bilinear skew-symmetric operation satisfying the defining properties of a Poisson bracket. ▲

Theorem 2.5.6 (Jacobi identity)
The Nambu bracket $\mathbb{R}^3$-bracket (2.5.12) satisfies the Jacobi identity.

Proof. The isomorphism $X_H = \{\,\cdot\,,\, H\}$ between the Lie algebra of divergenceless vector fields and functions under the $\mathbb{R}^3$-bracket is the key to proving this theorem. The Lie derivative among vector fields is identified with the Nambu bracket by

$$\mathcal{L}_{X_G} X_H = [X_G, X_H] = -X_{\{G,H\}}\,.$$

Repeating the Lie derivative produces,

$$\mathcal{L}_{X_F}(\mathcal{L}_{X_G} X_H) = [X_F, [X_G, X_H]\,] = X_{\{F,\{G,H\}\}}\,.$$

The result follows because both the left- and right-hand sides in this equation satisfy the Jacobi identity. ∎

Exercise. How is the $\mathbb{R}^3$ bracket related to the canonical Poisson bracket? Hint: Restrict to level surfaces of the function $c(\mathbf{x})$. ★

Exercise. (Casimirs of the $\mathbb{R}^3$ bracket) The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \text{for all } h(\mathbf{x})\,.$$

Suppose the function $c(\mathbf{x})$ is chosen so that the $\mathbb{R}^3$ bracket (2.5.10) defines a proper Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket (2.5.10)? Why? ★

Exercise. (Geometric interpretation of Nambu motion)

- Show that the Nambu motion equation (2.5.11)

  $$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h\,,$$

  for the $\mathbb{R}^3$ bracket (2.5.10) is invariant under a certain linear combination of the functions c and h. Interpret this invariance geometrically.

- Show that the rigid-body equations for

  $$\mathbb{I} = \mathrm{diag}\,(1,\ 1/2,\ 1/3)\,,$$

  may be interpreted as intersections in $\mathbb{R}^3$ of the spheres $x_1^2 + x_2^2 + x_3^2 = \text{constant}$ and the hyperbolic cylinders $x_1^2 - x_3^2 = \text{constant}$. See [HoMa1991] for more discussions of this geometric interpretation of solutions under the $\mathbb{R}^3$ bracket.

- A special case of the equations for three-wave interactions is [AlLuMaRo1998]

  $$\dot{x}_1 = s_1\gamma_1 x_2 x_3\,, \quad \dot{x}_2 = s_2\gamma_2 x_3 x_1\,, \quad \dot{x}_3 = s_3\gamma_3 x_1 x_2\,,$$

  for a set of constants $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and signs $s_1, s_2, s_3 = \pm 1$. Write these equations as a Nambu motion equation on $\mathbb{R}^3$ of the form (2.5.11). Interpret their solutions geometrically as intersections of level surfaces of quadratic functions for various values and signs of the γ's.

★

2.5.4 Clebsch variational principle for the rigid body

Proposition 2.5.7 (Clebsch variational principle)
The Euler rigid-body equations (2.4.2) on $T\mathbb{R}^3$ are equivalent to the ***constrained variational principle****,*

$$\delta S(\mathbf{\Omega}, \mathbf{Q}, \dot{\mathbf{Q}}; \mathbf{P}) = \delta \int_a^b l(\mathbf{\Omega}, \mathbf{Q}, \dot{\mathbf{Q}}; \mathbf{P})\, dt = 0, \tag{2.5.13}$$

for a ***constrained action integral***

$$\begin{aligned} S(\mathbf{\Omega}, \mathbf{Q}, \dot{\mathbf{Q}}) &= \int_a^b l(\mathbf{\Omega}, \mathbf{Q}, \dot{\mathbf{Q}})\, dt \qquad (2.5.14) \\ &= \int_a^b \frac{1}{2}\mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega} + \mathbf{P} \cdot \Big(\dot{\mathbf{Q}} + \mathbf{\Omega} \times \mathbf{Q} \Big)\, dt\,. \end{aligned}$$

Remark 2.5.8 (Reconstruction as constraint)

- *The first term in the Lagrangian (2.5.14),*

  $$l(\mathbf{\Omega}) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) = \frac{1}{2}\mathbf{\Omega}^T\mathbb{I}\mathbf{\Omega}\,, \tag{2.5.15}$$

  is again the (rotational) kinetic energy of the rigid body in (2.1.23).

- *The second term in the Lagrangian (2.5.14) introduces a the Lagrange multiplier* $\mathbf{P}$ *which imposes the constraint*

  $$\dot{\mathbf{Q}} + \mathbf{\Omega} \times \mathbf{Q} = 0\,.$$

  This ***reconstruction formula*** *has solution*

  $$\mathbf{Q}(t) = O^{-1}(t)\mathbf{Q}(0)\,,$$

  which satisfies

  $$\begin{aligned} \dot{\mathbf{Q}}(t) &= -\,(O^{-1}\dot{O})O^{-1}(t)\mathbf{Q}(0) \\ &= -\,\hat{\Omega}(t)\mathbf{Q}(t) = -\,\mathbf{\Omega}(t) \times \mathbf{Q}(t)\,. \qquad (2.5.16) \end{aligned}$$

Proof. The variations of S are given by

$$\begin{aligned}\delta S &= \int_a^b \Big(\frac{\delta l}{\delta \boldsymbol{\Omega}} \cdot \delta\boldsymbol{\Omega} + \frac{\delta l}{\delta \mathbf{P}} \cdot \delta\mathbf{P} + \frac{\delta l}{\delta \mathbf{Q}} \cdot \delta\mathbf{Q}\Big)\, dt \\ &= \int_a^b \Big[\Big(\mathbb{I}\boldsymbol{\Omega} - \mathbf{P}\times\mathbf{Q}\Big)\cdot \delta\boldsymbol{\Omega} \\ &\qquad + \delta\mathbf{P}\cdot\Big(\dot{\mathbf{Q}} + \boldsymbol{\Omega}\times\mathbf{Q}\Big) - \delta\mathbf{Q}\cdot\Big(\dot{\mathbf{P}} + \boldsymbol{\Omega}\times\mathbf{P}\Big)\Big]\, dt\,.\end{aligned}$$

Thus, stationarity of this ***implicit variational principle*** implies the following set of equations

$$\mathbb{I}\boldsymbol{\Omega} = \mathbf{P}\times\mathbf{Q}\,,\quad \dot{\mathbf{Q}} = -\,\boldsymbol{\Omega}\times\mathbf{Q}\,,\quad \dot{\mathbf{P}} = -\,\boldsymbol{\Omega}\times\mathbf{P}\,. \tag{2.5.17}$$

These ***symmetric equations*** for the rigid body first appeared in the theory of optimal control of rigid bodies [BlCrMaRa1998]. Euler's form of the rigid-body equations emerges from these, upon elimination of $\mathbf{Q}$ and $\mathbf{P}$, as

$$\begin{aligned}\mathbb{I}\dot{\boldsymbol{\Omega}} &= \dot{\mathbf{P}}\times\mathbf{Q} + \mathbf{P}\times\dot{\mathbf{Q}} \\ &= \mathbf{Q}\times(\boldsymbol{\Omega}\times\mathbf{P}) + \mathbf{P}\times(\mathbf{Q}\times\boldsymbol{\Omega}) \\ &= -\,\boldsymbol{\Omega}\times(\mathbf{P}\times\mathbf{Q}) = -\,\boldsymbol{\Omega}\times\mathbb{I}\boldsymbol{\Omega}\,,\end{aligned}$$

which are Euler's equations for the rigid body in $T\mathbb{R}^3$. ■

Remark 2.5.9 *The Clebsch variational principle for the rigid body is a natural approach in developing geometric algorithms for numerical integrations of rotating motion. Geometric integrators for rotations are derived using the Clebsch approach in [CoHo2007].*

Remark 2.5.10 *The Clebsch approach is also a natural path across to the Hamiltonian formulation of the rigid-body equations. This becomes clear in the course of the following exercise.*

> **Exercise.** Given that the canonical Poisson brackets in Hamilton's approach are
>
> $$\{Q_i, P_j\} = \delta_{ij} \quad \text{and} \quad \{Q_i, Q_j\} = 0 = \{P_i, P_j\}\,,$$
>
> what are the Poisson brackets for $\boldsymbol{\Pi} = \mathbf{P}\times\mathbf{Q} \in \mathbb{R}^3$ in (2.5.17)? Show that these Poisson brackets recover the rigid-body Poisson bracket (2.5.7). ★

Answer. The components of the angular momentum $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega}$ in (2.5.17) are

$$\Pi_a = \epsilon_{abc} P_b Q_c \,,$$

and their canonical Poisson brackets are (noting the similarity with the hat map)

$$\{\Pi_a, \Pi_i\} = \{\epsilon_{abc} P_b Q_c \,,\, \epsilon_{ijk} P_j Q_k\} = -\,\epsilon_{ail} \Pi_l \,.$$

Consequently, the derivative property of the canonical Poisson bracket yields,

$$\{f, h\}(\mathbf{\Pi}) = \frac{\partial f}{\partial \Pi_a} \{\Pi_a, \Pi_i\} \frac{\partial h}{\partial \Pi_b} = -\,\epsilon_{abc} \Pi_c \frac{\partial f}{\partial \Pi_a} \frac{\partial h}{\partial \Pi_b} \,, \qquad (2.5.18)$$

which is indeed the Lie-Poisson bracket in (2.5.7) on functions of the $\mathbf{\Pi}$'s. The correspondence with the hat map noted above shows that this Poisson bracket satisfies the Jacobi identity as a result of the Jacobi identity for the vector cross product on $\mathbb{R}^3$.

▲

Remark 2.5.11 *This exercise proves that the map $T^*\mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{\Pi} = \mathbf{P} \times \mathbf{Q} \in \mathbb{R}^3$ in (2.5.17) is Poisson. That is, the map takes Poisson brackets on one manifold into Poisson brackets on another manifold. Later we will recognise such an occurrence as one of the properties of a* ***momentum map****.*

Exercise. The Euler-Lagrange equations in matrix commutator form of Manakov's formulation of the rigid body on $SO(n)$ are

$$\frac{dM}{dt} = [\, M \,,\, \Omega \,] \,,$$

where the $n \times n$ matrices M, Ω are skew-symmetric. Show that these equations may be derived from Hamilton's principle $\delta S = 0$ with constrained action integral

$$S(\Omega, Q, P) = \int_a^b l(\Omega) + \mathrm{tr}\Big(P^T \left(\dot{Q} - Q\Omega \right) \Big) \, dt \,,$$

for which $M = \delta l/\delta\Omega = P^T Q - Q^T P$ and $Q, P \in SO(n)$ satisfy the following symmetric equations reminiscent of those in (2.5.17),

$$\dot{Q} = Q\Omega \quad \text{and} \quad \dot{P} = P\Omega\,, \tag{2.5.19}$$

as a result of the constraints. ★

Exercise. Write Manakov's deformation of the rigid-body equations (2.4.21) in the symmetric form (2.5.19). ★