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Solución Auxiliar 1

P1] a) Coordenadas cilíndricas: $\vec{r}(\rho,\varphi,z) = (\rho\cos(\varphi), \rho\,\mathrm{sen}(\varphi), z)$
Calculemos: $(\rho \geq 0,\ \varphi \in [0,2\pi],\ z \in \mathbb{R})$

- h_ρ: $\dfrac{\partial \vec{r}}{\partial \rho} = (\cos(\varphi), \mathrm{sen}(\varphi), 0) \Rightarrow h_\rho = \left\| \dfrac{\partial \vec{r}}{\partial \rho} \right\| = \sqrt{\cos^2(\varphi) + \mathrm{sen}^2(\varphi)} = 1$ //

- h_φ: $\dfrac{\partial \vec{r}}{\partial \varphi} = (-\rho\,\mathrm{sen}(\varphi), \rho\cos(\varphi), 0) \Rightarrow h_\varphi = \left\| \dfrac{\partial \vec{r}}{\partial \varphi} \right\| = \sqrt{\rho^2\cos^2(\varphi) + \rho^2\mathrm{sen}^2(\varphi)}$
$= \rho$ //

- h_z: $\dfrac{\partial \vec{r}}{\partial z} = (0,0,1) \Rightarrow h_z = \left\| \dfrac{\partial \vec{r}}{\partial z} \right\| = 1$ //

b) Calculemos:

- $\hat{\rho} = \dfrac{1}{h_\rho}\dfrac{\partial \vec{r}}{\partial \rho} = (\cos(\varphi), \mathrm{sen}(\varphi), 0)$ //

- $\hat{\varphi} = \dfrac{1}{h_\varphi}\dfrac{\partial \vec{r}}{\partial \varphi} = \dfrac{1}{\cancel{\rho}}(-\cancel{\rho}\,\mathrm{sen}(\varphi), \cancel{\rho}\cos(\varphi), 0) = (-\mathrm{sen}(\varphi), \cos(\varphi), 0)$

- $\hat{z} = \dfrac{1}{h_z}\dfrac{\partial \vec{r}}{\partial z} = (0,0,1)$ //

Veamos que $(\hat{\rho}, \hat{\varphi}, \hat{z})$ es un sistema ortogonal, ordenado positivamente.

- $\hat{\rho} \cdot \hat{\varphi} = \begin{pmatrix} \cos(\varphi) \\ \mathrm{sen}(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\mathrm{sen}(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} = -\cos(\varphi)\mathrm{sen}(\varphi) + \mathrm{sen}(\varphi)\cos(\varphi) = 0$

- $\hat{\rho} \cdot \hat{z} = \begin{pmatrix} \cos(\varphi) \\ \mathrm{sen}(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

- $\hat{\varphi} \cdot \hat{z} = \begin{pmatrix} -\mathrm{sen}(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \therefore (\hat{\rho}, \hat{\varphi}, \hat{z})$ es ortogonal.

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Ordenado positivamente (siguen regla de la mano derecha):

- $\hat{\rho} \times \hat{\varphi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos(\varphi) & \operatorname{sen}(\varphi) & 0 \\ -\operatorname{sen}(\varphi) & \cos(\varphi) & 0 \end{vmatrix} = [\cos^2(\varphi) + \operatorname{sen}^2(\varphi)]\hat{z} = \hat{z}$

- $\hat{\varphi} \times \hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\operatorname{sen}(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos(\varphi)\hat{x} + \operatorname{sen}(\varphi)\hat{y} = \hat{\rho}$

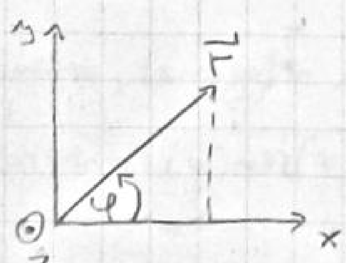
- $\hat{z} \times \hat{\rho} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \cos(\varphi) & \operatorname{sen}(\varphi) & 0 \end{vmatrix} = -\operatorname{sen}(\varphi)\hat{x} + \cos(\varphi)\hat{y} = \hat{\varphi}$

Concluimos que $(\hat{\rho}, \hat{\varphi}, \hat{z})$ es un sistema de coordenadas ortonormales (pues además, por definición, tienen norma 1) y están ordenados positivamente.

c) $$F(x,y,z) = \begin{pmatrix} \sqrt{x^2+y^2}\cos(\arctan(\frac{y}{x})) - \arctan(\frac{y}{x})\operatorname{sen}(\arctan(\frac{y}{x})) \\ \sqrt{x^2+y^2}\operatorname{sen}(\arctan(\frac{y}{x})) + \arctan(\frac{y}{x})\cos(\arctan(\frac{y}{x})) \\ z \end{pmatrix}$$

La idea es usar coordenadas cilíndricas para ahorrar matramka.

Notemos que, con estas coordenadas, $\rho = \sqrt{x^2+y^2}$, y además:



(vista desde arriba)

$$\tan(\varphi) = \frac{y}{x} \Rightarrow \varphi = \arctan\left(\frac{y}{x}\right)$$

Por lo que podemos reescribir F como:

$$F(\rho,\varphi,z) = \begin{pmatrix} \rho\cos(\varphi) - \varphi\operatorname{sen}(\varphi) \\ \rho\operatorname{sen}(\varphi) + \varphi\cos(\varphi) \\ z \end{pmatrix}$$

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Además, queremos escribir F de la forma:

$F = F_\rho \hat{\rho} + F_\varphi \hat{\varphi} + F_z \hat{z}$. Esto es: (o se pueden calcular: $F_u = F(\vec{r}(u,v,w)) \cdot \hat{u}(u,v,w)$)

$$\begin{pmatrix} \rho\cos(\varphi) - \varphi\,\text{sen}(\varphi) \\ \rho\,\text{sen}(\varphi) + \varphi\cos(\varphi) \\ z \end{pmatrix} = F_\rho \begin{pmatrix} \cos(\varphi) \\ \text{sen}(\varphi) \\ 0 \end{pmatrix} + F_\varphi \begin{pmatrix} -\text{sen}(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} + F_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Claramente, $F_z = z$. Luego, multiplicando la primera fila de la ecuación por $\cos(\varphi)$, y la segunda por $\text{sen}(\varphi)$, resulta:

$$\rho\cos^2(\varphi) - \varphi\,\text{sen}(\varphi)\cos(\varphi) = F_\rho\cos^2(\varphi) - F_\varphi\,\text{sen}(\varphi)\cos(\varphi)$$
$$\rho\,\text{sen}^2(\varphi) + \varphi\,\text{sen}(\varphi)\cos(\varphi) = F_\rho\,\text{sen}^2(\varphi) + F_\varphi\cos(\varphi)\,\text{sen}(\varphi).$$

Sumando ambas ecuaciones:

$\Rightarrow \rho = F_\rho$. Reemplazando en cualquier ecuación:

$$\rho\cancel{\cos^2(\varphi)} - \varphi\cancel{\text{sen}(\varphi)\cos(\varphi)} = \rho\cos^2(\varphi) - F_\varphi\cancel{\text{sen}(\varphi)\cos(\varphi)}$$

$\Rightarrow \varphi = F_\varphi$

Así, $F(\rho, \varphi, z) = \rho\hat{\rho} + \varphi\hat{\varphi} + z\hat{z}$

Calculemos: $(h_\rho = 1 = h_z \; ; \; h_\varphi = \rho)$

$$\cdot\, \text{div}(F) = \frac{1}{h_\rho h_\varphi h_z}\left(\frac{\partial}{\partial\rho}(F_\rho h_\varphi h_z) + \frac{\partial}{\partial\varphi}(F_\varphi h_\rho h_z) + \frac{\partial}{\partial z}(F_z h_\rho h_\varphi)\right)$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho^2) + \frac{\partial}{\partial\varphi}(\varphi) + \frac{\partial}{\partial z}(z\rho)\right)$$

$$= \frac{1}{\rho}(2\rho + 1 + \rho) = 3 + \frac{1}{\rho} = \boxed{3 + \frac{1}{\sqrt{x^2+y^2}}}$$

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$$\cdot\ \mathrm{rot}(F) = \frac{1}{h_\rho h_\varphi h_z}\begin{vmatrix} h_\rho\hat{\rho} & h_\varphi\hat{\varphi} & h_z\hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ F_\rho h_\rho & F_\varphi h_\varphi & F_z h_z \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ \rho & \varphi\rho & z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\underbrace{\left(\frac{\partial z}{\partial\varphi} - \frac{\partial(\varphi\rho)}{\partial z}\right)}_{0}\hat{\rho} - \underbrace{\left(\frac{\partial z}{\partial\rho} - \frac{\partial\rho}{\partial z}\right)}_{0}\rho\hat{\varphi} + \left(\underbrace{\frac{\partial(\varphi\rho)}{\partial\rho}}_{\varphi} - \underbrace{\frac{\partial\rho}{\partial\varphi}}_{0}\right)\hat{z}\right]$$

$$= \frac{\varphi}{\rho}\hat{z}\ \boxed{= \frac{\arctan(y/x)}{\sqrt{x^2+y^2}}\,\hat{z}}$$

LOL.

d) Sea $F = \nabla\times G = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ G_x & G_y & G_z \end{vmatrix}$

$$= \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\hat{x} - \left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z}\right)\hat{y}$$

$$+ \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\hat{z}$$

Luego: $\nabla\times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_y G_z - \partial_z G_y & \partial_z G_x - \partial_x G_z & \partial_x G_y - \partial_y G_x \end{vmatrix}$

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$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}G_y - \frac{\partial}{\partial y}G_x\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}G_x - \frac{\partial}{\partial x}G_z\right)\right)\hat{x}$$

$$- \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}G_y - \frac{\partial}{\partial y}G_x\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}G_x - \frac{\partial}{\partial x}G_z\right)\right)\hat{y}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}G_x - \frac{\partial}{\partial x}G_z\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}G_z - \frac{\partial}{\partial z}G_y\right)\right)\hat{z}$$

$$= \left(\frac{\partial^2 G_y}{\partial y \partial x} - \frac{\partial^2 G_x}{\partial y^2} - \frac{\partial^2 G_x}{\partial z^2} + \frac{\partial}{\partial z \partial x}G_x\right)\hat{x}$$

$$- \left(\frac{\partial^2}{\partial x^2}G_y - \frac{\partial}{\partial x \partial y}G_x - \frac{\partial^2}{\partial z^2}G_x + \frac{\partial^2}{\partial z \partial x}G_z\right)\hat{y}$$

$$+ \left(\frac{\partial^2}{\partial x \partial z}G_x - \frac{\partial^2}{\partial x^2}G_z - \frac{\partial^2}{\partial y^2}G_z + \frac{\partial^2}{\partial y \partial z}G_y\right)\hat{z}$$

Por otro lado: $\nabla \cdot G = \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right)$

$$\Rightarrow \nabla(\nabla \cdot G) = \frac{\partial}{\partial x}\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right)\hat{x} + \frac{\partial}{\partial y}\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right)\hat{y} + \frac{\partial}{\partial z}\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right)\hat{z}$$

$\therefore \nabla(\nabla \cdot G) - \mathrm{rot}(\mathrm{rot}(G)) = \left(\frac{\partial^2 G_x}{\partial x^2} + \frac{\partial^2 G_x}{\partial y^2} + \frac{\partial^2 G_x}{\partial z^2}\right)\hat{x}$
(cancelando y reordenando)

$$+ \left(\frac{\partial^2 G_y}{\partial x^2} + \frac{\partial^2 G_y}{\partial y^2} + \frac{\partial^2 G_y}{\partial z^2}\right)\hat{y} + \left(\frac{\partial^2 G_z}{\partial x^2} + \frac{\partial^2 G_z}{\partial y^2} + \frac{\partial^2 G_z}{\partial z^2}\right)\hat{z} = \Delta G$$

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P2)

a) $\operatorname{rot}(F) = \dfrac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{u} & h_v \hat{v} & h_w \hat{w} \\ \partial_u & \partial_v & \partial_w \\ F_u h_u & F_v h_v & F_w h_w \end{vmatrix}$

Tenemos que: $\varphi = \varphi_u \hat{u} + \varphi_v \hat{v} + \varphi_w \hat{w}$

$$\Rightarrow \int_a^b \varphi(\vec{r},t)dt = \int_a^b \varphi_u(\vec{r},t)\,dt\,\hat{u} + \int_a^b \varphi_v(\vec{r},t)\,dt\,\hat{v} + \int_a^b \varphi_w(\vec{r},t)\,dt\,\hat{w}$$

Ocupando la fórmula del rotor y haciendo producto escalar con $\hat{u}$: (pa ahorrar matraca)

$$\operatorname{rot}\left(\int_a^b \varphi(\vec{r},t)\,dt\right)\cdot\hat{u} = \frac{1}{h_v h_w}\left(\frac{\partial}{\partial v}\left(\int_a^b \varphi_w h_w\,dt\right) - \frac{\partial}{\partial w}\left(\int_a^b \varphi_v h_v\,dt\right)\right)$$

$$\overset{\text{Leibnitz}}{=} \frac{1}{h_v h_w}\left[\left(\int_a^b \frac{\partial}{\partial v}(\varphi_w h_w)\,dt\right) - \left(\int_a^b \frac{\partial}{\partial w}(\varphi_v h_v)\,dt\right)\right]$$

$$= \int_a^b \frac{1}{h_v h_w}\left(\frac{\partial}{\partial v}(\varphi_w h_w) - \frac{\partial}{\partial w}(\varphi_v h_v)\right)dt$$

$$= \int_a^b \operatorname{rot}(\varphi(\vec{r},t))\,dt\cdot\hat{u}$$

Para $\hat{v}$ y $\hat{w}$ se hace el proceso análogo.

Finalmente, como $\operatorname{rot}(F) = \operatorname{rot}(F)\cdot\hat{u} + \operatorname{rot}(F)\cdot\hat{v} + \operatorname{rot}(F)\cdot\hat{w}$, se concluye que:

$$\nabla\times\int_a^b \varphi(\vec{r},t)\,dt = \int_a^b \nabla\times\varphi(\vec{r},t)\,dt$$

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b). Como $F(\vec{r}) = g(r)\hat{\theta}$ (que sólo tiene componente $\hat{\theta}$)

$\Rightarrow F_r = F_\varphi = 0 \ , \ F_\theta = g(r)$

Así, $\operatorname{div}(F) = \dfrac{1}{h_r h_\theta h_\varphi}\left(\cancel{\dfrac{\partial}{\partial r}(F_r h_\theta h_\varphi)}^{0} + \dfrac{\partial}{\partial \theta}(F_\theta h_r h_\varphi) + \cancel{\dfrac{\partial}{\partial \varphi}(F_\varphi h_r h_\theta)}^{0}\right)$

$= \dfrac{1}{1 \cdot r\,\mathrm{sen}(\varphi) \cdot r}\Big(\underbrace{\dfrac{\partial}{\partial \theta}(g(r) \cdot 1 \cdot r)}_{0}\Big) = 0$,

0 ($g(r)r$ es constante para θ).

$\cdot\, (F(t\vec{r}) \times t\vec{r}) = (g(tr)\hat{\theta} \times tr\hat{r}) = trg(tr)\hat{\varphi} := G(r)$

Notemos que G sólo tiene componente en r.
Luego, su rotor (en esféricas) será:

$$\operatorname{rot}(G) = \frac{1}{r^2\,\mathrm{sen}(\varphi)}\begin{vmatrix} \hat{r} & r\hat{\varphi} & r\,\mathrm{sen}(\varphi)\hat{\theta} \\ \partial_r & \partial_\varphi & \partial_\theta \\ 0 & rG(r) & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\,\mathrm{sen}(\varphi)}\left[\cancel{\left(-\frac{\partial}{\partial \theta}(rG(r))\right)}^{0}\hat{r} + \left(\frac{\partial}{\partial r}(rG(r))\right) r\,\mathrm{sen}(\varphi)\hat{\theta}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(tr^2 g(rt))\hat{\theta} = \frac{1}{\cancel{r}}\left[2t\cancel{r}g(rt) + t^2 r^{\cancel{2}}\frac{\partial}{\partial r}g(rt)\right]\hat{\theta}$$

$$= \left(2tg(rt) + t^2 r\frac{\partial}{\partial r}g(rt)\right)\hat{\theta} \quad (*)$$

Por otro lado: $\dfrac{d}{dt}F(tr) = \dfrac{d}{dt}g(tr)\hat{\theta} = \Big(\dfrac{d}{dr}g(tr) \cdot \dfrac{d(tr)}{dt}$

$+ \underbrace{\dfrac{d}{d\theta}}_{0} g(tr)\dfrac{d}{dt}(tr) + \underbrace{\dfrac{d}{d\varphi}}_{0} g(tr) \cdot \dfrac{d}{dt}(tr)\Big)\hat{\theta} = \dfrac{d}{dr}g(tr) \cdot \dfrac{d(tr)}{dt}\hat{\theta}$

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$= \left[\frac{d}{dr} g(tr)\right] r\,\hat{\theta}$ (**) Luego, como $G = F(t\vec{r}) \times t\vec{r}$

$\overset{(*)}{\Rightarrow} rot(F(t\vec{r}) \times t\vec{r}) = \left(2t\,g(rt) + t^2 r \frac{\partial g(rt)}{\partial r}\right)\hat{\theta}$

$\overset{(**)}{=} 2t\,g(rt)\hat{\theta} + t^2 \frac{d}{dt} F(t\vec{r})$

$\overset{\text{Def } F}{=} 2t\,F(t\vec{r}) + t^2 \frac{d}{dt} F(t\vec{r})$ //

c) $rot(G) = \nabla \times \int_0^1 (F(t\vec{r}) \times t\vec{r})\,dt$

$\overset{(a)}{=} \int_0^1 \nabla \times (F(t\vec{r}) \times t\vec{r})\,dt$

(b), ya que $div(F) = 0$ en B

$= \int_0^1 2t\,F(t\vec{r})\,dt + \int_0^1 t^2 \frac{d}{dt} F(t\vec{r})\,dt \quad (*)$

Calculemos la segunda integral con vaca:

$u = t^2 \Rightarrow du = 2t\,dt$

$dv = \frac{d}{dt} F(t\vec{r})\,dt \Rightarrow v = F(t\vec{r})$

$\Rightarrow \int_0^1 t^2 \frac{d}{dt} F(t\vec{r})\,dt = t^2 F(t\vec{r})\Big|_0^1 - \int_0^1 F(t\vec{r}) \cdot 2t\,dt$

$= F(\vec{r}) - \int_0^1 2t\,F(t\vec{r})\,dt$

Finalmente:

$rot(G) \overset{(*)}{=} \cancel{\int_0^1 2t\,F(t\vec{r})\,dt} + F(\vec{r}) - \cancel{\int_0^1 2t\,F(t\vec{r})\,dt}$

$\boxed{\therefore \; rot(G(\vec{r})) = F(\vec{r})}$

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P3] Para las dos partes, $\vec{r}=(x,y,z)$, con los cambios de variables correspondientes

a) h_ε: $\dfrac{\partial \vec{r}}{\partial \varepsilon} = (\nu\cos(\varphi), \nu\,\text{sen}(\varphi), -\varepsilon)$

$$\Rightarrow h_\varepsilon = \left\| \frac{\partial \vec{r}}{\partial \varepsilon} \right\| = \sqrt{\nu^2 + \varepsilon^2}$$

h_ν: $\dfrac{\partial \vec{r}}{\partial \nu} = (\varepsilon\cos(\varphi), \varepsilon\,\text{sen}(\varphi), \nu)$

$$\Rightarrow h_\nu = \left\| \frac{\partial \vec{r}}{\partial \nu} \right\| = \sqrt{\varepsilon^2 + \nu^2}$$

h_φ: $\dfrac{\partial \vec{r}}{\partial \varphi} = (-\varepsilon\nu\,\text{sen}(\varphi), \varepsilon\nu\cos(\varphi), 0)$

$$\Rightarrow h_\varphi = \left\| \frac{\partial \vec{r}}{\partial \varphi} \right\| = \varepsilon\nu$$

Luego, el elemento de volumen es $dV = h_\varepsilon h_\nu h_\varphi \, d\varepsilon \, d\nu \, d\varphi$

$$\Rightarrow dV = (\nu^2 + \varepsilon^2)\varepsilon\nu \, d\varepsilon \, d\nu \, d\varphi_{//}$$

b) h_μ: $\dfrac{\partial \vec{r}}{\partial \mu} = \begin{pmatrix} a\cosh(\mu)\,\text{sen}(\nu)\cos(\varphi) \\ a\cosh(\mu)\,\text{sen}(\nu)\,\text{sen}(\varphi) \\ a\,\text{senh}(\mu)\cos(\nu) \end{pmatrix}$

$$\Rightarrow h_\mu = \left\| \frac{\partial \vec{r}}{\partial \mu} \right\| = a\sqrt{\cosh^2(\mu)\,\text{sen}^2(\nu) + \text{senh}^2(\mu)\cos^2(\nu)}$$

$$= a\sqrt{(1+\text{senh}^2(\mu))\,\text{sen}^2(\nu) + \text{senh}^2(\mu)\cos^2(\nu)}$$

$$= a\sqrt{\text{sen}^2(\nu) + \text{senh}^2(\mu)\,\text{sen}^2(\nu) + \text{senh}^2(\mu)\cos^2(\nu)}$$

$$= a\sqrt{\text{sen}^2(\nu) + \text{senh}^2(\mu)}_{//}$$

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h_ν: $\dfrac{\partial \vec{r}}{\partial \nu} = a\begin{pmatrix} \operatorname{senh}(\mu)\cos(\nu)\cos(\varphi) \\ \operatorname{senh}(\mu)\cos(\nu)\operatorname{sen}(\varphi) \\ -\cosh(\mu)\operatorname{sen}(\nu) \end{pmatrix}$

$$\Rightarrow h_\nu = \left\|\frac{\partial \vec{r}}{\partial \nu}\right\| = a\sqrt{\operatorname{senh}^2(\mu)\cos^2(\nu) + \cosh^2(\mu)\operatorname{sen}^2(\nu)}$$

$$= a\sqrt{\operatorname{senh}^2(\mu)\cos^2(\nu) + (1+\operatorname{senh}^2(\mu))\operatorname{sen}^2(\nu)}$$

$$= a\sqrt{\operatorname{senh}^2(\mu) + \operatorname{sen}^2(\nu)} \quad (= h_\mu) \quad //$$

h_φ: $\dfrac{\partial \vec{r}}{\partial \varphi} = a\begin{pmatrix} -\operatorname{senh}(\mu)\operatorname{sen}(\nu)\operatorname{sen}(\varphi) \\ \operatorname{senh}(\mu)\operatorname{sen}(\nu)\cos(\varphi) \\ 0 \end{pmatrix}$

$$\Rightarrow h_\varphi = \left\|\frac{\partial \vec{r}}{\partial \varphi}\right\| = a\operatorname{senh}(\mu)\operatorname{sen}(\nu) \quad //$$

Finalmente, el elemento de volumen es

$$dV = h_\mu h_\nu h_\varphi \, d\mu \, d\nu \, d\varphi$$

$$\therefore dV = a^3(\operatorname{senh}^2(\mu) + \operatorname{sen}^2(\nu))\operatorname{senh}(\mu)\operatorname{sen}(\nu)\, d\mu\, d\nu\, d\varphi \quad //$$

Obs: A estas coordenadas se les llama "esferoidales prolatas", que resultan de rotar un sistema elíptico (dicha rotación dada por los términos que relacionan a φ). Este cambio de coordenadas es útil para resolver algunas EDP's, como la ecuación de onda bajo ciertas condiciones (su wikipedia).