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# Solución Auxiliar 8

## P1

Utilizaremos las integrales sobre caminos parametrizados por una curva:

Si $r : [a,b] \to \mathbb{C}$ es una parametrización de $\gamma$, entonces:

$$\int_\gamma f(z)\,dz = \int_a^b f(r(t))\, r'(t)\, dt$$

(Tal como se hacía para calcular integrales de línea).

a) Una parametrización de esta circunferencia es

$$r(t) = 2e^{it}, \quad t \in [0, 2\pi].$$

$$\Rightarrow r'(t) = 2ie^{it}$$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_0^{2\pi} f(2e^{it}) \cdot 2ie^{it}\, dt$$

$$= \int_0^{2\pi} 2\cancel{e^{-it}} \cdot 2i\cancel{e^{it}}\, dt = 4i \int_0^{2\pi} dt = 8\pi i$$

b) La semicircunferencia unitaria se puede parametrizar como

$$r(t) = e^{it}, \text{ con } t \in [-\pi/2, \pi/2].$$

$$\Rightarrow r'(t) = ie^{it}$$

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Además, $Re(e^{it}) = Re(\cos(t) + i\,sen(t))$

$$= \cos(t) = \frac{e^{it} + e^{-it}}{2}$$

Así,

$$\int_{\gamma} f(z)\,dz = \int_{-\pi/2}^{\pi/2} \left(\frac{e^{it} + e^{-it}}{2}\right) \cdot ie^{it}\,dt$$

$$= \frac{i}{2} \int_{-\pi/2}^{\pi/2} (e^{2it} + 1)\,dt = \frac{i}{2}\left(\frac{e^{2it}}{2i} + t\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{i}{2}\left(\frac{e^{i\pi} - e^{-i\pi}}{2i} + \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{i}{2}\left(\cancel{sen(\pi)}^{\,0} + \pi\right) = \frac{\pi i}{2}$$

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c) Parametrizamos : $r(t) = e^{it}$, $t \in [0, 2\pi]$
$\Rightarrow r'(t) = ie^{it}$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_0^{2\pi} e^{i\alpha t} \cdot ie^{it}\,dt = i\int_0^{2\pi} e^{it(\alpha+1)}\,dt$$

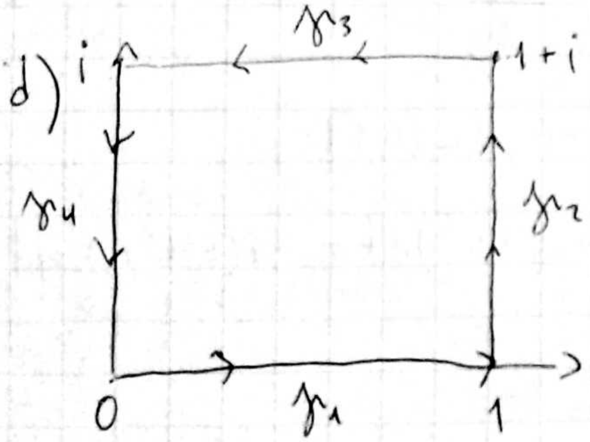
Caso 1: $\alpha \neq -1$

$$\Rightarrow \int_\gamma f(z)\,dz = \frac{\cancel{i}e^{it(\alpha+1)}}{\cancel{i}(\alpha+1)}\Bigg|_0^{2\pi} = \frac{e^{i2\pi(\alpha+1)} - 1}{\alpha+1}$$

$$= \frac{e^{i2\pi\alpha} - 1}{\alpha+1}$$

Si $\alpha = -1$: $\displaystyle\int_\gamma f(z)\,dz = i\int_0^{2\pi} 1\,dt = 2\pi i.$

d)




$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f$$

$\underline{\gamma_1}$: Una parametrización es $r(t) = t \Rightarrow r'(t) = 1$
$t \in [0, 1]$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_0^1 |t|^2 \cdot 1\,dt = \frac{t^3}{3}\Bigg|_0^1 = \frac{1}{3}$$

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$\underline{\gamma_2}$: Una parametrización es es
$r(t) = 1 + it \Rightarrow r'(t) = i,\ t \in [0,1]$

$$\Rightarrow \int_{\gamma_2} f(z)\,dz = \int_0^1 |1+it|^2 \cdot i\,dt = i\int_0^1 (1+t^2)\,dt$$

$$= i\left(t + \frac{t^3}{3}\right)\Big|_0^1 = \frac{4}{3}i$$

$\underline{\gamma_3}$: $r(t) = (1-t)+i \Rightarrow r'(t) = -1,\ t \in [0,1]$

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = -\int_0^1 |(1-t)+i|^2\,dt = -\int_0^1 ((1-t)^2+1)\,dt$$

$$= -\left(2t - t^2 + \frac{t^3}{3}\right)\Big|_0^1 = -\frac{4}{3}$$

$\underline{\gamma_4}$: $r(t) = (1-t)i \Rightarrow r'(t) = -i,\ t \in [0,1]$

$$\Rightarrow -i\int_{\gamma_4} |(1-t)i|^2\,dt = -i\int_0^1 (1-2t+t^2)\,dt = -i\left(t - t^2 + \frac{t^3}{3}\right)\Big|_0^1$$

$$= -\frac{i}{3}$$

$$\therefore \int_{\gamma} f(z)\,dz = \frac{1}{3} + \frac{4}{3}i - \frac{4}{3} - \frac{i}{3} = i - 1 \;//$$

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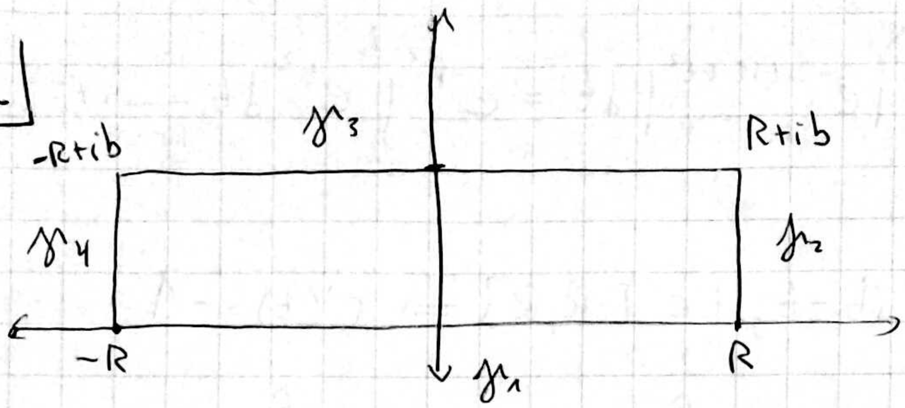
**P2**




Notemos que $f(z)=e^{-z^2}$ es holomorfa y que el rectángulo es una curva cerrada

$\Rightarrow$ Por Cauchy-Goursat: $\oint f(z)\,dz = 0$.

Calculemos cada parte:

$\gamma_1$: $r(t)=t,\ t\in[-R,R] \Rightarrow r'(t)=1$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-t^2}\,dt$$

Tomando límite cuando $R\to\infty$,

$$\int_{-R}^{R} e^{-t^2}\,dt \xrightarrow[R]{} \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

$\gamma_2$: $r(t)=R+it,\ t\in[0,b] \Rightarrow r'(t)=i$

Veamos que $\int_{\gamma_2} f \longrightarrow 0$ cuando $R\to\infty$:

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_0^b e^{-(R+it)^2}\, i\,dt\right| = \left|\int_0^b e^{-R^2}\cdot e^{-2itR}\cdot e^{t^2}\, i\,dt\right|$$

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$$\leq e^{-R^2}\int_0^b |e^{-2itR+t^2}|\,dt = e^{-R^2}\int_0^b e^{t^2}\,dt \xrightarrow[R\to\infty]{} 0$$

$\gamma_3$: $r(t) = ib - t$, $t\in[-R,R] \Rightarrow r'(t) = -1$

Luego:

$$\int_{\gamma_3} f(z)\,dz = -\int_{-R}^{R} e^{-(ib-t)^2}\,dt = -\int_{-R}^{R} e^{-(-b^2-2ibt+t^2)}\,dt$$

$$= -e^{b^2}\int_{-R}^{R} e^{-t^2}\cdot e^{2ibt}\,dt = -e^{b^2}\int_{-R}^{R} e^{-t^2}(\cos(2bt)+i\,\mathrm{sen}(2bt))\,dt$$

Notemos que, como $\mathrm{sen}(\cdot)$ es impar y $e^{-t^2}$ es par, entonces $e^{-t^2}\mathrm{sen}(2bt)$ es impar, y por lo tanto

$$\int_{-R}^{R} e^{-t^2}\,\mathrm{sen}(2bt)\,dt = 0$$

Así, $\displaystyle\int_{\gamma_3} f(z)\,dz = -e^{b^2}\int_{-R}^{R} e^{-t^2}\cos(2bt)\,dt$

$$\xrightarrow[R\to\infty]{} -e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt$$

$\gamma_4$: $r(t) = (b-t)i - R$, $t\in[0,b] \Rightarrow r'(t) = -i$

$$\Rightarrow \left|\int_{\gamma_4} f(z)\,dz\right| = \left|\int_0^b e^{-((b-t)i-R)^2}\,dt\right|$$

$$\leq e^{-R^2}\int_0^b e^{(b-t)^2}\underbrace{|e^{2(b-t)Ri}|}_{1}\,dt \xrightarrow[R\to\infty]{} 0$$

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$$\therefore \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f = 0 \quad \Big| \; R \to \infty$$

$$\Rightarrow \sqrt{\pi} + 0 - e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt + 0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = \sqrt{\pi}\,e^{-b^2}$$

Como $e^{-t^2}\cos(2bt)$ es par

$$\Rightarrow \int_{0}^{\infty} e^{-t^2}\cos(2bt)\,dt = \frac{\sqrt{\pi}\,e^{-b^2}}{2}. \quad //$$

P3| Claramente $f$ es holomorfa en el contorno $\Gamma$ (que no incluye al 0, donde $f$ explota).

Luego, por Cauchy-Goursat:

$$\int_{\Gamma} f(z)\,dz = 0 \Leftrightarrow \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2} + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{i}}{z^2} = 0$$

Veamos cada parte:

- $[-R,-\varepsilon] \cup [\varepsilon, R]$:

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx = \int_{-R}^{-\varepsilon} \frac{1-\cos(x)}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{1-\cos(x)}{x^2}\,dx$$
$$- i\left( \int_{-R}^{-\varepsilon} \frac{\operatorname{sen}(x)}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{\operatorname{sen}(x)}{x^2}\,dx \right)$$

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$$= -\int_R^{\varepsilon} \frac{1-\cos(u)}{u^2}\,du + \int_{\varepsilon}^{R} \frac{1-\cos(x)}{x^2}\,dx - i\left(-\int_{\varepsilon}^{R} \frac{\operatorname{sen}(u)}{u^2}du + \int_{\varepsilon}^{R}\frac{\operatorname{sen}(x)}{x^2}dx\right)$$

$\downarrow$ $u=-x$

$$= 2\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx$$

Como $\left|\frac{1-\cos(x)}{x^2}\right| \leq \frac{2}{x^2}$, el integrando es finito cuando tomamos $R\to\infty$.

Además, $\frac{1-\cos(x)}{x^2} \xrightarrow[x\to 0]{} \frac{1}{2}$

$\Rightarrow$ el integrando es integrable cerca del origen.

Sigue que $\lim\limits_{\substack{\varepsilon\to 0\\ R\to\infty}} 2\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx = 2\int_0^{\infty} \frac{1-e^{ix}}{x^2}\,dx$

· $\underline{\gamma_R^+}$: Veamos que $\lim\limits_{R\to\infty} \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0$.

Separando la integral: $\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = \underbrace{\int_{\gamma_R^+} \frac{1}{z^2}\,dz}_{(i)} - \underbrace{\int_{\gamma_R^+} \frac{e^{iz}}{z^2}\,dz}_{(ii)}$.

(i) Como $|z|=R$ en $\gamma_R^+$, tenemos que

$$\left|\int_{\gamma_R^+} \frac{dz}{z^2}\right| \leq \frac{1}{R^2} L(\gamma_R^+) = \frac{\pi R}{R^2} = \frac{\pi}{R} \longrightarrow 0.$$

(ii) $r(t) = Re^{it}$, $t\in[0,\pi]$. $r'(t) = Rie^{it}\,dt$

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$$\Rightarrow \int_{\gamma_R^+} \frac{e^{iz}}{z^2}\,dz = \int_0^{\pi} \frac{e^{iRe^{it}}}{(Re^{it})^2} Rie^{it}\,dt = \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\, i\,dt$$

$$= i\int_0^{\pi} \frac{e^{-R\,\mathrm{sen}(t)}\, e^{iR\cos(t)}}{Re^{it}}\,dt$$ , tomando módulo:

$$\left|\int_{\gamma_R^+} \frac{e^{iz}}{z^2}\,dz\right| \leq \int_0^{\pi} \frac{e^{-R\,\mathrm{sen}(t)}}{R}\,dt = 2\int_0^{\pi/2} \frac{e^{-R\,\mathrm{sen}(t)}}{R}\,dt$$

$$\leq \frac{2}{R}\int_0^{\pi/2} e^{-R\,\mathrm{sen}(0)}\,dt$$ $\rightsquigarrow$ función $\searrow$ en $[0, \pi/2]$.

$$= \frac{\cancel{2}}{R}\cdot\frac{\pi}{\cancel{2}} = \frac{\pi}{R} \xrightarrow[R\to\infty]{} 0.$$

$$\therefore \lim_{R\to\infty} \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0.$$

$\underline{\gamma_\varepsilon^+}$: Para $z\neq 0$: $f(z) = \dfrac{1-e^{iz}}{z^2} = \dfrac{1-\overbrace{(1+iz+(iz)^2 g(z))}^{\text{expansión serie de potencias}}}{z^2}$

$$= \frac{-i}{z} + h(z)$$ , $h$ continua.

Notemos que $\displaystyle\int_{\gamma_\varepsilon^+} \frac{-i}{z}\,dz = \int_{\pi}^{0} \frac{-i}{\cancel{\varepsilon}\cancel{e^{it}}}\, i\cancel{\varepsilon}\cancel{e^{it}}\,dt = -\pi.$

y como $h$ es continua sobre $\gamma_\varepsilon^+$, que es compacto, entonces $|h(z)| \leq M \ \forall z\in\gamma_\varepsilon^+$.

$$\Rightarrow \left|\int_{\gamma_\varepsilon^+} h(z)\,dz\right| \leq M\pi\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$$

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$\therefore \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon^+} f(z)\, dz = -\pi.$

Así, ~~tomando~~ tomando límite cuando $R \to \infty$, $\varepsilon \to 0$ en $\Gamma$:

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\Gamma f(z)\, dz = 2 \int_0^\infty \frac{1-\cos(x)}{x^2}\, dx - \pi = 0.$$

$$\therefore \int_0^\infty \frac{1-\cos(x)}{x^2}\, dx = \frac{\pi}{2} //$$

P4

Fórmula de Cauchy Generalizada:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Usaremos la fórmula para $f(z) = z^{1/m}$, $z_0 = 1$, $m-1 = n$.

$$\Rightarrow \int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(1)$$

Donde $f^{(m-1)}(z) = \frac{1}{m}\left(\frac{1}{m} - 1\right) \cdots \left(\frac{1}{m} - (m-2)\right) z^{\frac{1}{m} - (m-1)}$

$$\Rightarrow f^{(m-1)}(1) = \frac{1}{m}\left(\frac{1}{m} - 1\right) \cdots \left(\frac{1}{m} - (m-2)\right).$$

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