PAUTA AUXILIAR 13

P1 a) $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$

caso 1: $n=m$

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}dx = \int_{-\pi}^{\pi} \frac{1}{2}\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cancelto{0}{\cos(2nx)\,dx} = \frac{1}{2}x\Big|_{-\pi}^{\pi}$$

(pq' al integrar quedará $\text{sen}(kx)$ con algún $k\in\mathbb{Z}$ y $\text{sen}(kx)=0$ $\forall k$) (*)

$$= \frac{1}{2}(\pi - -\pi) = \pi \quad //$$

caso 2: $n \neq m$

notemos que $\int_{-\pi}^{\pi} \cos((n-m)x)\,dx = 0$ (mismo argumento que (*))

y además: $$\int_{-\pi}^{\pi} \cos((n-m)x)\,dx = \underbrace{\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx}_{I} + \int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx)\,dx$$ (identidad trigonométrica)

calculemos I integrando por partes

$$\int_{-\pi}^{\pi} \underset{\substack{f\\(u)}}{\cos(nx)}\,\underset{\substack{g'\\(dv)}}{\cos(mx)}\,dx = \underset{\substack{f\cdot g\\(u\ v)}}{\cancelto{0}{\frac{\cos(nx)\,\text{sen}(mx)}{m}}}\Big|_{-\pi}^{\pi} + \frac{n}{m}\underset{\substack{\int f'g\\(\int v\,du)}}{\int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx)\,dx}$$

$(\text{sen}(k\pi)=0)$

$$\Rightarrow \int_{-\pi}^{\pi} \cos((n-m)x)\,dx = 0 = \underbrace{\left(1+\frac{n}{m}\right)}_{\neq 0}\int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx)\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx) = 0$$

$$\Rightarrow 0 = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx + \int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx)\,dx = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0 \quad //$$

$$\therefore \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} \pi & \text{si } n=m \\ 0 & \text{si } n\neq m \end{cases} \quad //$$

b) $\int_{-\pi}^{\pi} sen(nx) sen(mx)\, dx$

caso 1: $n=m$

$$\int_{-\pi}^{\pi} sen^2(nx)\, dx = \int_{-\pi}^{\pi} 1 - \cos^2(nx)\, dx = \int_{-\pi}^{\pi} 1\, dx - \int_{-\pi}^{\pi} \overset{\pi \text{ (por (a))}}{\cancel{\cos^2(nx)\, dx}} = 2\pi - \pi = \pi \, //$$

caso 2: $n \neq m$

$$\int_{-\pi}^{\pi} sen(nx)\, sen(mx) = 0 \, // \quad \text{(en Desarrollo De parte (a))}$$

c) $\int_{-\pi}^{\pi} sen(nx) \cos(mx)\, dx$

Notemos que $sen(nx)$ es impar y $\cos(mx)$ es par, por lo que $sen(nx)\cos(mx)$ es impar.

$$\Rightarrow \int_{-\pi}^{\pi} sen(nx) \cos(mx) = 0 \, // \quad \text{(Función impar entre } -\ell \text{ y } \ell\text{)}$$

P2) recordemos que $\langle f, g \rangle = \int_{-\ell}^{\ell} f \cdot \bar{g}$

calculemos la serie de Fourier de f que es de la forma:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n sen(nx)$$

- $$a_0 = \frac{\langle f, \overset{1}{\cos(0)} \rangle}{\langle \cos(nx), \cos(nx) \rangle_{\to \pi \text{ (por P1)}}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{0}^{\pi} x\, dx = \frac{1}{\pi} \frac{x^2}{2} \Big|_0^{\pi} = \frac{\pi}{2} \, //$$

- $$a_n = \frac{\langle f, \cos(nx) \rangle}{\langle \cos(nx), \cos(nx) \rangle_{\to \pi}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \int_{0}^{\pi} \underset{\substack{f \\ (u)}}{x}\, \underset{\substack{g' \\ (dv)}}{\cos(nx)}$$

integramos por partes

$$\frac{1}{\pi} \Bigg[\underset{\substack{f \cdot g \\ (u \cdot v)}}{\overset{\to 0 \; (sen(k\pi)=0)}{\cancel{\frac{x\, sen(nx)}{n}}}} \Big|_0^{\pi} - \underset{\substack{\int f' g \\ (\int v\, du)}}{\int_0^{\pi} \frac{sen(nx)}{n}\, dx} \Bigg] = \frac{-1}{\pi n^2} (-\cos(nx)) \Big|_0^{\pi}$$

$$= \frac{-1}{\pi n^2} (-\cos(n\pi) + 1) = \frac{-1}{\pi n^2} (-(-1)^n + 1) = \frac{-1}{\pi n^2} ((-1)^{n+1} + 1) \, //$$

- $$b_n = \frac{\langle f, sen(nx) \rangle}{\langle sen(nx), sen(nx) \rangle_{\to \pi \text{ (por P1)}}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, sen(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} \underset{f}{x}\, \underset{g'}{sen(nx)} = \frac{1}{\pi} \Bigg[\frac{-x\cos(nx)}{n} \Big|_0^{\pi} + \overset{sen(k\pi)= \; 0}{\cancel{\int_0^{\pi} \frac{\cos(nx)}{n}\, dx}} \Bigg]$$

$$= \frac{1}{\cancel{\pi}} \left(\frac{-\cancel{\pi} \cos(\pi n)}{n} \right) = \frac{(-1)^{n+1}}{n} \, //$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \underbrace{\frac{1}{\pi n^2}\left((-1)^{n+1}+1\right)}_{\substack{=0 \text{ si } n \text{ par} \\ =\frac{2}{\pi n^2} \text{ si } n \text{ impar}}} \cos(nx) + \frac{(-1)^{n+1}}{n} \operatorname{sen}(nx)$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n \text{ impar}} \frac{2}{\pi n^2} \cos(nx) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{sen}(nx)$$

$$\Rightarrow f(0) = 0 = \frac{\pi}{4} + \sum_{n \text{ impar}} \frac{2}{\pi n^2}$$

$$\Rightarrow \sum_{n \text{ impar}} \frac{1}{n^2} = \frac{\pi^2}{8} \quad //$$

P4) Para calcular la serie de Fourier necesitamos que la función esté definida en un intervalo $[-\ell, \ell]$. Además, como nos piden la expansión en senos, necesitamos que sea impar. Luego extendemos f de forma tal que sea impar

$$\tilde{f} = \begin{cases} f(x) & \text{si } x \in [0,\pi] \\ -f(-x) & \text{si } x \in [-\pi, 0) \end{cases}$$

Notemos que f es impar y $\tilde{f}\big|_{[0,\pi]} = f$

Calculemos la serie de Fourier de $\tilde{f}$ en $[-\pi, \pi]$ y será la misma serie para f en $[0,\pi]$.

Como $\tilde{f}$ es impar, su serie es de la forma $\sum_{n=1}^{\infty} b_n \operatorname{sen}(nx)$

$$b_n = \frac{\langle \tilde{f}, \operatorname{sen}(nx) \rangle}{\underbrace{\langle \operatorname{sen}(nx), \operatorname{sen}(nx) \rangle}_{\pi \text{ (Por P1)}}} = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underset{\text{impar}}{\tilde{f}} \; \underset{\text{impar}}{\operatorname{sen}(nx)}}_{\text{par}} \, dx = \frac{2}{\pi} \int_0^{\pi} \tilde{f} \operatorname{sen}(nx) \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f \operatorname{sen}(nx) \, dx = \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \operatorname{sen}(nx) \, dx + \int_{\pi/2}^{\pi} 2 \operatorname{sen}(nx) \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos(nx)}{n} \Big|_0^{\pi/2} + 2 \left(\frac{-\cos(nx)}{n} \right) \Big|_{\pi/2}^{\pi} \right] = \frac{2}{\pi} \left[\frac{-\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{1}{n} + 2 \left(\frac{-(-1)^n}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n} \right) \right]$$

$$= \frac{2}{\pi n} \left[\cos\left(\frac{n\pi}{2}\right) + 1 + 2(-1)^{n+1} \right]$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \operatorname{sen}(nx) \left(\cos\left(\frac{n\pi}{2}\right) + 1 + 2(-1)^{n+1} \right) \quad //$$

$$S_N = \sum_{n=1}^{N} b_n \operatorname{sen}(nx) . \qquad S_N \rightarrow S_f = \begin{cases} f(x) & \text{si } x \in (0, \pi/2) \cup (\frac{\pi}{2}, \pi) \\ \frac{3}{2} & \text{si } x = \frac{\pi}{2} \quad \text{(La mitad del salto)} \\ 0 & \text{si } x = 0 \ \vee \ x = \pi \end{cases}$$