PAUTA AUXILIAR 4

P1

$$\vec{r}(\sigma,\tau,z) = \left(\sigma\tau, \frac{1}{2}(\tau^2-\sigma^2), z\right)$$

a) $$h_\sigma = \left\|\frac{\partial \vec{r}}{\partial \sigma}\right\| = \|(\tau,-\sigma,0)\| = \sqrt{\tau^2+\sigma^2}$$

$$h_\tau = \left\|\frac{\partial \vec{r}}{\partial \tau}\right\| = \|(\sigma,\tau,0)\| = \sqrt{\tau^2+\sigma^2}$$

$$h_z = \left\|\frac{\partial \vec{r}}{\partial z}\right\| = \|(0,0,1)\| = 1$$

b) $$\hat{\sigma} = \frac{1}{h_\sigma}\frac{\partial \vec{r}}{\partial \sigma} = \left(\frac{\tau}{\sqrt{\tau^2+\sigma^2}}, \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}, 0\right)$$

$$\hat{\tau} = \frac{1}{h_\tau}\frac{\partial \vec{r}}{\partial \tau} = \left(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}}, \frac{\tau}{\sqrt{\tau^2+\sigma^2}}, 0\right)$$

$$\hat{z} = \frac{1}{h_z}\frac{\partial \vec{r}}{\partial z} = (0,0,1)$$

→ veamos que son ortogonales (⟺ producto punto = 0)

- $\hat{\sigma}\cdot\hat{\tau} = \frac{\tau\sigma}{\tau^2+\sigma^2} - \frac{\tau\sigma}{\tau^2+\sigma^2} = 0$
- $\hat{\sigma}\cdot\hat{z} = 0+0+0 = 0$
- $\hat{\tau}\cdot\hat{z} = 0+0+0 = 0$

→ veamos que el orden positivo es $(\hat{\sigma},\hat{\tau},\hat{z})$, es decir, $\hat{\sigma}\times\hat{\tau}=\hat{z}$, $\hat{\tau}\times\hat{z}=\hat{\sigma}$ y $\hat{z}\times\hat{\sigma}=\hat{\tau}$

- $$\hat{\sigma}\times\hat{\tau} = \begin{pmatrix}\frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} \times \begin{pmatrix}\frac{\sigma}{\sqrt{\tau^2+\sigma^2}}\\ \frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{z}$$
- $$\hat{\tau}\times\hat{z} = \begin{pmatrix}\frac{\sigma}{\sqrt{\tau^2+\sigma^2}}\\ \frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} = \hat{\sigma}$$
- $$\hat{z}\times\hat{\sigma} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}\frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} = \begin{pmatrix}\frac{\sigma}{\sqrt{\tau^2+\sigma^2}}\\ \frac{\tau}{\sqrt{\tau^2+\sigma^2}}\\ 0\end{pmatrix} = \hat{\tau}$$

c) Notemos que $\vec{F} = z^2 \begin{pmatrix} \frac{z}{\sqrt{z^2+\sigma^2}} \\ \frac{-\sigma}{\sqrt{z^2+\sigma^2}} \\ 0 \end{pmatrix} + z \begin{pmatrix} \frac{\sigma}{\sqrt{z^2+\sigma^2}} \\ \frac{z}{\sqrt{z^2+\sigma^2}} \\ 0 \end{pmatrix} = z^2 \hat{\sigma} + z \hat{z}$

$$\Rightarrow \operatorname{div}(F) = \frac{1}{h_\sigma h_z h_z} \left[\frac{\partial}{\partial \sigma}\left(z^2 \cdot h_z h_z\right) + \frac{\partial}{\partial z}\left(z h_\sigma h_z\right) \right]$$

$$= \frac{1}{z^2+\sigma^2} \left[\frac{\partial}{\partial \sigma}\left(z^2 \sqrt{z^2+\sigma^2}\right) + \frac{\partial}{\partial z}\left(z\sqrt{z^2+\sigma^2}\right) \right]$$

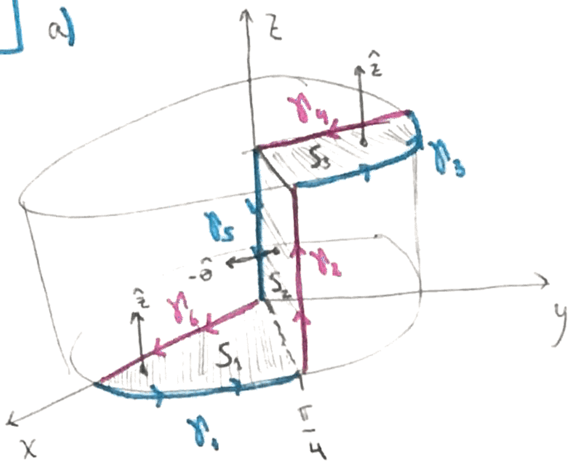
$$= \frac{1}{z^2+\sigma^2} \left[\frac{z^2 \cdot 2\sigma}{2\sqrt{z^2+\sigma^2}} + \frac{z \cdot 2z}{2\sqrt{z^2+\sigma^2}} \right] = \frac{1}{z^2+\sigma^2} \left[\frac{z^2 \sigma}{\sqrt{z^2+\sigma^2}} + \frac{z z}{\sqrt{z^2+\sigma^2}} \right] \quad //$$

$$\operatorname{rot}(F) = \frac{1}{z^2+\sigma^2} \begin{vmatrix} \sqrt{z^2+\sigma^2}\,\hat{\sigma} & \sqrt{z^2+\sigma^2}\,\hat{z} & \hat{z} \\ \frac{\partial}{\partial \sigma} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ z^2\sqrt{z^2+\sigma^2} & z\sqrt{z^2+\sigma^2} & 0 \end{vmatrix}$$

$$= \frac{1}{z^2+\sigma^2} \left[\sqrt{z^2+\sigma^2}\,\hat{\sigma}\left(-\sqrt{z^2+\sigma^2}\right) - \sqrt{z^2+\sigma^2}\,\hat{z}\,(0) + \hat{z}\left(\frac{z\sigma}{\sqrt{z^2+\sigma^2}} - \left(z^2 \cdot \frac{z}{\sqrt{z^2+\sigma^2}} + 2z\sqrt{z^2+\sigma^2} \right) \right) \right]$$

$$= \frac{1}{z^2+\sigma^2} \left[-\left(z^2+\sigma^2\right)\hat{\sigma} + \left(\frac{z\sigma - z^3}{\sqrt{z^2+\sigma^2}} - 2z\sqrt{z^2+\sigma^2} \right) \hat{z} \right] \quad //$$

P2 a)



	t (vector tangente)	F · t
γ_1	$\hat{\theta}$	0
γ_2	$\hat{z}$	ρ^2
γ_3	$\hat{\theta}$	0
γ_4	$-\hat{\rho}$	$-z\rho$
γ_5	$-\hat{z}$	$-\rho^2$
γ_6	$\hat{\rho}$	$z\rho$

$\vec{F} = \rho^2 \hat{z} + z\rho\hat{\rho}$

Pero en γ_5, $\rho = 0 \Rightarrow \vec{F}\cdot\hat{t} = 0$

y en γ_6, $z = 0 \Rightarrow \vec{F}\cdot\hat{t} = 0$

Luego $\displaystyle\int_\gamma \vec{F}\cdot\hat{t}\,dr = \underbrace{\int_{\gamma_1} \vec{F}\cdot\hat{t}\,dr}_{0} + \int_{\gamma_2} \vec{F}\cdot\hat{t}\,dr + \underbrace{\int_{\gamma_3} \vec{F}\cdot\hat{t}\,dr}_{0} + \int_{\gamma_4} \vec{F}\cdot\hat{t}\,dr + \underbrace{\int_{\gamma_5} \vec{F}\cdot\hat{t}\,dr}_{0} + \underbrace{\int_{\gamma_6} \vec{F}\cdot\hat{t}\,dr}_{0}$

$$= \int_{\gamma_2} \vec{F}\cdot\hat{t}\,dr + \int_{\gamma_4} \vec{F}\cdot\hat{t}\,dr = \underbrace{\int_0^H \rho^2\,dz}_{\text{en } \gamma_2 \text{, el parámetro que se mueve es } z} + \underbrace{\int_0^R -z\rho\,d\rho}_{\text{en } \gamma_4 \text{ el parámetro que se mueve es } \rho}$$

como ya está el signo − del $-\hat{\rho}$, los límites son 0 y R, no R y 0.

$$= H\rho^2 - \left(\frac{z\rho^2}{2}\right)\Big|_0^R = H\rho^2 - \frac{zR^2}{2} = \underbrace{HR^2}_{\text{en } \gamma_2,\ \rho = R} - \underbrace{\frac{HR^2}{2}}_{\text{en } \gamma_4,\ z = H} = \frac{1}{2}HR^2$$

b) $\displaystyle\oint_\gamma \vec{F}\cdot\hat{t}\,dr = \int_S \text{rot}(\vec{F})\cdot d\vec{s} = \int_S \text{rot}(\vec{F})\cdot\hat{n}\,ds$

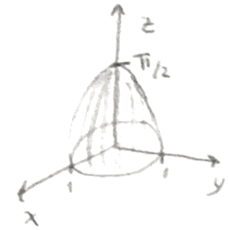
$$\text{rot}(\vec{F}) = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ z\rho & 0 & \rho^2 \end{vmatrix} = \frac{1}{\rho}\left[\hat{\rho}(0) - \rho\hat{\theta}(2\rho - \rho) + \hat{z}(0)\right] = -\rho\hat{\theta}$$

$$\int_S \text{rot}(\vec{F})\cdot\hat{n}\,ds = \int_{S_1} \text{rot}(\vec{F})\cdot\hat{n}\,ds + \int_{S_2} \text{rot}(\vec{F})\cdot\hat{n}\,ds + \int_{S_3} \text{rot}(\vec{F})\cdot\hat{n}\,ds$$

$$= \underbrace{\int_{S_1} -\rho\hat{\theta}\cdot\hat{z}\,ds}_{0} + \int_{S_2} -\rho\hat{\theta}\cdot -\hat{\theta}\,ds + \underbrace{\int_{S_3} -\rho\hat{\theta}\cdot\hat{z}\,ds}_{0} = \int_0^H\int_0^R \rho\,d\rho\,dz = \frac{HR^2}{2}$$

P3 a) $\vec{r}(\theta,z) = (\cos(z)\cos(\theta),\ \cos(z)\,\mathrm{sen}(\theta),\ z)$ $\quad \theta \in [0,2\pi]\quad z\in[0,\pi/2]$

$= \cos(z)\,\hat{\rho} + z\,\hat{z} \qquad \theta\in[0,2\pi],\ z\in[0,\pi/2]$



b) Queremos usar Gauss, pero S no es una superficie cerrada

I. La cerraremos con una tapa T, que será un círculo de radio 1 y altura 0.

$T: (\rho\cos(\theta),\ \rho\,\mathrm{sen}(\theta),\ 0) \overset{\rho=1}{=} (\cos\theta, \mathrm{sen}\,\theta, 0) \quad \theta\in[0,2\pi]$

$\Rightarrow \iint_{S\cup T} \vec{F}\cdot\hat{n}\,dS = \iiint_V \mathrm{div}(F)\,dV$

$\mathrm{div}(F) = \frac{1}{r^2\,\mathrm{sen}\phi}\left[\frac{\partial}{\partial\theta}\left((\exp(r^3)+\cos\phi)\cdot r\right)\right] = 0$

$\Rightarrow \iint_{S\cup T}\vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot\hat{n}\,dS + \iint_T \vec{F}\cdot\hat{n}\,dS = 0$

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = -\iint_T \vec{F}\cdot\hat{n}\,dS$

La normal exterior de la tapa es $-\hat{z}$.

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = -\iint_S (\exp(r^3)+\cos\phi)\,\hat{\theta}\cdot -\hat{z}\,dS = 0$

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = 0$ //

II. Usaremos una tapa T, que será la mitad de abajo de una esfera de radio 1

$T: (r\,\mathrm{sen}\phi\cos\theta,\ r\,\mathrm{sen}\phi\,\mathrm{sen}\theta,\ r\cos\phi) \overset{r=1}{=} (\mathrm{sen}\phi\cos\theta,\ \mathrm{sen}\phi\,\mathrm{sen}\theta,\ \cos\phi)$ $\quad \theta\in[0,2\pi]$, $\phi\in[\pi/2,\pi]$

Por la aplicación de Stokes en el campo eléctrico se tiene que

$\iint_{S\cup T}\vec{F}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0} \quad \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0} - \iint_T \vec{F}\cdot\hat{n}\,dS$

La normal exterior de la tapa es $\hat{r}$

$\Rightarrow \iint_T \vec{F}\cdot\hat{n}\,dS = \int_{\pi/2}^{\pi}\int_0^{2\pi} \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\cdot\hat{r}\,r^2\,\mathrm{sen}\phi\,d\theta\,d\phi = \frac{Q}{2\varepsilon_0}\int_{\pi/2}^{\pi}\mathrm{sen}\phi\,d\phi = \frac{Q}{2\varepsilon_0}(-\cos\phi)\Big|_{\pi/2}^{\pi} = \frac{Q}{2\varepsilon_0}$

Luego $\iint_S \vec{F}\cdot\hat{n}\, dS = \frac{Q}{\varepsilon_0} - \frac{Q}{2\varepsilon_0} = \frac{Q}{2\varepsilon_0}$ //.