Pauta Aux 1

P1) Queremos estimar $\theta = \mathbb{P}(\text{"Salga cara"})$

$X = 1$ si es cara

$X = 0$ si es sello $\Rightarrow X \sim \text{Bernoulli}(\theta)$

"Espacio muestral" $:= \mathcal{X} = \{0, 1\}$

"Espacio di parámetros" $:= \Theta = [0,1] \ni \theta$

"Espacio de probabilidades" $:= \mathcal{P} = \{\mathbb{P}_\theta : \theta \in [0,1]\}$ tal que $\mathbb{P}_\theta(X=1) = \theta$

$\mathbb{P}_\theta(X=0) = 1-\theta$

Ejemplo de estimador: $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$

P2) Sea $X = (X_1, \ldots, X_n)$ M.A.S. tal que $X_i \sim \mathcal{U}(0,\alpha)$; $\alpha > 0$ desconocido

a) $\hat{\alpha} = 2\bar{X}$. Demostrar que $\hat{\alpha}$ es insesgado

$$\mathbb{E}_\alpha(\hat{\alpha}) = 2\,\mathbb{E}_\alpha(\bar{X}) = \frac{2}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{2}{n}\cdot n\,\frac{\alpha}{2} = \alpha$$

$$ECM_\alpha(\hat{\alpha}) \overset{\hat{\alpha}\text{ insesgado}}{=} \text{Var}(\hat{\alpha}) = \frac{4}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{4}{n}\,\frac{\alpha^2}{12} = \frac{\alpha^2}{3n}$$

Error cuadrático medio

b) $\tilde{\alpha} = \underset{i \in \{1,\ldots,n\}}{\text{Max}}\{X_i\}$ Demostrar que es sesgado

$$\mathbb{E}_\alpha(\tilde{\alpha}) \overset{(*)}{=} \int_0^\alpha \mathbb{P}(\tilde{\alpha} > t)\,dt$$

(*) El resultado general dice

$E(X) = \int_0^\infty \mathbb{P}(X > t)\,dt$ si $X \geq 0$

¿Cómo llegar a esto?

$$= \int_0^\alpha dt - \int_0^\alpha \mathbb{P}(\tilde{\alpha} \leq t)\, dt$$

$$= \alpha - \int_0^\alpha \prod_{i=1}^n \mathbb{P}(X_i \leq t)\, dt$$

$$= \alpha - \int_0^\infty \left(\frac{t}{\alpha}\right)^n dt$$

$$= \alpha - \frac{t^{n+1}}{(n+1)\alpha^n} dt = \alpha\left(\frac{n}{n+1}\right) \neq \alpha \quad \forall n \in \mathbb{N}$$

Definiendo $\tilde{\tilde{\alpha}} = \frac{n+1}{n}\tilde{\alpha} \Rightarrow \mathbb{E}_\alpha(\tilde{\tilde{\alpha}}) = \alpha$ y es insesgado.

c) Calculemos $ECM_\alpha(\tilde{\alpha})$

$$ECM_\alpha(\tilde{\alpha}) = \mathbb{E}_\alpha((\tilde{\alpha} - \alpha)^2)$$

$$= \int_0^\alpha (t-\alpha)^2 f_{\tilde{\alpha}}(t)\, dt \qquad (\star) \text{ Recordando que } \mathbb{P}(\tilde{\alpha} \leq t) = F_{\tilde{\alpha}}(t)$$

$$\overset{(\star)}{=} \int_0^\alpha (t-\alpha)^2 \frac{n\, t^{n-1}}{\alpha^n} dt$$

$$= \int_0^\alpha (t^2 - 2t\alpha + \alpha^2) \frac{n\, t^{n-1}}{\alpha^n} dt$$

$$= \frac{n}{\alpha^n} \int_0^\alpha (t^{n+1} - 2\alpha t^n + \alpha^2 t^{n-1})\, dt$$

$$= \frac{2\alpha^2}{(n+1)(n+2)}$$

Luego $ECM_\lambda(\tilde{\lambda}) < ECM_\lambda(\hat{\lambda})$ y por tanto $\tilde{\lambda}$ es mejor que $\hat{\lambda}$ en el sentido del error cuadrático medio.

P3) Sea $X = (X_1, \ldots, X_n)$; $X_i \sim Poisson(\lambda)$. Interesa estimar

$$g(\lambda) = e^{-\lambda} = \mathbb{P}_\lambda(X_1 = 0). \text{ Considerar}$$

$$g_1(X) = e^{-\overline{X}} \quad g_2(X) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i=0\}} \quad g_3(X) = \left(1-\frac{1}{n}\right)^{n\overline{X}}$$

¿Cuáles son insesgados?

i) $$\mathbb{E}_\lambda(g_1) = \mathbb{E}_\lambda(e^{-\overline{X}})$$

$$= \sum_{x_n=0}^{\infty} \cdots \sum_{x_1=0}^{\infty} e^{-\frac{1}{n}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$= e^{-n\lambda} \sum_{x_n=0}^{\infty} \frac{e^{-\frac{x_n}{n}}\lambda^{x_n}}{x_n!} \cdots \sum_{x_1=0}^{\infty} \frac{e^{-\frac{x_1}{n}}\lambda^{x_1}}{x_1!}$$

$$= e^{-n\lambda} \prod_{i=1}^{n} e^{\lambda/e^n} = e^{-n\lambda} \cdot e^{n\lambda/e^n} \neq e^{-\lambda}$$

$\therefore$ g_1 es sesgado.

ii) $$\mathbb{E}_\lambda(g_2) = \mathbb{E}_\lambda\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i=0\}}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{\{X_i=0\}})$$

$$\overset{(*)}{=} \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(X_i = 0)$$

$$= \frac{1}{\cancel{n}} \cancel{n} e^{-\lambda} = e^{-\lambda}$$

(*) Recordar que si A evento $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$

$\therefore$ g_2 es insesgado.

iii) $\mathbb{E}_\lambda(g_3) = \mathbb{E}_\lambda\left(\left(1-\frac{1}{m}\right)^{m\bar{X}}\right)$

$$= \mathbb{E}_\lambda\left(\left(1-\frac{1}{m}\right)^{\sum_{i=1}^{m} X_i}\right)$$

$$= \sum_{x_m=0}^{\infty} \cdots \sum_{x_1=0}^{\infty} \left(1-\frac{1}{m}\right)^{\sum_{i=1}^{m} x_i} \prod_{i=1}^{m} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$= \left(\sum_{x_m=0}^{\infty} \cdots \sum_{x_1=0}^{\infty} \prod_{i=1}^{m} \frac{\left[\lambda\left(1-\frac{1}{m}\right)\right]^{x_i}}{x_i!}\right) e^{-m\lambda}$$

$$= \left(\prod_{i=1}^{m} \sum_{x=1}^{\infty} \frac{\left[\lambda\left(1-\frac{1}{m}\right)\right]^{x}}{x!}\right) e^{-m\lambda}$$

$$= \prod_{i=1}^{m} e^{\lambda\left(\frac{m-1}{m}\right)} e^{-m\lambda} = e^{-\lambda}$$

$\therefore g_3$ es insesgado.