Auxiliar #12

•) Background algebra lineal :

Sea $\vec{u} \in \mathbb{R}^n$, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \Rightarrow \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

≥ 0

Recuerdo "$\|\cdot\|$" norma eucliedeana es una función o aplicación que recibe un vecto y devuelve un valor real mayor o igual 0

•) Producto punto : Si $\vec{u}, \vec{v} \in \mathbb{R}^n$

$\Rightarrow$ 1) $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$

2) $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2 = \|u\|^2$

3) Si θ es el ángulo que hay entre $\vec{u}$ y $\vec{v}$

$$\Rightarrow \quad \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

En consecuencia, si $\vec{u} \perp \vec{v}$ $(\theta = 90° = \pi/2)$

$$\Rightarrow \quad \vec{u} \cdot \vec{v} = 0$$

El producto punto devuelve un real cualquiera puede ser <0, 0, >0

•) Producto vectorial: si $\vec{u}, \vec{v} \in \mathbb{R}^3$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} , \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Rightarrow \quad \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

$\Rightarrow$ 1) $\vec{u} \times \vec{u} = \vec{0}$

2) $\| \vec{u} \times \vec{v} \| = \| \vec{u} \| \cdot \| \vec{v} \| \cdot \operatorname{sen}(\theta)$

↳ Consecuencia, si $\vec{u}$ y $\vec{v}$ son paralelos

$\Rightarrow \| \vec{u} \times \vec{v} \| = 0 \iff \vec{u} \times \vec{v} = \vec{0}$

Es decir si $\vec{u} \parallel \vec{v} \Rightarrow \vec{u} \times \vec{v} = \vec{0}$

en particular $\vec{u} \times \vec{u} = \vec{0}$

3) (Anticonmutativa) $\vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$

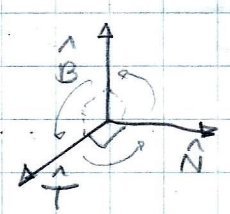
Si un vector en vez de escribirse $\vec{x}$ se escribe como $\hat{x}$

$$\Rightarrow \hat{x} = \frac{\vec{x}}{\|\vec{x}\|} \Big\} \text{ vector normalizado}$$

$\downarrow$

vector de norma 1

Aplicación a Frenet



Regla de la mano derecha

$T \times N = B$
$N \times T = -B$

$B \times T = N$
$T \times B = -N$

$N \times B = T$
$B \times N = -T$

$\boxed{P1}$ a) pdq: Γ es regular $\Leftarrow$ $\vec{r}(t)$ es C^1 y $\left\| \frac{d\vec{r}(t)}{dt} \right\| > 0$

$$\Rightarrow r(t) = \begin{pmatrix} \int_0^t \phi(u)\,\mathrm{sen}(u)\,du \\ \int_0^t \phi(u)\cos(u)\,du \\ \int_0^t \phi(u)\tan(u)\,du \end{pmatrix}$$

Cada componente es derivable por TFC

$$\Rightarrow \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} \phi(t)\,\text{sen}(t) \\ \phi(t)\cos(t) \\ \phi(t)\tan(t) \end{pmatrix} \Bigg\}$$

Cada componente es continua

Luego $\vec{r}$ es C^1

falta ver que $\left\| \frac{d\vec{r}}{dt}(t) \right\| > 0$

$$\Rightarrow \left\| \frac{d\vec{r}(t)}{dt} \right\| = \sqrt{(\phi(t))^2 (\text{sen}^2(t) + \cos^2(t) + \tan^2 t)}$$

$$= |\phi(t)| \cdot \sqrt{1 + \tan^2(t)}$$

$$= |\phi(t)| \cdot |\sec(t)|$$

$$= \phi(t)\sec(t)$$

$$> 0$$

Por enunciado $\phi(t) > 0$

y $\sec(t) \neq 0$ siempre

y entre 0 y $\frac{\pi}{2}$ $\sec(t) > 0$

$\therefore \Gamma$ es regular pues $\vec{r}$ es C^1 y $\left\| \frac{d\vec{r}}{dt}(t) \right\| > 0$

$$\rightarrow \hat{T}(t) = \frac{\frac{d\vec{r}(t)}{dt}}{\left\| \frac{d\vec{r}(t)}{dt} \right\|} = \frac{\cancel{\phi(t)} \begin{pmatrix} \text{sen}(t) \\ \cos(t) \\ \tan(t) \end{pmatrix}}{\cancel{\phi(t)}\sec(t)} = \cos(t) \begin{pmatrix} \text{sen}(t) \\ \cos(t) \\ \tan(t) \end{pmatrix}$$

$$= \begin{pmatrix} \text{sen}(t)\cos(t) \\ \cos^2(t) \\ \text{sen}(t) \end{pmatrix}$$

MÁS FACIL DE DERIVAR $\longrightarrow$ $= \begin{pmatrix} \frac{1}{2}\text{sen}(2t) \\ \cos^2(t) \\ \text{sen}(t) \end{pmatrix}$

$$N(t) = \frac{\frac{dT}{dt}(t)}{\left\|\frac{dT}{dt}(t)\right\|} = \frac{\begin{pmatrix} \cos(2t) \\ -\operatorname{sen}(2t) \\ \cos(t) \end{pmatrix}}{\sqrt{\cos^2(2t) + \operatorname{sen}^2(2t) + \cos^2(t)}}$$

$$= \frac{\begin{pmatrix} \cos(2t) \\ -\operatorname{sen}(2t) \\ \cos(t) \end{pmatrix}}{\sqrt{1+\cos^2(t)}} = \frac{\begin{pmatrix} \cos^2(t) - \operatorname{sen}^2(t) \\ -2\operatorname{sen}(t)\cos(t) \\ \cos(t) \end{pmatrix}}{\sqrt{1+\cos^2(t)}}$$

Usando la notación $\cos(t) = c$, $\operatorname{sen}(t) = s$

$$= \begin{pmatrix} c^2 - s^2 \\ -2sc \\ c \end{pmatrix} \cdot (1+c^2)^{-\frac{1}{2}}$$

$$\rightarrow K(t) = \frac{\sqrt{1+\cos^2(t)}}{\phi(t)\sec(t)} = \frac{\cos(t)\sqrt{1+\cos^2(t)}}{\phi(t)} = \frac{\left\|\frac{dT}{dt}(t)\right\|}{\left\|\frac{d\vec{r}}{dt}(t)\right\|}$$

$$\Rightarrow \text{Si } k(t) = 1 \Rightarrow \phi(t) = \cos(t)\sqrt{1+\cos^2(t)}$$

b) $\hat{B} = \hat{T} \times \hat{N}$

$$= \begin{pmatrix} \text{Sen}(t)\cos(t) \\ \cos^2(t) \\ \text{Sen}(t) \end{pmatrix} \times \left[\begin{pmatrix} \cos^2(t) - \text{Sen}^2(t) \\ -2\,\text{sen}(t)\cos(t) \\ \cos(t) \end{pmatrix} \frac{1}{\sqrt{1+\cos^2(t)}} \right]$$

Notación
$\text{sen}(t) = s$
$\cos(t) = c$

$$= \begin{pmatrix} sc \\ c^2 \\ s \end{pmatrix} \times \left[\begin{pmatrix} c^2 - s^2 \\ -2sc \\ c \end{pmatrix} (1+c^2)^{-1/2} \right]$$

factorizo

$$= (1+c^2)^{-1/2} \left[\begin{pmatrix} sc \\ c^2 \\ s \end{pmatrix} \times \begin{pmatrix} c^2 - s^2 \\ -2sc \\ c \end{pmatrix} \right]$$

$$= (1+c^2)^{-1/2} \begin{pmatrix} c^3 - (-2s^2c) \\ \cancel{sc^2} - s^3 - (\cancel{sc^2}) \\ -2s^2c^2 - (c^4 + s^2c^2) \end{pmatrix}$$

$$= (1+c^2)^{-1/2} \begin{pmatrix} c^3 + 2s^2c \\ -s^3 \\ -c^4 - s^2c^2 \end{pmatrix}$$

Calculemos la torsión τ sabiendo que

$$\frac{dB}{dt} = s(1+c^2)^{-3/2}\begin{pmatrix}(c^2-s^2)(2+c^2)\\ -2sc(2+c^2)\\ c(2+c^2)\end{pmatrix}$$

$$= s(1+c^2)^{-3/2}(2+c^2)\underbrace{\begin{pmatrix}c^2-s^2\\ -2sc\\ c\end{pmatrix}}_{\substack{\text{¿Quien es? :O}\\ \text{es } \hat{N} \text{ sin la "constante"}}}$$

$$= s(1+c^2)^{-3/2}(2+c^2)\cdot\hat{N}\,(1+c^2)^{1/2}$$

$$= s(1+c^2)^{-1}(2+c^2)\cdot\hat{N}$$

$$\rightarrow \tau = -\hat{N}(t)\bullet\frac{\frac{dB}{dt}}{\left\|\frac{d\vec{r}(t)}{dt}\right\|} = -\hat{N}(t)\bullet\frac{s(1+c^2)^{-1}(2+c^2)\hat{N}(t)}{\phi(t)\sec(t)}$$

Producto punto

$$= -\frac{s(1+c^2)^{-1}(2+c^2)}{\phi(t)\sec(t)}\left[\hat{N}(t)\cdot\hat{N}(t)\right]$$

factorizar

$$= -\frac{s(1+c^2)^{-1}(2+c^2)}{\phi(t)\sec(t)}\cancel{\|\hat{N}(t)\|^2}\,1^2$$

$$= -\frac{s(1+c^2)^{-1}(2+c^2)\,c}{\phi(t)}$$

$$\therefore\ \tau = -\frac{5c(1+c^2)^{-1}(2+c^2)}{\phi(t)}$$

Notar que $\|\hat{N}\| = 1$ pues $\hat{N}$ es unitario

Si $\tau = -1$

$$\Rightarrow \phi(t) = 5c(1+c^2)^{-1}(2+c^2)$$

$\boxed{P2}$ $\rho(x,y,z) = 2x + 9z$, $\vec{r}(t) = (t, t^2, t^3)$

$$\rightarrow M = \int_\Gamma \rho\, d\ell = \int_0^3 \rho(\vec{r}(t)) \left\| \frac{d\vec{r}(t)}{dt} \right\| dt$$

$$= \int_0^3 2t + 9t^3 \left\| \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \right\| dt$$

$$= \int_0^3 2t + 9t^3 \sqrt{1 + 4t^2 + 9t^4}\, dt$$

$u = 1 + 4t^2 + 9t^4$

$du = 8t + 36t^3$

$= 4(2t + 9t^3)$

$$= \int_{u(0)}^{u(3)} \frac{\sqrt{u}}{4} du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_{u(0)}^{u(3)}$$

$$= \frac{1}{4} \cdot \frac{2}{3} \left(1 + 4t^2 + 9t^4\right)\Big|_0^3$$

$$= \frac{1}{6}\left((766)^{3/2} - 1\right)$$

$$= M$$

el centro de masa queda expresado como

$$X_G = \frac{1}{M} \int_\Gamma x \rho \, dl = \frac{1}{M} \int_0^3 t(2t+9t^3)\sqrt{1+4t^2+9t^4}\, dt$$

$$Y_G = \frac{1}{M} \int_\Gamma y \rho \, dl = \frac{1}{M} \int_0^3 t^2(2t+9t^3)\sqrt{1+4t^2+9t^4}\, dt$$

$$Z_G = \frac{1}{M} \int_\Gamma z \rho \, dl = \frac{1}{M} \int_0^3 t^3(2t+9t^3)\sqrt{1+4t^2+9t^4}\, dt$$

P3 pdq: $\frac{dr}{ds} \cdot \left(\frac{d^2r}{ds} \times \frac{d^3r}{ds^3} \right) = \tau k^2$

r es lo que en el apunte sale como σ

$\Rightarrow \frac{dr}{ds} = T \quad , \quad \frac{d^2r}{ds} = \frac{d}{ds}\left(\frac{dr}{ds}\right)$, obs: $K \in \mathbb{R}$

P. punto

$$\Rightarrow \frac{dr}{ds} \cdot \left(\frac{d^2r}{ds} \times \frac{d^3r}{ds^3} \right) = T \cdot \left(\frac{dT}{ds} \times \frac{d^2T}{ds} \right)$$

$$= T \cdot \left(\underbrace{\frac{dT}{ds} \cdot \frac{\overbrace{\left\|\frac{dT}{ds}\right\|}^{K}}{\left\|\frac{dT}{ds}\right\|}}_{N} \times \frac{d^2T}{ds^2} \right)$$

$$= T \cdot \left(NK \times \frac{d}{ds}(NK) \right)$$

5

$$\Rightarrow \quad \ldots = \hat{T} \cdot \left(\hat{N}\kappa \times \frac{d}{ds}(\hat{N}\kappa) \right)$$

Regla del producto

$$= \hat{T} \cdot \left(\hat{N}\kappa \times \left(\kappa \frac{d\hat{N}}{ds} + \hat{N} \frac{d\kappa}{ds} \right) \right)$$

Distribución

$$= \hat{T} \cdot \left[\left(\hat{N}\kappa \times \kappa \frac{d\hat{N}}{ds} \right) + \left(\hat{N}\kappa \times \hat{N} \frac{d\kappa}{ds} \right) \right]$$

Saca κ como cte y formula de frenet

$$= \hat{T} \cdot \left[\kappa^2 \left(\hat{N} \times (-\kappa\hat{T} + \tau\hat{B}) \right) + \kappa \frac{d\kappa}{ds} \underbrace{\cancel{(\hat{N} \times \hat{N})}}_{\vec{0}} \right]$$

$$= \hat{T} \cdot \left\{ \kappa^2 \left[-\kappa (\hat{N} \times \hat{T}) + \tau (\hat{N} \times \hat{B}) \right] \right\}$$

ver BACK GROUND

$$= \hat{T} \cdot \left\{ \kappa^2 \left[-\kappa (-\hat{B}) + \tau \hat{T} \right] \right\}$$

$$= \kappa^2 \left\{ \hat{T} \cdot \left[\kappa\hat{B} + \tau \hat{T} \right] \right\}$$

Distrib

$$= \kappa^2 \left\{ \kappa (\hat{T} \cdot \hat{B}) + \tau (\hat{T} \cdot \hat{T}) \right\} \qquad T \perp B$$

Ver BACK GROUND

$$= \kappa^2 \left\{ \cancel{\kappa \cdot 0} + \tau \, \underbrace{\cancel{\|\hat{T}\|^2}}_{1} \right\} \qquad T \parallel T$$

$$= \kappa^2 \cdot \tau \quad \blacksquare$$