Q3 Solve the problem with LOG utility first.

(a) $\mathcal{L} = \alpha \log(c_1) + (1-\alpha)\log(l_1) + \beta\left[\alpha\log(c_2) + (1-\alpha)\log(l_2)\right]$

*Q3 $$+\lambda\left[-c_1 - \frac{c_2}{1+r} + w_1(1-l_1) + \frac{w_2(1-l_2)}{1+r}\right]$$

FOC's:

(c_1): $\frac{\alpha}{c_1} = \lambda$ (l_1): $\frac{1-\alpha}{l_1} = \lambda w_1$

(c_2): $\frac{\beta\alpha}{c_2} = \frac{\lambda}{1+r}$ (l_2): $\frac{\beta(1-\alpha)}{l_2} = \frac{\lambda w_2}{1+r}$

Insert c_1, c_2, l_1, l_2 from FOC's into the lifetime budget constraint

$$\frac{\alpha}{\lambda} + \frac{\beta\alpha}{\lambda} + \frac{1-\alpha}{\lambda} + \frac{\beta(1-\alpha)}{\lambda} = w_1 + \frac{w_2}{1+r} = I(r)$$

$$\frac{1+\beta}{\lambda} = I(r) \Rightarrow \frac{1}{\lambda} = \frac{I(r)}{1+\beta}$$

Insert $\frac{1}{\lambda}$ into FOC's

$$c_1 = \frac{\alpha}{\lambda} = \frac{\alpha}{1+\beta} I(r) \; ; \; c_2 = \frac{(1+r)\beta\alpha}{\lambda} = \frac{\alpha\beta(1+r)}{1+\beta} I(r).$$

$$l_1 = \frac{1}{\lambda}\frac{(1-\alpha)}{w_1} = \frac{I(r)(1-\alpha)}{(1+\beta)w_1} = \frac{(1-\alpha)}{1+\beta}\left[1 + \frac{w_2}{w_1}\frac{1}{1+r}\right]$$

$$l_2 = \frac{\beta(1-\alpha)(1+r)}{\lambda w_2} = \frac{\beta(1-\alpha)(1+r)}{(1+\beta)w_2}\left[w_1 + \frac{w_2}{1+r}\right]$$

$$l_2 = \frac{\beta(1-\alpha)}{1+\beta}\left((1+r)\frac{w_1}{w_2} + 1\right)$$

(b) Now solve the problem with ~~u(c,l) = α log(c) + (1-α) log(l)~~

* $$u(c,l) = \frac{(c^{\alpha} l^{1-\alpha})^{1-\sigma}}{1-\sigma}$$

Lagrangian:

$$\mathcal{L} = \frac{(c_1^{\alpha} l_1^{1-\alpha})^{1-\sigma}}{1-\sigma} + \beta\frac{(c_2^{\alpha} l_2^{1-\alpha})^{1-\sigma}}{1-\sigma}$$
$$+\lambda\left[-c_1 - \frac{c_2}{1+r} + w_1(1-l_1) + \frac{w_2(1-l_2)}{1+r}\right]$$

FOC's

(c_1): $$\frac{\alpha\cancel{(1-\sigma)}\, c_1^{\alpha(1-\sigma)-1} \cdot l_1^{(1-\alpha)(1-\sigma)}}{\cancel{1-\sigma}} = \lambda$$

$$\Rightarrow \alpha\, c_1^{\alpha(1-\sigma)-1} \cdot l_1^{(1-\alpha)(1-\sigma)} = \lambda$$

(l_1): $$(1-\alpha)\, c_1^{\alpha(1-\sigma)}\, l_1^{(1-\alpha)(1-\sigma)-1} = \lambda w_1$$

If we set $\sigma = 1$, we see that these FOC's become the same FOC's as in the LOG case. Therefore, solutions should be same when $\sigma = 1$.

Take the ratio of two FOC's.

$$\frac{1-\alpha}{\alpha}\frac{c_1}{l_1} = w_1 \Rightarrow w_1 l_1 = \frac{1-\alpha}{\alpha} c_1 \Rightarrow l_1 = \frac{1-\alpha}{\alpha}\frac{c_1}{w_1}$$

(c_2): $$\beta\alpha\, c_2^{\alpha(1-\sigma)-1}\, l_2^{(1-\alpha)(1-\sigma)} = \frac{\lambda}{1+r}$$

(l_2): $$\beta(1-\alpha)\, c_2^{\alpha(1-\sigma)}\, l_2^{(1-\alpha)(1-\sigma)-1} = \frac{\lambda w_2}{1+r}$$

Take ratio of these FOC's to obtain

$$w_2 l_2 = \frac{1-\alpha}{\alpha} c_2 \Rightarrow l_2 = \frac{1-\alpha}{\alpha}\frac{c_2}{w_2}$$

Use the FOCs for c_1 & c_2 to obtain the following Euler Equation (take the ratio of the two FOCs):

$$c_1^{\alpha(1-\sigma)-1} l_1^{(1-\alpha)(1-\sigma)} = \beta(1+r)\, c_2^{\alpha(1-\sigma)-1} l_2^{(1-\alpha)(1-\sigma)}$$

Insert l_1 as a fnc. of c_1 & l_2 as a fnc of c_2 obtained in the previous page to the Euler Equation above.

$$c_1^{\alpha(1-\sigma)-1}\left(\frac{1-\alpha}{\alpha}\frac{c_1}{w_1}\right)^{(1-\alpha)(1-\sigma)} = \beta(1+r)\, c_2^{\alpha(1-\sigma)-1}\cdot\left(\frac{1-\alpha}{\alpha}\frac{c_2}{w_2}\right)^{(1-\alpha)(1-\sigma)}$$

$$c_1^{-\sigma}\cdot\frac{1}{w_1^{(1-\alpha)(1-\sigma)}} = \beta(1+r)\, c_2^{-\sigma}\frac{1}{w_2^{(1-\alpha)(1-\sigma)}}$$

$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \beta(1+r)\left(\frac{w_1}{w_2}\right)^{(1-\alpha)(1-\sigma)}$$

$$c_2 = (\beta(1+r))^{1/\sigma}\left(\frac{w_1}{w_2}\right)^{\frac{(1-\alpha)(1-\sigma)}{\sigma}}\cdot c_1$$

Note that LBC reads

$$c_1 + \frac{c_2}{1+r} + w_1 l_1 + \frac{w_2 l_2}{1+r} = w_1 + \frac{w_2}{1+r} = I(r)$$

$$c_1 + \frac{c_2}{1+r} + \frac{1-\alpha}{\alpha}c_1 + \frac{1}{1+r}\frac{1-\alpha}{\alpha}c_2 = I(r)$$

$$\Rightarrow c_1 + \frac{c_2}{1+r} = \alpha\cdot I(r)$$

Set $w_1 = w_2 \Rightarrow c_2 = (\beta(1+r))^{1/\sigma} c_1$

$$c_1 + \frac{1}{1+r}c_1(\beta(1+r))^{1/\sigma} = \alpha I(r)$$

$$c_1\left(1+\beta^{1/\sigma}(1+r)^{\frac{1}{\sigma}-1}\right) = \alpha I(r)$$

$$c_1 = \frac{\alpha I(r)}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}$$

$$c_2 = \frac{\alpha(\beta(1+r))^{1/\sigma}}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\cancel{\alpha} I(r)$$

$$l_1 = \frac{1-\alpha}{\alpha}\frac{c_1}{w_1} = \frac{(1-\alpha)}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\left(1+\frac{1}{1+r}\right)$$

$$l_2 = \frac{1-\alpha}{\alpha}\frac{c_2}{w_2} = \frac{(1-\alpha)\beta(1+r)^{1/\sigma}}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\left(1+\frac{1}{1+r}\right)$$